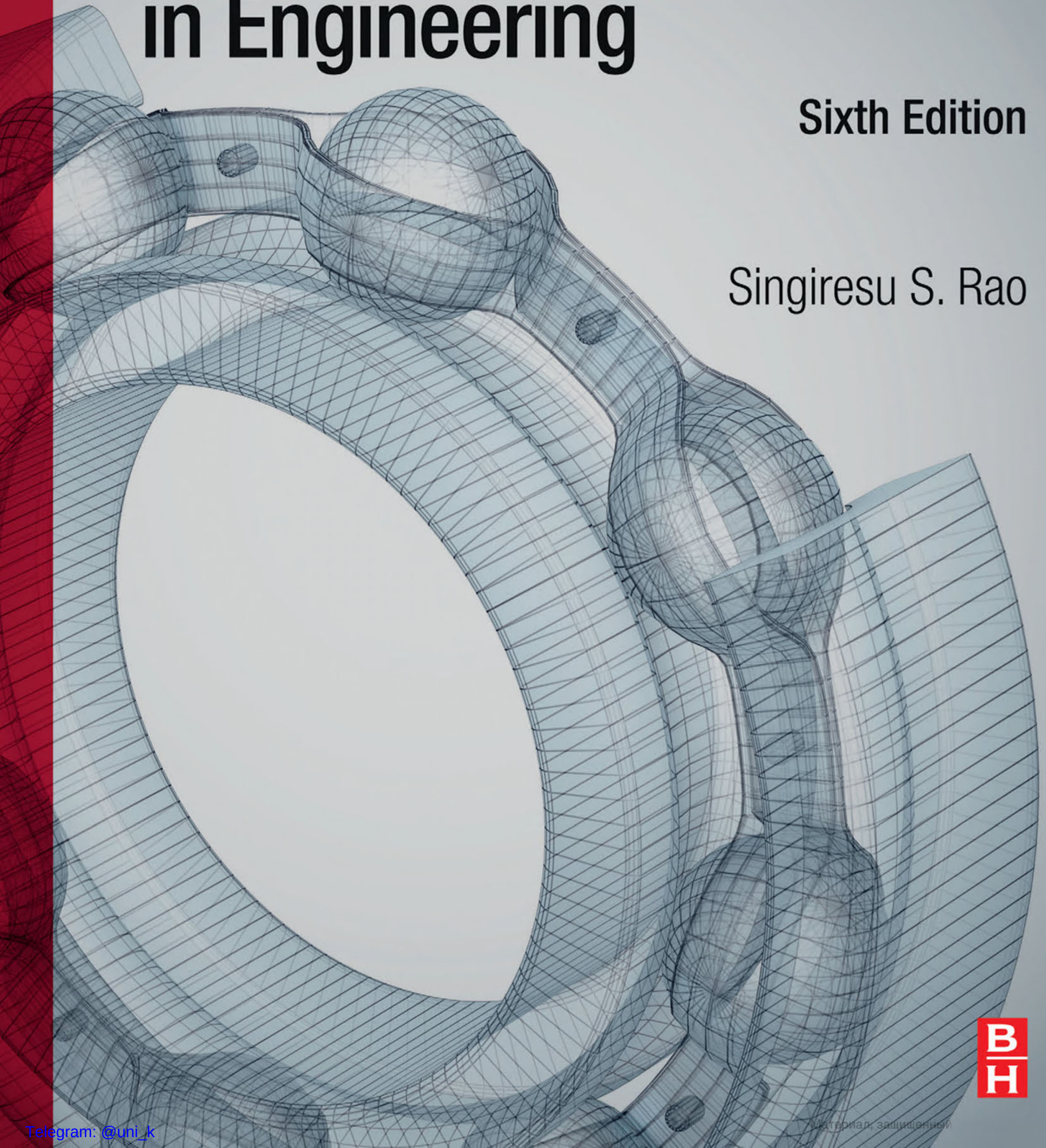
The Finite Element Method in Engineering

Sixth Edition



Singiresu S. Rao

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Singiresu S. Rao
Professor
Department of Mechanical and Aerospace Engineering
University of Miami, Coral Gables, FL, United States

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About the Author

Dr. S.S. Rao is a professor in the Department of Mechanical and Aerospace Engineering at University of Miami, Coral Gables, Florida. He was the chairman of the department during 1998−2011. Before that, he worked as a professor in the School of Mechanical Engineering at Purdue University, West Lafayette, Indiana; and a professor of mechanical engineering at the San Diego State University, San Diego, California, and the Indian Institute of Technology, Kanpur. He was a visiting research scientist for 2 years at NASA Langley Research Center.

Professor Rao is the author of eight textbooks: *The Finite Element Method in Engineering, Engineering Optimization, Mechanical Vibrations, Reliability-Based Design, Vibration of Continuous Systems, Reliability Engineering, Applied Numerical Methods for Engineers and Scientists,* and *Optimization Methods: Theory and Applications.* He coedited the three-volume *Encyclopedia of Vibration*. He edited four volumes of *Proceedings of the ASME Design Automation and Vibration Conferences*. He published over 200 journal papers in the areas of multiobjective optimization, structural dynamics and vibration, structural control, uncertainty modeling, analysis, design, and optimization using probability, fuzzy, interval, evidence, and gray system theories. Thirty-two doctoral students received degrees under the supervision of Professor S.S. Rao.

Professor Rao received numerous awards for academic and research achievements. He was awarded the Vepa Krishnamurti Gold Medal for University First Rank in all 5 years of the bachelors program among students of all branches of engineering in all colleges from Andhra University. He was awarded the Lazarus Prize for University First Rank among students of mechanical engineering in all colleges from Andhra University. He received the First Prize in the James F. Lincoln Design Contest open to all masters of science and doctoral students in the Untied States and Canada for a paper he wrote on “Automated optimum design of aircraft wing structures” from his doctoral dissertation. He received the Eliahu I. Jury Award for Excellence in Research from the College of Engineering, University of Miami in 2002. He was awarded the Distinguished Probabilistic Methods Educator Award from the Society of Automotive Engineers International for Demonstrated Excellence in Research Contributions in the Application of Probabilistic Methods to Diversified Fields, Including Aircraft Structures, Building Structures, Machine Tools, Air-conditioning and Refrigeration Systems, and Mechanisms in 1999; received the American Society of Mechanical Engineers (ASME) Design Automation Award for Pioneering Contributions to Design Automation, particularly in Multiobjective Optimization, and Uncertainty Modeling, Analysis, and Design Using Probability, Fuzzy, Interval, and Evidence Theories in 2012; and was awarded the ASME Worcester Reed Warner Medal for Outstanding Contributions to the Permanent Literature of Engineering, Particularly for His Highly Popular Books on Engineering Optimization, Reliability-Based Design, Vibrations, Finite Element Method, and Numerical Methods; and Numerous Trendsetting Research Papers in 2013.

Preface

The finite element method is a numerical method that can be used for the accurate solution of complex engineering problems. Although the origins of the method can be traced to several centuries ago, the method as currently used was originally presented by Turner, Clough, Martin, and Topp in 1956 in the context of the analysis of aircraft structures. Thereafter, within a decade, the potential of the method to solve different types of applied science and engineering problems was recognized. Over the years, the finite element technique has been so well established that today, it is considered one of the best methods to solve a wide variety of practical problems efficiently. In addition, the method has become an active research area not only for engineers but also for applied mathematicians. In fact, the finite element method appears to be the only technique developed by engineers and pursued by applied mathematicians. A main reason for the popularity of the method in different fields of engineering is that once a general computer program is written, it can be used to solve a variety of problems simply by changing the input data.

APPROACH OF THE BOOK

The objective of this book is to introduce the various aspects of the finite element method as applied to the solution of engineering problems in a systematic and simple manner. It develops each technique and idea from basic principles. New concepts are illustrated with simple examples whenever possible. An introduction to commercial software systems, ABAQUS and ANSYS, including some sample applications with images and output, is also presented in two separate chapters. In addition, several MATLAB programs are given along with examples to illustrate the use of the programs in a separate chapter. After studying the material presented in the book, the reader will not only be able to understand the current literature on the finite element method but will also be in a position to solve finite element problems using commercial software such as ABAQUS and ANSYS, use the MATLAB programs given in the book to solve a variety of finite element problems from different areas, and, if needed, be able to develop short programs to solve engineering problems.

FEATURES OF THIS EDITION

In this edition, some topics are modified and rewritten, and some new topics are added.

Most modifications and additions were suggested by users of the book and by reviewers. Some important features of the current edition are:

- 157 illustrative examples are given to illustrate the concepts, compared with 151 in the previous edition.
- 708 problems are included, compared with 680 in the previous edition. The solution to most problems is given in the Solutions Manual, which is available to instructors who use the book as a textbook.
- Expanded coverage is given of finite element applications to different areas of engineering.
- Answers to selected problems are given at the web site of the book.
- Solutions to most problems are available at the Web site of the book to instructors who use the book as a textbook.
- A comparison of the performance of the finite element method with those of other approximate analytical and numerical methods for a beam vibration problem is given in an Appendix.
- A total of 544 review questions are included at the end of chapters, with answers made available at the Web site of the book. These can be used by readers to review and test their understanding of the text material. The review questions are in the form of questions requiring brief answers, true–false questions, fill in the blank–type questions, multiple-choice questions, and matching-type questions.

- Ten MATLAB programs are given for the finite element analysis problems related to stress analysis, heat transfer, and fluid flow. The listings or codes of these programs are available at the Web site of the book for readers. These programs, with instructions given in the book (Chapter 23), can be used to solve a variety of finite element analysis problems.
- All of the concepts of the finite element method are explained with the help of illustrative examples. More than 200 references are given in the book.

ORGANIZATION

The book is divided into 23 chapters and two appendixes. Chapter 1 gives an introduction and overview of the finite element method. The basic approach and the generality of the method are illustrated through several simple examples. Chapters 2−7 describe the basic finite element procedure and the solution to the resulting equations. Finite element discretization and modeling, including considerations in selecting the number and types of elements, is discussed in Chapter 2. Interpolation models in terms of Cartesian and natural coordinate systems are given in Chapter 3. Chapter 4 discusses higher-order and isoparametric elements. The use of Lagrange and Hermite polynomials is also discussed in this chapter. The derivation of element characteristic matrices and vectors using variational and weighted residual approaches is given in Chapter 5. The assembly of element characteristic matrices and vectors and the derivation of system equations, including the various methods to incorporate boundary conditions, are indicated in Chapter 6. The solutions to finite element equations arising in equilibrium, eigenvalue, and propagation (transient or unsteady) problems are briefly outlined in Chapter 7.

Application of the finite element method to solid and structural mechanics problems is considered in Chapters 8−12. The basic equations of solid mechanics—namely, internal and external equilibrium equations, stress−strain relations, strain−displacement relations, and compatibility conditions—are summarized in Chapter 8. The analysis of trusses, beams, and frames is the topic of Chapter 9. The development of in-plane and bending plate elements is discussed in Chapter 10. The analysis of axisymmetric and three-dimensional solid bodies is considered in Chapter 11. Dynamic analysis, including free and forced vibration, of solid and structural mechanics problems is outlined in Chapter 12.

Chapters 13−16 are devoted to heat transfer applications. The basic equations of conduction, convection, and radiation heat transfer are summarized and finite element equations are formulated in Chapter 13. The solutions to one-, two-, and three-dimensional heat transfer problems are discussed in Chapters 14−16, respectively. Both steady-state and transient problems are considered. Application of the finite element method to fluid mechanics problems is discussed in Chapters 17−19. Chapter 17 gives a brief outline of the basic equations of fluid mechanics. The analysis of inviscid incompressible flows is considered in Chapter 18. The solution to incompressible viscous flows as well as non-Newtonian fluid flows is considered in Chapter 19. Chapter 20 presents the solution to the quasi-harmonic (Poisson) equation. Finally, the solution to engineering problems using the commercial finite element software systems ABAQUS and ANSYS, and also MATLAB programs is described in Chapters 21−23, respectively. A comparison of the finite element method with the exact solution, analytical approximation methods of Rayleigh, Rayleigh-Ritz, and Galerkin methods, and the finite difference method is given, with an illustrative example, for a typical eigenvalue problem (the natural frequency analysis of a beam) in Appendix A. The Green−Gauss theorem, which deals with integration by parts in two and three dimensions, is given in Appendix B.

The book is based on the author's experience in teaching the course to engineering students during the past several years. A basic knowledge of matrix theory is required in understanding the various topics presented in the book. More than enough material is included for a first course on the subject. Different parts of the book can be covered depending on the background of students and also on the emphasis to be given on specific areas, such as solid mechanics, heat transfer, and fluid mechanics. The student can be assigned a term project in which he or she is required to modify some of the established elements or to develop new finite elements and use them to solve a problem of his or her choice. The material of the book is also useful for self-study by practicing engineers who would like to learn the method.

RESOURCES FOR INSTRUCTORS

For instructors using this book as a textbook for their course, please visit *textbooks.elsevier.com* to register for the Instructor's Solutions Manual, Electronic Images from the text, MATLAB files, and other updated resources related to material presented in the text. Also available for readers of this book: m-files, answers to review questions, and other related resources, can also be accessed at *textbooks.elsevier.com*.

Acknowledgments

I express my appreciation to the many students who took my courses on the finite element method and helped me improve the presentation of the material. I would also like to thank the researchers, users, and faculty, whose comments and suggestions have helped me improve the book. I am also thankful to the reviewers of the revision plan of this revised edition, whose feedback helped improve the project.

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S.S. Rao
srao@miami.edu
July 2017

Part I

Introduction

Chapter 1

Overview of Finite Element Method

Chapter Outline

1.1 BASIC CONCEPT

The basic idea in the finite element method is to find the solution of a complicated problem by replacing it with a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution. The existing mathematical tools will not be sufficient to find the exact solution (and sometimes, even an approximate solution) of most of the practical problems. Thus, in the absence of any other convenient method to find even the approximate solution of a given problem, we have to prefer the finite element method. Moreover, in the finite element method, it will often be possible to improve or refine the approximate solution by spending more computational effort.

In the finite element method, the solution region is considered to be built of many small, interconnected subregions called elements. As an example of how a finite element model might be used to represent a complex geometrical shape, consider the milling machine structure shown in Fig. 1.1A. Since it is very difficult to find the exact response (like stresses and displacements) of the machine under any specified cutting (loading) condition, this structure is approximated as composed of several pieces as shown in Fig. 1.1B in the finite element method. In each piece or element, a convenient approximate solution is assumed and the conditions of overall equilibrium of the structure are derived. The satisfaction of these conditions will yield an approximate solution for the displacements and stresses. Fig. 1.2 shows the finite element idealization of a fighter aircraft.

1.2 HISTORICAL BACKGROUND

Although the name of the finite element method was introduced in 1960 by Clough [1.42], the concept dates back several centuries. For example, ancient mathematicians found the circumference of a circle by approximating it by the perimeter of a polygon as shown in Fig. 1.3. In terms of the present-day notation, each side of the polygon can be called an element. By considering the approximating polygon inscribed or circumscribed, we can obtain a lower bound $S^{(l)}$ or an upper bound $S^{(u)}$ for the true circumference S. Furthermore, as the number of sides of the polygon is increased, the approximate values

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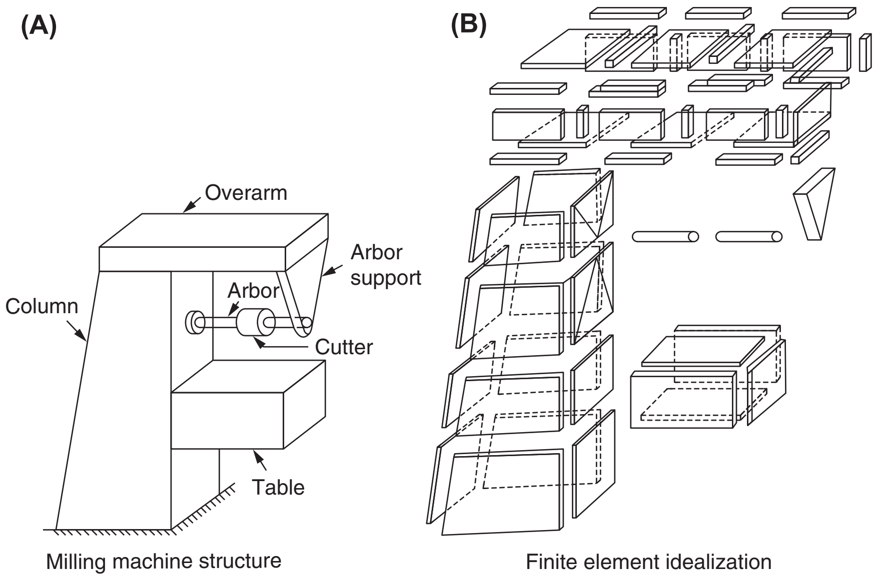


FIGURE 1.1 Representation of a milling machine structure by finite elements.

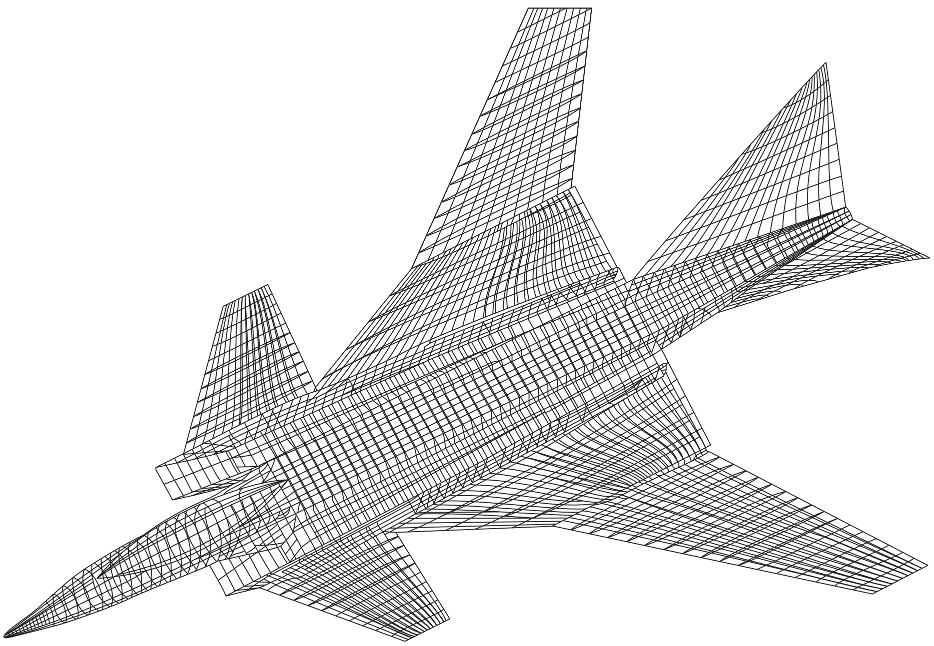


FIGURE 1.2 Finite element mesh of a fighter aircraft. *Reprinted with permission from Anamet Laboratories, Inc.*

converge to the true value. These characteristics, as will be seen later, will hold true in any general finite element application.

To find the differential equation of a surface of minimum area bounded by a specified closed curve, in 1851 Schellback discretized the surface into several triangles and used a finite difference expression to find the total discretized area [1.35]. In the current finite element method, a differential equation is solved by replacing it by a set of algebraic equations. Since the early 1900s, the behavior of structural frameworks, composed of several bars arranged in a regular pattern, has been approximated by that of an isotropic elastic body [1.36]. In 1943, Courant presented a method of determining the torsional rigidity of a hollow shaft by dividing the cross section into several triangles and using a linear variation of the stress function ϕ over each triangle in terms of the values of ϕ at net points (called nodes in present-day finite element terminology) [1.1]. This work is considered by some to be the origin of the present-day finite element method. Since the

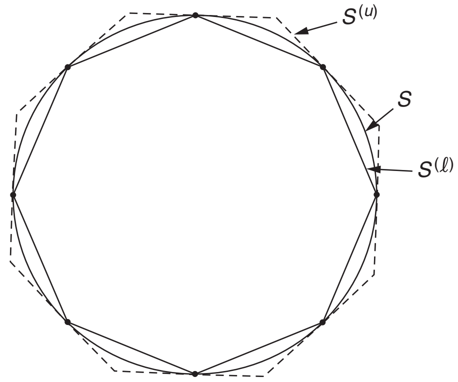


FIGURE 1.3 Lower and upper bounds to the circumference of a circle.

mid-1950s, engineers in the aircraft industry have worked on developing approximate methods for the prediction of stresses induced in aircraft wings. In 1956, Turner et al. [1.2] presented a method for modeling the wing skin using three-node triangles. At about the same time, Argyris and Kelsey presented several papers outlining matrix procedures, which contained some of the finite element ideas, for the solution of structural analysis problems [1.3]. The study by Turner et al. [1.2] is considered one of the key contributions in the development of the finite element method.

The name finite element was coined, for the first time, by Clough in 1960 [1.42]. Although the finite element method was originally developed based mostly on intuition and physical argument, the method was recognized as a form of the classic Rayleigh-Ritz method in the early 1960s. Once the mathematical basis of the method was recognized, the developments of new finite elements for different types of problems and the popularity of the method started to grow almost exponentially [1.37–1.40]. The digital computer provided a rapid means of performing the many calculations involved in the finite element analysis (FEA) and made the method practically viable. Along with the development of high-speed digital computers, the application of the finite element method also progressed at a very impressive rate. Zienkiewicz and his associates [1.4,1.6] presented the broad interpretation of the method and its applicability to any general field problem. The book by Przemieniecki [1.5] presents the finite element method as applied to the solution of stress analysis problems.

With this broad interpretation of the finite element method, it has been found that the finite element equations can also be derived by using a weighted residual method such as the Galerkin method or the least squares approach. This led to widespread interest among applied mathematicians in applying the finite element method for the solution of linear and nonlinear differential equations. Traditionally, mathematicians developed techniques such as matrix theory and solution methods for differential equations, and engineers used those methods to solve engineering analysis problems. Only in the case of the finite element method, engineers developed and perfected the technique; applied mathematicians use the method for the solution of complex ordinary and partial differential equations. Today, it has become an industry standard to solve practical engineering problems using the finite element method. Millions of degrees of freedom (dof) are being used in the solution of some important practical problems.

A brief history of the beginning of the finite element method was presented by Gupta and Meek [1.7]. Books that deal with the basic theory, mathematical foundations, mechanical design, structural, fluid flow, heat transfer, electromagnetic and manufacturing applications, and computer programming aspects are given at the end of the chapter [1.8–1.30]. The rapid progress of the finite element method can be seen by noting that annually about 3800 papers were being published; a total of about 56,000 papers, 380 books, and 400 conference proceedings have been published as estimated in 1995 [1.40]. With all the progress, today the finite element method is considered one of the well-established and convenient analysis tools by engineers and applied scientists.

EXAMPLE 1.1

The circumference of a circle (S) is approximated by the perimeters of inscribed and circumscribed n-sided polygons as shown in Fig. 1.3. Prove the following:

$$\lim_{n\to\infty} S^{(l)} = S \quad \text{and} \quad \lim_{n\to\infty} S^{(u)} = S$$

where $S^{(l)}$ and $S^{(u)}$ denote the perimeters of the inscribed and circumscribed polygons, respectively.

