

Thomas Calculus Early Transcendentals 14th Edition Hass **SOLUTIONS** **MANUAL**

CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{28-9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$
2. (a) $\frac{\Delta g}{\Delta x} = \frac{g(3)-g(1)}{3-1} = \frac{3-(-1)}{2} = 2$ (b) $\frac{\Delta g}{\Delta x} = \frac{g(4)-g(-2)}{4-(-2)} = \frac{8-8}{6} = 0$
3. (a) $\frac{\Delta h}{\Delta t} = \frac{h(\frac{3\pi}{4})-h(\frac{\pi}{4})}{\frac{3\pi}{4}-\frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h(\frac{\pi}{2})-h(\frac{\pi}{6})}{\frac{\pi}{2}-\frac{\pi}{6}} = \frac{0-\sqrt{3}}{\frac{\pi}{3}} = -\frac{3\sqrt{3}}{\pi}$
4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(0)}{\pi-0} = \frac{(2-1)-(2+1)}{\pi-0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(-\pi)}{\pi-(-\pi)} = \frac{(2-1)-(2-1)}{2\pi} = 0$
5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2)-R(0)}{2-0} = \frac{\sqrt{8+1}-1}{2} = \frac{3-1}{2} = 1$
6. $\frac{\Delta P}{\Delta \theta} = \frac{P(2)-P(1)}{2-1} = \frac{(8-16+10)-(1-4+5)}{1} = 2-2 = 0$
7. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-5)-(2^2-5)}{h} = \frac{4+4h+h^2-5+1}{h} = \frac{4h+h^2}{h} = 4+h$. As $h \rightarrow 0$, $4+h \rightarrow 4 \Rightarrow$ at $P(2, -1)$ the slope is 4.
(b) $y - (-1) = 4(x - 2) \Rightarrow y + 1 = 4x - 8 \Rightarrow y = 4x - 9$
8. (a) $\frac{\Delta y}{\Delta x} = \frac{(7-(2+h)^2)-(7-2^2)}{h} = \frac{7-4-4h-h^2-3}{h} = \frac{-4h-h^2}{h} = -4-h$. As $h \rightarrow 0$, $-4-h \rightarrow -4 \Rightarrow$ at $P(2, 3)$ the slope is -4.
(b) $y - 3 = (-4)(x - 2) \Rightarrow y - 3 = -4x + 8 \Rightarrow y = -4x + 11$
9. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-2(2+h)-3)-(2^2-2(2)-3)}{h} = \frac{4+4h+h^2-4-2h-3-(-3)}{h} = \frac{2h+h^2}{h} = 2+h$. As $h \rightarrow 0$, $2+h \rightarrow 2 \Rightarrow$ at $P(2, -3)$ the slope is 2.
(b) $y - (-3) = 2(x - 2) \Rightarrow y + 3 = 2x - 4 \Rightarrow y = 2x - 7$.
10. (a) $\frac{\Delta y}{\Delta x} = \frac{((1+h)^2-4(1+h))-(1^2-4(1))}{h} = \frac{1+2h+h^2-4-4h-(-3)}{h} = \frac{h^2-2h}{h} = h-2$. As $h \rightarrow 0$, $h-2 \rightarrow -2 \Rightarrow$ at $P(1, -3)$ the slope is -2.
(b) $y - (-3) = (-2)(x - 1) \Rightarrow y + 3 = -2x + 2 \Rightarrow y = -2x - 1$.

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 2^3}{h} = \frac{8+12h+4h^2+h^3-8}{h} = \frac{12h+4h^2+h^3}{h} = 12+4h+h^2$. As $h \rightarrow 0$, $12+4h+h^2 \rightarrow 12$, $\Rightarrow$ at $P(2, 8)$ the slope is 12.
 (b) $y-8=12(x-2) \Rightarrow y-8=12x-24 \Rightarrow y=12x-16$.
12. (a) $\frac{\Delta y}{\Delta x} = \frac{2-(1+h)^3-(2-1^3)}{h} = \frac{2-1-3h-3h^2-h^3-1}{h} = \frac{-3h-3h^2-h^3}{h} = -3-3h-h^2$. As $h \rightarrow 0$, $-3-3h-h^2 \rightarrow -3$, $\Rightarrow$ at $P(1, 1)$ the slope is -3 .
 (b) $y-1=(-3)(x-1) \Rightarrow y-1=-3x+3 \Rightarrow y=-3x+4$.

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$$13. (a) \frac{\Delta y}{\Delta x} = \frac{(1+h)^3 - 12(1+h) - (1^3 - 12(1))}{h} = \frac{1+3h+3h^2+h^3-12-12h-(-11)}{h} = \frac{-9h+3h^2+h^3}{h} = -9+3h+h^2.$$

As $h \rightarrow 0$, $-9+3h+h^2 \rightarrow -9 \Rightarrow$ at $P(1, -11)$ the slope is -9 .

$$(b) y - (-11) = (-9)(x - 1) \Rightarrow y + 11 = -9x + 9 \Rightarrow y = -9x - 2.$$

$$14. (a) \frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 3(2+h)^2 + 4 - (2^3 - 3(2)^2 + 4)}{h} = \frac{8+12h+6h^2+h^3-12-12h-3h^2+4-0}{h} = \frac{3h^2+h^3}{h} = 3h+h^2.$$

As $h \rightarrow 0$, $3h+h^2 \rightarrow 0 \Rightarrow$ at $P(2, 0)$ the slope is 0 .

$$(b) y - 0 = 0(x - 2) \Rightarrow y = 0.$$

$$15. (a) \frac{\Delta y}{\Delta x} = \frac{\frac{1}{-2+h} - \frac{1}{-2}}{h} = \frac{2+(-2+h)}{2(-2+h)} \cdot \frac{1}{h} = \frac{1}{2(-2+h)}.$$

As $h \rightarrow 0$, $\frac{1}{2(-2+h)} \rightarrow \frac{-1}{4}$, $\Rightarrow$ at $P(-2, \frac{1}{2})$ the slope is $\frac{-1}{4}$.

$$(b) y - \left(\frac{1}{2}\right) = \frac{-1}{4}(x - (-2)) \Rightarrow y + \frac{1}{2} = \frac{-1}{4}x - \frac{1}{2} \Rightarrow y = \frac{-1}{4}x - 1$$

$$16. (a) \frac{\Delta y}{\Delta x} = \frac{\frac{(4+h)}{2-(4+h)} - \frac{4}{-2-4}}{h} = \left(\frac{4+h}{-2-h} + \frac{2}{1}\right) \cdot \frac{1}{h} = \frac{4+h+2(-2-h)}{-2-h} \cdot \frac{1}{h} = \frac{-1}{-2-h} = \frac{1}{2+h}.$$

As $h \rightarrow 0$, $\frac{1}{2+h} \rightarrow \frac{1}{2}$, $\Rightarrow$ at $P(4, -2)$ the slope is $\frac{1}{2}$.

$$(b) y - (-2) = \frac{1}{2}(x - 4) \Rightarrow y + 2 = \frac{1}{2}x - 2 \Rightarrow y = \frac{1}{2}x - 4$$

$$17. (a) \frac{\Delta y}{\Delta x} = \frac{\sqrt{4+h} - \sqrt{4}}{h} = \frac{4+h-2}{\sqrt{h}} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$$

As $h \rightarrow 0$, $\frac{1}{\sqrt{4+h}+2} \rightarrow \frac{1}{\sqrt{4}+2} = \frac{1}{4}$, $\Rightarrow$ at $P(4, 2)$ the slope is $\frac{1}{4}$.

$$(b) y - 2 = \frac{1}{4}(x - 4) \Rightarrow y - 2 = \frac{1}{4}x - 1 \Rightarrow y = \frac{1}{4}x + 1$$

$$18. (a) \frac{\Delta y}{\Delta x} = \frac{\sqrt{7-(-2+h)} - \sqrt{7-(-2)}}{h} = \frac{9-h-3}{\sqrt{h}} = \frac{9-h-3}{\sqrt{h}} \cdot \frac{\sqrt{9-h}+3}{\sqrt{9-h}+3} = \frac{(9-h)-9}{h(\sqrt{9-h}+3)} = \frac{-1}{\sqrt{9-h}+3}.$$

As $h \rightarrow 0$, $\frac{-1}{\sqrt{9-h}+3} \rightarrow \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$, $\Rightarrow$ at $P(-2, 3)$ the slope is $\frac{-1}{6}$.

$$(b) y - 3 = \frac{-1}{6}(x - (-2)) \Rightarrow y - 3 = \frac{-1}{6}x - \frac{1}{3} \Rightarrow y = \frac{-1}{6}x + \frac{8}{3}$$

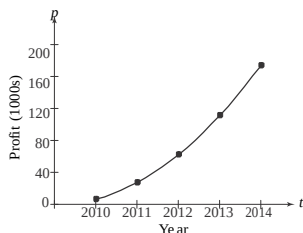
19. (a)	Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
	$Q_1(10, 225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
	$Q_2(14, 375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
	$Q_3(16.5, 475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
	$Q_4(18, 550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

20. (a)	Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
	$Q_1(5, 20)$	$\frac{80-20}{10-5} = 12$ m/sec
	$Q_2(7, 39)$	$\frac{80-39}{10-7} = 13.7$ m/sec
	$Q_3(8.5, 58)$	$\frac{80-58}{10-8.5} = 14.7$ m/sec
	$Q_4(9.5, 72)$	$\frac{80-72}{10-9.5} = 16$ m/sec

(b) Approximately 16 m/sec

21. (a)



$$(b) \frac{\Delta p}{\Delta t} = \frac{174-62}{2014-2012} = \frac{112}{2} = 56 \text{ thousand dollars per year}$$

$$(c) \text{ The average rate of change from 2011 to 2012 is } \frac{\Delta p}{\Delta t} = \frac{62-27}{2012-2011} = 35 \text{ thousand dollars per year.}$$

$$\text{The average rate of change from 2012 to 2013 is } \frac{\Delta p}{\Delta t} = \frac{111-62}{2013-2012} = 49 \text{ thousand dollars per year.}$$

So, the rate at which profits were changing in 2012 is approximately $\frac{1}{2}(35 + 49) = 42$ thousand dollars per year.

22. (a) $F(x) = (x+2)/(x-2)$

x	1.2	1.1	1.01	1.001	1.0001	1
$F(x)$	-4.0	-3.4	-3.04	-3.004	-3.0004	-3

$$\frac{\Delta F}{\Delta x} = \frac{-4.0 - (-3)}{1.2 - 1} = -5.0;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.4 - (-3)}{1.1 - 1} = -4.4;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.04 - (-3)}{1.01 - 1} = -4.04;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.004 - (-3)}{1.001 - 1} = -4.004;$$

$$\frac{\Delta F}{\Delta x} = \frac{-3.0004 - (-3)}{1.0001 - 1} = -4.0004;$$

(b) The rate of change of $F(x)$ at $x = 1$ is -4 .

$$23. (a) \frac{\Delta g}{\Delta x} = \frac{g(2) - g(1)}{2-1} = \frac{2-1}{\sqrt{2}-1} \approx 0.414213 \quad \frac{\Delta g}{\Delta x} = \frac{g(1.5) - g(1)}{1.5-1} = \frac{1.5-1}{\sqrt{0.5}} \approx 0.449489$$

$$\frac{\Delta g}{\Delta x} = \frac{g(1+h) - g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$$

(b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.0000005	1.00000005
$(\sqrt{1+h}-1)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of $g(x)$ at $x = 1$ is 0.5 .(d) The calculator gives $\lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

$$24. (a) \text{ i) } \frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{-\frac{1}{6}}{1} = -\frac{1}{6}$$

$$\text{ii) } \frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2-T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = -\frac{1}{2T}, T \neq 2$$

T	2.1	2.01	2.001	2.0001	2.00001	2.000001
$f(T)$	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
$(f(T)-f(2))/(T-2)$	-0.2381	-0.2488	-0.2500	-0.2500	-0.2500	-0.2500

(c) The table indicates the rate of change is -0.25 at $t = 2$.

$$(d) \lim_{T \rightarrow 2} \left(\frac{1}{-2T} \right) = -\frac{1}{4}$$

NOTE: Answers will vary in Exercises 25 and 26.

$$25. (a) [0, 1]: \frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15 \text{ mph}; [1, 2.5]: \frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3} \text{ mph}; [2.5, 3.5]: \frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10 \text{ mph}$$

- (b) At $P(\frac{1}{2}, 7.5)$: Since the portion of the graph from $t = 0$ to $t = 1$ is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr. At $P(2, 20)$: Since the portion of the graph from $t = 2$ to $t = 2.5$ is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at $t = 2$ is 0 mi/hr.

At $P(3, 22)$:

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$	Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr	$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr	$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr	$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3$ is about 7 mi/hr.

- (c) It appears that the curve is increasing the fastest at $t = 3.5$. Thus for $P(3.5, 30)$

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$	Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr	$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr	$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr	$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3.5$ is about 20 mi/hr.

26. (a) $[0, 3]: \frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67 \frac{\text{gal}}{\text{day}}$; $[0, 5]: \frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2 \frac{\text{gal}}{\text{day}}$; $[7, 10]: \frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5 \frac{\text{gal}}{\text{day}}$

- (b) At $P(1, 14)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$	Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day	$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day	$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day	$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is about -1.45 gal/day.