Solution

Approach: Express the perimeters of polygons in terms of the radius of the circle R and the number of sides of the polygons n and find their limiting values as $n\to\infty$.

If the radius of the circle is R, each side of the inscribed and the circumscribed polygon (Fig. 1.4) can be expressed as

Continued

EXAMPLE 1.1 —cont'd

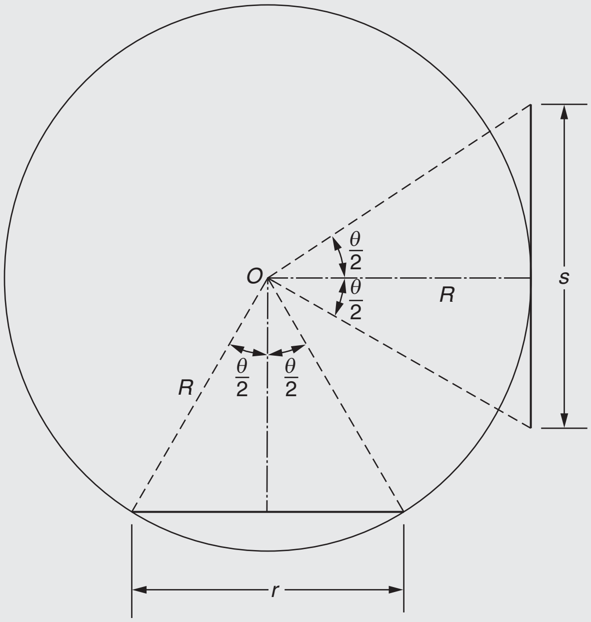


FIGURE 1.4 Sides of inscribed and circumscribed polygons.

$$r = 2R\sin\frac{\pi}{n}, \quad s = 2R\tan\frac{\pi}{n} \tag{E.1}$$

Thus, the perimeters of the inscribed and circumscribed polygons are given by

$$S^{(l)} = nr = 2nR\sin\frac{\pi}{n}, \quad S^{(u)} = ns = 2nR\tan\frac{\pi}{n} \tag{E.2}$$

which can be rewritten as

$$S^{(l)} = 2\pi R\left[\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right], \quad S^{(u)} = 2\pi R\left[\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}\right] \tag{E.3}$$

As $n \to \infty, \frac{\pi}{n} \to 0$, and hence

$$S^{(l)} \to 2\pi R = S, \quad S^{(u)} \to 2\pi R = S \tag{E.4}$$

1.3 GENERAL APPLICABILITY OF THE METHOD

Although the method has been extensively used in the field of structural mechanics, it has been successfully applied to solve several other types of engineering problems, such as heat conduction, fluid dynamics, seepage flow, and electric and magnetic fields. These applications prompted mathematicians to use this technique for the solution of complicated boundary value and other problems. In fact, it has been established that the method can be used for the numerical solution of ordinary and partial differential equations. The general applicability of the finite element method can be seen by observing the strong similarities that exist between various types of engineering problems. For illustration, let us consider the following phenomena.

1.3.1 One-Dimensional Heat Transfer

Consider the thermal equilibrium of an element of a heated one-dimensional body as shown in Fig. 1.5A. The rate at which heat enters the left face can be written as

$$q_x = -kA\frac{\partial T}{\partial x} \tag{1.1}$$

where k is the thermal conductivity of the material, A is the area of cross section through which heat flows (measured perpendicular to the direction of heat flow), and $\partial T/\partial x$ is the rate of change of temperature T with respect to the axial direction [1.30].

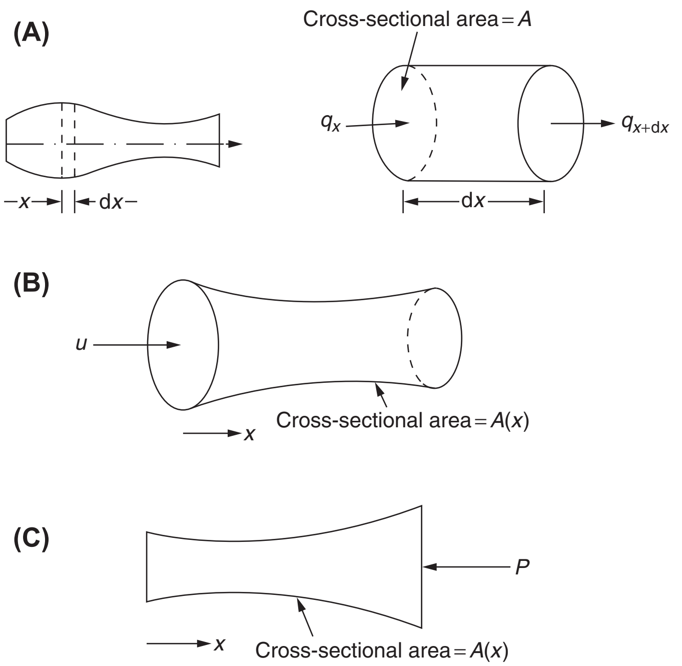


FIGURE 1.5 One-dimensional problems.

The rate at which heat leaves the right face can be expressed as (by retaining only two terms in the Taylor's series expansion)

$$q_{x+\mathrm{d}x} = q_x + \frac{\partial q_x}{\partial x}\mathrm{d}x = -kA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)\mathrm{d}x \tag{1.2}$$

The energy balance for the element for a small time dt is given by

Heat inflow in time dt	+	Heat generated by internal sources in time dt	=	Heat outflow in time dt	+	Change in internal energy during time dt

That is,

$$q_x\mathrm{d}t + \dot{q}A\,\mathrm{d}x\,\mathrm{d}t = q_{x+\mathrm{d}x}\mathrm{d}t + c\rho\frac{\partial T}{\partial t}\mathrm{d}x\,\mathrm{d}t \tag{1.3}$$

where $\dot{q}$ is the rate of heat generation per unit volume (by the heat source), c is the specific heat, ρ is the density, and $\frac{\partial T}{\partial t}\mathrm{d}t = \mathrm{d}T$ is the temperature change of the element in time dt. Eq. (1.3) can be simplified to obtain

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{q}A = c\rho\frac{\partial T}{\partial t} \tag{1.4}$$

SPECIAL CASES

If the heat source $\dot{q} = 0$, we get the Fourier equation

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = c\rho\frac{\partial T}{\partial t} \tag{1.5}$$

If the system is in a steady state, we obtain the Poisson equation

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{q}A = 0 \tag{1.6}$$

If the heat source is zero and the system is in steady state, we get the Laplace equation

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = 0 \tag{1.7}$$

If the thermal conductivity and area of cross section are constant, Eq. (1.7) reduces to

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{1.8}$$

1.3.2 One-Dimensional Fluid Flow

In the case of one-dimensional fluid flow [Fig. 1.5B], we have the net mass flow the same at every cross section; that is,

$$\rho A u = \text{constant} \tag{1.9}$$

where ρ is the density, A is the cross-sectional area, and u is the flow velocity. Eq. (1.9) can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(\rho A u) = 0 \tag{1.10}$$

If the fluid is inviscid, there exists a potential function $\phi(x)$ such that [1.31].

$$u = \frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{1.11}$$

and hence Eq. (1.10) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho A \frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = 0 \tag{1.12}$$

1.3.3 Solid Bar Under Axial Load

For the solid rod shown in Fig. 1.5C, we have at any section x,

$$\begin{aligned} \text{Reaction force} &= (\text{area})(\text{stress}) = (\text{area})(E)(\text{strain}) \\ &= AE\frac{\partial u}{\partial x} = \text{applied force} \end{aligned} \tag{1.13}$$

where E is the Young's modulus, u is the axial displacement, and A is the cross-sectional area. If the applied load is constant, we can write Eq. (1.13) as

$$\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) = 0 \tag{1.14}$$

COMMONALITY OF EQUATIONS

A comparison of Eqs. (1.7), (1.12), and (1.14) indicates that a solution procedure applicable to any one of the problems can be used to solve the others also. We shall see how the finite element method can be used to solve Eqs. (1.7), (1.12), and (1.14) with appropriate boundary conditions in Section 1.5 and also in subsequent chapters.

1.4 ENGINEERING APPLICATIONS OF THE FINITE ELEMENT METHOD

As stated earlier, the finite element method was developed originally for the analysis of aircraft structures. However, the general nature of its theory makes it applicable to a wide variety of boundary and initial value problems in engineering. A boundary value problem is one in which a solution is sought in the domain (or region) of a body subject to the satisfaction of prescribed boundary (edge) conditions on the dependent variables or their derivatives. Table 1.1 gives specific applications of the finite element in the three major categories of boundary value problems, namely (1) equilibrium or steady-state or time-independent problems, (2) eigenvalue problems, and (3) propagation or transient problems.

In an equilibrium problem, we need to find the steady-state displacement or stress distribution if it is a solid mechanics problem, temperature or heat flux distribution if it is a heat transfer problem, and pressure or velocity distribution if it is a fluid mechanics problem.

In eigenvalue problems also, time will not appear explicitly. They may be considered as extensions of equilibrium problems in which critical values of certain parameters are to be determined in addition to the corresponding steady-state configurations. In these problems, we need to find the natural frequencies or buckling loads and mode shapes if it is a solid mechanics or structures problem, stability of laminar flows if it is a fluid mechanics problem, and resonance characteristics if it is an electrical circuit problem.

TABLE 1.1 Engineering Applications of the Finite Element Method

Area of Study	Equilibrium Problems	Eigenvalue Problems	Propagation Problems
1. Civil engineering structures	Static analysis of trusses, frames, folded plates, shell roofs, shear walls, bridges, and prestressed concrete structures	Natural frequencies and modes of structures; stability of structures	Propagation of stress waves; response of structures to aperiodic loads
2. Aircraft structures	Static analysis of aircraft wings, fuselages, fins, rockets, spacecraft, and missile structures	Natural frequencies, flutter, and stability of aircraft, rocket, spacecraft, and missile structures	Response of aircraft structures to random loads, and dynamic response of aircraft and spacecraft to aperiodic loads
3. Heat conduction	Steady-state temperature distribution in solids and fluids	—	Transient heat flow in rocket nozzles, internal combustion engines, turbine blades, fins, and building structures
4. Geomechanics	Analysis of excavations, retaining walls, underground openings, rock joints, and soil—structure interaction problems; stress analysis in soils, dams, layered piles, and machine foundations	Natural frequencies and modes of dam—reservoir systems and soil—structure interaction problems	Time-dependent soil—structure interaction problems; transient seepage in soils and rocks; stress wave propagation in soils and rocks
5. Hydraulic and water resources engineering; hydrodynamics	Analysis of potential flows, free surface flows, boundary layer flows, viscous flows, transonic aerodynamic problems; analysis of hydraulic structures and dams	Natural periods and modes of shallow basins, lakes, and harbors; sloshing of liquids in rigid and flexible containers	Analysis of unsteady fluid flow and wave propagation problems; transient seepage in aquifers and porous media; rarefied gas dynamics; magnetohydrodynamic flows
6. Nuclear engineering	Analysis of nuclear pressure vessels and containment structures; steady-state temperature distribution in reactor components	Natural frequencies and stability of containment structures; neutron flux distribution	Response of reactor containment structures to dynamic loads; unsteady temperature distribution in reactor components; thermal and viscoelastic analysis of reactor structures
7. Biomedical engineering	Stress analysis of eyeballs, bones, and teeth; load-bearing capacity of implant and prosthetic systems; mechanics of heart valves	—	Impact analysis of skull; dynamics of anatomical structures
8. Mechanical design	Stress concentration problems; stress analysis of pressure vessels, pistons, composite materials, linkages, and gears	Natural frequencies and stability of linkages, gears, and machine tools	Crack and fracture problems under dynamic loads
9. Electrical machines and electromagnetics	Steady-state analysis of synchronous and induction machines, eddy current, and core losses in electric machines, magnetostatics	—	Transient behavior of electromechanical devices such as motors and actuators, magnetodynamics

The propagation or transient problems are time-dependent problems. This type of problem arises, for example, whenever we are interested in finding the response of a body under time-varying force in the area of solid mechanics and under sudden heating or cooling in the field of heat transfer.

1.5 GENERAL DESCRIPTION OF THE FINITE ELEMENT METHOD

In the finite element method, the actual continuum or body of matter, such as a solid, liquid, or gas, is represented as an assemblage of subdivisions called elements. These elements are considered to be interconnected at specified joints called nodes or nodal points. The nodes usually lie on the element boundaries where adjacent elements are considered to be connected. Since the actual variation of the field variable (e.g., displacement, stress, temperature, pressure, or velocity) inside the continuum is not known, we assume that the variation of the field variables inside a finite element can be approximated by a simple function. These approximating functions (also called interpolation models) are defined in terms of the values of the field variables at the nodes. When field equations (like equilibrium equations) for the whole continuum are written, the new unknowns will be the nodal values of the field variable. By solving the finite element equations, which are generally in the form of matrix equations, the nodal values of the field variable will be known. Once these are known, the approximating functions define the field variable throughout the assemblage of elements.

The solution of a general continuum problem by the finite element method always follows an orderly step-by-step process. With reference to static structural problems, the step-by-step procedure can be stated as follows:

Step 1: Divide the structure into discrete elements (discretization).
The first step in the finite element method is to divide the structure or solution region into subdivisions or elements. Hence, the structure is to be modeled with suitable finite elements. The number, type, size, and arrangement of the elements are to be decided.

Step 2: Select a proper interpolation or displacement model.
Since the displacement solution of a complex structure under any specified load conditions cannot be predicted exactly, we assume some suitable solution within an element to approximate the unknown solution. The assumed solution must be simple from a computational standpoint, but it should satisfy certain convergence requirements. In general, the solution or the interpolation model is taken in the form of a polynomial.

Step 3: Derive element stiffness matrices and load vectors.
From the assumed displacement model, the stiffness matrix $[K^{(e)}]$ and the load vector $\vec{P}^{(e)}$ of element e are to be derived by using a suitable variational principle, a weighted residual approach (such as the Galerkin method), or equilibrium conditions. The method of deriving the stiffness matrix and load vector using a suitable variational principle is illustrated in Section 1.6, while the derivation based on equilibrium conditions (also called the direct method) is illustrated in Section 1.8. The derivation of the element stiffness matrix and load vector using a weighted residual approach (such as the Galerkin method) is presented in Chapter 5.

Step 4: Assemble element equations to obtain the overall equilibrium equations.
Since the structure is composed of several finite elements, the individual element stiffness matrices and load vectors are to be assembled in a suitable manner and the overall equilibrium equations have to be formulated as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{P}} \tag{1.15}$$

where $[\underset{\sim}{K}]$ is the assembled stiffness matrix, Φ is the vector of nodal displacements, and $\underset{\sim}{\vec{P}}$ is the vector of nodal forces for the complete structure.

Step 5: Solve for the unknown nodal displacements.
The overall equilibrium equations have to be modified to account for the boundary conditions of the problem. After the incorporation of the boundary conditions, the equilibrium equations can be expressed as

$$[K]\vec{\Phi} = \vec{P} \tag{1.16}$$

For linear problems, the vector $\vec{\Phi}$ can be solved very easily. However, for nonlinear problems, the solution has to be obtained in a sequence of steps, with each step involving the modification of the stiffness matrix $[K]$ and/or the load vector $\vec{P}$.

Step 6: Compute element strains and stresses.
From the known nodal displacements $\vec{\Phi}$, if required, the element strains and stresses can be computed by using the necessary equations of solid or structural mechanics.

The terminology used in the previous six steps has to be modified if we want to extend the concept to other fields. For example, we have to use the term continuum or domain in place of structure, field variable in place of displacement, characteristic matrix in place of stiffness matrix, and element resultants in place of element strains.

1.6 ONE-DIMENSIONAL PROBLEMS WITH LINEAR INTERPOLATION MODEL

The application of the six steps of the FEA is illustrated with the help of the following one-dimensional examples based on linear interpolation models.

EXAMPLE 1.2

Find the stresses induced in the axially loaded stepped bar subjected to an axial load $P = 1$ N at the right end as shown in Fig. 1.6A. The cross-sectional areas of the two steps of the bar are 2 cm^2 and 1 cm^2 over the lengths l_1 and l_2, respectively, with $l_i = l^{(i)} = 10$ cm, $i = 1, 2$. The Young's modulus of the material is given by $E = 2 \times 10^7$ N/cm^2.

Solution

Approach: Apply the six steps of the finite element method (using the minimization of the potential energy of the bar to derive the finite element equations).

Step 1: Idealize bar.

The bar is idealized as an assemblage of two elements, one element for each step of the bar as shown in Fig. 1.6B. Each element is assumed to have nodes at the ends so that the stepped bar will have a total of three nodes. Since the load is applied in the axial direction, the axial displacements of the three nodes are considered as the nodal unknown dof of the system, and are denoted as Φ_1,Φ_2, and Φ_3 as shown in Fig. 1.6B.

Step 2: Develop interpolation or displacement model.

Since the two end displacements of element e, $\Phi_1^{(e)}$ and $\Phi_2^{(e)}$, are considered the dof, the axial displacement, $\phi(x)$,within the element e is assumed to vary linearly as (Fig. 1.6C):

$$\phi(x) = a + bx \tag{E.1}$$

where a and b are constants that can be expressed in terms of the end (nodal) displacements of the element $\Phi_1^{(e)}$ and $\Phi_2^{(e)}$, as follows. Since $\phi(x)$ must be equal to $\Phi_1^{(e)}$ at $x = 0$ and $\Phi_2^{(e)}$, at $x = l^{(e)}$, we obtain

$$\phi(x = 0) = a \equiv \Phi_1^{(e)}, \phi\left(x = l^{(e)}\right) = a + bl^{(e)} = \Phi_2^{(e)} \tag{E.2}$$

Eq. (E.2) yields the solution

$$a = \Phi_1^{(e)}, b = \left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}}\right) \tag{E.3}$$

Thus the axial displacement of the element e, Eq. (E.1), can be expressed as

$$\phi(x) = \Phi_1^{(e)} + \left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}}\right)x \tag{E.4}$$

Step 3: Derive element stiffness matrix and element load vector.[1]

The element stiffness matrices can be derived from the principle of minimum potential energy. For this, we write the potential energy of the bar (I) under axial deformation

$$\begin{aligned} I &= \text{strain energy—work done by external forces} \\ &= \pi^{(1)} + \pi^{(2)} - W_p \end{aligned} \tag{E.5}$$

where $\pi^{(e)}$ represents the strain energy of element e, and W_p denotes the work done by external forces acting on the bar. For the element shown in Fig. 1.6C, the strain energy $\pi^{(e)}$ can be written as

$$\pi^{(e)} = \iiint_{V^{(e)}} \pi_0^{(e)} \mathrm{d}V \tag{E.6}$$

where $V^{(e)}$ is the volume of element e and $\pi_0^{(e)}$ is the strain energy density given by the area under the stress–strain curve shown in Fig. 1.6D:

$$\pi_0^{(e)} = \frac{1}{2}\sigma^{(e)}\varepsilon^{(e)}$$

Continued

1. The element load vector need not be found if loads applied to the stepped bar (structure or system) are in the form of concentrated forces applied only at the nodes of the system.

EXAMPLE 1.2 —cont'd

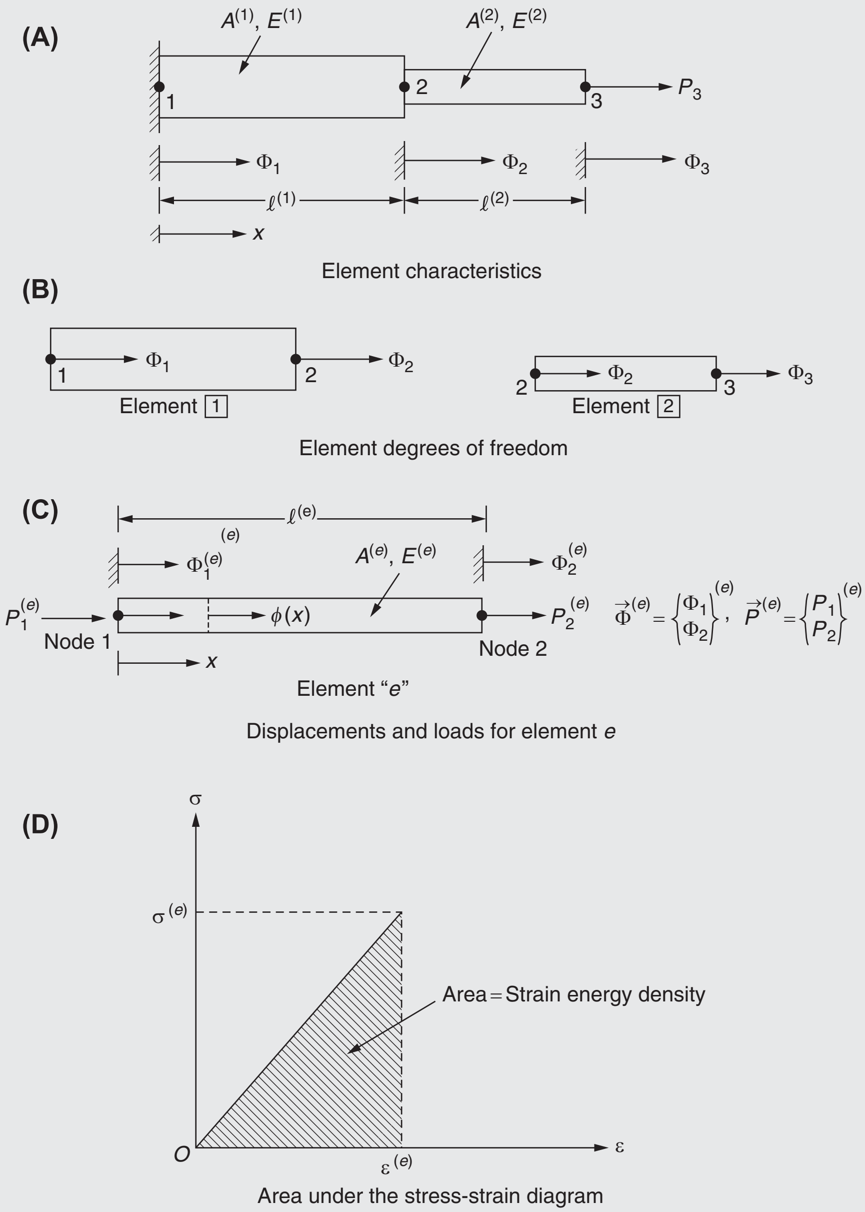


FIGURE 1.6 A stepped bar under axial load.

where $\sigma^{(e)}$ is the stress in element e and $\varepsilon^{(e)}$ is the strain in element e. Using $dV = A^{(e)}dx$ and $\sigma^{(e)} = \varepsilon^{(e)}E^{(e)}$, the strain energy of element e, given by Eq. (E.6) can be expressed as

$$\pi^{(e)} = A^{(e)} \int_0^{l(e)} \frac{1}{2}\sigma^{(e)} \cdot \varepsilon^{(e)} \cdot \mathrm{d}x = \frac{A^{(e)}E^{(e)}}{2} \int_0^{l(e)} \varepsilon^{(e)^2}\, \mathrm{d}x \tag{E.7}$$

where $A^{(e)}$ is the cross-sectional area of element e, $l^{(e)}$ is the length of element e, $\sigma^{(e)}$ is the stress in element e, $\varepsilon^{(e)}$ is the strain in element e, and $E^{(e)}$ is the Young's modulus of element e. From the expression of $\phi(x)$, we can write

$$\varepsilon^{(e)} = \frac{\partial \phi}{\partial x} = \frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}}$$

EXAMPLE 1.2 —cont'd

and hence

$$\pi^{(e)} = \frac{A^{(e)}E^{(e)}}{2}\int_0^{l^{(e)}}\left\{\frac{\Phi_2^{(e)^2}+\Phi_1^{(e)^2}-2\Phi_1^{(e)}\Phi_2^{(e)}}{l^{(e)2}}\right\}dx$$
$$= \frac{A^{(e)}E^{(e)}}{2l^{(e)}}\left(\Phi_1^{(e)^2}+\Phi_2^{(e)^2}-2\Phi_1^{(e)}\Phi_2^{(e)}\right) \tag{E.8}$$

This expression for $\pi^{(e)}$ can always be written in matrix form as

$$\pi^{(e)} = \frac{1}{2}\vec{\Phi}^{(e)^T}\left[K^{(e)}\right]\vec{\Phi}^{(e)} \tag{E.9}$$

where $\vec{\Phi}^{(e)} = \begin{Bmatrix}\Phi_1^{(e)}\\ \Phi_2^{(e)}\end{Bmatrix}$ is the vector of nodal displacements of element e

$\equiv \begin{Bmatrix}\Phi_1\\ \Phi_2\end{Bmatrix}$ for $e = 1$ and $\begin{Bmatrix}\Phi_2\\ \Phi_3\end{Bmatrix}$ for $e = 2$, and

$$\left[K^{(e)}\right] = \frac{A^{(e)}E^{(e)}}{l^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ is called the stiffness matrix of element } e.^2 \tag{E.10}$$

Since there are only concentrated loads acting at the nodes of the bar (and no distributed load acts on the bar), the work done by external forces can be expressed as

$$W_p = \Phi_1P_1+\Phi_2P_2+\Phi_3P_3 \equiv \underset{\sim}{\vec{\Phi}}^T\underset{\sim}{\vec{P}} \tag{E.11}$$

where $\Phi = \begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{Bmatrix}$ is the vector of nodal dof and $\underset{\sim}{\vec{P}} = \begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix}$ is the vector of nodal loads of the complete system. A tilde below a symbol indicates that the boundary conditions have not been incorporated yet. In the present case, $\Phi_1 = 0$ since node 1 is fixed while the loads applied externally at the nodes 1, 2, and 3 in the directions of Φ_1, Φ_2, and Φ_3, respectively, are P_1 = unknown (denotes the reaction at the fixed node 1 where the displacement Φ_1 is zero), $P_2 = 0$, and $P_3 = 1$ N.