At $P(4, 6)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$	Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day	$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day	$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day	$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -3 gal/day.

(solution continues on next page)

At $P(8, 1)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -0.55 gal/day.

- (c) It appears that the curve (the consumption) is decreasing the fastest at $t = 3.5$. Thus for $P(3.5, 7.8)$

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at $t = 3.5$ is about -4 gal/day.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

- Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \rightarrow 1$.
 - 1
 - 0
 - 0.5
- 0
 - 1
 - Does not exist. As t approaches 0 from the left, $f(t)$ approaches -1. As t approaches 0 from the right, $f(t)$ approaches 1. There is no single number L that $f(t)$ gets arbitrarily close to as $t \rightarrow 0$.
 - 1
- True
 - True
 - True
 - True
 - True
 - False
 - False
 - False
 - True
 - True
- False
 - False
 - True
 - True
 - True
 - True
 - True
 - True
 - True
 - False
- $\lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \rightarrow 0$.
- As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no number L that all the function values get arbitrarily close to as $x \rightarrow 1$, so $\lim_{x \rightarrow 1} \frac{1}{x-1}$ does not exist.
- Nothing can be said about $f(x)$ because the existence of a limit as $x \rightarrow x_0$ does not depend on how the function is defined at x_0 . In order for a limit to exist, $f(x)$ must be arbitrarily close to a single real number L when x is

close enough to x_0 . That is, the existence of a limit depends on the values of $f(x)$ for x near x_0 , not on the definition of $f(x)$ at x_0 itself.

8. Nothing can be said. In order for $\lim_{x \rightarrow 0} f(x)$ to exist, $f(x)$ must close to a single value for x near 0 regardless of the value $f(0)$ itself.
9. No, the definition does not require that f be defined at $x = 1$ in order for a limiting value to exist there. If $f(1)$ is defined, it can be any real number, so we can conclude nothing about $f(1)$ from $\lim_{x \rightarrow 1} f(x) = 5$.
10. No, because the existence of a limit depends on the values of $f(x)$ when x is near 1, not on $f(1)$ itself. If $\lim_{x \rightarrow 1} f(x)$ exists, its value may be some number other than $f(1) = 5$. We can conclude nothing about $\lim_{x \rightarrow 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of $f(1)$ alone.
11. $\lim_{x \rightarrow -3} (x^2 - 13) = (-3)^2 - 13 = 9 - 13 = -4$
12. $\lim_{x \rightarrow 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$
13. $\lim_{t \rightarrow 6} 8(t - 5)(t - 7) = 8(6 - 5)(6 - 7) = -8$
14. $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$
15. $\lim_{x \rightarrow 2} \frac{2x+5}{11-x^3} = \frac{2(2)+5}{11-(2)^3} = \frac{9}{3} = 3$
16. $\lim_{t \rightarrow 2/3} (8-3s)(2s-1) = \left(8-5\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right)-1\right) = (8-2)\left(\left(\frac{4}{3}\right)-1\right) = (6)\left(\frac{1}{3}\right) = 2$
17. $\lim_{x \rightarrow -1/2} 4x(3x+4)^2 = 4\left(-\frac{1}{2}\right)\left(3\left(-\frac{1}{2}\right)+4\right)^2 = (-2)\left(-\frac{3}{2}+4\right)^2 = (-2)\left(\frac{5}{2}\right)^2 = -\frac{25}{2}$
18. $\lim_{y \rightarrow 2} \frac{y+2}{y^2+5y+6} = \frac{2+2}{(2)^2+5(2)+6} = \frac{4}{4+10+6} = \frac{4}{20} = \frac{1}{5}$
19. $\lim_{y \rightarrow -3} (5-y)^{4/3} = [5-(-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$
20. $\lim_{z \rightarrow 4} \sqrt{z^2-10} = \sqrt{4^2-10} = \frac{16-10}{\sqrt{\quad}} = \frac{6}{\sqrt{\quad}}$
21. $\lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{3(0)+1+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$
22. $\lim_{h \rightarrow 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \lim_{h \rightarrow 0} \frac{(5h+4)-4}{h(\sqrt{5h+4}+2)} = \lim_{h \rightarrow 0} \frac{5h}{h(\sqrt{5h+4}+2)} = \lim_{h \rightarrow 0} \frac{5}{\sqrt{5h+4}+2} = \frac{5}{\sqrt{4}+2} = \frac{5}{4}$
23. $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \rightarrow 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$
24. $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3} = \lim_{x \rightarrow -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x \rightarrow -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$

$$25. \lim_{x \rightarrow -5} \frac{x^2 + 3x - 10}{x + 5} = \lim_{x \rightarrow -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \rightarrow -5} (x-2) = -5-2 = -7$$

$$26. \lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x \rightarrow 2} (x-5) = 2-5 = -3$$

$$27. \lim_{t \rightarrow 1} \frac{t^2 + t - 2}{t^2 - 1} = \lim_{t \rightarrow 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \rightarrow 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$$

$$28. \lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2} = \lim_{t \rightarrow -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t \rightarrow -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$$

$$29. \lim_{x \rightarrow -2} \frac{-2x-4}{x^3 + 2x^2} = \lim_{x \rightarrow -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \rightarrow -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$30. \lim_{y \rightarrow 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2} = \lim_{y \rightarrow 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \rightarrow 0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$$

$$31. \lim_{x \rightarrow 1} \frac{x^{-1}-1}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x \rightarrow 1} \left(\frac{1-x}{x} \cdot \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} -\frac{1}{x} = -1$$

$$32. \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} - \frac{1}{x+1}}{x} = \lim_{x \rightarrow 0} \frac{\frac{(x+1)-(x-1)}{(x-1)(x+1)}}{x} = \lim_{x \rightarrow 0} \left(\frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$$

$$33. \lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} = \lim_{u \rightarrow 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \rightarrow 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$$

$$34. \lim_{v \rightarrow 2} \frac{v^3 - 8}{v^4 - 16} = \lim_{v \rightarrow 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v \rightarrow 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$$

$$35. \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$$36. \lim_{x \rightarrow 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x \rightarrow 4} x(2+\sqrt{x}) = 4(2+2) = 16$$

$$37. \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x \rightarrow 1} (\sqrt{x+3}+2) = \sqrt{4}+2 = 4$$

$$38. \lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \rightarrow -1} \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \rightarrow -1} \frac{x^2}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x \rightarrow -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} \\ = \lim_{x \rightarrow -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$$

$$39. \lim_{x \rightarrow 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x \rightarrow 2} \frac{(\sqrt{x^2+12}-4)(\sqrt{x^2+12}+4)}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x \rightarrow 2} \frac{x^2-16}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+12}+4)} \\ = \lim_{x \rightarrow 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$$

$$\lim_{x \rightarrow 2} \sqrt{x^2 + 12} + 4 = \sqrt{16 + 4} + 4 = 2 + 4 = 6$$

$$40. \lim_{x \rightarrow -2} \frac{x+2}{\sqrt{x^2+5}-3} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(x^2+5)-9} = \lim_{x \rightarrow -2} \frac{(x+2)(\sqrt{x^2+5}+3)}{(x+2)(x-2)}$$

$$= \lim_{x \rightarrow -2} \frac{\sqrt{x^2+5}+3}{x-2} = \frac{\sqrt{9+5}+3}{-4} = -\frac{3}{2}$$

$$41. \lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim_{x \rightarrow -3} \frac{(2-\sqrt{x^2-5})(2+\sqrt{x^2-5})}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{4-(x^2-5)}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{9-x^2}{(x+3)(2+\sqrt{x^2-5})}$$

$$= \lim_{x \rightarrow -3} \frac{(3-x)(3+x)}{(x+3)(2+\sqrt{x^2-5})} = \lim_{x \rightarrow -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{6}{2+\sqrt{4}} = \frac{3}{2}$$

$$42. \lim_{x \rightarrow 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(5-\sqrt{x^2+9})(5+\sqrt{x^2+9})} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{25-(x^2+9)} = \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2}$$

$$= \lim_{x \rightarrow 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(4-x)(4+x)} = \lim_{x \rightarrow 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5+\sqrt{25}}{8} = \frac{5}{4}$$

$$43. \lim_{x \rightarrow 0} (2 \sin x - 1) = 2 \sin 0 - 1 = 0 - 1 = -1$$

$$44. \lim_{x \rightarrow 0} \sin^2 x = \left(\lim_{x \rightarrow 0} \sin x \right)^2 = (\sin 0)^2 = 0^2 = 0$$

$$45. \lim_{x \rightarrow 0} \sec x = \lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$46. \lim_{x \rightarrow 0} \tan x = \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$47. \lim_{x \rightarrow 0} \frac{1+x+\sin x}{3 \cos x} = \frac{1+0+\sin 0}{3 \cos 0} = \frac{1+0+0}{3} = \frac{1}{3}$$

$$48. \lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x) = (0^2 - 1)(2 - \cos 0) = (-1)(2 - 1) = (-1)(1) = -1$$

$$49. \lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi) = \lim_{x \rightarrow -\pi} \sqrt{x+4} \cdot \lim_{x \rightarrow -\pi} \cos(x+\pi) = \sqrt{-\pi+4} \cdot \cos 0 = \sqrt{4-\pi} \cdot 1 = \sqrt{4-\pi}$$

$$50. \lim_{x \rightarrow 0} \sqrt{7 + \sec^2 x} = \sqrt{\lim_{x \rightarrow 0} (7 + \sec^2 x)} = \sqrt{7 + \lim_{x \rightarrow 0} \sec^2 x} = \sqrt{7 + \sec^2 0} = \sqrt{7 + (1)^2} = 2\sqrt{2}$$

51. (a) quotient rule

(b) difference and power rules

(c) sum and constant multiple rules

52. (a) quotient rule

(b) power and product rules

(c) difference and constant multiple rules

$$53. (a) \lim_{x \rightarrow c} f(x)g(x) = \left[\lim_{x \rightarrow c} f(x) \right] \left[\lim_{x \rightarrow c} g(x) \right] = (5)(-2) = -10$$

$$(b) \lim_{x \rightarrow c} 2f(x)g(x) = 2 \left[\lim_{x \rightarrow c} f(x) \right] \left[\lim_{x \rightarrow c} g(x) \right] = 2(5)(-2) = -20$$

$$(c) \lim_{x \rightarrow c} [f(x) + 3g(x)] = \lim_{x \rightarrow c} f(x) + 3 \lim_{x \rightarrow c} g(x) = 5 + 3(-2) = -1$$

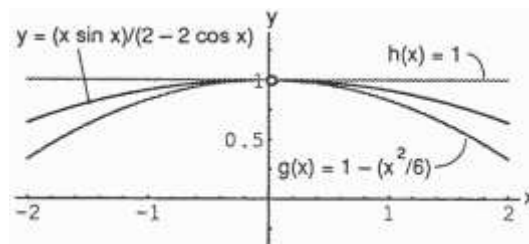
$$\frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{5}{-2} = -\frac{5}{2}$$

$$(d) \lim_{x \rightarrow c} f(x) - g(x) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x) = 5 - (-2) = 7$$