If the bar as a whole is in equilibrium under the loads $\underset{\sim}{\vec{P}} = \begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix}$, the principle of minimum potential energy gives

Continued

2. The element and system stiffness matrices in solid/structural mechanics are always symmetric because of Maxwell's theorem of reciprocal displacements, also known as the Maxwell–Betti reciprocity theorem [1.41]. Thus noting that $\vec{\Phi}^{(e)} = \begin{Bmatrix}\Phi_1^{(e)}\\ \Phi_2^{(e)}\end{Bmatrix}$ is a 2 × 1 matrix (or a two-component column vector) and $\left[K^{(e)}\right] = \begin{bmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{bmatrix}$ is a symmetric matrix of order 2 × 2, the right-hand side of Eq. (E.9) can be expanded as

$$\frac{1}{2}\left\{\Phi_1^{(e)}\quad \Phi_2^{(e)}\right\}\begin{bmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{bmatrix}\begin{Bmatrix}\Phi_1^{(e)}\\ \Phi_2^{(e)}\end{Bmatrix} = \frac{1}{2}\left\{\Phi_1^{(e)}\quad \Phi_2^{(e)}\right\}\begin{Bmatrix}K_{11}\Phi_1^{(e)}+K_{12}\Phi_2^{(e)}\\ K_{12}\Phi_1^{(e)}+K_{22}\Phi_2^{(e)}\end{Bmatrix}$$
$$= \frac{1}{2}\left\{K_{11}\Phi_1^{(e)^2}+2K_{12}\Phi_1^{(e)}\Phi_2^{(e)}+K_{22}\Phi_2^{(e)^2}\right\} \tag{a}$$

By comparing the coefficients of the terms involving $\Phi_1^{(e)^2}$, $\Phi_1^{(e)}\Phi_2^{(e)}$, and $\Phi_2^{(e)^2}$ in Eq. (a) with the corresponding terms on the right-hand side of Eq. (E.8), we identify the elements of the matrix $[K^{(e)}]$ as

$$K_{11} = \frac{A^{(e)}E^{(e)}}{l^{(e)}},\ K_{12} = -\frac{A^{(e)}E^{(e)}}{l^{(e)}},\ \text{and}\quad K_{22} = \frac{A^{(e)}E^{(e)}}{l^{(e)}} \tag{b}$$

EXAMPLE 1.2 —cont'd

$$\frac{\partial I}{\partial \Phi_i} = 0, \quad i = 1,2,3 \tag{E.12}$$

This equation can be rewritten as

$$\frac{\partial I}{\partial \Phi_i} = \frac{\partial}{\partial \Phi_i}\left(\sum_{e=1}^{2} \pi^{(e)} - W_p\right) = 0, \quad i = 1,2,3 \tag{E.13}$$

where the summation sign indicates the addition of the strain energies (scalars) of the elements. In general, when W_p is composed of work done by the externally applied distributed forces, Eq. (E.13) can be written as

$$\sum_{e=1}^{2}\left(\left[K^{(e)}\right]\vec{\Phi}^{(e)} - \vec{P}^{(e)}\right) = \vec{0} \tag{E.14}$$

where the summation sign indicates the assembly of vectors (not the addition of vectors) in which only the elements corresponding to a particular dof in different vectors are added.

Step 4: Assemble element stiffness matrices and element load vectors, and derive system equations.

This step includes the assembly of element stiffness matrices $[K^{(e)}]$ and element load vectors $\vec{P}^{(e)}$ to obtain the overall or global equilibrium equations. Eq. (E.14) can be rewritten as

$$\left[\underset{\sim}{K}\right]\underset{\sim}{\vec{\Phi}} - \underset{\sim}{\vec{P}} = \vec{0} \tag{E.15}$$

where $\left[\underset{\sim}{K}\right]$ is the assembled or global stiffness matrix $= \sum_{e=1}^{2}\left[K^{(e)}\right]$, and $\Phi = \begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{Bmatrix}$ is the vector of global displacements. For the data given, the element matrices would be

$$[K^{(1)}] = \frac{A^{(1)}E^{(1)}}{l^{(1)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^6 \begin{array}{cc} \Phi_1 & \Phi_2 \\ \begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix} \end{array} \begin{array}{c}\Phi_1\\ \Phi_2\end{array} \tag{E.16}$$

$$[K^{(2)}] = \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^6 \begin{array}{cc} \Phi_2 & \Phi_3 \\ \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} \end{array} \begin{array}{c}\Phi_2\\ \Phi_3\end{array} \tag{E.17}$$

Since the displacements of the left and right nodes of the first element are Φ_1 and Φ_2 the rows and columns of the stiffness matrix corresponding to these unknowns are identified as indicated in Eq. (E.16). Similarly, the rows and columns of the stiffness matrix of the second element corresponding to its nodal unknowns Φ_2 and Φ_3 are also identified as indicated in Eq. (E.17).
The overall stiffness matrix of the bar can be obtained by assembling the two element stiffness matrices. Since there are three nodal displacement unknowns (Φ_1, Φ_2,and Φ_3), the global stiffness matrix, $\left[\underset{\sim}{K}\right]$, will be of order three. To obtain $\left[\underset{\sim}{K}\right]$, the elements of $[K^{(1)}]$ and $[K^{(2)}]$ corresponding to the unknowns Φ_1, Φ_2, and Φ_3 are added as shown below:

$$\left[\underset{\sim}{K}\right] = 10^6 \begin{array}{ccc} \Phi_1 & \Phi_2 & \Phi_3 \\ \begin{bmatrix}4 & -4 & 0\\ -4 & 4+2 & -2\\ 0 & -2 & 2\end{bmatrix} \end{array} \begin{array}{c}\Phi_1\\ \Phi_2\\ \Phi_3\end{array} = 2\times 10^6 \begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix} \tag{E.18}$$

In the present case, external loads act only at the node points; as such, there is no need to assemble the element load vectors. The overall or global load vector can be written as

$$\underset{\sim}{\vec{P}} = \begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix} = \begin{Bmatrix}P_1\\ 0\\ 1\end{Bmatrix}$$

where P_1 denotes the reaction at node 1 (considered to be unknown). Thus, the overall equilibrium equations (E.15) become

$$2\times 10^6 \begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{Bmatrix} = \begin{Bmatrix}P_1\\ 0\\ 1\end{Bmatrix} \tag{E.19}$$

Note that a systematic step-by-step finite element procedure has been used to derive Eq. (E.19). If a step-by-step procedure is not followed, Eq. (E.19) can be derived in a much simpler way, in this example, as follows.

EXAMPLE 1.2 —cont'd

The potential energy of the stepped bar, Eq. (E.5), can be expressed using Eqs. (E.9) and (E.11) as

$$I = \pi^{(1)} + \pi^{(2)} - W_p$$
$$= \frac{1}{2}\frac{A^{(1)}E^{(1)}}{l^{(1)}}\left(\Phi_1^2 + \Phi_2^2 - 2\Phi_1\Phi_2\right) + \frac{1}{2}\frac{A^{(2)}E^{(2)}}{l^{(2)}}\left(\Phi_2^2 + \Phi_3^2 - 2\Phi_2\Phi_3\right) - P_1\Phi_1 - P_2\Phi_2 - P_3\Phi_3 \tag{E.20}$$

Eqs. (E.12) and (E.20) yield

$$\frac{\partial I}{\partial \Phi_1} = \frac{A^{(1)}E^{(1)}}{l^{(1)}}(\Phi_1 - \Phi_2) - P_1 = 0 \tag{E.21}$$

$$\frac{\partial I}{\partial \Phi_2} = \frac{A^{(1)}E^{(1)}}{l^{(1)}}(\Phi_2 - \Phi_1) + \frac{A^{(2)}E^{(2)}}{l^{(2)}}(\Phi_2 - \Phi_3) - P_2 = 0 \tag{E.22}$$

$$\frac{\partial I}{\partial \Phi_3} = \frac{A^{(2)}E^{(2)}}{l^{(2)}}(\Phi_3 - \Phi_2) - P_3 = 0 \tag{E.23}$$

For the given data, Eqs. (E.21)–(E.23), in matrix form, are identical to Eq. (E.19).

Step 5: Solve for displacements after incorporating the boundary conditions.

If we try to solve Eq. (E.19) for the unknowns Φ_1, Φ_2, and Φ_3, we will not be able to do it since the matrix $\left[\underset{\sim}{K}\right]$, given by Eq. (E.18), is singular. This is because we have not incorporated the known geometric boundary condition, namely $\Phi_1 = 0$. We can incorporate this by setting $\Phi_1 = 0$ or by deleting the row and column corresponding to Φ_1 in Eq. (E.19). The final equilibrium equations can be written as

$$[K]\,\vec{\Phi} = \vec{P}$$

or

$$2 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{E.24}$$

The solution of Eq. (E.24) gives

$$\Phi_2 = 0.25 \times 10^{-6} \text{ cm} \quad \text{and} \quad \Phi_3 = 0.75 \times 10^{-6} \text{ cm}$$

Step 6: Derive element strains and stresses.

Once the displacements are computed, the strains in the elements can be found as

$$\varepsilon^{(1)} = \frac{\partial \phi}{\partial x} \text{ for element 1 } = \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} \equiv \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 0.25 \times 10^{-7}$$

and

$$\varepsilon^{(2)} = \frac{\partial \phi}{\partial x} \text{ for element 2 } = \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} \equiv \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 0.50 \times 10^{-7}$$

The stresses in the elements are given by

$$\sigma^{(1)} = E^{(1)}\varepsilon^{(1)} = \left(2 \times 10^7\right)\left(0.25 \times 10^{-7}\right) = 0.5 \text{ N/cm}^2$$

and

$$\sigma^{(2)} = E^{(2)}\varepsilon^{(2)} = \left(2 \times 10^7\right)\left(0.50 \times 10^{-7}\right) = 1.0 \text{ N/cm}^2$$

EXAMPLE 1.3

Find the distribution of temperature in the one-dimensional fin shown in Fig. 1.7A.

The differential equation governing the steady-state temperature distribution $T(x)$ along a uniform fin is given by

$$\left.\begin{aligned} &kA\frac{d^2T}{dx^2} - hp(T - T_\infty) = 0 \\ \text{or} \quad &\frac{d^2T}{dx^2} - \frac{hp}{kA}(T - T_\infty) = 0 \\ &\text{with the boundary condition } T(x = 0) = T_0 \end{aligned}\right\} \tag{E.1}$$

Continued

EXAMPLE 1.3 —cont'd

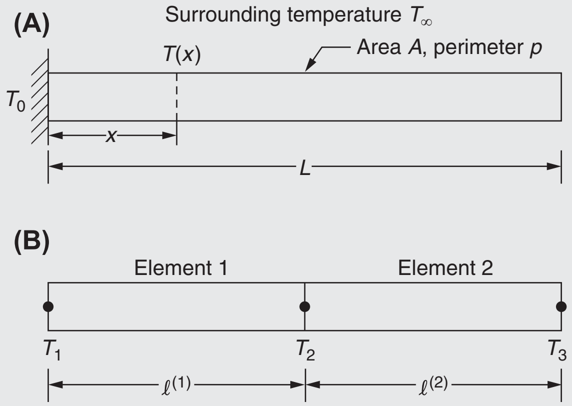


FIGURE 1.7 A one-dimensional fin.

where h is the convection heat transfer coefficient, p is the perimeter, k is the thermal conductivity, A is the cross-sectional area, T_∞ is the surrounding temperature, and T_0 is the temperature at the root of the fin [1.30]. The derivation of Eq. (E.1) is similar to that of Eq. (1.4) except that convection term is also included in the derivation of Eq. (E.1) along with the assumption of $\dot{q} = \partial T/\partial t = 0$. The problem stated in Eq. (E.1) is equivalent to [1.8].

$$\left.\begin{aligned}&\text{Minimize}\quad I = \frac{1}{2}\int_{x=0}^{L}\left[\left(\frac{dT}{dx}\right)^2 + \frac{hp}{kA}\left(T^2 - 2TT_\infty\right)\right]\\ &\quad\text{with the boundary condition } T(x=0) = T_0.\end{aligned}\right\} \tag{E.2}$$

Assume the following data: $h = 10$ W/cm^2 °C, $k = 70$ W/cm °C, $T_\infty = 40$°C, $T_0 = 140$°C, and $L = 5$ cm, and the cross-section of fin is circular with a radius of 1 cm.

Note

The temperature distribution in the fin (i.e., the solution of the problem) can be found either by solving the governing differential Eq. (E.1) using the given boundary condition, or by minimizing or extremizing the functional I (a function of another function is called a functional) using the given boundary condition. The functional I is used in this example to illustrate the method of deriving the element matrices and element load vectors using a variational principle. Although the functional I has no physical meaning, it is similar to the potential energy functional used for stress analysis (in Example 1.2).

Solution

Approach: Apply the six steps of the finite element method (using the minimization of the functional of Eq. (E.2) to derive the finite element equations).

Terminology: Since the present problem is a heat transfer problem, the terms used in the case of solid mechanics problems, such as solid body, displacement, strain, stiffness matrix, load vector, and equilibrium equations, have to be replaced by terms such as body, temperature, gradient of temperature, characteristic matrix, characteristic vector, and governing equations, respectively.

Step 1: Idealize into a finite number of elements.

Let the fin be idealized into two finite elements as shown in Fig. 1.7B. If the temperatures of the nodes are taken as the unknowns, there will be three nodal temperature unknowns, namely T_1, T_2, and T_3, in the problem.

Step 2: Select the interpolation (temperature distribution) model.

In each element e ($e = 1, 2$), the temperature (T) is assumed to vary linearly as

$$\boldsymbol{T(x) = a + bx} \tag{E.3}$$

where a and b are constants. If the nodal temperatures $T_1^{(e)}$(T at $x = 0$) and $T_2^{(e)}\left(T \text{ at } x = l^{(e)}\right)$ of element e are taken as unknowns, the constants a and b can be expressed as $a = T_1^{(e)}$ and $b = \left(T_2^{(e)} - T_1^{(e)}\right)/l^{(e)}$, where $l^{(e)}$ is the length of element e. Thus,

EXAMPLE 1.3 —cont'd

$$T(x) = T_1^{(e)} + \left(T_2^{(e)} - T_1^{(e)}\right)\frac{x}{l^{(e)}} \tag{E.4}$$

Step 3: Identify element characteristic matrices and vectors.

The element characteristic matrices and vectors can be identified by expressing the functional I in matrix form. When the integral in I is evaluated over the length of element e, we obtain

$$I^{(e)} = \frac{1}{2}\int_{x=0}^{l^{(e)}}\left[\left(\frac{dT}{dx}\right)^2 + \frac{hp}{kA}\left(T^2 - 2T_\infty T\right)\right]dx \tag{E.5}$$

Substitution of Eq. (E.4) into (E.5) leads to

$$I^{(e)} = \frac{1}{2}\int_{x=0}^{l^{(e)}}\left[\left(\frac{T_2^{(e)} - T_1^{(e)}}{l^{(e)}}\right)^2 + \frac{hp}{kA}\left\{T_1^{(e)} + \left(T_2^{(e)} - T_1^{(e)}\right)\frac{x}{l^{(e)}}\right\}^2 - \frac{2hpT_\infty}{kA}\left\{T_1^{(e)} + \left(T_1^{(e)} - T_l^{(e)}\right)\frac{x}{l^{(e)}}\right\}\right]dx \tag{E.6}$$

Eq. (E.6) can be expressed, after evaluating the integral, in matrix notation as

$$I^{(e)} = \frac{1}{2}\vec{T}^{(e)^T}\left[K^{(e)}\right]\vec{T}^{(e)} - \vec{T}^{(e)^T}\vec{P}^{(e)} \tag{E.7}$$

where $\vec{T}^{(e)} = \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \end{Bmatrix}$ is the vector of nodal temperatures of element $e = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$ for $e = 1$ and $\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix}$ for $e = 2$, $[K^{(e)}]$ is the characteristic matrix of element e

$$= \frac{1}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hpl^{(e)}}{6kA}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{E.8}$$

and $\vec{P}^{(e)} = \begin{Bmatrix} P_1^{(e)} \\ P_2^{(e)} \end{Bmatrix}$ is the characteristic vector of element e

$$\begin{aligned} &= \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \text{for } e = 1 \quad \text{and} \quad \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix} \text{for } e = 2 \\ &= \frac{hpT_\infty l^{(e)}}{2kA}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \end{aligned} \tag{E.9}$$

Step 4: Assemble element matrices and vectors and derive governing equations.

As stated in Eq. (E.2), the nodal temperatures can be determined by minimizing the functional I. The conditions for the minimum of I are given by

$$\frac{\partial I}{\partial T_i} = \frac{\partial\left(\sum_{e=1}^{2} I^{(e)}\right)}{\partial T_i} = \sum_{e=1}^{2}\frac{\partial I^{(e)}}{\partial T_i} = 0, \quad i = 1, 2, 3 \tag{E.10}$$

where I has been replaced by the sum of elemental contributions, $I^{(e)}$. Eq. (E.10) can also be stated as

$$\sum_{e=1}^{2}\frac{\partial I^{(e)}}{\partial \vec{T}^{(e)}} = \sum_{e=1}^{2}\left(\left[K^{(e)}\right]\vec{T}^{(e)} - \vec{P}^{(e)}\right) = [\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} - \underset{\sim}{\vec{P}} = \underset{\sim}{\vec{0}} \tag{E.11}$$

where $[\underset{\sim}{K}] = \sum_{e=1}^{2}\left[K^{(e)}\right]$ is the assembled characteristic matrix, $\underset{\sim}{\vec{P}} = \sum_{e=1}^{2}\vec{P}^{(e)}$ is the assembled characteristic vector, and $\underset{\sim}{T}$ is the assembled or overall nodal temperature vector $= \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$. Eq. (E.11) gives the governing matrix equations as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{E.12}$$

Continued

EXAMPLE 1.3 —cont'd

From the given data we can obtain

$$[K^{(1)}] = \frac{1}{2.5}\begin{array}{c}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{array}{cc} T_1 & T_2 \end{array}\end{array} + \frac{10\times 2\pi \times 2.5}{6\times 70\times \pi}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.6382 & -0.2809 \\ -0.2809 & 0.6382 \end{bmatrix}\begin{array}{c} T_1 \\ T_2 \end{array} \tag{E.13}$$

$$[K^{(2)}] = \begin{array}{c}\begin{array}{cc} T_2 & T_3 \end{array} \\ \begin{bmatrix} 0.6382 & -0.2809 \\ -0.2809 & 0.6382 \end{bmatrix}\end{array}\begin{array}{c} T_2 \\ T_3 \end{array} \tag{E.14}$$

$$\vec{P}^{(1)} = \frac{10 \times 2\pi \times 40 \times 2.5}{2 \times 70 \times \pi}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 14.29\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\begin{array}{c} T_1 \\ T_2 \end{array} \tag{E.15}$$

$$\vec{P}^{(2)} = 14.29\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\begin{array}{c} T_2 \\ T_3 \end{array} \tag{E.16}$$

where the nodal unknowns associated with each row and column of the element matrices and vectors are also indicated in Eqs. (E.13)–(E.16). The overall characteristic matrix of the fin can be obtained by adding the elements of $[K^{(1)}]$ and $[K^{(2)}]$ corresponding to the unknowns T_1, T_2, and T_3:

$$[\underset{\sim}{K}] = \begin{array}{c}\begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{bmatrix} 0.6382 & -0.2809 & 0 \\ -0.2809 & (0.6382 + 0.6382) & -0.2809 \\ 0 & -0.2809 & 0.6382 \end{bmatrix}\end{array}\begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \tag{E.17}$$

Similarly, the overall characteristic vector of the fin can be obtained as

$$\vec{\underset{\sim}{P}} = \begin{Bmatrix} 14.29 \\ (14.29 + 14.29) \\ 14.29 \end{Bmatrix}\begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \tag{E.18}$$

Thus, the governing finite element equation of the fin, Eq. (E.12), becomes

$$\begin{bmatrix} 0.6382 & -0.2809 & 0 \\ -0.2809 & 1.2764 & -0.2809 \\ 0 & -0.2809 & 0.6382 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 14.29 \\ 28.58 \\ 14.29 \end{Bmatrix} \tag{E.19}$$

Step 5. Solve for nodal temperatures after incorporating boundary conditions.
Eq. (E.19) has to be solved after applying the boundary condition, namely, T (at node 1) $= T_1 = T_0 = 140°C$. For this, the first equation of (E.19) is replaced by $T_1 = T_0 = 140$ and the remaining two equations are written in scalar form as

$$\begin{aligned} -0.2809T_1 + 1.2764T_2 - 0.2809T_3 &= 28.58 \\ -0.2809T_2 + 0.6382T_3 &= 14.29 \end{aligned}$$

or

$$\left.\begin{aligned} 1.2764T_2 - 0.2809T_3 &= 28.58 + 0.2809 \times 140 = 67.906 \\ -0.2809T_2 + 0.6382T_3 &= 14.29 \end{aligned}\right\} \tag{E.20}$$

The solution of Eq. (E.20) gives the nodal temperatures as $T_2 = 64.39°C$ and $T_3 = 50.76°C$.

Note

There is no need to use Step 6 in this example. Step 6 is required if information such as temperature gradient (similar to strains and stresses in a stress analysis problem) in the fin is to be computed.

EXAMPLE 1.4

Find the velocity distribution of an inviscid fluid flowing through the tube shown in Fig. 1.8A. The differential equation governing the velocity distribution $u(x)$ is given by Eq. (1.12) with the boundary condition $u(x = 0) = u_0$. This problem is equivalent to

$$\left.\begin{aligned}&\text{Minimize}\quad I = \frac{1}{2}\int_{x=0}^{L} \rho A\left(\frac{d\phi}{dx}\right)^2 \cdot dx\\ &\text{with the boundary condition } u(x = 0) = u_0\end{aligned}\right\} \tag{E.1}$$

where $\phi(x)$ is the potential function that gives the velocity of the fluid, $u(x)$, as $u(x) = d\phi(x)/dx$. Assume the area of cross section of the tube as $A(x) = A_0 \bullet e^{-(x/L)}$.

Note

The variation of the potential function along the tube (i.e., the solution of the problem) can be found either by solving the governing differential Eq. (1.12) using the given boundary condition, or by minimizing or extremizing the functional I using the given boundary condition. The functional I is used in this example to illustrate the method of deriving the element matrices and element load vectors using a variational principle. Although the functional I has no physical meaning, it is similar to the potential energy functional used for stress analysis (in Example 1.2).

Solution

Approach: Apply the six steps of the finite element method (using the minimization of the functional of Eq. (E.1) to derive the finite element equations).

Terminology: In this case the terminology of solid mechanics, such as solid body, displacement, stiffness matrix, load vector, and equilibrium equations, has to be replaced by the terms continuum, potential function, characteristic matrix, characteristic vector, and governing equations.

Step 1: Idealize into a finite number of elements.

Divide the continuum into two elements as shown in Fig. 1.8B. If the values of the potential function at the various nodes are taken as the unknowns, there will be three quantities, namely Φ_1, Φ_2, and Φ_3, to be determined in the problem.

Step 2: Select the interpolation (potential function) model.

The potential function, $\phi(x)$, is assumed to vary linearly within an element e ($e = 1, 2$) as (see Fig. 1.8C):

$$\phi(x) = a + bx \tag{E.2}$$

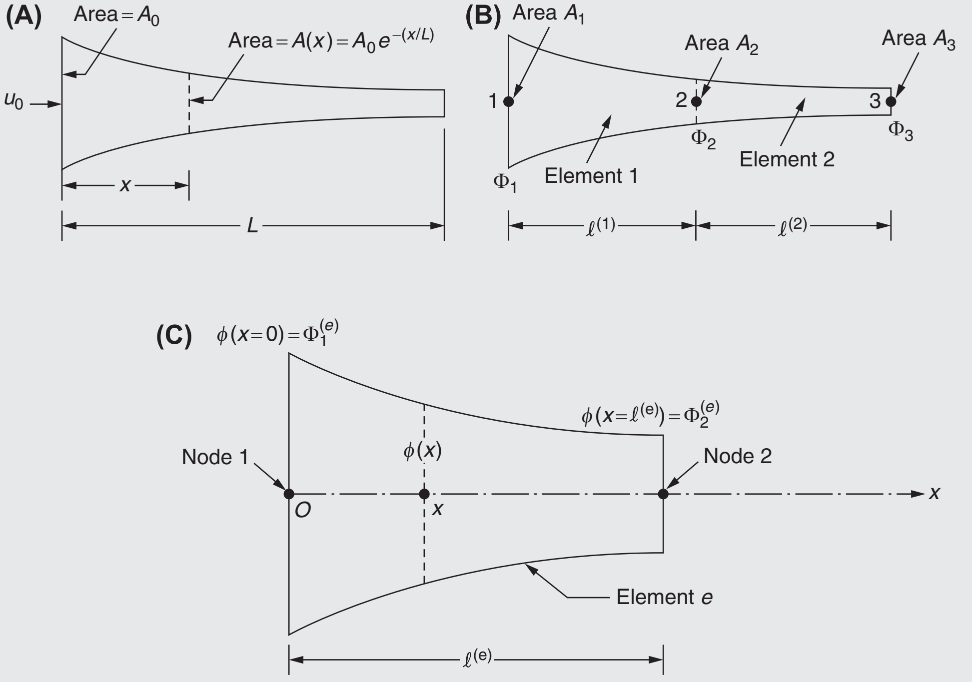


FIGURE 1.8 A one-dimensional tube of varying cross section.

Continued

EXAMPLE 1.4 —cont'd

where the constants a and b can be evaluated using the nodal conditions $\phi(x = 0) = \Phi_1^{(e)}$ and $\phi(x = l^{(e)}) = \Phi_2^{(e)}$ to obtain

$$\phi(x) = \Phi_1^{(e)} + \left(\Phi_2^{(e)} - \Phi_1^{(e)}\right)\frac{x}{l^{(e)}} \tag{E.3}$$

where $l^{(e)}$ is the length of element e.

Step 3: Derive element characteristic matrices.

The functional I corresponding to element e can be expressed as

$$\begin{aligned} I^{(e)} &= \frac{1}{2}\int_{x=0}^{l^{(e)}} \rho A\left(\frac{d\phi}{dx}\right)^2 dx = \frac{1}{2}\int_{x=0}^{l^{(e)}} \rho A\left(\frac{\Phi_2^{(e)} - \Phi_1^{(e)}}{l^{(e)}}\right)^2 dx \\ &= \frac{1}{2}\vec{\Phi}^{(e)^T}\left[K^{(e)}\right]\vec{\Phi}^{(e)} \end{aligned} \tag{E.4}$$

where $[K^{(e)}]$ is the characteristic matrix of element e.

$$= \frac{\rho A^{(e)}}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{E.5}$$

$A^{(e)}$ is the cross-sectional area of element e (which can be taken as $(A_1 + A_2)/2$ for $e = 1$ and $(A_2 + A_3)/2$ for $e = 2$ for simplicity), and $\vec{\Phi}^{(e)}$ is the vector of nodal unknowns of element e.

$$= \begin{Bmatrix} \vec{\Phi}_1^{(e)} \\ \vec{\Phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix} \text{for } e = 1 \quad \text{and} \quad \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} \text{for } e = 2.$$

Step 4: Assemble element matrices and derivation of system equations.

The overall equations can be written as

$$\begin{bmatrix} \frac{\rho A^{(1)}}{l^{(1)}} & -\frac{\rho A^{(1)}}{l^{(1)}} & 0 \\ -\frac{\rho A^{(1)}}{l^{(1)}} & \left(\frac{\rho A^{(1)}}{l^{(1)}} + \frac{\rho A^{(2)}}{l^{(2)}}\right) & -\frac{\rho A^{(2)}}{l^{(2)}} \\ 0 & -\frac{\rho A^{(2)}}{l^{(2)}} & \frac{\rho A^{(2)}}{l^{(2)}} \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 = -\rho A_1 u_0 \\ Q_2 = 0 \\ Q_3 = \rho A_3 u_3 \end{Bmatrix} \tag{E.6}$$

where Q_i is the mass flow rate across section i ($i = 1, 2, 3$) and is nonzero when fluid is either added to or subtracted from the tube with $Q_1 = -\rho A_1 u_1$ (negative since u_1 is opposite to the outward normal to Section 1), $Q_2 = 0$, and $Q_3 = \rho A_3 u_3$. Since $u_1 = u_0$ is given, Q_1 is known, whereas Q_3 is unknown.

Step 5: Solve system equations after incorporating boundary conditions.

In the third equation of (E.6), both Φ_3 and Q_3 are unknowns and thus the given system of equations cannot be solved. Hence, we set $\Phi_3 = 0$ as a reference value and try to find the values of Φ_1 and Φ_2 with respect to this value. The first two equations of (E.6) can be expressed in scalar form as

$$\frac{\rho A^{(1)}}{l^{(1)}}\Phi_1 - \frac{\rho A^{(1)}}{l^{(1)}}\Phi_2 = Q_1 = -\rho A_1 u_0 \tag{E.7}$$

and

$$-\frac{\rho A^{(1)}}{l^{(1)}}\Phi_1 + \left(\frac{\rho A^{(1)}}{l^{(1)}} + \frac{\rho A^{(2)}}{l^{(2)}}\right)\Phi_2 = \frac{\rho A^{(2)}}{l^{(2)}}\Phi_3 = 0 \tag{E.8}$$

By substituting $A^{(1)} \simeq (A_1 + A_2)/2 = 0.8032A_0$, $A^{(2)} \simeq (A_2 + A_3)/2 = 0.4872A_0$, and $l^{(1)} = l^{(2)} = L/2$, Eqs. (E.7) and (E.8) can be written as

$$0.8032\Phi_1 - 0.8032\Phi_2 = -u_0 L/2 \tag{E.9}$$

and

$$-0.8032\Phi_1 + 1.2904\Phi_2 = 0 \tag{E.10}$$

EXAMPLE 1.4 —cont'd

The solution of Eqs. (E.9) and (E.10) is given by

$$\Phi_1 = -1.650\, u_0 L \quad \text{and} \quad \Phi_2 = -1.027\, u_0 L$$

Step 6: Computation of fluid velocities.
The velocities of the fluid in elements 1 and 2 can be found as

$$u \text{ in element } 1 = u^{(1)} = \frac{d\phi}{dx}(\text{element } 1)$$

$$= \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 1.246 u_0$$

and

$$u \text{ in element } 2 = u^{(2)} = \frac{d\phi}{dx}(\text{element } 2)$$

$$= \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 2.054 u_0$$

These velocities will be constant along the elements in view of the linear relationship assumed for $\phi(x)$ within each element. The velocity of the fluid at node 2 can be approximated as

$$u_2 = \left(u^{(1)} + u^{(2)}\right)/2 = 1.660 u_0.$$

The third equation of (E.6) can be written as

$$-\frac{\rho A^{(2)}}{l^{(2)}}\Phi_2 + \frac{\rho A^{(2)}}{l^{(2)}}\Phi_3 = Q_3$$

or

$$\frac{\rho(0.4872 A_0)}{(L/2)}(-\Phi_2 + \Phi_3) = Q_3$$

or

$$Q_3 = \rho A_0 u_0.$$

This shows that the mass flow rate is the same at nodes 1 and 3, which proves the principle of conservation of mass.

1.7 ONE-DIMENSIONAL PROBLEMS WITH A CUBIC INTERPOLATION MODEL

The application of the six steps of the FEA is illustrated with the help of a one-dimensional (beam) example based on a cubic interpolation or displacement model.

A beam is a one-dimensional member that is subjected to forces applied in a transverse direction (perpendicular to the length direction). For any type of end supports, the deflection of the beam will be a curve (not linear). Hence, both the deflection and the rate of change of deflection in the length direction (slope or rotation) become important. As such, for a beam element with two end nodes as shown in Fig. 1.9, both the transverse displacement and its derivative (or slope or

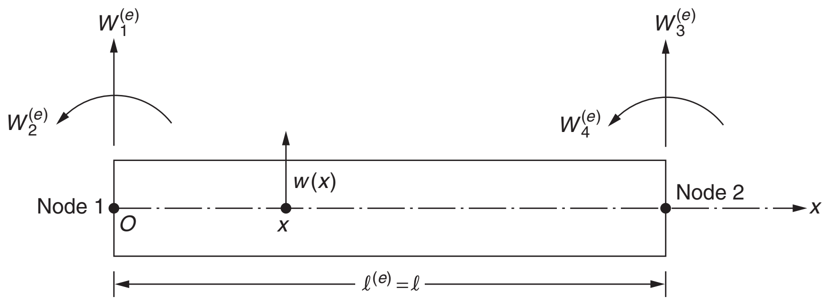


FIGURE 1.9 A beam element.

rotation) are taken as the unknown dof at each node. Because there are a total of four unknown dof at the two nodes, a displacement model for the deflection of the beam element is assumed to be a cubic polynomial (which has four unknown constants). These four constants can be expressed in terms of the four nodal dof shown in Fig. 1.9 so that the displacement model of the element can be expressed as (see Problem 1.42)

$$\begin{aligned} w(x) &= W_1^{(e)}N_1(x) + W_2^{(e)}N_2(x) + W_3^{(e)}N_3(x) + W_4^{(e)}N_4(x) \\ &\equiv W_1^{(e)}\frac{1}{l^3}\left(2x^3 - 3lx^2 + l^3\right) + W_2^{(e)}\frac{1}{l^2}\left(x^3 - 2lx^2 + l^2x\right) + W_3^{(e)}\frac{1}{l^3}\left(3lx^2 - 2x^3\right) + W_4^{(e)}\frac{1}{l^2}\left(x^3 - lx^2\right) \end{aligned} \tag{1.17}$$

where $W_1^{(e)}$ to $W_4^{(e)}$ denote the displacements at the ends of the element e.

The stiffness matrix of any finite element (in structural analysis) can be identified by expressing the strain energy of the element in matrix form. The strain energy of a beam element in bending ($\pi^{(e)}$) can be expressed in matrix form [1.32] as

$$\pi^{(e)} = \frac{1}{2}\int_0^{l^{(e)}} E^{(e)}I^{(e)}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x = \frac{1}{2}\vec{W}^{(e)^T}\left[K^{(e)}\right]\vec{W}^{(e)} \tag{1.18}$$

where $E^{(e)}$ is the Young's modulus, $I^{(e)}$ is the area moment of inertia of the cross section, $l^{(e)}$ is the length, and $\vec{W}^{(e)}$ is the vector of nodal degrees of the beam element e:

$$\vec{W}^{(e)} = \begin{Bmatrix} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \end{Bmatrix}$$

By substituting Eq. (1.17) into Eq. (1.18), the stiffness matrix of the beam element can be derived as

$$\left[K^{(e)}\right] = \frac{2E^{(e)}I^{(e)}}{l^{(e)^3}} \begin{array}{c} \begin{array}{cccc} W_1^{(e)} & W_2^{(e)} & W_3^{(e)} & W_4^{(e)} \end{array} \\ \begin{bmatrix} 6 & 3l^{(e)} & -6 & 3l^{(e)} \\ 3l^{(e)} & 2l^{(e)^2} & -3l^{(e)} & l^{(e)^2} \\ -6 & -3l^{(e)} & 6 & -3l^{(e)} \\ 3l^{(e)} & l^{(e)^2} & -3l^{(e)} & 2l^{(e)^2} \end{bmatrix} \end{array} \begin{array}{c} W_1^{(e)} \\ W_2^{(e)} \\ W_3^{(e)} \\ W_4^{(e)} \end{array} \tag{1.19}$$

The use of the displacement model, Eq. (1.17), and the stiffness matrix, Eq. (1.19), in the deflection and stress analysis of a beam is illustrated through the following example.

EXAMPLE 1.5

A beam of uniform rectangular cross section with width 1 cm and depth 2 cm and length 60 cm is subjected to a vertical concentrated load of 1000 N as shown in Fig. 1.10A. If the beam is fixed at both the ends, find the stresses in the beam using a two-element idealization. Assume the Young's modulus of the beam as $E = 10^7$ N/cm^2.

Solution

Approach: Apply the six steps of the finite element method. For the stress analysis, use the stress–strain and strain–displacement relations of a beam.

Step 1: Idealize using finite elements.

By assuming the two fixed ends of the beam as the end nodes and introducing an additional node at the point of application of the load, the beam can be replaced by a two-element idealization as shown in Fig. 1.10A and B. The global dof of the beam are indicated in Fig. 1.10A so that the vector of displacement dof of the system (beam) is given by

$$\underset{\sim}{\vec{W}} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} \tag{E.1}$$

EXAMPLE 1.5 —cont'd

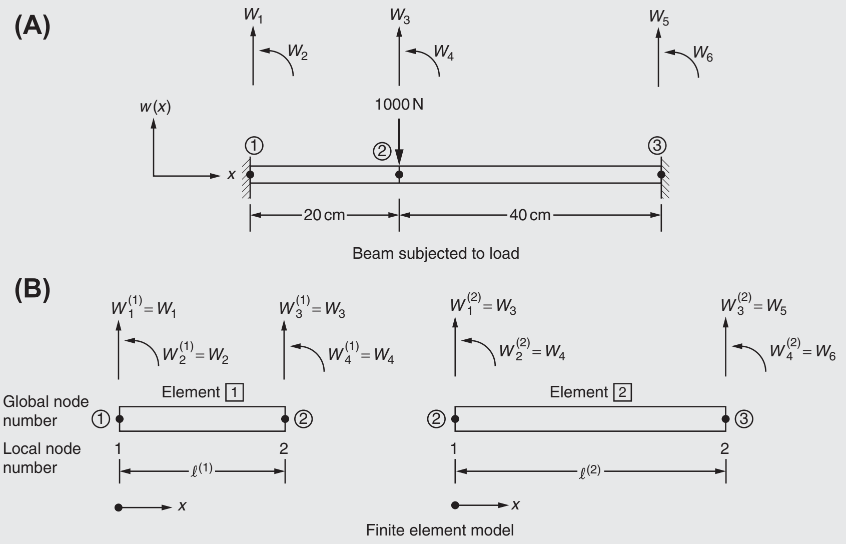


FIGURE 1.10 Finite element idealization of a beam subjected to load.

The element nodal dof can be identified from Fig. 1.10B as

$$\vec{W}^{(1)} = \begin{Bmatrix} W_1^{(1)} \\ W_2^{(1)} \\ W_3^{(1)} \\ W_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}, \quad \vec{W}^{(2)} = \begin{Bmatrix} W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{Bmatrix} = \begin{Bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} \tag{E.2}$$

Step 2: Select interpolation or displacement model.

In each of the beam elements, a cubic displacement model is assumed as shown in Eq. (1.17).

Step 3: Derive element stiffness matrix.

The element stiffness matrix given by Eq. (1.19) can be derived by expressing the strain energy of the element in matrix form. Using $E^{(1)} = E^{(2)} = 10^7$ N/cm^2, $I^{(1)} = I^{(2)} = (1)\,(2^3)/12 = 2/3$ cm^4, $l^{(1)} = 20$ cm, and $l^{(2)} = 40$ cm in Eq. (1.19), we obtain

$$[K^{(1)}] = 10^4 \begin{array}{cc} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} & \\ \begin{bmatrix} 1 & 10 & -1 & 10 \\ 10 & \frac{400}{3} & -10 & \frac{200}{3} \\ -1 & -10 & 1 & -10 \\ 10 & \frac{200}{3} & -10 & \frac{400}{3} \end{bmatrix} & \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array} \end{array} \tag{E.3}$$

$$[K^{(2)}] = 10^4 \begin{array}{cc} \begin{array}{cccc} W_3 & W_4 & W_5 & W_6 \end{array} & \\ \begin{bmatrix} \frac{1}{8} & \frac{5}{2} & \frac{-1}{8} & \frac{5}{2} \\ \frac{5}{2} & \frac{200}{3} & \frac{-5}{2} & \frac{100}{3} \\ \frac{-1}{8} & \frac{-5}{2} & \frac{1}{8} & \frac{-5}{2} \\ \frac{5}{2} & \frac{100}{3} & \frac{-5}{2} & \frac{200}{3} \end{bmatrix} & \begin{array}{c} W_3 \\ W_4 \\ W_5 \\ W_6 \end{array} \end{array} \tag{E.4}$$

Note that the nodal dof associated with the various rows and columns of the matrices $[K^{(e)}]$ are also indicated in Eqs. (1.19), (E.3), and (E.4).

Continued

EXAMPLE 1.5 —cont'd

Step 4: Assemble element stiffness matrices and derive system equations.
The assembled stiffness matrix of the beam can be obtained as

$$[\underset{\sim}{K}] = 10^4 \begin{array}{c} \begin{matrix} W_1 & \quad W_2 & \quad\quad W_3 & \quad\quad W_4 & \quad\quad W_5 & \quad W_6 \end{matrix} \\ \left[\begin{matrix} 1 & 10 & -1 & 10 & & \\ 10 & \frac{400}{3} & -10 & \frac{200}{3} & & \\ -1 & -10 & \left(1+\frac{1}{8}\right) & \left(-10+\frac{5}{2}\right) & -\frac{1}{8} & \frac{5}{2} \\ 10 & \frac{200}{3} & \left(-10+\frac{5}{2}\right) & \left(\frac{400}{3}+\frac{200}{3}\right) & -\frac{5}{2} & \frac{100}{3} \\ & & -\frac{1}{8} & -\frac{5}{2} & \frac{1}{8} & -\frac{5}{2} \\ & & \frac{5}{2} & \frac{100}{3} & -\frac{5}{2} & \frac{200}{3} \end{matrix}\right] \end{array} \begin{matrix} \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix} \tag{E.5}$$

The equilibrium equations of the system can be expressed as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{W}} = \underset{\sim}{\vec{P}} \tag{E.6}$$

where $[\underset{\sim}{K}]$ is given by Eq. (E.5), the system nodal displacement vector by Eq. (E.1), and the system nodal load vector by

$$\underset{\sim}{\vec{P}} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} \tag{E.7}$$

where a tilde below a symbol indicates that the boundary conditions have not been incorporated yet. In Eq. (E.7), the loads P_1, P_2, P_5, and P_6 denote the reactions at the fixed ends of the beam and $P_3 = -1000 =$ external load applied along the dof W_3 and $P_4 = 0 =$ external moment applied along the dof W_4.

Step 5: Solve for displacements.
Noting that nodes 1 and 3 are fixed, we have $W_1 = W_2 = W_5 = W_6 = 0$ and hence by deleting the rows and columns corresponding to these dof in Eq. (E.5), we obtain the final stiffness matrix of the beam as

$$[K] = 10^4 \begin{array}{c} \begin{matrix} W_3 & \quad W_4 \end{matrix} \\ \left[\begin{matrix} \frac{9}{8} & -\frac{15}{2} \\ -\frac{15}{2} & 200 \end{matrix}\right] \end{array} \begin{matrix} \\ W_3 \\ W_4 \end{matrix} \tag{E.8}$$

Since the externally applied vertical load at node 2 (in the direction of W_3) is -1000 N and the rotational load (bending moment) at node 2 (in the direction of W_4) is 0, the load vector of the beam corresponding to the dof W_3 and W_4 can be expressed as

$$\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 0 \end{Bmatrix} \tag{E.9}$$

Thus, the equilibrium equations of the beam are given by

$$[K]\vec{W} = \vec{P}$$

or

$$10^4 \begin{bmatrix} \frac{9}{8} & -\frac{15}{2} \\ -\frac{15}{2} & 200 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 0 \end{Bmatrix} \tag{E.10}$$

EXAMPLE 1.5 —cont'd

The solution of Eq. (E.10) gives

$$W_3 = -0.1185 \text{ cm}, \quad W_4 = -0.0044 \text{ rad} \tag{E.11}$$

Step 6: Compute stresses.