54. (a) $\lim_{x \rightarrow 4} [g(x) + 3] = \lim_{x \rightarrow 4} g(x) + \lim_{x \rightarrow 4} 3 = -3 + 3 = 0$
 (b) $\lim_{x \rightarrow 4} xf(x) = \lim_{x \rightarrow 4} x \cdot \lim_{x \rightarrow 4} f(x) = (4)(0) = 0$
 (c) $\lim_{x \rightarrow 4} [g(x)]^2 = \left[\lim_{x \rightarrow 4} g(x) \right]^2 = [-3]^2 = 9$
 (d) $\lim_{x \rightarrow 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x \rightarrow 4} g(x)}{\lim_{x \rightarrow 4} f(x) - \lim_{x \rightarrow 4} 1} = \frac{-3}{0-1} = 3$
55. (a) $\lim_{x \rightarrow b} [f(x) + g(x)] = \lim_{x \rightarrow b} f(x) + \lim_{x \rightarrow b} g(x) = 7 + (-3) = 4$
 (b) $\lim_{x \rightarrow b} f(x) \cdot g(x) = \left[\lim_{x \rightarrow b} f(x) \right] \left[\lim_{x \rightarrow b} g(x) \right] = (7)(-3) = -21$
 (c) $\lim_{x \rightarrow b} 4g(x) = \left[\lim_{x \rightarrow b} 4 \right] \left[\lim_{x \rightarrow b} g(x) \right] = (4)(-3) = -12$
 (d) $\lim_{x \rightarrow b} f(x)/g(x) = \lim_{x \rightarrow b} f(x) / \lim_{x \rightarrow b} g(x) = \frac{7}{-3} = -\frac{7}{3}$
56. (a) $\lim_{x \rightarrow -2} [p(x) + r(x) + s(x)] = \lim_{x \rightarrow -2} p(x) + \lim_{x \rightarrow -2} r(x) + \lim_{x \rightarrow -2} s(x) = 4 + 0 + (-3) = 1$
 (b) $\lim_{x \rightarrow -2} p(x) \cdot r(x) \cdot s(x) = \left[\lim_{x \rightarrow -2} p(x) \right] \left[\lim_{x \rightarrow -2} r(x) \right] \left[\lim_{x \rightarrow -2} s(x) \right] = (4)(0)(-3) = 0$
 (c) $\lim_{x \rightarrow -2} [-4p(x) + 5r(x)]/s(x) = \frac{-4 \lim_{x \rightarrow -2} p(x) + 5 \lim_{x \rightarrow -2} r(x)}{\lim_{x \rightarrow -2} s(x)} = \frac{-4(4) + 5(0)}{-3} = \frac{16}{3}$
57. $\lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow 0} \frac{1+2h+h^2-1}{h} = \lim_{h \rightarrow 0} \frac{h(2+h)}{h} = \lim_{h \rightarrow 0} (2+h) = 2$
58. $\lim_{h \rightarrow 0} \frac{(-2+h)^2 - (-2)^2}{h} = \lim_{h \rightarrow 0} \frac{4-4h+h^2-4}{h} = \lim_{h \rightarrow 0} \frac{h(h-4)}{h} = \lim_{h \rightarrow 0} (h-4) = -4$
59. $\lim_{h \rightarrow 0} \frac{[3(2+h)-4] - [3(2)-4]}{h} = \lim_{h \rightarrow 0} \frac{3h}{h} = 3$
60. $\lim_{h \rightarrow 0} \frac{(-2+h)^2 - (-2)^2}{h} = \lim_{h \rightarrow 0} \frac{-2+h}{-2h} = \lim_{h \rightarrow 0} \frac{-2-(-2+h)}{-2h(-2+h)} = \lim_{h \rightarrow 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$
61. $\lim_{h \rightarrow 0} \frac{\sqrt{7+h} - \sqrt{7}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{7+h}-\sqrt{7})(\sqrt{7+h}+\sqrt{7})}{h(\sqrt{7+h}+\sqrt{7})} = \lim_{h \rightarrow 0} \frac{(7+h)-7}{h(\sqrt{7+h}+\sqrt{7})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{7+h}+\sqrt{7})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$
62. $\lim_{h \rightarrow 0} \frac{\sqrt{3(0+h)+1} - \sqrt{3(0)+1}}{h} = \lim_{h \rightarrow 0} \frac{(\sqrt{3h+1}-1)(\sqrt{3h+1}+1)}{h(\sqrt{3h+1}+1)} = \lim_{h \rightarrow 0} \frac{(3h+1)-1}{h(\sqrt{3h+1}+1)} = \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3h+1}+1)} = \lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{2}$
63. $\lim_{x \rightarrow 0} \sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$ and $\lim_{x \rightarrow 0} \sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5}$; by the sandwich theorem, $\lim_{x \rightarrow 0} f(x) = \sqrt{5}$
64. $\lim_{x \rightarrow 0} (2-x^2) = 2-0 = 2$ and $\lim_{x \rightarrow 0} 2 \cos x = 2(1) = 2$; by the sandwich theorem, $\lim_{x \rightarrow 0} g(x) = 2$
65. (a) $\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{6}\right) = 1 - \frac{0}{6} = 1$ and $\lim_{x \rightarrow 0} 1 = 1$; by the sandwich theorem, $\lim_{x \rightarrow 0} \frac{x \sin x}{2-2 \cos x} = 1$

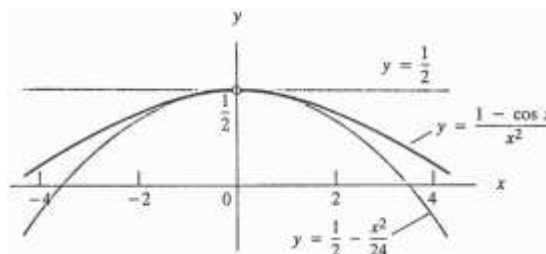
(b)

- (b) For $x \neq 0$, $y = (x \sin x)/(2 - 2 \cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \rightarrow 0$.



66. (a) $\lim_{x \rightarrow 0} \left(\frac{1}{2} - \frac{x^2}{24} \right) = \lim_{x \rightarrow 0} \frac{1}{2} - \lim_{x \rightarrow 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$; by the sandwich theorem, $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

- (b) For all $x \neq 0$, the graph of $f(x) = (1 - \cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \rightarrow 0$.



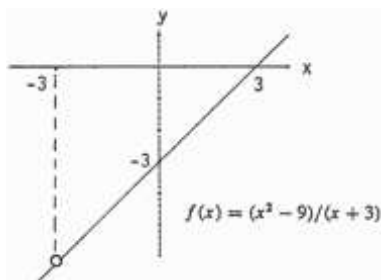
67. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	-3.1	-3.01	-3.001	-3.0001	-3.00001	-3.000001
$f(x)$	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001

x	-2.9	-2.99	-2.999	-2.9999	-2.99999	-2.999999
$f(x)$	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999

The estimate is $\lim_{x \rightarrow -3} f(x) = -6$.

(b)

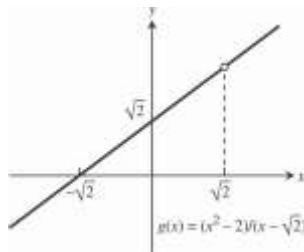


- (c) $f(x) = \frac{x^2 - 9}{x + 3} = \frac{(x + 3)(x - 3)}{x + 3} = x - 3$ if $x \neq -3$, and $\lim_{x \rightarrow -3} (x - 3) = -3 - 3 = -6$.

68. (a) $g(x) = (x^2 - 2)/(x - \sqrt{2})$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
$g(x)$	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



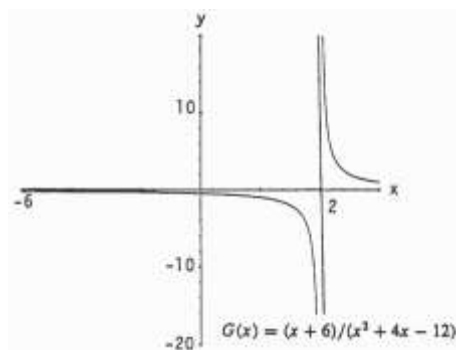
$$(c) \quad g(x) = \frac{x^2 - 2}{x - \sqrt{2}} = \frac{(x + \sqrt{2})(x - \sqrt{2})}{(x - \sqrt{2})} = x + \sqrt{2} \text{ if } x \neq \sqrt{2}, \text{ and } \lim_{x \rightarrow \sqrt{2}} (x + \sqrt{2}) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

69. (a) $G(x) = (x+6)/(x^2 + 4x - 12)$

x	-5.9	-5.99	-5.999	-5.9999	-5.99999	-5.999999
$G(x)$	-.126582	-.1251564	-.1250156	-.1250015	-.1250001	-.1250000

x	-6.1	-6.01	-6.001	-6.0001	-6.00001	-6.000001
$G(x)$	-.123456	-.124843	-.124984	-.124998	-.124999	-.124999

(b)



$$(c) \quad G(x) = \frac{x+6}{x^2 + 4x - 12} = \frac{x+6}{(x+6)(x-2)} = \frac{1}{x-2} \text{ if } x \neq -6, \text{ and } \lim_{x \rightarrow -6} \frac{1}{x-2} = \frac{1}{-6-2} = -\frac{1}{8} = -0.125.$$

70. (a) $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
$h(x)$	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$h(x)$	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)

$$(c) \quad h(x) = \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{x+1}{x-1} \text{ if } x \neq 3, \text{ and } \lim_{x \rightarrow 3} \frac{x+1}{x-1} = \frac{3+1}{3-1} = \frac{4}{2} = 2.$$

72



continuity

Section 2.2 Limit of a Function and Limit Laws

72

$$3^{x-1} \quad (b) \quad 3^{-1} = 2 =$$

$x \rightarrow$

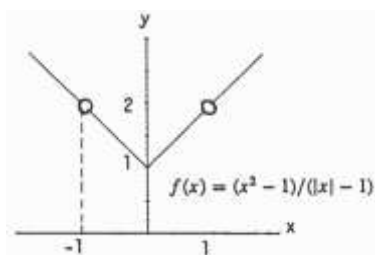
(b)

71. (a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	-0.9	-0.99	-0.999	-0.9999	-0.99999	-0.999999
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



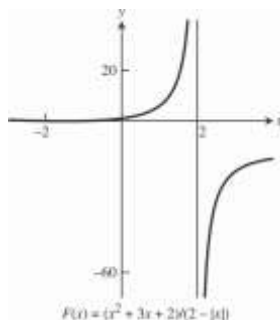
(c) $f(x) = \frac{x^2 - 1}{|x| - 1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \geq 0 \text{ and } x \neq 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \neq -1 \end{cases}$, and $\lim_{x \rightarrow -1} (1-x) = 1 - (-1) = 2$.

72. (a) $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
$F(x)$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

x	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
$F(x)$	-0.9	-0.99	-0.999	-0.9999	-0.99999	-0.999999

(b)



(c) $F(x) = \frac{x^2 + 3x + 2}{2 - |x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \geq 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \neq -2 \end{cases}$, and $\lim_{x \rightarrow -2} (x+1) = -2+1 = -1$.

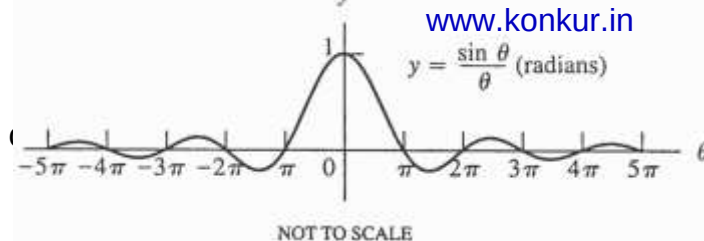
73. (a) $g(\theta) = (\sin \theta)\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	-.1	-.01	-.001	-.0001	-.00001	-.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \rightarrow 0} g(\theta) = 1$

74



of a Function and Limit Laws

74

(b)

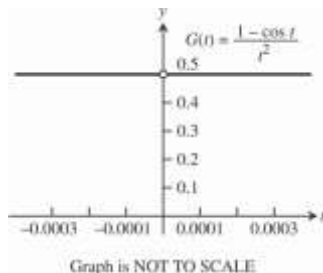
74. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

t	-.1	-.01	-.001	-.0001	-.00001	-.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$$\lim_{t \rightarrow 0} G(t) = 0.5$$

(b)



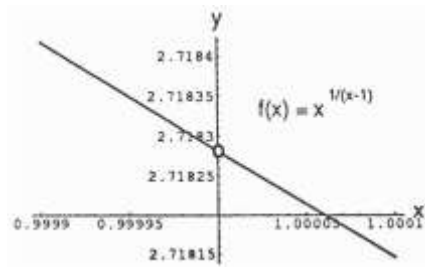
75. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$$\lim_{x \rightarrow 1} f(x) \approx 0.36788$$

(b)



Graph is NOT TO SCALE. Also, the intersection of the axes is not the origin: the axes intersect at the point (1, 2.71820).

(b)

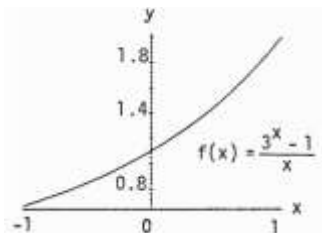
76. (a) $f(x) = (3^x - 1)/x$

x	.1	.01	.001	.0001	.00001	.000001
$f(x)$	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

x	-.1	-.01	-.001	-.0001	-.00001	-.000001
$f(x)$	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$$\lim_{x \rightarrow 1} f(x) \approx 1.0986$$

(b)



77. $\lim_{x \rightarrow c} f(x)$ exists at those points c where $\lim_{x \rightarrow c} x^4 = \lim_{x \rightarrow c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1 - c^2) = 0 \Rightarrow c = 0, 1, \text{ or } -1$.

Moreover, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 = 0$ and $\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} f(x) = 1$.

78. Nothing can be concluded about the values of f , g , and h at $x = 2$. Yes, $f(2)$ could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim_{x \rightarrow 2} f(x) = -5 \neq 0$.

79. $1 = \lim_{x \rightarrow 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x \rightarrow 4} f(x) - \lim_{x \rightarrow 4} 5}{\lim_{x \rightarrow 4} x - \lim_{x \rightarrow 4} 2} = \frac{\lim_{x \rightarrow 4} f(x) - 5}{4 - 2} \Rightarrow \lim_{x \rightarrow 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x \rightarrow 4} f(x) = 2 + 5 = 7$.

80. (a) $1 = \lim_{x \rightarrow -2} \frac{f(x)}{x^2} = \frac{\lim_{x \rightarrow -2} f(x)}{\lim_{x \rightarrow -2} x^2} = \frac{\lim_{x \rightarrow -2} f(x)}{4} \Rightarrow \lim_{x \rightarrow -2} f(x) = 4$.

(b) $1 = \lim_{x \rightarrow -2} \frac{f(x)}{x^2} = \left[\lim_{x \rightarrow -2} \frac{f(x)}{x} \right] \left[\lim_{x \rightarrow -2} \frac{1}{x} \right] = \left[\lim_{x \rightarrow -2} \frac{f(x)}{x} \right] \left(\frac{1}{-2} \right) \Rightarrow \lim_{x \rightarrow -2} \frac{f(x)}{x} = -2$.