The bending stress, $\sigma_{xx}^{(e)}(x, y)$, induced at a point (fiber) located at a distance x from node 1 (left-side node) in a horizontal direction and y from the neutral axis in a vertical direction in element e is given by (see Section 9.3):

$$\sigma_{xx}^{(e)} = \sigma_{xx}^{(e)}(x,y) = [E]\varepsilon_{xx}^{(e)}(x,y) \equiv [E][B]\overrightarrow{W}^{(e)} \equiv -yE^{(e)}\frac{d^2w^{(e)}(x)}{dx^2} \tag{E.12}$$

where $w^{(e)}(x)$ is given by Eq. (1.17) so that

$$\frac{d^2w^{(e)}(x)}{dx^2} = \frac{1}{l^{(e)^3}}\left(12x - 6l^{(e)}\right)W_1^{(e)} + \frac{1}{l^{(e)^2}}\left(6x - 4l^{(e)}\right)W_2^{(e)} + \frac{1}{l^{(e)^3}}\left(6l^{(e)} - 12x\right)W_3^{(e)} + \frac{1}{l^{(e)^2}}\left(6x - 2l^{(e)}\right)W_4^{(e)} \tag{E.13}$$

Thus, the stresses in the elements can be determined as follows:

Element 1: Using $E^{(1)} = 10^7$, $l^{(1)} = 20$, $W_1^{(1)} = W_1 = 0$, $W_2^{(1)} = W_2 = 0$, $W_3^{(1)} = W_3 = -0.11851852$, and $W_4^{(1)} = W_4 = -0.00444444$, Eq. (E.12) gives

$$\sigma_{xx}^{(1)}(x,y) = 1777.7778(10 - x)y + 222.2222(3x - 20)y \tag{E.14}$$

The stress induced in the top fibers of element 1 is given by (with $y = +1$ cm)

$$\sigma_{xx}^{(1)}(x) = 1777.7778(10 - x) - 222.2222(20 - 2x) \tag{E.15}$$

For instance, the maximum stresses induced at $x = 0$ (fixed end) and $x = 20$ cm (point of load application) will be

$$\sigma_{xx}^{(1)}(0) = 13{,}333.334 \text{ N/cm}^2 \quad \text{and} \quad \sigma_{xx}^{(1)}(20) = -8{,}888.890 \text{ N/cm}^2$$

Element 2: Using

$$E^{(2)} = 10^7, l^{(2)} = 40, W_1^{(2)} = W_3 = -0.11851852, W_2^{(2)} = W_4 = -0.00444444, W_3^{(2)} = W_5 = 0,$$

and

$$W_4^{(2)} = W_6 = 0, \tag{E.12}$$

gives

$$\sigma_{xx}^{(2)}(x,y) = 222.2222(x - 20)y + 55.5550(3x - 80)y \tag{E.16}$$

The stress induced in the top fibers of element 2 is given by (with $y = +1$ cm)

$$\sigma_{xx}^{(2)}(x,y) = 222.2222(x - 20) + 55.5550(3x - 80) \tag{E.17}$$

For instance, the maximum stresses induced at $x = 0$ (point of load application) and $x = 40$ cm (fixed end) will be

$$\sigma_{xx}^{(2)}(0) = -8{,}888.844 \text{ N/cm}^2 \quad \text{and} \quad \sigma_{xx}^{(2)}(40) = 6{,}666.644 \text{ N/cm}^2$$

1.8 DERIVATION OF FINITE ELEMENT EQUATIONS USING A DIRECT APPROACH

The element stiffness (or characteristic) matrices and load (characteristic) vectors and the finite element equations can be derived by using a direct approach. In this method, direct physical reasoning relevant to the problem (such as consideration of equilibrium of the system) is used to establish the element properties (characteristic matrices and vectors) in terms of pertinent variables. The direct approach is applicable only to problems involving simple types of elements; hence most practical (complex) problems cannot be solved using this approach. However, a study of direct methods enhances our understanding of the physical interpretation of the finite element method. The direct approach is presented in this section by considering simple problems from the areas of elastic systems, fluid flow, heat transfer, and electrical circuits.

1.8.1 Bar Element Under Axial Load

Consider a stepped bar as shown in Fig. 1.11A. The different steps are assumed to have different lengths, areas of cross section, and Young's moduli. The way to discretize this system into finite elements is immediately obvious. If we define each step as an element, the system consists of three elements and four nodes (Fig. 1.11B).

The force-displacement equations of a step constitute the required element equations. To derive these equations for a typical element e, we isolate the element as shown in Fig. 1.11C. In this figure, a force (P) and a displacement (u) are defined at each of the two nodes in the positive direction of the x axis. The field variable ϕ is the deflection u. The element equations can be expressed in matrix form as

$$\left[K^{(e)}\right]\vec{u} = \vec{P} \tag{1.20}$$

or

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{1.21}$$

where $[K^{(e)}]$ is called the stiffness or characteristic matrix, $\vec{u}$ is the vector of nodal displacements, and $\vec{P}$ is the vector of nodal forces of the element. We shall derive the element stiffness matrix from the basic definition of the stiffness coefficient, and for this no assumed interpolation polynomials are needed. In structural mechanics [1.33], the stiffness influence coefficient k_{ij} is defined as the force needed at node i (in the direction of u_i) to produce a unit displacement at node j ($u_j = 1$) while all other nodes are restrained. This definition can be used to generate the matrix $[K^{(e)}]$. For example, when we apply a unit displacement to node 1 and restrain node 2 as shown in Fig. 1.11D, we induce a force (k_{11}) equal to[3] $k_{11} = (A_e E_e / l_e)$ at node 1 and a force (k_{21}) equal to $-(A_e E_e / l_e)$ at node 2. Similarly, we can obtain the values of k_{22} and

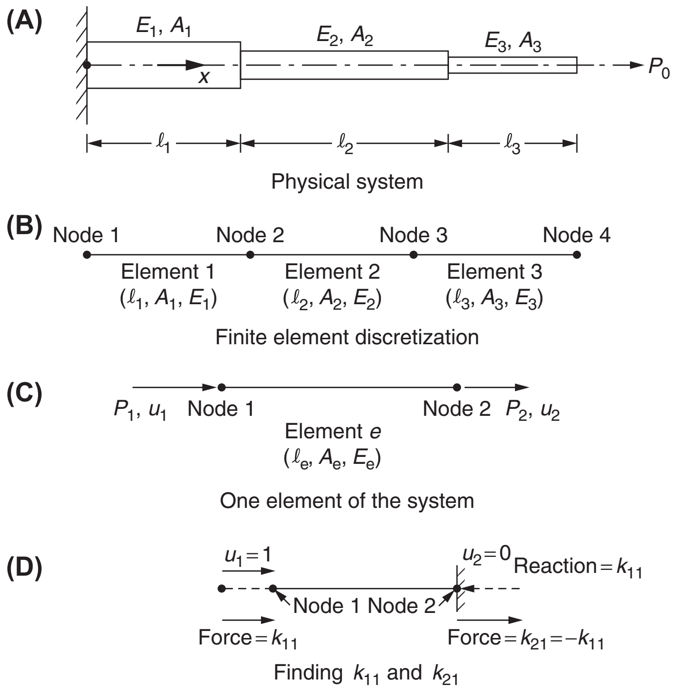


FIGURE 1.11 A stepped bar under axial load.

3. Force = stress × area of cross section = strain × Young's modulus × area of cross section = (change in length/original length) × Young's modulus × area of cross section = $(1/l_e) \cdot E_e \cdot A_e = (A_e E_e / l_e)$.

k_{12} as (A_eE_e/l_e) and $-(A_eE_e/l_e)$, respectively, by giving a unit displacement to node 2 and restraining node 1. Thus, the characteristic (stiffness) matrix of the element is given by

$$[K^{(e)}] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} (A_eE_e/l_e) & -(A_eE_e/l_e) \\ -(A_eE_e/l_e) & (A_eE_e/l_e) \end{bmatrix} = \frac{A_eE_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.22}$$

Notes

1. Eq. (1.20) denotes the element equations regardless of the type of problem, the complexity of the element, or the way in which the element, characteristic matrix or $[K^{(e)}]$, is derived.
2. The stiffness matrix $[K^{(e)}]$ obeys the Maxwell–Betti reciprocal theorem [1.33], which states that all stiffness matrices of linear structures are symmetric.

1.8.2 Spring Element

Consider a system of springs connected in series as shown in Fig. 1.12A. The analysis of the system (to find the nodal displacements under a prescribed set of loads) can be conducted using the finite element method. For this, the equilibrium relations of a typical spring element are to be derived first. Let the stiffnesses of the springs be denoted $k_1, k_2, \ldots, k_n$. Let F_i and F_j be the forces applied at nodes i and j, and u_i and u_j be the displacements of nodes i and j, respectively (Fig. 1.12B). The relations between the nodal forces and nodal displacements of a typical spring element e can be derived using the relation:

$$\text{Force} = \text{Spring stiffness} \times \text{Net deformation of the spring}$$

Thus, the forces at nodes i and j can be expressed as

$$F_i = k_e(u_i - u_j) \tag{1.23}$$

(where u_i is assumed to be larger than u_j so that the compressive force F_i leads to a decrease in the length of the spring), and

$$F_j = k_e(u_j - u_i) \tag{1.24}$$

(where u_j is assumed to be larger than u_i so that the tensile force F_j leads to an increase in the length of the spring). Eqs. (1.23) and (1.24) can be expressed in matrix form as

$$k_e\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{1.25}$$

From Eq. (1.25), the characteristic (stiffness) matrix of the element can be identified as

$$[K^{(e)}] = k_e \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} i \\ j \end{matrix} \\ \begin{matrix} i & \quad j \end{matrix} \end{array} \tag{1.26}$$

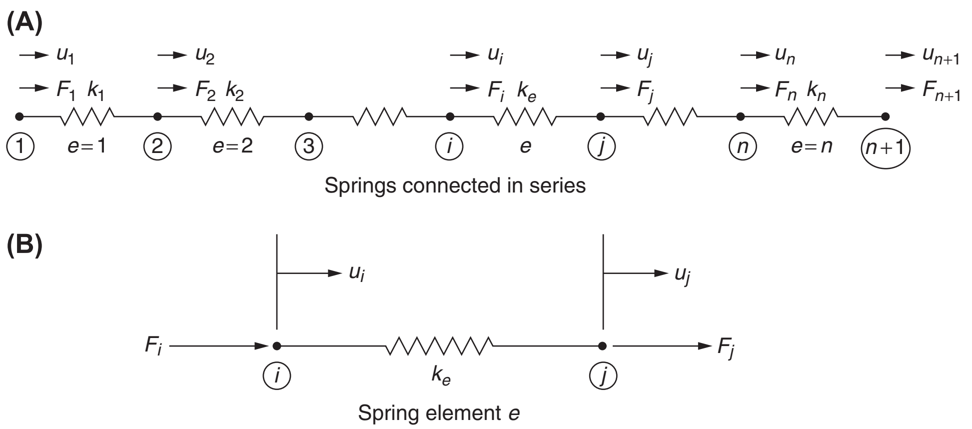


FIGURE 1.12 A system of n springs.

where the dof numbers corresponding to the rows and columns of the stiffness matrix $\left[K^{(e)}\right]$ are also indicated. Eq. (1.25) can be expressed as

$$\left[K^{(e)}\right]\vec{u} = \vec{F} \tag{1.27}$$

where $\vec{u}$ and $\vec{F}$ denote the vectors of nodal displacements and nodal forces, respectively.

1.8.3 Line Element for Heat Flow

Consider a composite (layered) wall through which heat flows in only the x direction (Fig. 1.13A). The left face is assumed to be at a uniform temperature higher than that of the right face. Each layer is assumed to be a nonhomogeneous material whose thermal conductivity is a known function of x. Since heat flows only in the x direction, the problem can be treated as one-dimensional, with each layer considered a finite element. The nodes for any element will be the bounding planes of the layer. Thus, there are four elements and five nodes in the system. The field variable ϕ is temperature T in this problem. Thus, the nodal unknowns denote the temperatures that are uniform over the bounding planes.

We can derive the element equations by considering the basic relation between the heat flow and the temperature gradient without using any interpolation polynomials. The quantity of heat crossing a unit area per unit time in the x direction (q) is given by Eq. (1.1) as

$$q = -k(x)\cdot\frac{\mathrm{d}T}{\mathrm{d}x} \tag{1.28}$$

where $k(x)$ is the thermal conductivity of the material and $(\mathrm{d}T/\mathrm{d}x)$ is the temperature gradient.

Eq. (1.28) can be integrated over the thickness of any element to obtain a relation between the nodal heat fluxes and the nodal temperatures. The integration can be avoided if we assume the thermal conductivity of a typical element e to be a constant as $k(x) = k'_e$. The temperature gradient at node 1 can be written as $(\mathrm{d}T/\mathrm{d}x)$ (at node 1) $= (T_2 - T_1)/(x_2 - x_1) = (T_2 - T_1)/t_e$ and the temperature gradient at node 2 as $(\mathrm{d}T/\mathrm{d}x)$ (at node 2) $= -(\mathrm{d}T/\mathrm{d}x)$ (at node 1) $= -(T_2 - T_1)/t_e$. Thus, the heat fluxes entering nodes 1 and 2 can be written as (see Fig. 1.13B)

$$\left.\begin{aligned} F_1 &= q(\text{at node }1) = -k'_e(T_2 - T_1)/t_e \\ F_2 &= q(\text{at node }2) = -k'_e(T_2 - T_1)/t_e \end{aligned}\right\} \tag{1.29}$$

By defining $k_e = k'_e/t_e$ Eq. (1.29) can be rewritten as

$$\left.\begin{aligned} F_1 &= -k_e(T_2 - T_1) \\ F_2 &= -k_e(T_1 - T_2) \end{aligned}\right\} \tag{1.30}$$

Eq. (1.30) can be expressed in matrix notation as

$$\left[K^{(e)}\right]\vec{T} = \vec{F} \tag{1.31}$$

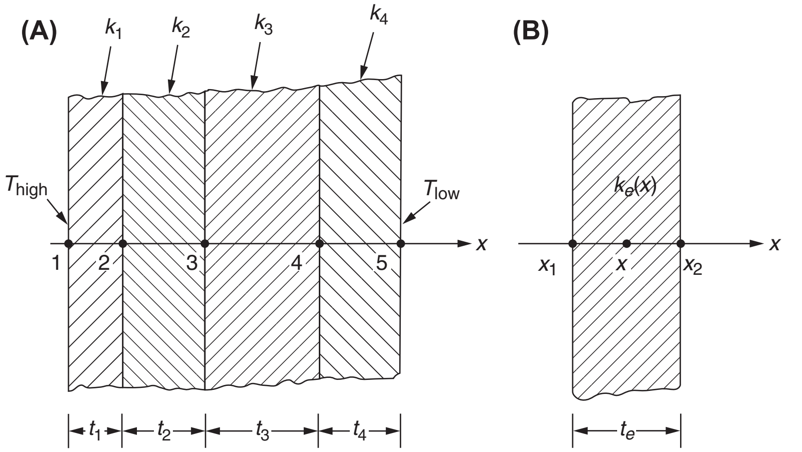


FIGURE 1.13 Heat flow through a composite (layered) wall.

where

$$\left[K^{(e)}\right] = \begin{bmatrix} k_e & -k_e \\ -k_e & k_e \end{bmatrix} = \text{element conductivity(characteristic)matrix}$$

$$\vec{T} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \text{vector of nodal temperatures}$$

and

$$\vec{F} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \text{vector of nodal heat fluxes}$$

Note that Eq. (1.31) has the same form as Eq. (1.20).

1.8.4 Pipe Element (Fluid Flow)

Consider a network of pipes for fluid flow as shown in Fig. 1.14A. Usually, there is a source such as a pump at some point in the network and when the inflow rate of the fluid is known, along with the fluid pressures at inlet and outlet(s), the problem involves the determination of the fluid flow rates through the various segments of the pipe network. The finite element method can be used for this purpose. For this, first we discretize the network into several finite elements, each element representing the flow path between any two connected nodes or junctions. Thus, the network shown in Fig. 1.14A has seven nodes and seven finite elements. In this case, the pressure loss−flow rate relations constitute the element equations and can be derived from the basic principles of fluid mechanics without any need for interpolation models.

To develop a relationship between the fluid flow rates and fluid pressures at the nodes of a typical pipe element, we start from the fluid flow rate Q (volume per unit time) in a pipe that is proportional to the pressure gradient, $\frac{\mathrm{d}p}{\mathrm{d}x}$ [1.34],

$$Q = -\frac{\pi d^4}{128\mu} \frac{\mathrm{d}p}{\mathrm{d}x} \tag{1.32}$$

where the pressure gradient at node i (Fig. 1.14B) can be expressed as

$$\left.\frac{\mathrm{d}p}{\mathrm{d}x}\right|_i = \frac{p_j - p_i}{l^{(e)}} \tag{1.33}$$

where p_i and p_j denote the fluid pressures at nodes i and j, respectively, and $l^{(e)}$ is the length of the pipe element. Note that $\frac{\mathrm{d}p}{\mathrm{d}x}$ needs to be negative in order for the fluid to flow from node i to node j. Note that Eq. (1.32) assumes that the flow is laminar, which implies that the Reynolds number, R_e, is less than 2000:

$$R_e = \frac{\rho v d}{\mu} < 2000 \tag{1.34}$$

where ρ is the density, v is the velocity, μ is the viscosity of the fluid, and d is the diameter of the pipe. The fluid inflow rate at node i can be expressed using Eqs. (1.32) and (1.33) as

$$Q_i = -\frac{\pi d^4}{128\mu l^{(e)}}(p_i - p_j) \tag{1.35}$$

Similarly, the fluid outflow rate at node j can be expressed as

$$Q_j = -\frac{\pi d^4}{128\mu l^{(e)}}(-p_i + p_j) \tag{1.36}$$

Eqs. (1.35) and (1.36) can be used to express the fluid flow rate−pressure relationship for pipe element e as

$$\begin{Bmatrix} Q_i \\ Q_j \end{Bmatrix} = \frac{\pi d^{(\mathrm{e})^4}}{128\mu^{(e)} l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} p_i \\ p_j \end{Bmatrix} \tag{1.37}$$

Eq. (1.37) can be written as

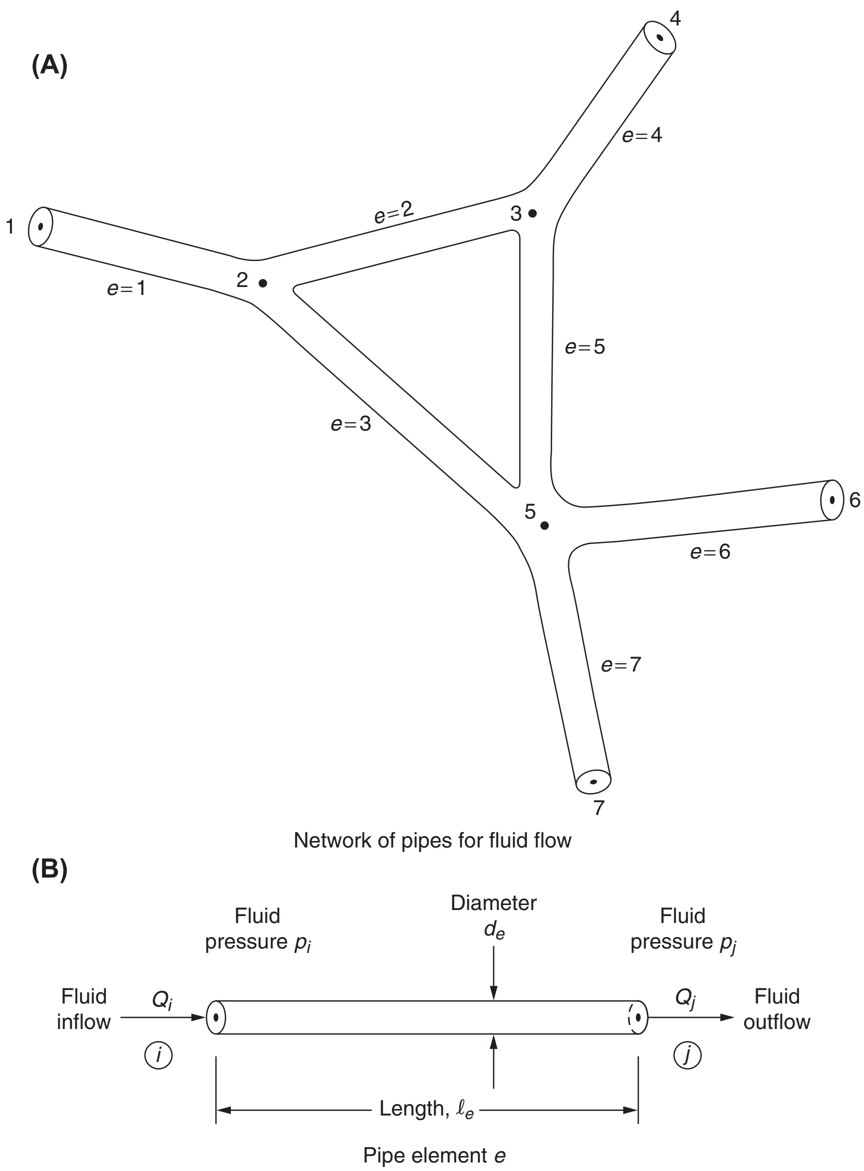


FIGURE 1.14 A network of pipes.

$$[K^{(e)}]\vec{p} = \vec{Q} \tag{1.38}$$

where $\vec{p}$ is the vector of nodal pressures, $\vec{Q}$ is the vector of nodal flows, and $[K^{(e)}]$ is the element characteristic (fluidity) matrix given by

$$[K^{(e)}] = \frac{\pi\, d^{(e)^4}}{128\,\mu^{(e)}\, l^{(e)}} \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc|c} \multicolumn{2}{c}{} & \\ \end{array} \begin{array}{c} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]\begin{array}{c} i \\ j \end{array} \\ \begin{array}{cc} i & \quad j \end{array} \end{array} \tag{1.39}$$

where the dof numbers corresponding to the rows and columns of the characteristic matrix $[K^{(e)}]$ are also indicated.