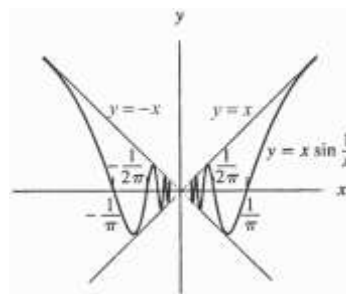
81. (a) $0 = 3 \cdot 0 = \left[\lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \right] \left[\lim_{x \rightarrow 2} (x-2) \right] = \lim_{x \rightarrow 2} \left[\left(\frac{f(x)-5}{x-2} \right) (x-2) \right] = \lim_{x \rightarrow 2} [f(x) - 5]$
 $= \lim_{x \rightarrow 2} f(x) - 5 \Rightarrow \lim_{x \rightarrow 2} f(x) = 5$.

(b) $0 = 4 \cdot 0 = \left[\lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \right] \left[\lim_{x \rightarrow 2} (x-2) \right] \Rightarrow \lim_{x \rightarrow 2} f(x) = 5$ as in part (a).

82. (a) $0 = 1 \cdot 0 = \left[\lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[\lim_{x \rightarrow 0} x^2 \right] = \left[\lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[\lim_{x \rightarrow 0} x^2 \right] = \lim_{x \rightarrow 0} \left[\frac{f(x)}{x^2} \cdot x^2 \right] = \lim_{x \rightarrow 0} f(x)$.
 That is, $\lim_{x \rightarrow 0} f(x) = 0$.

(b) $0 = 1 \cdot 0 = \left[\lim_{x \rightarrow 0} \frac{f(x)}{x^2} \right] \left[\lim_{x \rightarrow 0} x \right] = \lim_{x \rightarrow 0} \left[\frac{f(x)}{x^2} \cdot x \right] = \lim_{x \rightarrow 0} \frac{f(x)}{x}$. That is, $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$.

83. (a) $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

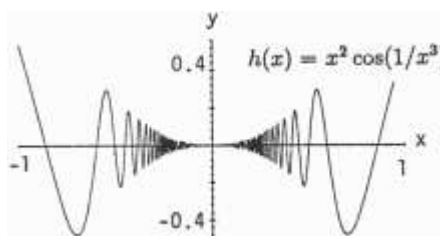


(b) $-1 \leq \sin \frac{1}{x} \leq 1$ for $x \neq 0$:

$x > 0 \Rightarrow -x \leq x \sin \frac{1}{x} \leq x \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem;

$x < 0 \Rightarrow -x \geq x \sin \frac{1}{x} \geq x \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ by the sandwich theorem.

84. (a) $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$



(b) $-1 \leq \cos\left(\frac{1}{x^3}\right) \leq 1$ for $x \neq 0 \Rightarrow -x^2 \leq x^2 \cos\left(\frac{1}{x^3}\right) \leq x^2 \Rightarrow \lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x^3}\right) = 0$ by the sandwich theorem since $\lim_{x \rightarrow 0} x^2 = 0$.

85–90. Example CAS commands:

Maple:

```
f := x -> (x^4 - 16)/(x - 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
      title = "Section 2.2, #85(a)" );
limit( f(x), x = x0 );
```

In Exercise 87, note that the standard cube root, $x^{1/3}$, is not defined for $x < 0$ in many CASs. This can be overcome in Maple by entering the function as $f := x \rightarrow (\text{surd}(x+1, 3) - 1)/x$.

Mathematica: (assigned function and values for x_0 and h may vary)

```
Clear[f, x]
f[x_]:= (x^3 - x^2 - 5x - 3)/(x + 1)^2
x0 = -1; h = 0.1;
Plot[f[x], {x, x0 - h, x0 + h}]
Limit[f[x], x -> x0]
```

2.3 THE PRECISE DEFINITION OF A LIMIT



Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: $\delta + 5 = 7 \Rightarrow \delta = 2$, or $-\delta + 5 = 1 \Rightarrow \delta = 4$.

The value of δ which assures $|x - 5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

| |



Step 1: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$

Step 2: $-\delta + 2 = 1 \Rightarrow \delta = 1$, or $\delta + 2 = 7 \Rightarrow \delta = 5$.

The value of δ which assures $|x - 2| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 1$.



Step 1: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$

Step 2: $-\delta - 3 = -\frac{7}{2} \Rightarrow \delta = \frac{1}{2}$, or $\delta - 3 = -\frac{1}{2} \Rightarrow \delta = \frac{5}{2}$.

The value of δ which assures $|x - (-3)| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$ is the smaller value, $\delta = \frac{1}{2}$.



Step 1: $|x - (-\frac{3}{2})| < \delta \Rightarrow -\delta < x + \frac{3}{2} < \delta \Rightarrow -\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$

Step 2: $-\delta - \frac{3}{2} = -\frac{7}{2} \Rightarrow \delta = 2$, or $\delta - \frac{3}{2} = -\frac{1}{2} \Rightarrow \delta = 1$.

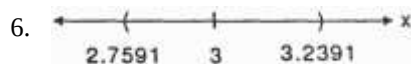
The value of δ which assures $|x - (-\frac{3}{2})| < \delta \Rightarrow -\frac{7}{2} < x < -\frac{1}{2}$ is the smaller value, $\delta = 1$.



Step 1: $|x - \frac{1}{2}| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$

Step 2: $-\delta + \frac{1}{2} = \frac{4}{9} \Rightarrow \delta = \frac{1}{18}$, or $\delta + \frac{1}{2} = \frac{4}{7} \Rightarrow \delta = \frac{1}{14}$.

The value of δ which assures $|x - \frac{1}{2}| < \delta \Rightarrow \frac{4}{9} < x < \frac{4}{7}$ is the smaller value, $\delta = \frac{1}{18}$.



Step 1: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$

Step 2: $-\delta + 3 = 2.7591 \Rightarrow \delta = 0.2409$, or $\delta + 3 = 3.2391 \Rightarrow \delta = 0.2391$.

The value of δ which assures $|x - 3| < \delta \Rightarrow 2.7591 < x < 3.2391$ is the smaller value, $\delta = 0.2391$.

7. Step 1: $|x - 5| < \delta \Rightarrow -\delta < x - 5 < \delta \Rightarrow -\delta + 5 < x < \delta + 5$

Step 2: From the graph, $-\delta + 5 = 4.9 \Rightarrow \delta = 0.1$, or $\delta + 5 = 5.1 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$ in either case.

8. Step 1: $|x - (-3)| < \delta \Rightarrow -\delta < x + 3 < \delta \Rightarrow -\delta - 3 < x < \delta - 3$

Step 2: From the graph, $-\delta - 3 = -3.1 \Rightarrow \delta = 0.1$, or $\delta - 3 = -2.9 \Rightarrow \delta = 0.1$; thus $\delta = 0.1$.

9. Step 1: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$

Step 2: From the graph, $-\delta + 1 = \frac{9}{16} \Rightarrow \delta = \frac{7}{16}$, or $\delta + 1 = \frac{25}{16} \Rightarrow \delta = \frac{9}{16}$; thus $\delta = \frac{7}{16}$.

10. Step 1: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$

Step 2: From the graph, $-\delta + 3 = 2.61 \Rightarrow \delta = 0.39$, or $\delta + 3 = 3.41 \Rightarrow \delta = 0.41$; thus $\delta = 0.39$.

11. Step 1: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$
 Step 2: From the graph, $-\delta+2 = \sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$, or $\delta+2 = \sqrt{5} \Rightarrow \delta = \sqrt{5}-2 \approx 0.2361$; thus $\delta = \sqrt{5}-2$.
12. Step 1: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$
 Step 2: From the graph, $-\delta-1 = -\frac{\sqrt{5}}{2} \Rightarrow \delta = \frac{\sqrt{5}-2}{2} \approx 0.118$ or $\delta-1 = -\frac{\sqrt{3}}{2} \Rightarrow \delta = \frac{2-\sqrt{3}}{2} \approx 0.1340$; thus $\delta = \frac{\sqrt{5}-2}{2}$.
13. Step 1: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$
 Step 2: From the graph, $-\delta-1 = -\frac{16}{9} \Rightarrow \delta = \frac{7}{9} \approx 0.77$, or $\delta-1 = -\frac{16}{25} \Rightarrow \frac{9}{25} = 0.36$; thus $\delta = \frac{9}{25} = 0.36$.
14. Step 1: $|x-\frac{1}{2}| < \delta \Rightarrow -\delta < x-\frac{1}{2} < \delta \Rightarrow -\delta+\frac{1}{2} < x < \delta+\frac{1}{2}$
 Step 2: From the graph, $-\delta+\frac{1}{2} = \frac{1}{2.01} \Rightarrow \delta = \frac{1}{2} - \frac{1}{2.01} \approx 0.00248$, or $\delta+\frac{1}{2} = \frac{1}{1.99} \Rightarrow \delta = \frac{1}{1.99} - \frac{1}{2} \approx 0.00251$; thus $\delta = 0.00248$.
15. Step 1: $|(x+1)-5| < 0.01 \Rightarrow |x-4| < 0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
 Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta = 0.01$.
16. Step 1: $|(2x-2)-(-6)| < 0.02 \Rightarrow |2x+4| < 0.02 \Rightarrow -0.02 < 2x+4 < 0.02 \Rightarrow -4.02 < 2x < -3.98 \Rightarrow -2.01 < x < -1.99$
 Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2 \Rightarrow \delta = 0.01$.
17. Step 1: $x+1-1 < 0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
 $\Rightarrow -0.19 < x < 0.21$
 Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta$. Then, $-\delta = -0.19 \Rightarrow \delta = 0.19$ or $\delta = 0.21$; thus, $\delta = 0.19$.
18. Step 1: $|\sqrt{x}-\frac{1}{2}| < 0.1 \Rightarrow -0.1 < \sqrt{x}-\frac{1}{2} < 0.1 \Rightarrow 0.4 < \sqrt{x} < 0.6 \Rightarrow 0.16 < x < 0.36$
 Step 2: $|x-\frac{1}{4}| < \delta \Rightarrow -\delta < x-\frac{1}{4} < \delta \Rightarrow$
 Then $-\delta+\frac{1}{4} = 0.16 \Rightarrow \delta = 0.09$ or $\delta+\frac{1}{4} = 0.36 \Rightarrow \delta = 0.11$; thus $\delta = 0.09$.
19. Step 1: $|\sqrt{19-x}-3| < 1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$
 $\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
 Step 2: $|x-10| < \delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10$.
 Then $-\delta+10 = 3 \Rightarrow \delta = 7$, or $\delta+10 = 15 \Rightarrow \delta = 5$; thus $\delta = 5$.
20. Step 1: $|\sqrt{x-7}-4| < 1 \Rightarrow -1 < \sqrt{x-7}-4 < 1 \Rightarrow 3 < \sqrt{x-7} < 5 \Rightarrow 9 < x-7 < 25 \Rightarrow 16 < x < 32$
 Step 2: $|x-23| < \delta \Rightarrow -\delta < x-23 < \delta \Rightarrow -\delta+23 < x < \delta+23$.
 Then $-\delta+23 = 16 \Rightarrow \delta = 7$, or $\delta+23 = 32 \Rightarrow \delta = 9$; thus $\delta = 7$.
21. Step 1: $|\frac{1}{x}-\frac{1}{4}| < 0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
 Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.
 Then $-\delta+4 = \frac{10}{3}$ or $\delta = \frac{2}{3}$, or $\delta+4 = 5$ or $\delta = 1$; thus $\delta = \frac{2}{3}$.

22. Step 1: $|x^2 - 3| < 0.1 \Rightarrow -0.1 < x^2 - 3 < 0.1 \Rightarrow 2.9 < x^2 < 3.1 \Rightarrow \sqrt{2.9} < x < \sqrt{3.1}$
 Step 2: $|x - \sqrt{3}| < \delta \Rightarrow -\delta < x - \sqrt{3} < \delta \Rightarrow -\delta + \sqrt{3} < x < \delta + \sqrt{3}$.
 Then $-\delta + \sqrt{3} = \sqrt{2.9} \Rightarrow \delta = \sqrt{3} - \sqrt{2.9} \approx 0.0291$, or $\delta + \sqrt{3} = \sqrt{3.1} \Rightarrow \delta = \sqrt{3.1} - \sqrt{3} \approx 0.0286$;
 thus $\delta = 0.0286$.
23. Step 1: $|x^2 - 4| < 0.5 \Rightarrow -0.5 < x^2 - 4 < 0.5 \Rightarrow 3.5 < x^2 < 4.5 \Rightarrow \sqrt{3.5} < x < \sqrt{4.5}$
 for x near -2 .
 Step 2: $|x - (-2)| < \delta \Rightarrow -\delta < x + 2 < \delta \Rightarrow -\delta - 2 < x < \delta - 2$.
 Then $-\delta - 2 = -\sqrt{4.5} \Rightarrow \delta = \sqrt{4.5} - 2 \approx 0.1213$, or $\delta - 2 = -\sqrt{3.5} \Rightarrow \delta = 2 - \sqrt{3.5} \approx 0.1292$;
 thus $\delta = \sqrt{4.5} - 2 \approx 0.12$.
24. Step 1: $\left|\frac{1}{x} - (-1)\right| < 0.1 \Rightarrow -0.1 < \frac{1}{x} + 1 < 0.1 \Rightarrow -\frac{11}{10} < \frac{1}{x} < -\frac{9}{10} \Rightarrow -\frac{10}{11} > x > -\frac{10}{9}$ or $-\frac{10}{9} < x < -\frac{10}{11}$
 Step 2: $|x - (-1)| < \delta \Rightarrow -\delta < x + 1 < \delta \Rightarrow -\delta - 1 < x < \delta - 1$.
 Then $-\delta - 1 = -\frac{10}{9} \Rightarrow \delta = \frac{1}{9}$, or $\delta - 1 = -\frac{10}{11} \Rightarrow \delta = \frac{1}{11}$; thus $\delta = \frac{1}{11}$.
25. Step 1: $|(x^2 - 5) - 11| < 1 \Rightarrow |x^2 - 16| < 1 \Rightarrow -1 < x^2 - 16 < 1 \Rightarrow 15 < x^2 < 17 \Rightarrow \sqrt{15} < x < \sqrt{17}$.
 Step 2: $|x - 4| < \delta \Rightarrow -\delta < x - 4 < \delta \Rightarrow -\delta + 4 < x < \delta + 4$.
 Then $-\delta + 4 = \sqrt{15} \Rightarrow \delta = 4 - \sqrt{15} \approx 0.1270$, or $\delta + 4 = \sqrt{17} \Rightarrow \delta = \sqrt{17} - 4 \approx 0.1231$; thus
 $\delta = \sqrt{17} - 4 \approx 0.12$.
26. Step 1: $\left|\frac{120}{x} - 5\right| < 1 \Rightarrow -1 < \frac{120}{x} - 5 < 1 \Rightarrow 4 < \frac{120}{x} < 6 \Rightarrow \frac{1}{4} > \frac{x}{120} > \frac{1}{6} \Rightarrow 30 > x > 20$ or $20 < x < 30$.
 Step 2: $|x - 24| < \delta \Rightarrow -\delta < x - 24 < \delta \Rightarrow -\delta + 24 < x < \delta + 24$.
 Then $-\delta + 24 = 20 \Rightarrow \delta = 4$, or $\delta + 24 = 30 \Rightarrow \delta = 6$; thus $\delta = 4$.
27. Step 1: $|mx - 2m| < 0.03 \Rightarrow -0.03 < mx - 2m < 0.03 \Rightarrow -0.03 + 2m < mx < 0.03 + 2m \Rightarrow 2 - \frac{0.03}{m} < x < 2 + \frac{0.03}{m}$.
 Step 2: $|x - 2| < \delta \Rightarrow -\delta < x - 2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2$.
 Then $-\delta + 2 = 2 - \frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$, or $\delta + 2 = 2 + \frac{0.03}{m} \Rightarrow \delta = \frac{0.03}{m}$. In either case, $\delta = \frac{0.03}{m}$.
28. Step 1: $|mx - 3m| < c \Rightarrow -c < mx - 3m < c \Rightarrow -c + 3m < mx < c + 3m \Rightarrow 3 - \frac{c}{m} < x < 3 + \frac{c}{m}$
 Step 2: $|x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3$.
 Then $-\delta + 3 = 3 - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$, or $\delta + 3 = 3 + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$. In either case, $\delta = \frac{c}{m}$.
29. Step 1: $|(mx + b) - \left(\frac{2}{m} + b\right)| < c \Rightarrow -c < mx - \frac{2}{m} < c \Rightarrow -c + \frac{2}{m} < mx < c + \frac{2}{m} \Rightarrow \frac{2}{m} - \frac{c}{m} < x < \frac{2}{m} + \frac{c}{m}$.
 Step 2: $|x - \frac{1}{2}| < \delta \Rightarrow -\delta < x - \frac{1}{2} < \delta \Rightarrow -\delta + \frac{1}{2} < x < \delta + \frac{1}{2}$.
 Then $-\delta + \frac{1}{2} = \frac{2}{m} - \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$, or $\delta + \frac{1}{2} = \frac{2}{m} + \frac{c}{m} \Rightarrow \delta = \frac{c}{m}$. In either case, $\delta = \frac{c}{m}$.
30. Step 1: $|(mx + b) - (m + b)| < 0.05 \Rightarrow -0.05 < mx - m < 0.05 \Rightarrow -0.05 + m < mx < 0.05 + m$
 $\Rightarrow 1 - \frac{0.05}{m} < x < 1 + \frac{0.05}{m}$.
 Step 2: $|x - 1| < \delta \Rightarrow -\delta < x - 1 < \delta \Rightarrow -\delta + 1 < x < \delta + 1$.
 Then $-\delta + 1 = 1 - \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$, or $\delta + 1 = 1 + \frac{0.05}{m} \Rightarrow \delta = \frac{0.05}{m}$. In either case, $\delta = \frac{0.05}{m}$.
31. $\lim_{x \rightarrow 3} (3 - 2x) = 3 - 2(3) = -3$
 Step 1: $|(3 - 2x) - (-3)| < 0.02 \Rightarrow -0.02 < 6 - 2x < 0.02 \Rightarrow -6.02 < -2x < -5.98 \Rightarrow 3.01 > x > 2.99$ or
 $2.99 < x < 3.01$.

Step 2: $0 < |x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$.

Then $-\delta+3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta+3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

32. $\lim_{x \rightarrow -1} (-3x-2) = (-3)(-1)-2 = 1$

Step 1: $|(-3x-2)-1| < 0.03 \Rightarrow -0.03 < -3x-3 < 0.03 \Rightarrow 0.01 > x+1 > -0.01 \Rightarrow -1.01 < x < -0.99$.

Step 2: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$.

Then $-\delta-1 = -1.01 \Rightarrow \delta = 0.01$, or $\delta-1 = -0.99 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

33. $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \rightarrow 2} (x+2) = 2+2 = 4, x \neq 2$

Step 1: $\left| \left(\frac{x^2-4}{x-2} \right) - 4 \right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)} - 4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05, x \neq 2$
 $\Rightarrow 1.95 < x < 2.05, x \neq 2$.

Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$.

Then $-\delta+2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta+2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

34. $\lim_{x \rightarrow -5} \frac{x^2+6x+5}{x+5} = \lim_{x \rightarrow -5} \frac{(x+5)(x+1)}{(x+5)} = \lim_{x \rightarrow -5} (x+1) = -4, x \neq -5$.

Step 1: $\left| \left(\frac{x^2+6x+5}{x+5} \right) - (-4) \right| < 0.05 \Rightarrow -0.05 < \frac{(x+5)(x+1)}{(x+5)} + 4 < 0.05 \Rightarrow -4.05 < x+1 < -3.95, x \neq -5$
 $\Rightarrow -5.05 < x < -4.95, x \neq -5$.

Step 2: $|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta-5 < x < \delta-5$.

Then $-\delta-5 = -5.05 \Rightarrow \delta = 0.05$, or $\delta-5 = -4.95 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

35. $\lim_{x \rightarrow -3} \sqrt{1-5x} = \sqrt{1-5(-3)} = \sqrt{16} = 4$

Step 1: $|\sqrt{1-5x}-4| < 0.5 \Rightarrow -0.5 < \sqrt{1-5x}-4 < 0.5 \Rightarrow 3.5 < \sqrt{1-5x} < 4.5 \Rightarrow 12.25 < 1-5x < 20.25$
 $\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < 2.25$.

Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$.

Then $-\delta-3 = -3.85 \Rightarrow \delta = 0.85$, or $\delta-3 = -2.25 \Rightarrow 0.75$; thus $\delta = 0.75$.

36. $\lim_{x \rightarrow 2} \frac{4}{x} = \frac{4}{2} = 2$

Step 1: $\left| \frac{4}{x} - 2 \right| < 0.4 \Rightarrow -0.4 < \frac{4}{x} - 2 < 0.4 \Rightarrow 1.6 < \frac{4}{x} < 2.4 \Rightarrow \frac{10}{16} > \frac{x}{4} > \frac{10}{24} \Rightarrow \frac{10}{4} > x > \frac{10}{6}$ or $\frac{5}{3} < x < \frac{5}{2}$

Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$.

Then $-\delta+2 = \frac{5}{3} \Rightarrow \delta = \frac{1}{3}$, or $\delta+2 = \frac{5}{2} \Rightarrow \delta = \frac{1}{2}$; thus $\delta = \frac{1}{3}$.

37. Step 1: $|(9-x)-5| < \square \Rightarrow -\square < 4-x < \square \Rightarrow -\square-4 < -x < \square-4 \Rightarrow \square+4 > x > 4-\square \Rightarrow 4-\square < x < 4+\square$

Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.

Then $-\delta+4 = -\square+4 \Rightarrow \delta = \square$, or $\delta+4 = \square+4 \Rightarrow \delta = \square$. Thus choose $\delta = \square$.

38. Step 1: $|(3x-7)-2| < \square \Rightarrow -\square < 3x-9 < \square \Rightarrow 9-\square < 3x < 9+\square \Rightarrow 3-\frac{\square}{3} < x < 3+\frac{\square}{3}$.

Step 2: $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$.

Then $-\delta+3 = 3-\frac{\square}{3} \Rightarrow \delta = \frac{\square}{3}$, or $\delta+3 = 3+\frac{\square}{3} \Rightarrow \delta = \frac{\square}{3}$. Thus choose $\delta = \frac{\square}{3}$.

39. Step 1: $|\sqrt{x-5}-2| < \square \Rightarrow -\square < \sqrt{x-5}-2 < \square \Rightarrow 2-\square < \sqrt{x-5} < 2+\square \Rightarrow (2-\square)^2 < x-5 < (2+\square)^2$
 $\Rightarrow (2-\square)^2 + 5 < x < (2+\square)^2 + 5$.

Step 2: $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta+9 < x < \delta+9$.

Then $-\delta + 9$

$$= \frac{1}{4} - 4\frac{1}{4}$$

$$9 \Rightarrow \delta = 4\frac{1}{4}$$

$$- \frac{1}{4}, \text{ or } \delta +$$

$$9 = \frac{1}{4} + 4\frac{1}{4}$$

$$+ 9 \Rightarrow \delta =$$

$$4\frac{1}{4} + \frac{1}{4}.$$

Thus

choose the

smaller

distance, δ

$$= 4\frac{1}{4} - \frac{1}{4}.$$

40. Step 1: $|\sqrt{4-x}-2| < \square \Rightarrow -\square < \sqrt{4-x}-2 < \square \Rightarrow 2-\square < \sqrt{4-x} < 2+\square \Rightarrow (2-\square)^2 < 4-x < (2+\square)^2$
 $\Rightarrow -(2+\square)^2 < x-4 < -(2-\square)^2 \Rightarrow -(2+\square)^2+4 < x < -(2-\square)^2+4.$
 Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta.$
 Then $-\delta = -(2+\square)^2+4 = -\square^2-4\square \Rightarrow \delta = 4\square+\square^2$, or $\delta = -(2-\square)^2+4 = 4\square-\square^2$. Thus choose the smaller distance, $\delta = 4\square-\square^2$.
41. Step 1: For $x \neq 1$, $|x^2-1| < \square \Rightarrow -\square < x^2-1 < \square \Rightarrow 1-\square < x^2 < 1+\square \Rightarrow \sqrt{1-\square} < |x| < \sqrt{1+\square}$
 $\Rightarrow \sqrt{1-\square} < x < \sqrt{1+\square}$ near $x=1$.
 Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow -\delta+1 < x < \delta+1.$
 Then $-\delta+1 = \sqrt{1-\square} \Rightarrow \delta = 1-\sqrt{1-\square}$, or $\delta+1 = \sqrt{1+\square} \Rightarrow \delta = \sqrt{1+\square}-1$. Choose $\delta = \min\{1-\sqrt{1-\square}, \sqrt{1+\square}-1\}$, that is, the smaller of the two distances.
42. Step 1: For $x \neq -2$, $|x^2-4| < \square \Rightarrow -\square < x^2-4 < \square \Rightarrow 4-\square < x^2 < 4+\square \Rightarrow \sqrt{4-\square} < |x| < \sqrt{4+\square} \Rightarrow -\sqrt{4+\square} < x < -\sqrt{4-\square}$ near $x=-2$.
 Step 2: $|x-(-2)| < \delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2.$
 Then $-\delta-2 = -\sqrt{4+\square} \Rightarrow \delta = \sqrt{4+\square}-2$, or $\delta-2 = -\sqrt{4-\square} \Rightarrow \delta = 2-\sqrt{4-\square}$. Choose $\delta = \min\{\sqrt{4+\square}-2, 2-\sqrt{4-\square}\}$.
43. Step 1: $|\frac{1}{x}-1| < \square \Rightarrow -\square < \frac{1}{x}-1 < \square \Rightarrow 1-\square < \frac{1}{x} < 1+\square \Rightarrow \frac{1}{1+\square} < x < \frac{1}{1-\square}$
 Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
 Then $1-\delta = \frac{1}{1+\square} \Rightarrow \delta = 1-\frac{1}{1+\square} = \frac{\square}{1+\square}$, or $1+\delta = \frac{1}{1-\square} \Rightarrow \delta = \frac{1}{1-\square}-1 = \frac{\square}{1-\square}$. Choose $\delta = \frac{\square}{1+\square}$, the smaller of the two distances.
44. Step 1: $|\frac{1}{x^2}-\frac{1}{3}| < \square \Rightarrow -\square < \frac{1}{x^2}-\frac{1}{3} < \square \Rightarrow \frac{1}{3}-\square < \frac{1}{x^2} < \frac{1}{3}+\square \Rightarrow \frac{1-3\square}{3} < \frac{1}{x^2} < \frac{1+3\square}{3}$
 $\Rightarrow \frac{3}{1+3\square} > x^2 > \frac{3}{1-3\square} \Rightarrow \frac{3}{1+3\square} < x < \sqrt{\frac{3}{1-3\square}}$ or $\sqrt{\frac{3}{1+3\square}} < x < \sqrt{\frac{3}{1-3\square}}$ for x near $\sqrt{3}$.
 Step 2: $|x-\sqrt{3}| < \delta \Rightarrow -\delta < x-\sqrt{3} < \delta \Rightarrow \sqrt{3}-\delta < x < \sqrt{3}+\delta.$
 Then $\sqrt{3}-\delta = \sqrt{\frac{3}{1+3\square}} \Rightarrow \delta = \sqrt{3}-\sqrt{\frac{3}{1+3\square}}$, or $\sqrt{3}+\delta = \sqrt{\frac{3}{1-3\square}} \Rightarrow \delta = \sqrt{\frac{3}{1-3\square}}-\sqrt{3}$. Choose $\delta = \min\{\sqrt{3}-\sqrt{\frac{3}{1+3\square}}, \sqrt{\frac{3}{1-3\square}}-\sqrt{3}\}$.
45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right| < \square \Rightarrow -\square < (x-3)+6 < \square, x \neq -3 \Rightarrow -\square < x+3 < \square \Rightarrow -\square-3 < x < \square-3.$
 Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3.$
 Then $-\delta-3 = -\square-3 \Rightarrow \delta = \square$ or $\delta-3 = \square-3 \Rightarrow \delta = \square$. Choose $\delta = \square$.
46. Step 1: $\left|\left(\frac{x^2-1}{x-1}\right)-2\right| < \square \Rightarrow -\square < (x+1)-2 < \square, x \neq 1 \Rightarrow 1-\square < x < 1+\square$
 Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
 Then $1-\delta = 1-\square \Rightarrow \delta = \square$, or $1+\delta = 1+\square \Rightarrow \delta = \square$. Choose $\delta = \square$.
47. Step 1: $x < 1: |(4-2x)-2| < \square \Rightarrow 0 < 2-2x < \square$ since $x < 1$. Thus, $1-\frac{\square}{2} < x < 0$;
 $x \geq 1: |(6x-4)-2| < \square \Rightarrow 0 \leq 6x-6 < \square$ since $x \geq 1$. Thus, $1 \leq x < 1+\frac{\square}{6}$.
 Step 2: $|x-1| < \delta \Rightarrow -\delta < x-1 < \delta \Rightarrow 1-\delta < x < 1+\delta.$
 Then $1-\delta = 1-\frac{\square}{2} \Rightarrow \delta = \frac{\square}{2}$, or $1+\delta = 1+\frac{\square}{6} \Rightarrow \delta = \frac{\square}{6}$. Choose $\delta = \frac{\square}{6}$.