1.8.5 Electrical Resistor Element (Line Element for Current Flow)

Consider an electrical circuit, consisting of several resistors and a voltage source (battery) as shown in Fig. 1.15A. The analysis of the circuit can be conducted using the finite element method similar to that used for the analysis of networks of springs and pipes. For this, the governing equations (Ohm's law) for a typical resistor element *e* shown in Fig. 1.15B are to be developed. Using the relation

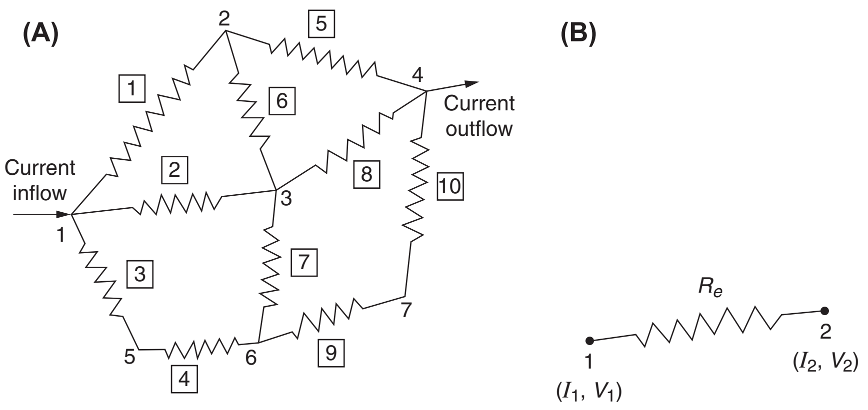


FIGURE 1.15 An electrical network.

$$I = \frac{V}{R} \tag{1.40}$$

where I is the current (amperes), R is the resistance of the resistor (ohms), and V is the voltage drop (volts), we obtain, at node i,

$$I_i = \frac{1}{R^{(e)}}(V_i - V_j) \tag{1.41}$$

where $V_i - V_j$ denotes the voltage drop across the resistor (difference in the voltage between the nodes i and j) and $R^{(e)}$ is the resistance of element e. Similarly, Eq. (1.40) can be expressed for node j as

$$I_j = \frac{1}{R^{(e)}}(V_j - V_i) \tag{1.42}$$

Thus, the current−voltage relationship of element e can be obtained from Eqs. (1.41) and (1.42) as

$$\vec{I} = \left[K^{(e)}\right]\vec{V} \text{ or } \begin{Bmatrix} I_i \\ I_j \end{Bmatrix} = \frac{1}{R^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} V_i \\ V_j \end{Bmatrix} \tag{1.43}$$

Equation (1.43) shows that the element characteristic matrix is given by

$$\left[K^{(e)}\right] = \frac{1}{R^{(e)}}\begin{array}{c c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} i \\ i \end{matrix} \\ \begin{matrix} i \quad & \quad j \end{matrix} & \end{array} \tag{1.44}$$

where the dof numbers corresponding to the rows and columns of the characteristic matrix $[K^{(e)}]$ are also indicated.

EXAMPLE 1.6

Five springs, having stiffnesses $k_1 = 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, $k_3 = 3 \times 10^5$ N/m, $k_4 = 4 \times 10^5$ N/m, and $k_5 = 5 \times 10^5$ N/m are connected as a parallel-series system, which is subjected to a load $P = 1000$ N at node 4 as shown in Fig. 1.16. Determine the displacements of nodes 2, 3, and 4 using the finite element method. State the assumptions made in your solution.

Solution

Assumption: The vertical bars with nodes 2, 3, and 4 are rigid and massless and move horizontally.

Using a spring finite element for each of the five springs, the element stiffness matrices can be obtained as follows:

$$\left[K^{(1)}\right] = \begin{array}{c c} \begin{matrix} U_1 & U_2 \end{matrix} & \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{matrix} U_1 \\ U_2 \end{matrix} \end{array} \quad \textbf{(corresponding degrees of freedom: } \boldsymbol{U_1} \textbf{ and } \boldsymbol{U_2}\textbf{)}$$

$$\left[K^{(2)}\right] = \begin{array}{c c} \begin{matrix} U_1 & U_2 \end{matrix} & \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{matrix} U_1 \\ U_2 \end{matrix} \end{array} \quad \textbf{(corresponding degrees of freedom: } \boldsymbol{U_1} \textbf{ and } \boldsymbol{U_2}\textbf{)}$$

Continued

EXAMPLE 1.6 —cont'd

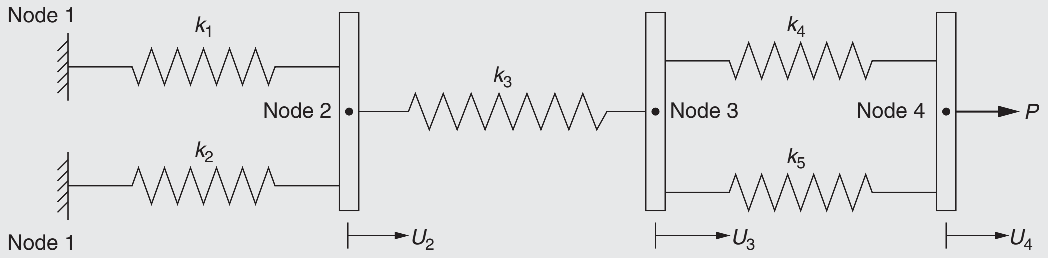


FIGURE 1.16 Springs in combination.

$$\begin{array}{c} \\ [K^{(3)}] = \end{array} \begin{array}{cc} U_2 & U_3 \\ \left[\begin{array}{cc} k_3 & -k_3 \\ -k_3 & k_3 \end{array}\right] \end{array} \begin{array}{c} \\ U_2 \\ U_3 \end{array}$$ **(corresponding degrees of freedom: U_2 and U_3)**

$$\begin{array}{c} \\ [K^{(4)}] = \end{array} \begin{array}{cc} U_3 & U_4 \\ \left[\begin{array}{cc} k_4 & -k_4 \\ -k_4 & k_4 \end{array}\right] \end{array} \begin{array}{c} \\ U_3 \\ U_4 \end{array}$$ **(corresponding degrees of freedom: U_3 and U_4)**

$$\begin{array}{c} \\ [K^{(5)}] = \end{array} \begin{array}{cc} U_3 & U_4 \\ \left[\begin{array}{cc} k_5 & -k_5 \\ -k_5 & k_5 \end{array}\right] \end{array} \begin{array}{c} \\ U_3 \\ U_4 \end{array}$$ **(corresponding degrees of freedom: U_3 and U_4)**

The assembled stiffness matrix of the system, including all the dof, can be determined as

$$\begin{array}{c} \\ [K] = \end{array} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \\ \left[\begin{array}{cccc} k_1+k_2 & -(k_1+k_2) & 0 & 0 \\ -(k_1+k_2) & k_1+k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4+k_5 & -(k_4+k_5) \\ 0 & 0 & -(k_4+k_5) & k_4+k_5 \end{array}\right] \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} \tag{E.1}$$

where the dof corresponding to the different rows and columns are also indicated. The nodal load vector and the vector of displacement dof of the system, considering all the dof, can be expressed as

$$\vec{P} = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ P \end{Bmatrix}, \vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} \tag{E.2}$$

where P_1 denotes the reaction (considered as unknown) at node 1 with zero displacement. The equilibrium equations of the system, before applying the boundary conditions, can be written as

$$[\underset{\sim}{K}]\vec{\underset{\sim}{U}} = \vec{\underset{\sim}{P}} \tag{E.3}$$

Using the given data, $k_1 = 1 \times 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, $k_3 = 3 \times 10^5$ N/m, $k_4 = 4 \times 10^5$ N/m, $k_5 = 5 \times 10^5$N/m, and $P = 1000$ N, Eq. (E.3) can be expressed as

$$10^5 \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 12 & -9 \\ 0 & 0 & -9 & 9 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \tag{E.4}$$

EXAMPLE 1.6 —cont'd

After applying the known boundary condition ($U_1 = 0$), by deleting the row and column corresponding to U_1 in Eq. (E.4), we obtain the final system equations as

$$10^5 \begin{bmatrix} 6 & -3 & 0 \\ -3 & 12 & -9 \\ 0 & -9 & 9 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix} \tag{E.5}$$

The solution of Eq. (E.5) gives the desired displacements:

$$\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 3.33 \\ 6.67 \\ 7.78 \end{Bmatrix} \times 10^{-3}\text{m} \tag{E.6}$$

Note

The deletion of row and column corresponding to U_1in the matrix Eq. (E.4) is possible because the value of $U_1$is specified as zero. When a nonzero value is specified for a nodal value, a different procedure is to be used for incorporating the condition (as indicated in the following examples).

EXAMPLE 1.7

Consider a network of four pipes with five nodes as shown in Fig. 1.17. A fluid of viscosity $\mu = 2.0 \times 10^{-6}$ lb-sec/in^2 and density $\rho = 2.2$ slug/ft^3 flows through the network. If the pressures at the four exterior nodes are given by $p_1 = 30$ psi, $p_3 = 22$ psi, $p_4 = 20$ psi, and $p_5 = 25$ psi, determine the flow rates through the various pipe segments.

Approach: Use the finite element approach, Eq. (1.38).

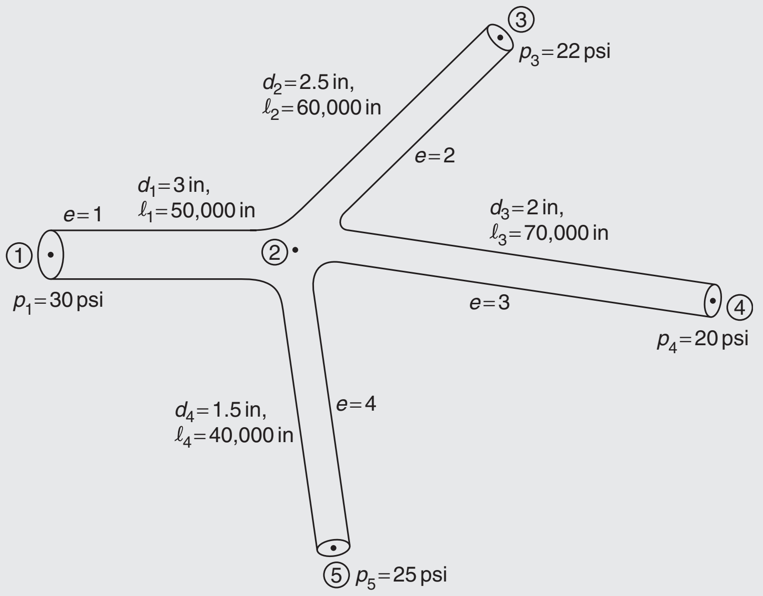


FIGURE 1.17 A network of four pipes.

Solution

The element characteristic matrix of a pipe element is given by Eq. (1.39):

$$\left[K^{(e)}\right] = \frac{\pi\, d_e^4}{128\mu_e l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{E.1}$$

where

$$c_1 = \frac{\pi\, d_1^4}{128\mu_1 l_1} = \frac{\pi(3)^4}{128(2.0\text{x}10^{-6})(50,000)} = 19.88 \text{ in}^5/\text{lb} - \text{sec for } e = 1$$

Continued

EXAMPLE 1.7 —cont'd

$$c_2 = \frac{\pi\, d_2^4}{128\mu_2 l_2} = \frac{\pi(2.5)^4}{128(2.0\text{x}10^{-6})(60,000)} = 7.99 \text{ in}^5/\text{lb} - \text{sec for } e = 2$$
$$c_3 = \frac{\pi\, d_3^4}{128\mu_3 l_3} = \frac{\pi(2.0)^4}{128(2.0\text{x}10^{-6})(70,000)} = 2.803 \text{ in}^5/\text{lb} - \text{sec for } e = 3$$
$$c_4 = \frac{\pi\, d_4^4}{128\mu_4 l_4} = \frac{\pi(1.5)^4}{128(2.0\text{x}10^{-6})(40,000)} = 1.553 \text{ in}^5/\text{lb} - \text{sec for } e = 4$$

Thus, the element characteristic matrices are given by

$$[K^{(1)}]=19.88\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}, \quad [K^{(2)}]=7.99\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ 3 \end{array}$$

$$[K^{(3)}]=2.803\begin{array}{c} \begin{array}{cc} 2 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ 4 \end{array}, \quad [K^{(4)}]=1.553\begin{array}{c} \begin{array}{cc} 2 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ 5 \end{array}$$

where the numbers identified with the columns and rows of the element characteristic matrices denote the numbers corresponding to nodal unknowns (pressures). The system or assembled characteristic matrix can be obtained as

$$[K]=\begin{array}{c}\begin{bmatrix} 19.88 & -19.88 & 0 & 0 & 0 \\ -19.88 & \begin{pmatrix}19.88+7.99+\\ 2.803+1.553\end{pmatrix} & -7.99 & -2.803 & -1.553 \\ 0 & -7.99 & 7.99 & 0 & 0 \\ 0 & -2.803 & 0 & 2.803 & 0 \\ 0 & -1.553 & 0 & 0 & 1.553 \end{bmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array} = \begin{array}{c}\begin{bmatrix} 19.88 & -19.88 & 0 & 0 & 0 \\ -19.88 & 32.226 & -7.99 & -2.803 & -1.553 \\ 0 & -7.99 & 7.99 & 0 & 0 \\ 0 & -2.803 & 0 & 2.803 & 0 \\ 0 & -1.553 & 0 & 0 & 1.553 \end{bmatrix} \\ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array} \tag{E.2}$$

Thus, the system equations can be expressed as

$$[K]\,\vec{p} = \vec{Q} \tag{E.3}$$

where $[K]$ is given by Eq. (E.2) with

$$\vec{P} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{Bmatrix} \text{ and } \vec{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix}$$

where Q_i denotes the external flow rate at node i (Q_i, $i = 1, 3, 4, 5$ are not known). Since the pressure at nodes $i = 1, 3, 4, 5$ are known, we retain the diagonal coefficients in $[K]$ corresponding to the unknown pressure p_2 (i.e., $k_{22} = 32.226$ is retained) and move all other coefficients of the second row of $[K]$ multiplied by their respective known pressures to the second term of the right-hand side vector. This leads to the modified form of Eq. (E.3) as

$$\begin{bmatrix} 19.88 & 0 & 0 & 0 & 0 \\ 0 & 32.226 & 0 & 0 & 0 \\ 0 & 0 & 7.99 & 0 & 0 \\ 0 & 0 & 0 & 2.803 & 0 \\ 0 & 0 & 0 & 0 & 1.553 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{Bmatrix} = \begin{Bmatrix} 19.88p_1 = 596.4 \\ 19.88p_1 + 7.99p_3 + 2.803p_4 + 1.553p_5 = 867.065 \\ 7.99p_3 = 175.78 \\ 2.803p_4 = 56.06 \\ 1.553p_5 = 38.825 \end{Bmatrix} \tag{E.4}$$

EXAMPLE 1.7 —cont'd

The solution of Eq. (E.4) gives $p_2 = 26.9057$ psi along with the known values, $p_1 = 30$ psi, $p_3 = 22$ psi, $p_4 = 20$ psi, and $p_5 = 25$ psi. The flow rates through each of the pipe elements can be determined as

$$\begin{aligned}
Q_1 &= c_1(p_1 - p_2) = 19.88(30.0 - 26.9057) = 61.5147\ \text{in}^3/\text{sec}\\
Q_2 &= c_2(p_2 - p_3) = 7.99(26.9057 - 22.0) = 39.1965\ \text{in}^3/\text{sec}\\
Q_3 &= c_3(p_2 - p_4) = 2.803(26.9057 - 20.0) = 19.3567\ \text{in}^3/\text{sec}\\
Q_4 &= c_4(p_2 - p_5) = 1.553(26.9057 - 25.0) = 2.9595\ \text{in}^3/\text{sec}
\end{aligned}$$

It can be seen that the inflow rate (Q_1) is equal to the total outflow rate ($Q_2 + Q_3 + Q_4$).

EXAMPLE 1.8

An electrical circuit is composed of six resistors and a battery as shown in Fig. 1.18. Determine the following: (a) voltages at the node points, and (b) current flows in the resistors.

Approach: Use the finite element Eq. (1.43) and the principle: the sum of currents meeting at any node i must be equal to zero.

Solution

Using Eq. (1.43), the element characteristic matrices of the various resistor elements can be found as

$$[K^{(1)}] = \frac{1}{10}\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \\ \begin{matrix} \ 1 & \quad 2 \end{matrix} & \end{array} = 0.1\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \\ \begin{matrix} \ 1 & \quad 2 \end{matrix} & \end{array}$$

$$[K^{(2)}] = \frac{1}{25}\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 3 \end{matrix} \\ \begin{matrix} \ 1 & \quad 3 \end{matrix} & \end{array} = 0.04\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 3 \end{matrix} \\ \begin{matrix} \ 1 & \quad 3 \end{matrix} & \end{array}$$

$$[K^{(3)}] = \frac{1}{20}\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 2 \\ 4 \end{matrix} \\ \begin{matrix} \ 2 & \quad 4 \end{matrix} & \end{array} = 0.05\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 2 \\ 4 \end{matrix} \\ \begin{matrix} \ 2 & \quad 4 \end{matrix} & \end{array}$$

$$[K^{(4)}] = \frac{1}{15}\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 3 \\ 5 \end{matrix} \\ \begin{matrix} \ 3 & \quad 5 \end{matrix} & \end{array} = 0.0667\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 3 \\ 5 \end{matrix} \\ \begin{matrix} \ 3 & \quad 5 \end{matrix} & \end{array}$$

$$[K^{(5)}] = \frac{1}{30}\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 4 \\ 6 \end{matrix} \\ \begin{matrix} \ 4 & \quad 6 \end{matrix} & \end{array} = 0.0333\begin{array}{c@{}l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{matrix} 4 \\ 6 \end{matrix} \\ \begin{matrix} \ 4 & \quad 6 \end{matrix} & \end{array}$$

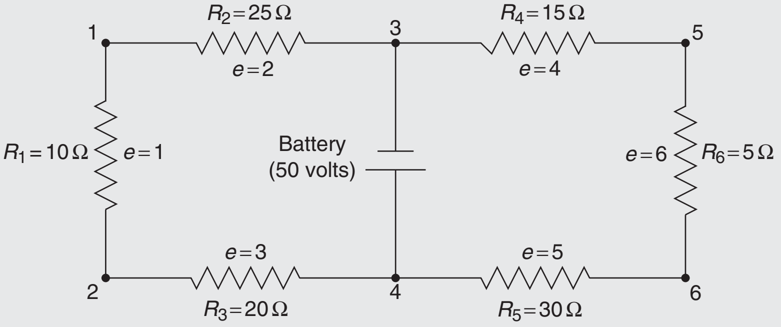


FIGURE 1.18 An electrical circuit.

Continued

EXAMPLE 1.8 —cont'd

$$[K^{(6)}]=\frac{1}{5}\begin{array}{cc}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} & \begin{matrix}5\\6\end{matrix}\\ \begin{matrix}5 & \;\;6\end{matrix} & \end{array}=0.02\begin{array}{cc}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} & \begin{matrix}5\\6\end{matrix}\\ \begin{matrix}5 & \;\;6\end{matrix} & \end{array}$$

where the numbers identified with the columns and rows of the element characteristic matrices denote the corresponding nodal unknowns (voltages). The system or assembled characteristic matrix can be obtained as

$$[K]=\begin{array}{cc}\begin{bmatrix}(0.1+0.04) & -0.1 & -0.04 & 0 & 0 & 0\\ -0.1 & (0.1+0.05) & 0 & -0.05 & 0 & 0\\ -0.04 & 0 & (0.04+0.0667) & 0 & -0.0667 & 0\\ 0 & -0.05 & 0 & (0.05+0.0333) & 0 & -0.0333\\ 0 & 0 & -0.0667 & 0 & (0.0667+0.2) & -0.2\\ 0 & 0 & 0 & -0.0333 & -0.2 & (0.0333+0.2)\end{bmatrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\\ \begin{matrix}1 & \quad\quad 2 & \quad\quad 3 & \quad\quad 4 & \quad\quad 5 & \quad\quad 6\end{matrix} & \end{array}\tag{E.1}$$

$$=\begin{array}{cc}\begin{bmatrix}0.14 & -0.1 & -0.04 & 0 & 0 & 0\\ -0.1 & 0.15 & 0 & -0.05 & 0 & 0\\ -0.04 & 0 & 0.1067 & 0 & -0.0667 & 0\\ 0 & -0.05 & 0 & 0.0833 & 0 & -0.0333\\ 0 & 0 & -0.0667 & 0 & 0.2667 & -0.2\\ 0 & 0 & 0 & -0.0333 & -0.2 & 0.2333\end{bmatrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix}\\ \begin{matrix}1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6\end{matrix} & \end{array}$$

Thus, the system equations can be written in matrix form as

$$[K]\,\vec{V}=\vec{I}\tag{E.2}$$

where $[K]$ is given by Eq. (E.1),

$$\vec{V}=\begin{Bmatrix}V_1\\V_2\\V_3\\V_4\\V_5\\V_6\end{Bmatrix},\text{ and }\vec{I}=\begin{Bmatrix}I_1\\I_2\\I_3\\I_4\\I_5\\I_6\end{Bmatrix}$$

with V_i denoting the voltage at node i and I_i representing the current flow through the resistor element i, $i=1, 2, \ldots, 6$.

1. To incorporate the known values $V_4=50$ V and $V_3=0$, the system Eq. (E.2) is modified as follows. Since no value of I_j in the vector $\vec{I}$ is known, it can be assumed to be a zero vector to start with. All the nonzero elements in column 3 of the matrix $[K]$, except the diagonal element, are multiplied by zero (specified value of V_3) and moved to the third element of the vector $\vec{I}$ by changing their signs. Similarly, all the nonzero elements in column 4 of the matrix $[K]$, except the diagonal element, are multiplied by 60 (specified value of V_4) and moved to the fourth element of the vector $\vec{I}$ by changing their signs. This results in the following system of equations.
   (Note: The procedure used for incorporating the specified values of V_j is also explained in Section 6.3.2.)

$$\begin{bmatrix}0.14 & -0.1 & -0.04 & 0 & 0 & 0\\ -0.1 & 0.15 & 0 & 0 & 0 & 0\\ 0 & 0 & 0.1067 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.0833 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.2667 & -0.2\\ 0 & 0 & 0 & 0 & -0.2 & 0.2333\end{bmatrix}\begin{Bmatrix}V_1\\V_2\\V_3\\V_4\\V_5\\V_6\end{Bmatrix}=\begin{Bmatrix}0\\2.5\\0\\4.165\\0\\1.665\end{Bmatrix}\tag{E.3}$$

The solution of Eq. (E.3) is given by $V_1=22.7273$ volts, $V_2=31.8182$ volts, $V_3=0$, $V_4=50$ volts, $V_5=14.9865$ volts, and $V_6=19.9845$ volts.

EXAMPLE 1.8 —cont'd

2. The current flows in the various resistors can be computed as

$$I_1 = \frac{V_1 - V_2}{R_1} = \frac{22.7273 - 31.8182}{10} = -0.9091 \text{ ampere}$$

$$I_2 = \frac{V_1 - V_3}{R_2} = \frac{22.7273 - 0}{25} = 0.9091 \text{ ampere}$$

$$I_3 = \frac{V_2 - V_4}{R_3} = \frac{31.8182 - 50}{20} = -0.9091 \text{ ampere}$$

$$I_4 = \frac{V_3 - V_5}{R_4} = \frac{0 - 14.9865}{15} = -0.9991 \text{ ampere}$$

$$I_5 = \frac{V_4 - V_6}{R_5} = \frac{50 - 19.9845}{30} = 1.0005 \text{ ampere}$$

$$I_6 = \frac{V_5 - V_6}{R_6} = \frac{14.9865 - 19.9845}{5} = -0.9996 \text{ ampere}$$

Note

A negative sign for I_e denotes that current flows from node j to node i instead of from node i to node j.