48. Step 1: $x < 0: |2x - 0| < \epsilon \Rightarrow -\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0;$

$x \geq 0: |x - 0| < \epsilon \Rightarrow 0 \leq x < \epsilon$

Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta.$

Then $-\delta = -\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $\delta = 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = \frac{\epsilon}{2}$.

49. By the figure, $-x \leq x \sin \frac{1}{x} \leq x$ for all $x > 0$ and $-x \geq x \sin \frac{1}{x} \geq x$ for $x < 0$. Since $\lim_{x \rightarrow 0} (-x) = \lim_{x \rightarrow 0} x = 0$, then by the sandwich theorem, in either case, $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

50. By the figure, $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ for all x except possibly at $x = 0$. Since $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} x^2 = 0$, then by the sandwich theorem, $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$.

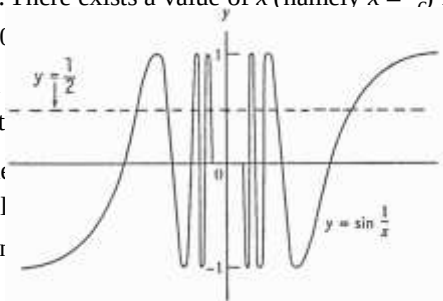
51. As x approaches the value 0, the values of $g(x)$ approach k . Thus for every number $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |g(x) - k| < \epsilon$

52. Write $x = h + c$. Then $0 < |x - c| < \delta \Leftrightarrow -\delta < x - c < \delta, x \neq c \Leftrightarrow -\delta < (h + c) - c < \delta, h + c \neq c \Leftrightarrow -\delta < h < \delta, h \neq 0 \Leftrightarrow 0 < |h - 0| < \delta$.

Thus, $\lim_{x \rightarrow c} f(x) = L \Leftrightarrow$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta \Leftrightarrow |f(h + c) - L| < \epsilon$ whenever $0 < |h - 0| < \delta \Leftrightarrow \lim_{h \rightarrow 0} f(h + c) = L$.

53. Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim_{x \rightarrow 0} f(x) = 0$, not -1 . The function $f(x) = x^2$ never gets arbitrarily close to -1 for x near 0.

54. Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely $x = \frac{\pi}{6}$) for which $|\sin x - \frac{1}{2}| < \epsilon$ for any given $\epsilon > 0$. However, $\lim_{x \rightarrow 0} \sin x = 0$.
As another example, let $g(x) = \sin \frac{1}{x}$. As you can see from the figure that there are always values of x near 0 such that $\sin \frac{1}{x} = \frac{1}{2}$. The wrong statement does not require all values of x in $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are always values of x near 0 such that $\sin \frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$, we cannot satisfy the inequality $|\sin \frac{1}{x} - \frac{1}{2}| < \epsilon$ for $x_0 = 0$.



55. $|A - 9| \leq 0.01 \Rightarrow -0.01 \leq \pi \left(\frac{x}{4} \right)^2 - 9 \leq 0.01 \Rightarrow 8.99 \leq \frac{x^2}{4} \leq 9.01 \Rightarrow (8.99) \leq x^2 \leq (9.01)$

$\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \leq x \leq 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \leq x \leq 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

$$56. V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left| \frac{V}{R} - 5 \right| \leq 0.1 \Rightarrow -0.1 \leq \frac{120}{R} - 5 \leq 0.1 \Rightarrow 4.9 \leq \frac{120}{R} \leq 5.1 \Rightarrow \frac{10}{49} \geq \frac{R}{120} \geq \frac{10}{51} \\ \Rightarrow \frac{(120)(10)}{51} \leq R \leq \frac{(120)(10)}{49} \Rightarrow 23.53 \leq R \leq 24.48.$$

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x)-2| = |x-2| = 2-x > 2-1=1$. That is, $|f(x)-2| \geq 1 \geq \frac{1}{2}$ no matter how small δ is taken when $1-\delta < x < 1 \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 2$.
- (b) $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$. Then $|f(x)-1| = |(x+1)-1| = |x| = x > 1$. That is, $|f(x)-1| \geq 1$ no matter how small δ is taken when $1 < x < 1+\delta \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 1$.
- (c) $-\delta < x-1 < 0 \Rightarrow 1-\delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x)-1.5| = |x-1.5| = 1.5-x > 1.5-1=0.5$. Also, $0 < x-1 < \delta \Rightarrow 1 < x < 1+\delta \Rightarrow f(x) = x+1$. Then $|f(x)-1.5| = |(x+1)-1.5| = |x-0.5| = x-0.5 > 1-0.5=0.5$. Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x-1 < \delta$ but $|f(x)-1.5| \geq \frac{1}{2} \Rightarrow \lim_{x \rightarrow 1} f(x) \neq 1.5$.
58. (a) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-4| = 2$. Thus for $\square < 2$, $|h(x)-4| \geq \square$ whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 4$.
- (b) For $2 < x < 2+\delta \Rightarrow h(x) = 2 \Rightarrow |h(x)-3| = 1$. Thus for $\square < 1$, $|h(x)-3| \geq \square$ whenever $2 < x < 2+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 3$.
- (c) For $2-\delta < x < 2 \Rightarrow h(x) = x^2$ so $|h(x)-2| = |x^2-2|$. No matter how small $\delta > 0$ is chosen, x^2 is close to 4 when x is near 2 and to the left on the real line $\Rightarrow |x^2-2|$ will be close to 2. Thus if $\square < 1$, $|h(x)-2| \geq \square$ whenever $2-\delta < x < 2$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 2} h(x) \neq 2$.
59. (a) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-4| \geq 0.8$. Thus for $\square < 0.8$, $|f(x)-4| \geq \square$ whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 4$.
- (b) For $3 < x < 3+\delta \Rightarrow f(x) < 3 \Rightarrow |f(x)-4.8| \geq 1.8$. Thus for $\square < 1.8$, $|f(x)-4.8| \geq \square$ whenever $3 < x < 3+\delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 4.8$.
- (c) For $3-\delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x)-3| \geq 1.8$. Again, for $\square < 1.8$, $|f(x)-3| \geq \square$ whenever $3-\delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x \rightarrow 3} f(x) \neq 3$.
60. (a) No matter how small we choose $\delta > 0$, for x near -1 satisfying $-1-\delta < x < -1+\delta$, the values of $g(x)$ are near $1 \Rightarrow |g(x)-2|$ is near 1 . Then, for $\square = \frac{1}{2}$ we have $|g(x)-2| \geq \frac{1}{2}$ for some x satisfying $-1-\delta < x < -1+\delta$, or $0 < |x+1| < \delta \Rightarrow \lim_{x \rightarrow -1} g(x) \neq 2$.
- (b) Yes, $\lim_{x \rightarrow -1} g(x) = 1$ because from the graph we can find a $\delta > 0$ such that $|g(x)-1| < \square$ if $0 < |x-(-1)| < \delta$.

61–66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3); x0 := 3;
```

```
plot( f(x), x=x0-1..x0+1, color=black, # (a)
```

```
title="Section 2.3, #61(a)");
```

```
L := limit( f(x), x=x0 ); # (b)
```

```
epsilon := 0.2; # (c)
```

```
plot( [f(x), L-epsilon, L+epsilon], x=x0-0.01..x0+0.01,
color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)");
```

```

q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1);      # (d)
delta := abs(x0-q);
plot( [f(x), L-epsilon, L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)");
for eps in [0.1, 0.005, 0.001] do                          # (e)
q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1);
delta := abs(x0-q);
head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f\n", eps, delta);
print(plot( [f(x), L-eps, L+eps], x=x0-delta..x0+delta,
color=black, linestyle=[1, 3, 3], title=head ));
end do;

```

Mathematica (assigned function and values for x0, eps and del may vary):