EXAMPLE 1.9

The wall of an industrial furnace is composed of a 6 cm thick inner layer made of silica brick and a 3 cm outer layer made of masonry brick as shown in Fig. 1.19A. The temperature of the inside surface of the wall (silica brick surface) is 30°C and the temperature of the air surrounding the brick surface of the wall is −2°C. The thermal conductivities of silica and masonry bricks are 0.08 W/(cm °C) and 0.25 W/(cm °C), respectively. If the heat transfer coefficient at the outer (masonry brick) surface is

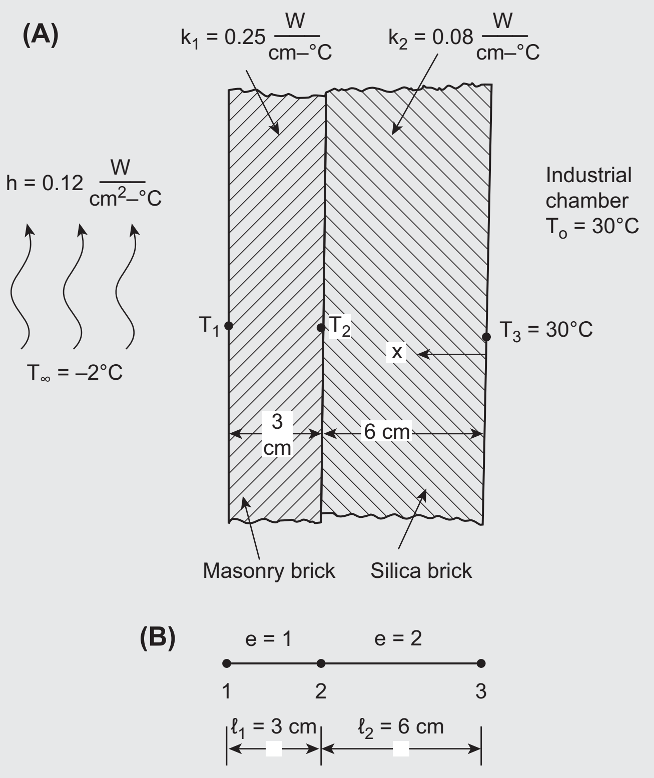


FIGURE 1.19 Heat flow through two bricks.

Continued

EXAMPLE 1.9 —cont'd

0.12 W/(cm^2 °C), determine the temperatures at the interface of the two brick surfaces and the outside surface of the masonry wall.

Solution

By assuming the heat transfer through the wall to be one-dimensional (along the x-direction), the system is modeled by two finite elements, one for each material, as shown in Fig. 1.19B. By using the notation $k_e = k'_e/l^{(e)}$, the conductivity matrices of elements 1 (masonry brick) and 2 (silica brick) are given by

$$\left[k^{(1)}\right] = k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{k'_1 A_1}{l^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(0.25)(1)}{3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.08333\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathrm{W/^\circ C} \tag{E.1}$$

$$\left[k^{(2)}\right] = k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{k'_2 A_2}{l^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(0.08)(1)}{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.01333\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathrm{W/^\circ C} \tag{E.2}$$

where the areas of wall surfaces through which heat flows are assumed to be 1 cm^2($A_1 = A_2 = 1$). The right-hand side vectors (heat flux vectors) of the elements in a heat transfer problem (similar to the nodal force vectors in a solid mechanics problem) are given by

$$\vec{F}^{(1)} = \begin{Bmatrix} hAT_\infty \\ 0 \end{Bmatrix} = \begin{Bmatrix} (0.12)(1)(-2) \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.24 \\ 0 \end{Bmatrix} \mathrm{W} \tag{E.3}$$

$$\vec{F}^{(2)} = \begin{Bmatrix} 0 \\ Q_3 \end{Bmatrix} \mathrm{W} \tag{E.4}$$

These nodal force vectors use the known information that node 1 (of element 1) is exposed to $T_\infty = -2°C$, has a heat flux of $h\,A\,T_\infty$ (area A is assumed to be 1 cm^2), and no heat transfer coefficient is specified ($h = 0$) at node 2. At node 3, there will be an unknown heat flux Q_3 to satisfy the heat flow equation or thermal equilibrium of the system (similar to the reaction at a fixed end in a solid mechanics problem). The assembled matrix and assembled right-hand side vector of the system (before applying the boundary condition) are given by

$$[K] = \begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_3 \end{array} \\ \begin{bmatrix} 0.08333 & -0.08333 & 0 \\ -0.08333 & (0{,}08333 + 0.01333) & -0.01333 \\ 0 & -0.01333 & 0.01333 \end{bmatrix} \end{array} \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \quad \frac{W}{{}^0C} \tag{E.5}$$

$$\vec{F} = \begin{Bmatrix} -0.24 \\ 0 + 0 \\ Q_3 \end{Bmatrix} \mathrm{W} \tag{E.6}$$

To incorporate the boundary condition (T_3 is specified as 30°C), we delete the third row in Eqs. (E.5) and (E.6), and move the coefficient $K_{23} = -0.01333$ multiplied by the known temperature ($T_3 = 30$) to the right-hand side element F_2 as $(+0.01333 \times 30 = 0.3999)$. This process is similar to transferring a term to the right-hand side from the left-hand side in a linear equation. Then we delete the third column of the matrix $[K]$ to obtain the matrix equations, after applying the known boundary condition, as

$$\begin{bmatrix} 0.08333 & -0.08333 \\ -0.08333 & 0.09666 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} -0.24 \\ 0.03999 \end{Bmatrix} \tag{E.7}$$

The solution of Eq. (E.7) gives the temperatures of nodes 1 and 2 as $T_1 = 9.1154°C$ and $T_2 = 11.9955°C$.

1.9 COMMERCIAL FINITE ELEMENT PROGRAM PACKAGES

The general applicability of the finite element method makes it a powerful and versatile tool for a wide range of problems. Hence, a number of computer program packages have been developed for the solution of a variety of structural and solid mechanics problems. Some of the programs have been developed in such a general manner that the same program can be used for the solution of problems belonging to different branches of engineering with little or no modification.

Many of these packages represent large programs that can be used for solving real complex problems. For example, the NASTRAN (National Aeronautics and Space Administration Structural Analysis) program package contains approximately 150,000 FORTRAN statements and can be used to analyze physical problems of practically any size, including a complete aircraft, an automobile, and a space shuttle.

The availability of supercomputers has made a strong impact on the finite element technology. In order to realize the full potential of these supercomputers in finite element computation, special parallel numerical algorithms, programming strategies, and programming languages are being developed. The use of personal computers and workstations in engineering analysis and design is becoming increasingly popular as the price of hardware is decreasing dramatically. Many finite element programs, especially suitable for the personal computer and workstation environment, have been developed. Among the main advantages are a user-friendly environment and inexpensive graphics.

1.10 SOLUTIONS USING FINITE ELEMENT SOFTWARE

Three Steps of Finite Element Solution

The solution of any engineering analysis problem using commercial FEA software involves the following three steps:

Preprocessing:
In this step, the geometry, material properties, loads (actions), and boundary conditions are given as input data. In-built automatic mesh generation modules develop the finite element mesh with minimal input from the analyst on the type of elements and mesh density to be used. The analyst can display the data as well as the finite element mesh generated for visual inspection and verification for correctness.
Numerical analysis:
The software automatically generates the element characteristic (stiffness) matrices and characteristic (load) vectors, assembles them to generate the system equations, implements the specified boundary conditions, solves the equations to find the nodal values of the field variable (displacements), and computes the element resultants (stresses and strains).
Postprocessing:
The solution of the problem, such as nodal displacements and element stresses, can be displayed either numerically in tabular form or graphically (two- or three-dimensional plots of deformed shape or stress variation). The analyst can choose the mode of display for the results.

Checking the Results of FEA

It is extremely important to check the results given by the FEA software. Usually a simpler version of the actual problem is to be solved using the software so that the results can be compared with known solutions (obtained by other methods such as a simplified analysis technique). In addition, the analyst must ensure that the results agree with engineering intuition and behavior. Also, we need to verify whether the solution satisfies the specified boundary and symmetry conditions. If necessary, the problem needs to be solved by changing the boundary conditions, loads, or materials to find whether the resulting FEA solutions behave as per engineering intuition and expectations.

Software Applications

This book presents the applications of two commercial/computer program packages, ANSYS and ABAQUS, for the solution of engineering analysis problems. In addition, several MATLAB programs are given for the solution of solid and structural mechanics, heat transfer and fluid flow problems, along with illustrative examples to demonstrate their use.

ANSYS is general-purpose FEA software; ANSYS Mechanical is for mechanical and structural systems and ANSYS Multiphysics is for general field problems. The ANSYS software includes modules for the creation of geometry and finite element mesh; solution and postprocessing in a unified graphical user interface. The FEA can be performed either in batch or interactive modes. In the batch mode, an input file of commands will be executed from the command line. In the interactive mode, user actions (to be picked from a menu or typed as commands) are required in a graphics window. The use of ANSYS for the finite element solution of several problems is illustrated using the interactive mode in this book.

ABAQUS finite element system includes ABAQUS/Standard, a general-purpose finite element program; ABAQUS/Explicit, an explicit dynamics finite element program; and ABAQUS/CAE, an interactive environment for constructing, analyzing, and visualizing results. In ABAQUS, two methods can be used to construct the models — one is to use ABAQUS/CAE interactive environment and the other is to write script files. The ABAQUS examples presented in this book are based on writing script files. An input (script) file can be written in any text editor tool such as WordPad. The input file contains information including the kind of problem, the geometry of the structure, the properties of the structure, the analysis parameters, and the output requirements.

MATLAB is popular software that can be used for the solution of a variety of scientific and engineering problems. It contains a library of programs or m-files that can be used for the solution of finite element equations (in Step 5 of the six-step FEA procedure). For example, the programs/commands for the solution of simultaneous linear algebraic equations can be used in the solution of static or steady-state problems. The programs/commands for matrix manipulation, such as decomposition, inversion, multiplication, and eigenvalue solution can be used in the formulation and solution of matrix eigenvalue problems related to engineering systems. Similarly, the programs/commands for the solution of ordinary and partial differential equations can be used in the solution of dynamic or transient or unsteady state problems related to a variety of field problems. The MATLAB finite element programs given in the book include the development of the model, and formulation and solution of structural, heat transfer, and fluid flow problems.

This book presents several MATLAB finite element programs for the solution of typical analysis problems from the areas of solid/structural mechanics, heat transfer, and fluid flow. These programs are written in the MATLAB programming language and are self-contained in the sense that they do not use any of the MATLAB library of programs or m-files. The listings of all these programs are available at the website of the book. The use of each program is illustrated with an example along with the input data required, output/results given by the program, and the development of the main program or m-file required for the solution.

REVIEW QUESTIONS

1.1 Give brief answers to the following questions.

1. What is discretization?
2. What is the purpose of the interpolation model in the finite element method?
3. Define the potential energy of a body subjected to loads.
4. What is strain energy?
5. Why is the stiffness matrix always symmetric?
6. Why is a cubic interpolation model assumed for a beam element?
7. Give names of two popular commercial finite element software packages.
8. Who coined the term finite element?
9. Who published the first finite element paper for application to aircraft structures?
10. State two reasons for the widespread use of the finite element method.
11. State the equation for the strain energy of a solid body of volume V.
12. Define Reynolds number for fluid flow in a pipe.
13. How can you use the finite element method for the analysis of mechanisms used in industry?

1.2 Fill in the blank with a suitable word.

1. The stiffness matrix of a bar element under axial load, $[k^{(e)}]$, is given by ________________.
2. The stiffness matrix of a bar or beam element denotes the relation between nodal forces and nodal ________________________________.
3. The composite wall of a building having brick, insulation, and plaster layers can be represented by three heat flow elements, one for each layer using their respective thermal ________________________________.
4. The linear finite elements used for the analysis of electrical circuits use ________________________________ ________ law.

1.3 Indicate whether each of the following statements is true or false.

1. The finite element method can be used to obtain both lower and upper bounds on the exact solution of an engineering mechanics problem.
2. We can use a quadratic interpolation model for a two-noded beam element.
3. In a fluid flow (in a pipe) element, the fluidity matrix relates velocity of the fluid to the pressure gradient.

1.4 Select the most appropriate answer from the multiple choices given.

1. The direct approach of deriving the finite element equations for a structural problem requires the application of
   (a) Equilibrium equations (b) Variational equations (c) Energy equations
2. The stiffness matrix of a bar element of length l with cross-sectional area A and Young's modulus E is given by

   $$\left[k^{(e)}\right] = c\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ with } c \text{ equal to}$$

   (a) $\frac{A\,E}{l}$ (b) $\frac{l}{A\,E}$ (c) $\frac{A\,l}{E}$
3. The element conductivity matrix in heat transfer indicates the relation between
   (a) nodal conductivities and nodal temperatures
   (b) nodal heat fluxes and nodal temperatures

(c) nodal heat fluxes and nodal conductivities

4. In an electrical circuit, the element characteristic matrix, $[k^{(e)}]$, denotes the relation between
(a) nodal currents and nodal resistances
(b) nodal resistances and nodal voltages
(c) nodal currents and nodal voltages

5. The element fluidity (characteristic) matrix derived in this chapter is valid for
(a) any fluid flow (b) laminar flow (c) turbulent flow

PROBLEMS

1.1 If $S^{(l)}$ and $S^{(u)}$ denote the perimeters of the inscribed and circumscribed polygons, respectively, as shown in Fig. 1.3, prove that

$$S^{(l)} \le S \le S^{(u)}$$

where S is the circumference of the circle.

1.2 Find the values of the perimeters of the inscribed and circumscribed polygons ($S^{(l)}$ and $S^{(u)}$) of a circle of radius 1 for polygons with the number of sides ranging from $n = 3$ to 12.

1.3 Suggest a procedure, similar to the one described in Example 1.1 for the circumference of a circle, for finding (a) the bounds on the area of a circle, and (b) the exact area of a circle.

1.4 Using a one-beam element idealization, find the stress distribution under a load of P for the uniform cantilever beam shown in Fig. 1.20.

1.5 Find the stress distribution in the cantilever beam shown in Fig. 1.21 using one beam element.

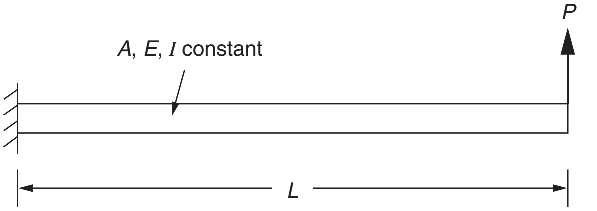


FIGURE 1.20 A uniform cantilever beam.

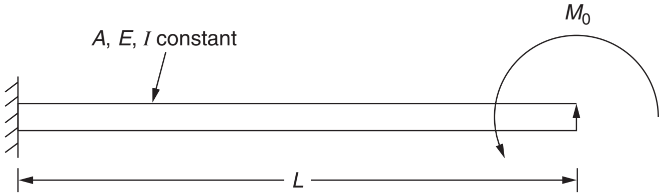


FIGURE 1.21 A cantilever beam subjected to an end moment.

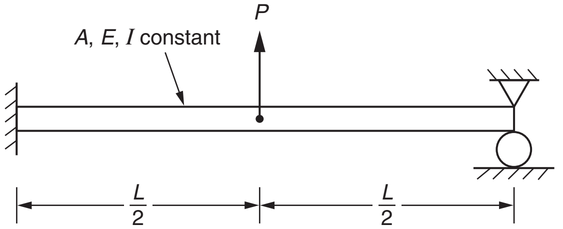


FIGURE 1.22 A fixed-pinned beam.

1.6 Find the stress distribution in the beam shown in Fig. 1.22 using two beam elements.
1.7 Find the stress distribution in the beam shown in Fig. 1.23 using two beam elements.
1.8 Find the stress distribution in the beam shown in Fig. 1.24 using two beam elements.
1.9 For the tapered bar shown in Fig. 1.25, the area of cross section changes along the length as $A(x) = A_0 e^{-(x/l)}$, where A_0 is the cross-sectional area at $x = 0$, and l is the length of the bar. By expressing the strain and kinetic energies of the bar in matrix forms, identify the stiffness and mass matrices of a typical element. Assume a linear model for the axial displacement of the bar element.

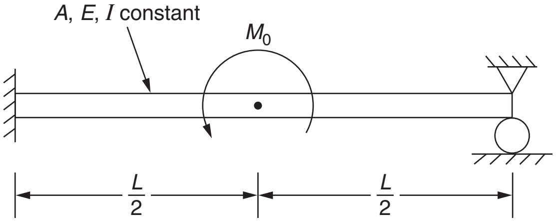


FIGURE 1.23 A fixed-pinned beam subjected to a moment.

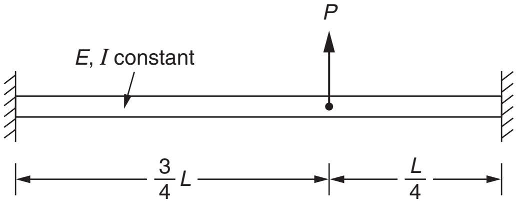


FIGURE 1.24 A fixed-fixed beam.

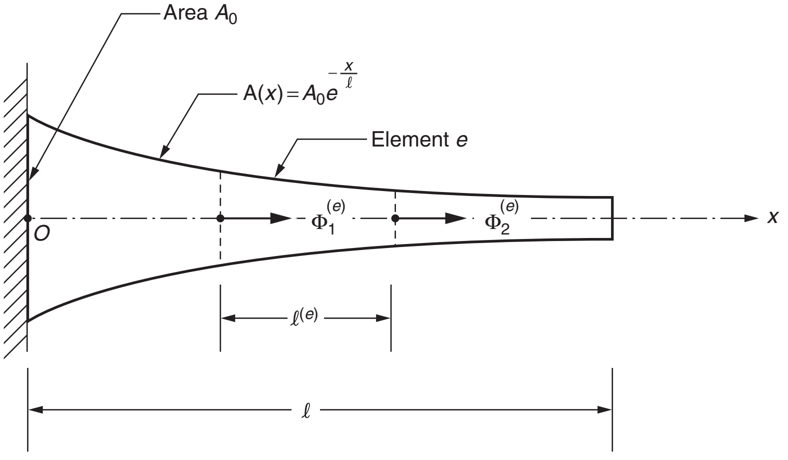
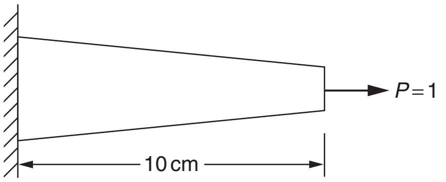


FIGURE 1.25 A tapered bar.



Cross-sectional area at root = 2 cm^2
Cross-sectional area at end = 1 cm^2
Young's modulus = 2×10^7 N/cm^2

FIGURE 1.26 A tapered bar subjected to axial load.

1.10 Find the stress distribution in the tapered bar shown in Fig. 1.26 using two finite elements under an axial load of $P = 1$ N.

1.11 Consider the flow of an incompressible fluid through the network of three pipes as shown in Fig. 1.27. The viscosity of the fluid is $\mu = 1.6 \times 10^{-6}\,\mathrm{lb_f\text{-}sec/in^2}$ and the density is $\rho = 1.9\,\mathrm{slugs/ft^3}$. Determine the following:

a. Fluid pressure at node 2.
b. Volume flow rate in each of the pipes.
c. Reynolds number in each of the pipes.
d. Whether the flow is laminar or turbulent in each of the pipes.

1.12 Show that the stiffness matrix of a spring element, with a spring constant k, shown in Fig. 1.28, is given by

$$\left[K^{(e)}\right] = k \begin{array}{c} \begin{array}{cc} U_1^{(e)} & U_2^{(e)} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_2^{(e)} \\ U_1^{(e)} \end{array}$$

Hint: Express the strain energy of the spring in matrix form.

1.13 Two springs, having stiffnesses $k_1 = 10^5$ N/m and $k_2 = 5 \times 10^5$ N/m, are connected in series as shown in Fig. 1.29. Determine the displacements of nodes 2 and 3 when an axial load of $P = 1000$ N is applied at node 3 using the finite element method.

1.14 Three springs, having stiffnesses $k_1 = 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, and $k_3 = 3 \times 10^5$ N/m, are connected in series as shown in Fig. 1.30. Determine the displacements of nodes 2, 3, and 4 when an axial load of $P = 1000$ N is applied at node 4 using the finite element method.

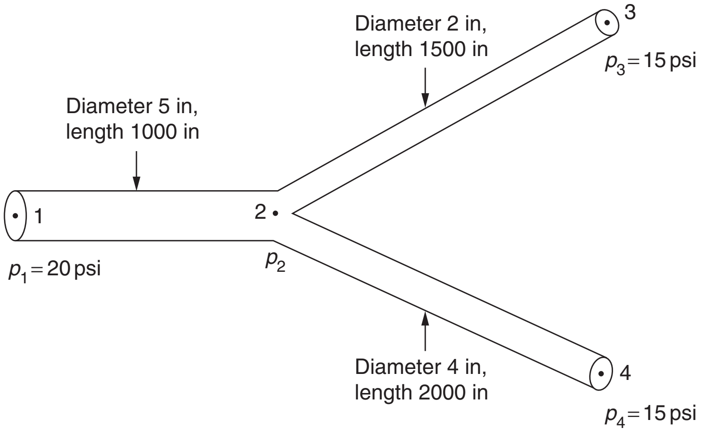


FIGURE 1.27 A network of three pipes.

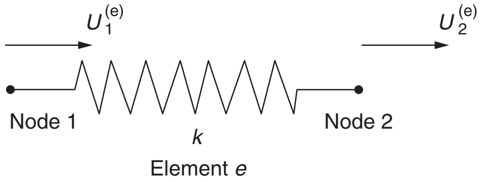


FIGURE 1.28 A spring element.

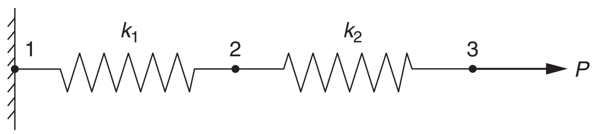


FIGURE 1.29 Two springs in series.

1.15 Three springs, having stiffnesses $k_1 = 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, and $k_3 = 3 \times 10^5$ N/m, are connected in series as shown in Fig. 1.31. Determine the displacements of nodes 2, 3, and 4 when an axial load of $P = 1000$ N is applied at node 2 using the finite element method.

1.16 Three springs, having stiffnesses $k_1 = 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, and $k_3 = 3 \times 10^5$ N/m, are connected in series as shown in Fig. 1.32. Determine the displacements of nodes 2, 3, and 4 when an axial load of $P = 1000$ N is applied at node 2 using the finite element method.