```

Clear[f, x]
y1:= L - eps; y2:= L + eps; x0 = 1;
f[x_]:= (3x^2 - (7x + 1)Sqrt[x] + 5)/(x - 1)
Plot[f[x], {x, x0 - 0.2, x0 + 0.2}]
L:= Limit[f[x], x -> x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2}, {x, x0 - del, x0 + del}, PlotRange -> {L - 2eps, L + 2eps}]

```

2.4 ONE-SIDED LIMITS

- | | | | |
|-----------|-----------|-----------|-----------|
| (a) True | (b) True | (c) False | (d) True |
| (e) True | (f) True | (g) False | (h) False |
| (i) False | (j) False | (k) True | (l) False |
- | | | | |
|----------|-----------|-----------|----------|
| (a) True | (b) False | (c) False | (d) True |
| (e) True | (f) True | (g) True | (h) True |
| (i) True | (j) False | (k) True | |
- (a) $\lim_{x \rightarrow 2^+} f(x) = \frac{2}{2} + 1 = 2$, $\lim_{x \rightarrow 2^-} f(x) = 3 - 2 = 1$

(b) No, $\lim_{x \rightarrow 2} f(x)$ does not exist because $\lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$

(c) $\lim_{x \rightarrow 4^-} f(x) = \frac{4}{2} + 1 = 3$, $\lim_{x \rightarrow 4^+} f(x) = \frac{4}{2} + 1 = 3$

(d) Yes, $\lim_{x \rightarrow 4} f(x) = 3$ because $3 = \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x)$
- (a) $\lim_{x \rightarrow 2^+} f(x) = \frac{2}{2} = 1$, $\lim_{x \rightarrow 2^-} f(x) = 3 - 2 = 1$, $f(2) = 2$

(b) Yes, $\lim_{x \rightarrow 2} f(x) = 1$ because $1 = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$

(c) $\lim_{x \rightarrow -1^-} f(x) = 3 - (-1) = 4$, $\lim_{x \rightarrow -1^+} f(x) = 3 - (-1) = 4$

(d) Yes, $\lim_{x \rightarrow -1} f(x) = 4$ because $4 = \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$
- (a) No, $\lim_{x \rightarrow 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0

(b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 0 = 0$

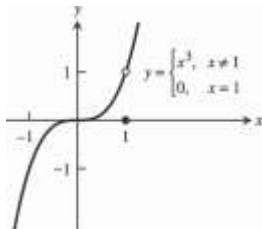
(c) $\lim_{x \rightarrow 0} f(x)$ does not exist because

$\lim_{x \rightarrow 0^+}$

$f(x)$
does
not
exist

6. (a) Yes, $\lim_{x \rightarrow 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \leq g(x) \leq \sqrt{x}$ when $x > 0$
 (b) No, $\lim_{x \rightarrow 0^-} g(x)$ does not exist since $\sqrt{x}$ is not defined for $x < 0$
 (c) Yes, $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0^+} g(x) = 0$ since $x = 0$ is a boundary point of the domain

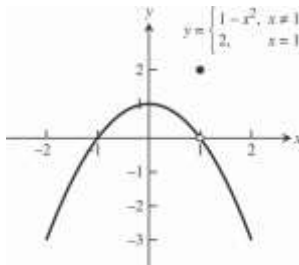
7. (a)



$$(b) \lim_{x \rightarrow 1^-} f(x) = 1 = \lim_{x \rightarrow 1^+} f(x)$$

- (c) Yes, $\lim_{x \rightarrow 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

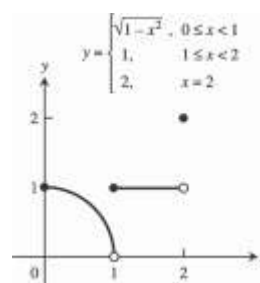
8. (a)



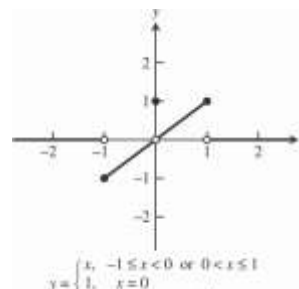
$$(b) \lim_{x \rightarrow 1^+} f(x) = 0 = \lim_{x \rightarrow 1^-} f(x)$$

- (c) Yes, $\lim_{x \rightarrow 1} f(x) = 0$ since the right-hand and left-hand limits exist and equal 0

9. (a) domain: $0 \leq x \leq 2$
 range: $0 < y \leq 1$ and $y = 2$
 (b) $\lim_{x \rightarrow c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
 (c) $x = 2$
 (d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
 range: $-1 \leq y \leq 1$
 (b) $\lim_{x \rightarrow c} f(x)$ exists for c belonging to $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
 (c) none
 (d) none



$$11. \lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5-1}} = \frac{3/2}{1/2} = \sqrt{3}$$

$$12. \lim_{x \rightarrow 1^+} \frac{x-1}{x+2} = \frac{1-1}{1+2} = \frac{0}{3} = 0$$

$$13. \lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x+x} \right) = \left(\frac{2}{2+1} \right) \left(\frac{2(-2)+5}{-2+-2} \right) = \left(\frac{2}{3} \right) \left(\frac{-1}{-2} \right) = \frac{1}{3}$$

$$14. \lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right) = \left(\frac{1}{1+1} \right) \left(\frac{1+6}{1} \right) \left(\frac{3-1}{7} \right) = \left(\frac{1}{2} \right) \left(\frac{7}{1} \right) \left(\frac{2}{7} \right) = 1$$

$$15. \lim_{h \rightarrow 0^+} \frac{h+4h+5-5}{\sqrt{h^2+4h+5}-5} = \lim_{h \rightarrow 0^+} \left(\frac{h+4h+5-5}{\sqrt{h^2+4h+5}-5} \right) \left(\frac{h+4h+5+5}{h+4h+5+5} \right) = \lim_{h \rightarrow 0^+} \frac{(h+4h+5)-5}{h(\sqrt{h^2+4h+5}+\sqrt{5})}$$

$$= \lim_{h \rightarrow 0^+} \frac{h(h+4)}{h(\sqrt{h^2+4h+5}+\sqrt{5})} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$16. \lim_{h \rightarrow 0^-} \frac{\sqrt{6-5h+11h+6}}{\sqrt{h^2+4h+5}-5} = \lim_{h \rightarrow 0^-} \left(\frac{6-5h+11h+6}{\sqrt{h^2+4h+5}-5} \right) \left(\frac{6+5h+11h+6}{6+5h+11h+6} \right)$$

$$= \lim_{h \rightarrow 0^-} \frac{6-(5h^2+11h+6)}{h(\sqrt{6+5h+11h+6}-\sqrt{6+5h+11h+6})} = \lim_{h \rightarrow 0^-} \frac{-h(5h+11)}{h(\sqrt{6+5h+11h+6}-\sqrt{6+5h+11h+6})} = \frac{-(0+11)}{6+\sqrt{6}} = -\frac{11}{6+ \sqrt{6}}$$

$$17. (a) \lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} = \lim_{x \rightarrow -2^+} (x+3) \frac{(x+2)}{(x+2)} \quad (|x+2| = (x+2) \text{ for } x > -2)$$

$$= \lim_{x \rightarrow -2^+} (x+3) = ((-2)+3) = 1$$

$$(b) \lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2} = \lim_{x \rightarrow -2^-} (x+3) \frac{-(x+2)}{(x+2)} \quad (|x+2| = -(x+2) \text{ for } x < -2)$$

$$= \lim_{x \rightarrow -2^-} (x+3)(-1) = -((-2)+3) = -1$$

$$18. (a) \lim_{x \rightarrow 1^+} \frac{\sqrt{2x(x-1)}}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{\sqrt{2x(x-1)}}{(x-1)} \quad (|x-1| = x-1 \text{ for } x > 1)$$

$$= \lim_{x \rightarrow 1^+} \sqrt{2x} = \sqrt{2}$$

$$(b) \lim_{x \rightarrow 1^-} \frac{\sqrt{2x(x-1)}}{|x-1|} = \lim_{x \rightarrow 1^-} \frac{\sqrt{2x(x-1)}}{-(x-1)} \quad (|x-1| = -(x-1) \text{ for } x < 1)$$

$$= \lim_{x \rightarrow 1^-} -\sqrt{2x} = -\sqrt{2}$$

$$19. (a) \text{ If } 0 < x < \frac{\pi}{2}, \text{ then } \sin x > 0, \text{ so that } \lim_{x \rightarrow 0^+} \frac{|\sin x|}{\sin x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{\sin x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$(b) \text{ If } -\frac{\pi}{2} < x < 0, \text{ then } \sin x < 0, \text{ so that } \lim_{x \rightarrow 0^-} \frac{|\sin x|}{\sin x} = \lim_{x \rightarrow 0^-} \frac{-\sin x}{\sin x} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$20. (a) \text{ If } 0 < x < \frac{\pi}{2}, \text{ then } \cos x < 1, \text{ so that } \lim_{x \rightarrow 0^+} \frac{1-\cos x}{|\cos x-1|} = \lim_{x \rightarrow 0^+} \frac{1-\cos x}{-(\cos x-1)} = \lim_{x \rightarrow 0^+} \frac{1-\cos x}{1-\cos x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$(b) \text{ If } -\frac{\pi}{2} < x < 0, \text{ then } \cos x < 1, \text{ so that } \lim_{x \rightarrow 0^-} \frac{\cos x-1}{|\cos x-1|} = \lim_{x \rightarrow 0^-} \frac{\cos x-1}{-(\cos x-1)} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$21. (a) \lim_{\theta \rightarrow 3^+} \frac{t-\lfloor t \rfloor}{\theta} = \frac{3}{3} = 1$$

$$(b) \lim_{\theta \rightarrow 3^-} \frac{t-\lfloor t \rfloor}{\theta} = \frac{2}{3}$$

$$22. (a) \lim_{t \rightarrow 4^+} (t - \lfloor t \rfloor) = 4 - 4 = 0$$

$$(b) \lim_{t \rightarrow 4^-} (t - \lfloor t \rfloor) = 4 - 3 = 1$$

$$23. \lim_{x \rightarrow 0} \frac{\sin 2\theta}{\theta} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (\text{where } x = \sqrt{2}\theta)$$

$$\lim_{\theta \rightarrow 0} \frac{\sqrt{2\theta}}{\sqrt{2\theta}} = \lim_{x \rightarrow 0} x$$

$$24. \lim_{t \rightarrow 0} \frac{\sin kt}{t} = \lim_{t \rightarrow 0} \frac{k \sin kt}{kt} = \lim_{\theta \rightarrow 0} \frac{k \sin \theta}{\theta} = k \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = k \cdot 1 = k \quad (\text{where } \theta = kt)$$

$$25. \lim_{y \rightarrow 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \rightarrow 0} \frac{3 \sin 3y}{3y} = \frac{3}{4} \lim_{y \rightarrow 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{3}{4} \quad (\text{where } \theta = 3y)$$

$$26. \lim_{h \rightarrow 0^-} \frac{\sin 3}{\frac{h}{h}} = \lim_{h \rightarrow 0^-} \left(3 \cdot \frac{\sin 3}{h} \right) = 3 \lim_{h \rightarrow 0^-} \left(\frac{\sin 3}{\frac{1}{3h}} \right) = 3 \lim_{h \rightarrow 0^-} \left(\frac{\sin 3}{\frac{1}{3h}} \right) = 3 \lim_{\theta \rightarrow 0^-} \left(\frac{\sin \theta}{\frac{1}{3\theta}} \right) = 3 \cdot 1 = 3 \quad (\text{where } \theta = 3h)$$

$$27. \lim_{x \rightarrow 0} \frac{\tan 2x}{x} = \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{\cos 2x} \cdot \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x \cos 2x} = \left(\lim_{x \rightarrow 0} \frac{1}{\cos 2x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \right) = 1 \cdot 2 = 2$$

$$28. \lim_{t \rightarrow 0} \frac{2t}{\tan t} = 2 \lim_{t \rightarrow 0} \left(\frac{t}{\frac{\sin t}{\cos t}} \right) = 2 \lim_{t \rightarrow 0} \frac{t \cos t}{\sin t} = 2 \left(\lim_{t \rightarrow 0} \cos t \right) \left(\lim_{t \rightarrow 0} \frac{t}{\sin t} \right) = 2 \cdot 1 \cdot 1 = 2$$

$$29. \lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x} \right) = \left(\lim_{x \rightarrow 0} \frac{x}{\sin 2x} \right) \left(\lim_{x \rightarrow 0} \frac{1}{\cos 5x} \right) = \left(\frac{1}{2} \cdot 1 \right) (1) = \frac{1}{2}$$

$$30. \lim_{x \rightarrow 0} 6x^2 (\cot x)(\csc 2x) = \lim_{x \rightarrow 0} \frac{6x^2 \cos x}{\sin x \sin 2x} = \lim_{x \rightarrow 0} \left(3 \cos x \cdot \frac{x}{\sin x} \cdot \frac{2x}{\sin 2x} \right) = 3 \cdot 1 \cdot 1 = 3$$

$$31. \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x} \right) = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \cdot \frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \frac{x}{\sin x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\frac{\sin x}{x}} \cdot \frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = (1)(1) + 1 = 2$$

$$32. \lim_{x \rightarrow 0} \frac{x - x + \sin x}{2x} = \lim_{x \rightarrow 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{1}{2} \left(\frac{\sin x}{x} \right) \right) = 0 - \frac{1}{2} + \frac{1}{2} (1) = 0$$

$$33. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(2 \cos \theta)(1 + \cos \theta)} = \frac{0}{(2)(2)} = 0$$

$$34. \lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{\frac{x(1 - \cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{9x}}{\left(\frac{\sin 3x}{3x} \right)^2} = \frac{\frac{1}{9} \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$$

$$35. \lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = 1 - \cos t \rightarrow 0 \text{ as } t \rightarrow 0$$

$$36. \lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = \sin h \rightarrow 0 \text{ as } h \rightarrow 0$$

$$37. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta} \right) = \frac{1}{2} \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta} \right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

$$38. \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x} = \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{\sin 4x} \cdot \frac{4x}{5x} \cdot \frac{5}{4} \right) = \frac{5}{4} \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \cdot \frac{4x}{\sin 4x} \right) = \frac{5}{4} \cdot 1 \cdot 1 = \frac{5}{4}$$

39. $\lim_{\theta \rightarrow 0} \theta \cos \theta = 0 \cdot 1 = 0$

40. $\lim_{\theta \rightarrow 0} \sin \theta \cot 2\theta = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{2 \sin \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\cos 2\theta}{2 \cos \theta} = \frac{1}{2}$

41. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x} = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8} \right)$
 $= \frac{3}{8} \lim_{x \rightarrow 0} \left(\frac{1}{\cos 3x} \right) \left(\frac{\sin 3x}{3x} \right) \left(\frac{8x}{\sin 8x} \right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$
42. $\lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y} = \lim_{y \rightarrow 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim_{y \rightarrow 0} \left(\frac{\sin 3y}{y} \right) \left(\frac{\sin 4y}{\cos 4y} \right) \left(\frac{\cos 5y}{\sin 5y} \right) \left(\frac{3 \cdot 4 \cdot 5y}{3 \cdot 4 \cdot 5y} \right)$
 $= \lim_{y \rightarrow 0} \left(\frac{\sin 3y}{3y} \right) \left(\frac{\sin 4y}{4y} \right) \left(\frac{5y}{\sin 5y} \right) \left(\frac{\cos 5y}{\cos 4y} \right) \left(\frac{3 \cdot 4}{5} \right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12}{5} = \frac{12}{5}$
43. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta}}{\theta^2 \frac{\cos 3\theta}{\sin 3\theta}} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta^2 \cos \theta \cos 3\theta} = \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right) \left(\frac{1}{\theta} \right) \left(\frac{1}{\cos \theta \cos 3\theta} \right) = (1)(1) \left(\frac{1}{1 \cdot 1} \right) = 1$
44. $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2 \theta \frac{\cos^2 2\theta}{\sin^2 2\theta}} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta}{\sin^2 \theta \cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta (2 \sin \theta \cos \theta)^2}{\sin^2 \theta \cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \frac{\theta \cos 4\theta (4 \sin^2 \theta \cos^2 \theta)}{\sin^2 \theta \cos^2 2\theta \sin 4\theta}$
 $= \lim_{\theta \rightarrow 0} \frac{4\theta \cos 4\theta \cos^2 \theta}{\cos^2 2\theta \sin 4\theta} = \lim_{\theta \rightarrow 0} \left(\frac{4\theta}{\sin 4\theta} \right) \left(\frac{\cos 4\theta \cos^2 \theta}{\cos^2 2\theta} \right) = \lim_{\theta \rightarrow 0} \left(\frac{1}{\frac{\sin 4\theta}{4\theta}} \right) \left(\frac{\cos 4\theta \cos^2 \theta}{\cos^2 2\theta} \right) = \left(\frac{1}{1} \right) \left(\frac{1 \cdot 1^2}{1^2} \right) = 1$
45. $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x} = \lim_{x \rightarrow 0} \frac{1 - \cos 3x}{2x} \cdot \frac{1 + \cos 3x}{1 + \cos 3x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 3x}{2x(1 + \cos 3x)} = \lim_{x \rightarrow 0} \frac{\sin^2 3x}{2x(1 + \cos 3x)} = \lim_{x \rightarrow 0} \frac{3}{2} \cdot \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{1 + \cos 3x}$
 $= \lim_{\theta \rightarrow 0} \frac{3}{2} \cdot \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} = \frac{3}{2} (1) \left(\frac{0}{1+1} \right) = 0 \quad (\text{where } \theta = 3x)$
46. $\lim_{x \rightarrow 0} \frac{\cos^2 x - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x (\cos x - 1)}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x (\cos x - 1)}{x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x \rightarrow 0} \frac{\cos x (\cos^2 x - 1)}{x^2 (\cos x + 1)} = \lim_{x \rightarrow 0} \frac{\cos x (-\sin^2 x)}{x^2 (\cos x + 1)}$
 $= \lim_{x \rightarrow 0} \left\{ -\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{\cos x}{\cos x + 1} \right\} = -(1)(1) \cdot \frac{1}{1+1} = -\frac{1}{2}$
47. Yes. If $\lim_{x \rightarrow a^+} f(x) = L = \lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x) = L$. If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.
48. Since $\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{x \rightarrow c^+} f(x) = L$ and $\lim_{x \rightarrow c^-} f(x) = L$, then $\lim_{x \rightarrow c} f(x)$ can be found by calculating $\lim_{x \rightarrow c^+} f(x)$.
49. If f is an odd function of x , then $f(-x) = -f(x)$. Given $\lim_{x \rightarrow 0^+} f(x) = 3$, then $\lim_{x \rightarrow 0^-} f(x) = -3$.
50. If f is an even function of x , then $f(-x) = f(x)$. Given $\lim_{x \rightarrow 2^-} f(x) = 7$ then $\lim_{x \rightarrow -2^+} f(x) = 7$. However, nothing can be said about $\lim_{x \rightarrow -2^-} f(x)$ because we don't know $\lim_{x \rightarrow 2^+} f(x)$.
51. $I = (5, 5 + \delta) \Rightarrow 5 < x < 5 + \delta$. Also, $\sqrt{x-5} < \square \Rightarrow x-5 < \square^2 \Rightarrow x < 5 + \square^2$. Choose $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 5^+} \sqrt{x-5} = 0$.
52. $I = (4 - \delta, 4) \Rightarrow 4 - \delta < x < 4$. Also, $\sqrt{4-x} < \square \Rightarrow 4-x < \square^2 \Rightarrow x > 4 - \square^2$. Choose $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 4^-} \sqrt{4-x} = 0$.

53. As $x \rightarrow 0^-$ the number x is always negative. Thus, $\left| \frac{x}{|x|} - (-1) \right| < \square \Rightarrow \left| \frac{x}{-x} + 1 \right| < \square \Rightarrow 0 < \square$ which is always true independent of the value of x . Hence we can choose any $\delta > 0$ with $-\delta < x < 0 \Rightarrow \lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$.
54. Since $x \rightarrow 2^+$ we have $x > 2$ and $|x - 2| = x - 2$. Then, $\left| \frac{x-2}{|x-2|} - 1 \right| = \left| \frac{x-2}{x-2} - 1 \right| < \square \Rightarrow 0 < \square$ which is always true so long as $x > 2$. Hence we can choose any $\delta > 0$, and thus $2 < x < 2 + \delta \Rightarrow \left| \frac{x-2}{|x-2|} - 1 \right| < \square$. Thus, $\lim_{x \rightarrow 2^+} \frac{x-2}{|x-2|} = 1$.
55. (a) $\lim_{x \rightarrow 400^+} \lfloor x \rfloor = 400$. Just observe that if $400 < x < 401$, then $\lfloor x \rfloor = 400$. Thus if we choose $\delta = 1$, we have for any number $\square > 0$ that $400 < x < 400 + \delta \Rightarrow \lfloor x \rfloor - 400 = |400 - 400| = 0 < \square$.
- (b) $\lim_{x \rightarrow 400^-} \lfloor x \rfloor = 399$. Just observe that if $399 < x < 400$ then $\lfloor x \rfloor = 399$. Thus if we choose $\delta = 1$, we have for any number $\square > 0$ that $400 - \delta < x < 400 \Rightarrow \lfloor x \rfloor - 399 = |399 - 399| = 0 < \square$.
- (c) Since $\lim_{x \rightarrow 400^+} \lfloor x \rfloor \neq \lim_{x \rightarrow 400^-} \lfloor x \rfloor$ we conclude that $\lim_{x \rightarrow 400} \lfloor x \rfloor$ does not exist.
56. (a) $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{x} = \sqrt{0} = 0$; $|\sqrt{x} - 0| < \square \Rightarrow -\square < \sqrt{x} < \square \Rightarrow 0 < x < \square^2$ for x positive. Choose $\delta = \square^2 \Rightarrow \lim_{x \rightarrow 0^+} f(x) = 0$.
- (b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the sandwich theorem since $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$ for all $x \neq 0$. Since $|x^2 - 0| = |-x^2 - 0| = x^2 < \square$ whenever $|x| < \sqrt{\square}$, we choose $\delta = \sqrt{\square}$ and obtain $\left| x^2 \sin\left(\frac{1}{x}\right) - 0 \right| < \square$ if $-\delta < x < 0$.
- (c) The function f has limit 0 at $x_0 = 0$ since both the right-hand and left-hand limits exist and equal 0.

2.5 CONTINUITY

- No, discontinuous at $x = 2$, not defined at $x = 2$
 - No, discontinuous at $x = 3$, $1 = \lim_{x \rightarrow 3^-} g(x) \neq g(3) = 1.5$
 - Continuous on $[-1, 3]$
 - No, discontinuous at $x = 1$, $1.5 = \lim_{x \rightarrow 1^-} k(x) \neq \lim_{x \rightarrow 1^+} k(x) = 0$
- Yes
 - Yes
 - Yes, $\lim_{x \rightarrow -1^+} f(x) = 0$
 - Yes
 - Yes, $f(1) = 1$
 - No
 - Yes, $\lim_{x \rightarrow 1} f(x) = 2$
 - No
 - No
 - No
 - $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$
 - $f(2) = 0$, since $\lim_{x \rightarrow 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \rightarrow 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x \rightarrow 1} f(x)$
11. Nonremovable discontinuity at $x = 1$ because $\lim_{x \rightarrow 1} f(x)$ fails to exist ($\lim_{x \rightarrow 1^-} f(x) = 1$ and $\lim_{x \rightarrow 1^+} f(x) = 0$).
Removable discontinuity at $x = 0$ by assigning the number $\lim_{x \rightarrow 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.
12. Nonremovable discontinuity at $x = 1$ because $\lim_{x \rightarrow 1} f(x)$ fails to exist ($\lim_{x \rightarrow 1^-} f(x) = 2$ and $\lim_{x \rightarrow 1^+} f(x) = 1$).
Removable discontinuity at $x = 2$ by assigning the number $\lim_{x \rightarrow 2} f(x) = 1$ to be the value of $f(2)$ rather than $f(2) = 2$.
13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$
14. Discontinuous only when $(x + 2)^2 = 0 \Rightarrow x = -2$
15. Discontinuous only when $x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0 \Rightarrow x = 3$ or $x = 1$
16. Discontinuous only when $x^2 - 3x - 10 = 0 \Rightarrow (x - 5)(x + 2) = 0 \Rightarrow x = 5$ or $x = -2$
17. Continuous everywhere. ($|x - 1| + \sin x$ defined for all x ; limits exist and are equal to function values.)
18. Continuous everywhere. ($|x| + 1 \neq 0$ for all x ; limits exist and are equal to function values.)
19. Discontinuous only at $x = 0$
20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n - 1)\frac{\pi}{2}$, n an integer, but continuous at all other x .
21. Discontinuous when $2x$ is an integer multiple of π , i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x .
22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2} = (2n - 1)\frac{\pi}{2}$, n an integer $\Rightarrow x = 2n - 1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.
23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n - 1)\frac{\pi}{2}$, n an integer, but continuous at all other x .
24. Continuous everywhere since $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq \sin^2 x \leq 1 \Rightarrow 1 + \sin^2 x \geq 1$; limits exist and are equal to the function values.
25. Discontinuous when $2x + 3 < 0$ or $x < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.
26. Discontinuous when $3x - 1 < 0$ or $x < \frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3}, \infty\right)$.
27. Continuous everywhere: $(2x - 1)^{1/3}$ is defined for all x ; limits exist and are equal to function values.
28. Continuous everywhere: $(2 - x)^{1/5}$ is defined for all x ; limits exist and are equal to function values.
29. Continuous everywhere since $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 2)}{x - 3} = \lim_{x \rightarrow 3} (x + 2) = 5 = g(3)$

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30. Discontinuous at $x = -2$ since $\lim_{x \rightarrow -2} f(x)$ does not exist while $f(-2) = 4$.
31. Discontinuous at $x = 1$; $\lim_{x \rightarrow 1^+} (x^2 + 2) = 3$, but $\lim_{x \rightarrow 1^-} e^x = e$, so that $\lim_{x \rightarrow 1} f(x)$ does not exist while $f(1) = e$;
and $\lim_{x \rightarrow 0^-} (1 - x) = 1 = \lim_{x \rightarrow 0^+} e^x$, so that $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$
32. Discontinuous at $x = \ln 2$, since $2 - e^x = 0 \Rightarrow e^x = 2 \Rightarrow \ln e^x = \ln 2 \Rightarrow x = \ln 2$
33. $\lim_{x \rightarrow \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$
34. $\lim_{t \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan t)\right) = \sin\left(\frac{\pi}{2} \cos(\tan(0))\right) = \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.
35. $\lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1) = \lim_{y \rightarrow 1} \sec(y \sec^2 y - \sec^2 y) = \lim_{y \rightarrow 1} \sec((y - 1) \sec^2 y) = \sec((1 - 1) \sec^2 1) = \sec 0 = 1$, and function continuous at $y = 1$.
36. $\lim_{x \rightarrow 0} \tan\left[\frac{\pi}{4} \cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4} \cos(\sin(0))\right] = \tan\left(\frac{\pi}{4} \cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.
37. $\lim_{t \rightarrow 0} \cos\left[\frac{\pi}{\sqrt{19-3 \sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19-3 \sec 0}}\right] = \cos \frac{\pi}{\sqrt{16}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.
38. $\lim_{x \rightarrow \frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3} \tan x} = \sqrt{\csc^2\left(\frac{\pi}{6}\right) + 5\sqrt{3} \tan\left(\frac{\pi}{6}\right)} = \sqrt{4 + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{9} = 3$, and function continuous at $x = \frac{\pi}{6}$.
39. $\lim_{x \rightarrow 0^+} \sin\left(\frac{\pi}{2} e^{\sqrt{x}}\right) = \sin\left(\frac{\pi}{2} e^0\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and the function is continuous at $x = 0$.
40. $\lim_{x \rightarrow 1} \cos^{-1}(\ln \sqrt{x}) = \cos^{-1}(\ln \sqrt{1}) = \cos^{-1}(0) = \frac{\pi}{2}$, and the function is continuous at $x = 1$.
41. $g(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = x+3, x \neq 3 \Rightarrow g(3) = \lim_{x \rightarrow 3} (x+3) = 6$
42. $h(t) = \frac{t^2+3t-10}{t-2} = \frac{(t+5)(t-2)}{t-2} = t+5, t \neq 2 \Rightarrow h(2) = \lim_{t \rightarrow 2} (t+5) = 7$
43. $f(s) = \frac{s^3-1}{s^3-1} = \frac{(s^2+s+1)(s-1)}{(s+1)(s-1)} = \frac{s^2+s+1}{s+1}, s \neq 1 \Rightarrow f(1) = \lim_{s \rightarrow 1} \left(\frac{s^2+s+1}{s+1}\right) = \frac{3}{2}$
44. $g(x) = \frac{x^2-16}{x^2-3x-4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1}, x \neq 4 \Rightarrow g(4) = \lim_{x \rightarrow 4} \left(\frac{x+4}{x+1}\right) = \frac{8}{5}$
45. As defined, $\lim_{x \rightarrow 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim_{x \rightarrow 3^+} (2a)(3) = 6a$. For $f(x)$ to be continuous we must have $6a = 8 \Rightarrow a = \frac{4}{3}$.