1.17 Five springs, having stiffnesses $k_1 = 10^5$ N/m, $k_2 = 2 \times 10^5$ N/m, $k_3 = 3 \times 10^5$ N/m, $k_4 = 4 \times 10^5$ N/m, and $k_5 = 5 \times 10^5$ N/m, are connected as a parallel-series system and are subjected to a load $P = 1000$ N at node 4 as shown in Fig. 1.33. Determine the displacements of nodes 2, 3, and 4 using the finite element method. State the assumptions made in your solution.

1.18 Two links, made up of aluminum and steel, are connected by a hinge joint and an axial load $P = 1000$ N is applied at node 3 as shown in Fig. 1.34. Determine the stresses developed in the two links using the finite element method.

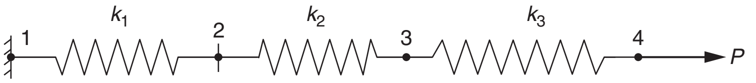


FIGURE 1.30 Three springs in series.

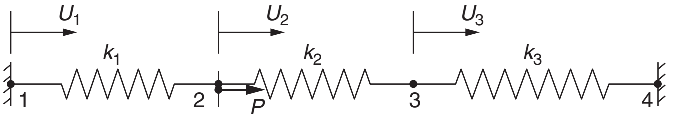


FIGURE 1.31 Three springs in series with fixed ends.

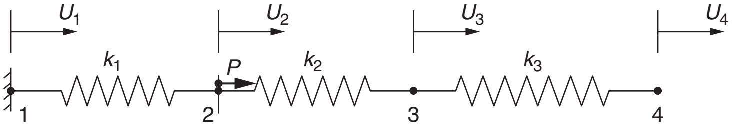


FIGURE 1.32 Three springs in series with one end fixed.

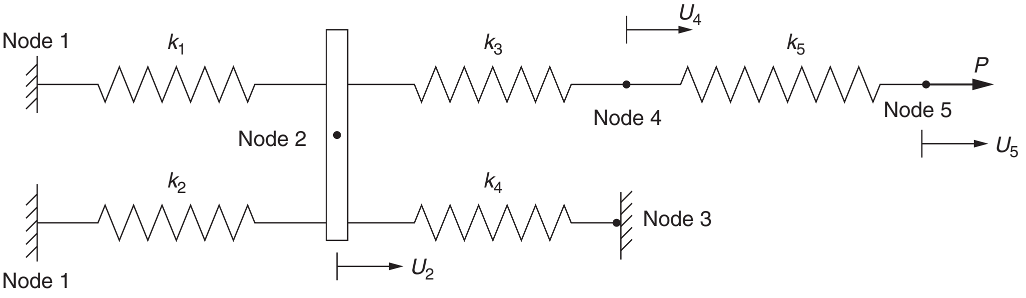


FIGURE 1.33 Parallel-series system of springs.

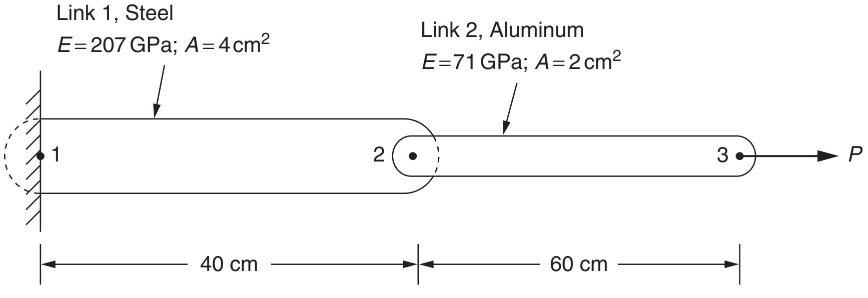


FIGURE 1.34 Two links subjected to an axial load.

1.19 A stepped bar is subjected to an axial load of $P = 2000$ N as shown in Fig. 1.35. If the material of the stepped bar is steel with a Young's modulus of 207×10^9 Pa, find the stresses developed in each of the steps.

1.20 Suggest a method of finding the stresses in the frame shown in Fig. 1.36 using the finite element method.

1.21 Derive the stiffness matrix of a beam element, given by Eq. (1.19), using Eqs. (1.17) and (1.18).

1.22 A typical stiffness coefficient, k_{ij}, in the stiffness matrix, $[K]$, denotes the force along the dof i that results in a unit displacement along the dof j when the displacements along all other dof are zero. Using this definition, beam deflection relations, and static equilibrium equations generate expressions for k_{11}, k_{21}, k_{31}, and k_{41} for the uniform beam element shown in Fig. 1.37. The Young's modulus, area moment of inertia, and length of the element are given by E, I, and l, respectively.

1.23 For the beam element considered in Problem 1.22, generate expressions for the stiffness coefficients k_{12}, k_{22}, k_{32}, and k_{42}.

1.24 For the beam element considered in Problem 1.22, generate expressions for the stiffness coefficients k_{13}, k_{23}, k_{33}, and k_{43}.

1.25 For the beam element considered in Problem 1.22, generate expressions for the stiffness coefficients k_{14}, k_{24}, k_{34}, and k_{44}.

1.26 Derive the stiffness matrix of a tapered bar, with linearly varying area of cross section (Fig. 1.38), using a direct approach.

1.27 The heat transfer in the tapered fin shown in Fig. 1.39 can be assumed to be one-dimensional due to the large value of W compared to L. Derive the element characteristic matrix of the fin using the direct approach.

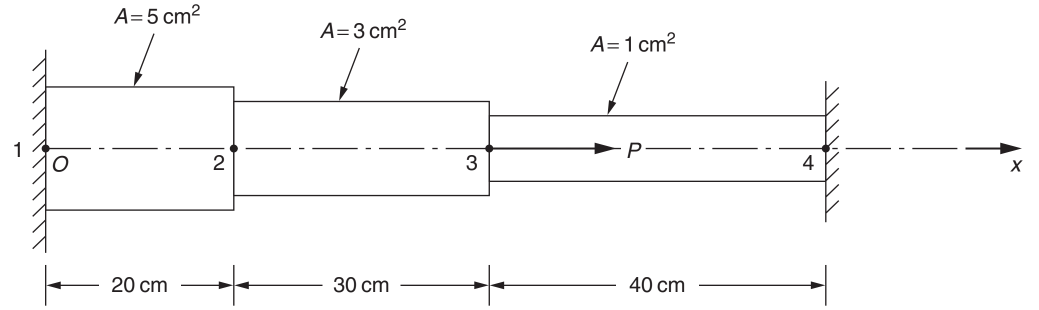


FIGURE 1.35 A stepped bar with fixed ends.

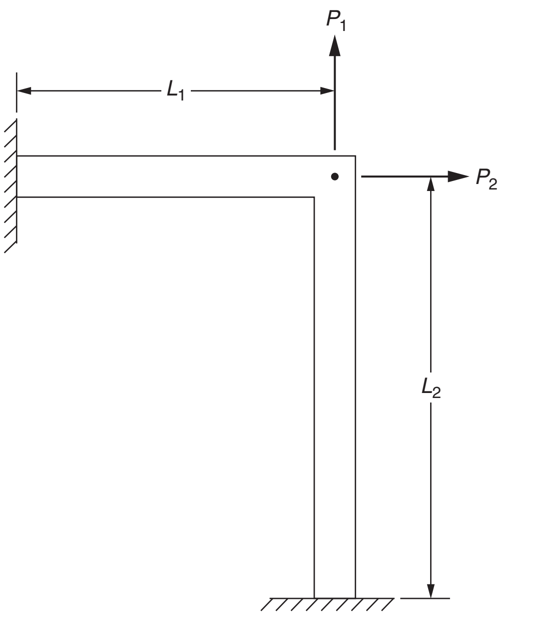


FIGURE 1.36 A frame.

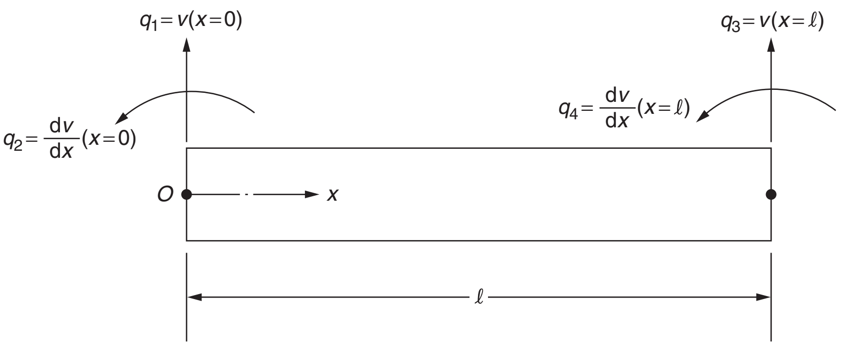


FIGURE 1.37 A beam element with nodal displacement.

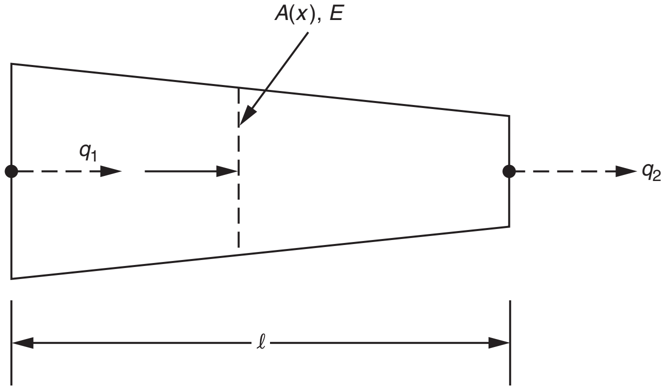


FIGURE 1.38 A tapered bar.

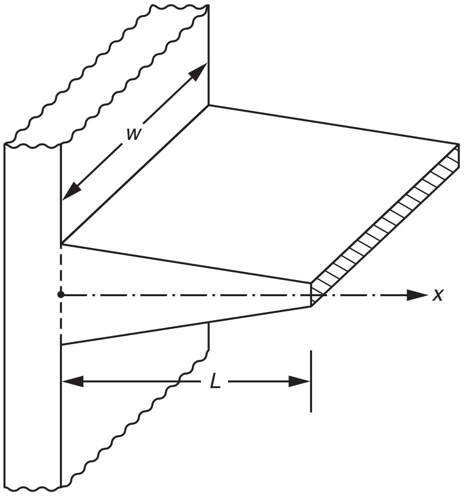


FIGURE 1.39 A tapered fin.

1.28 Two springs, with stiffnesses 500 N/mm and 800 N/mm, are connected in series. The ends of the system are fixed and an axial load of magnitude 1000 N is applied at the middle node as shown in Fig. 1.40. Determine the displacement of the middle node and the reactions at the fixed ends.

1.29 Two springs, of stiffnesses 400 N/mm and 600 N/mm, are connected in series. One end of the system is fixed and the middle point is subjected to an axial load of 800 N as shown in Fig. 1.41. Determine the nodal displacements and the reaction at the fixed end.

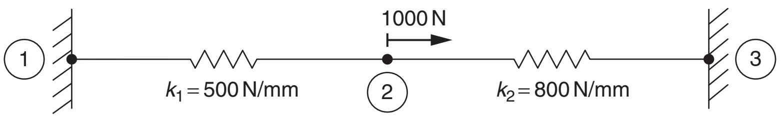


FIGURE 1.40 Two springs in series.

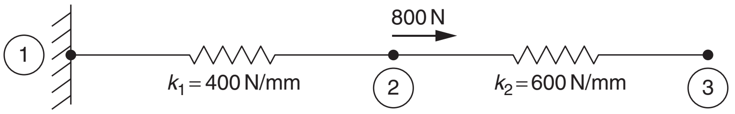


FIGURE 1.41 A system of two springs with one end fixed.

1.30 Fig. 1.42 shows a system of three springs in series subjected to two axial loads. The stiffnesses of the springs and the axial loads applied are given by $k_1 = 40$ N/mm, $k_2 = 20$ N/mm, $k_3 = 60$ N/mm, $P_2 = -100$ N, and $P_3 = 50$ N. Determine the displacements of the nodes of the system.

1.31 A fluid of viscosity $\mu = 2 \times 10^{-6}$ lbf-sec/in^2 and density $\rho = 2.0$ slugs/ft^3 flows through the pipe network shown in Fig. 1.43. The diameters and lengths of the various pipe segments and the pressures at the exterior nodes are as indicated in Fig. 1.43. Determine the following:

a. Whether the flow is laminar in each pipe segment.
b. Pressure at the interior node 3.
c. Volume flow rates in each pipe segment.

1.32 Water, with viscosity $\mu = 1.5 \times 10^{-6}$ lbf-sec/in^2 and density $\rho = 1.93$ slugs/ft^3, flows through the pipe network shown in Fig. 1.44. The pipe segments data are shown in the following table:

Pipe Segment or Element e	Diameter (inch)	Length (inch)
1	3	5000
2	2	7000
3	1.5	4000
4	2	6000
5	2.5	8000
6	1	5000
7	2	3000
8	3	2000
9	2.5	6000
10	3	4000

Determine the following:

a. Whether the flow is laminar in each pipe segment.
b. Pressure at the interior node 3.
c. Volume flow rates in each pipe segment.

1.33 The electrical circuit shown in Fig. 1.45 consists of four resistors and a battery. Determine the voltages at various node points and the current flows through the various resistors.

1.34 The electrical circuit shown in Fig. 1.46 consists of five resistors and two batteries. Determine the voltages at various node points and the current flows through the various resistors.

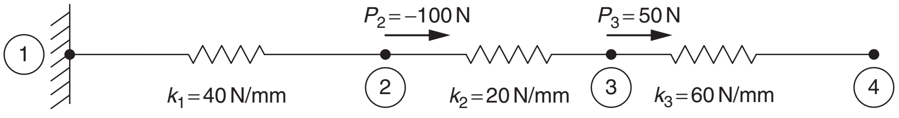


FIGURE 1.42 A system of three springs with one end fixed.

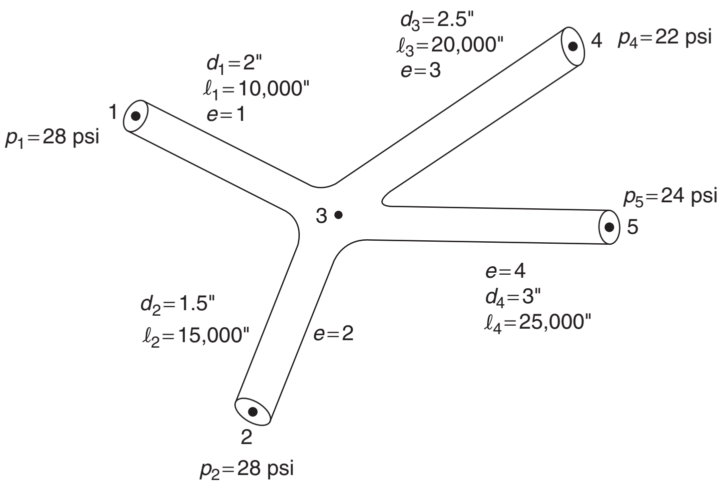


FIGURE 1.43 A pipe network with four segments.

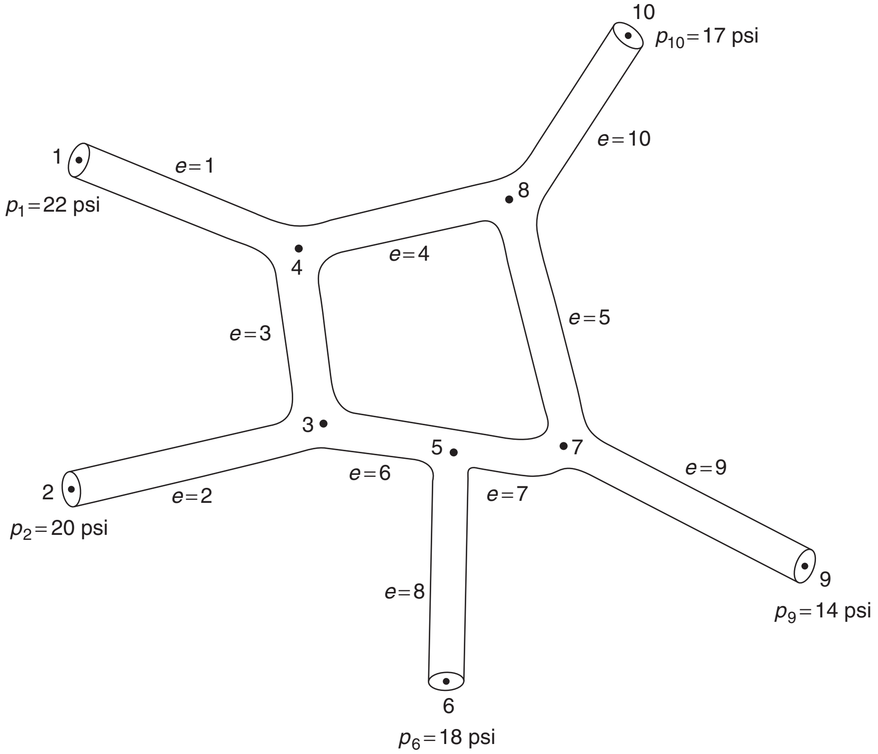


FIGURE 1.44 A pipe network with nine segments.

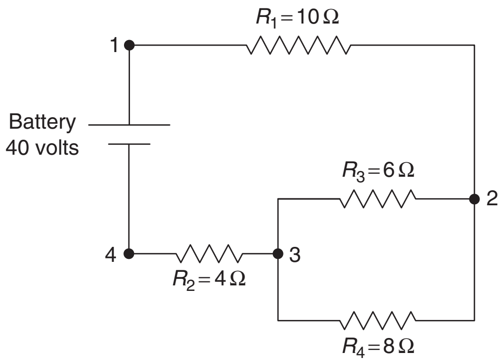


FIGURE 1.45 An electrical circuit with four resistors and one battery.

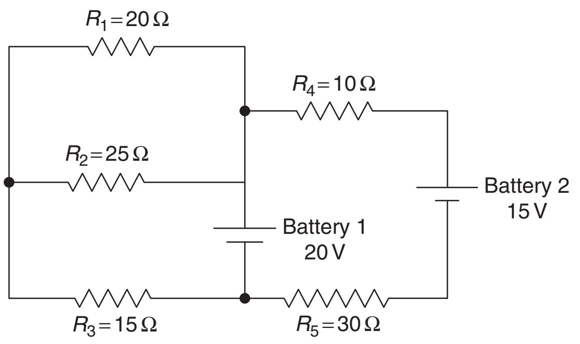


FIGURE 1.46 An electrical circuit with five resistors and two batteries.

1.35 A stepped bar has a linearly varying area of cross section between the nodes 1 and 2 and a constant area of cross section between the nodes 2 and 3 as shown in Fig. 1.47. If the Young's modulus of the stepped bar is 30×10^6 psi, determine the nodal displacements and stresses induced in the two steps of the bar when an axial load of magnitude $P = 100$ lb is applied at node 3. Use a two-element idealization.

1.36 Three bars, made up of different materials, are connected to carry axial loads of $P_2 = 1000$ N and $P_4 = -500$ N as shown in Fig. 1.48. Determine the displacements of nodes 2, 3, and 4 and the stresses in the three bars.

1.37 A composite wall, made up of two materials, is shown in Fig. 1.49. The temperatures on the left and right faces are maintained at constant values of 100 °F and 70 °F, respectively. Find the temperature distribution in the wall using a linear finite element for each of the two materials.

1.38 A composite wall, made up of four materials, is shown in Fig. 1.13. The temperatures on the left and right faces are maintained at constant values of $T_1 = T_{\text{high}} = 200°\text{C}$ and $T_5 = T_{\text{low}} = 50°\text{C}$. Find the temperature distribution in the various materials of the wall using a linear finite element for each of the four materials
Data: $t_1 = 5$ mm, $t_2 = 10$ mm, $t_3 = 15$ mm, $t_4 = 8$ mm, $k_1 = 0.4$ W/mm °C, $k_2 = 0.3$ W/mm °C, $k_3 = 0.2$ W/mm °C, $k_4 = 0.1$ W/mm °C.

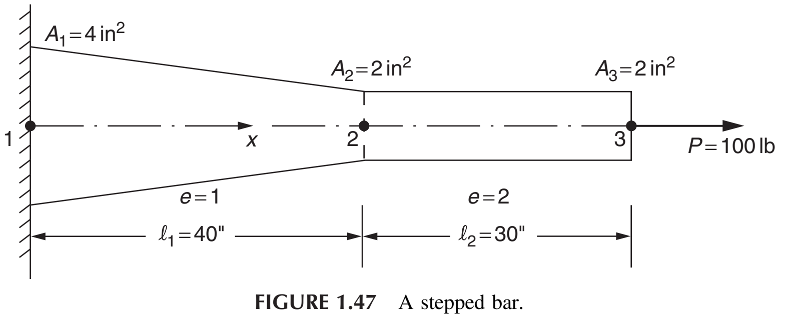


FIGURE 1.47 A stepped bar.

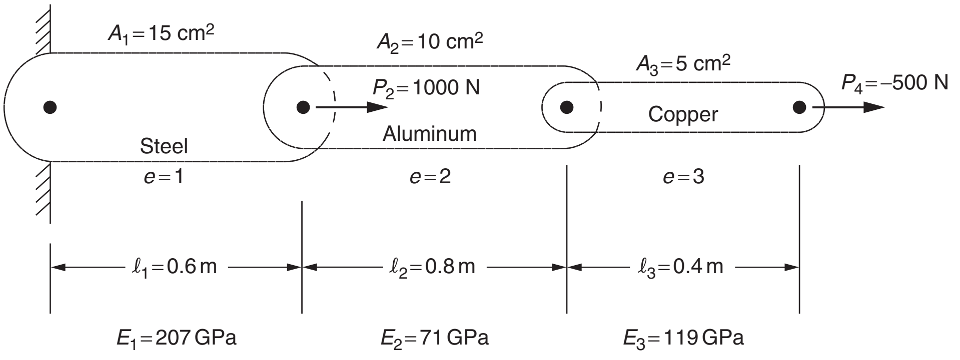


FIGURE 1.48 Three bars in series.

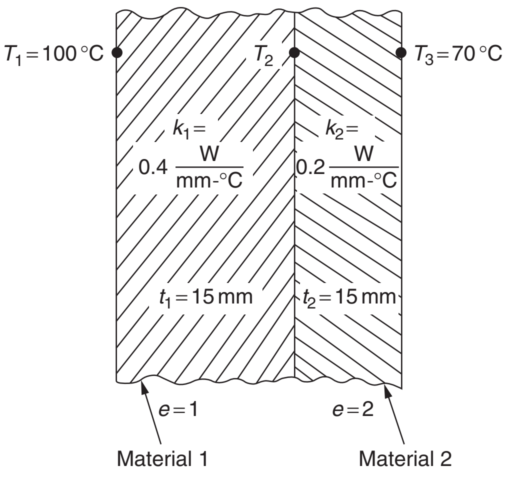


FIGURE 1.49 A composite wall.

1.39 Find the temperature distribution in the stepped fin shown in Fig. 1.50 using two finite elements.

1.40 An irrigation system consists of a source (water pumped from a well) and four branch pipes as shown in Fig. 1.51. The pressures at different points, and the lengths and diameters of pipes are:
$p_i = 40,\ 20,\ 30,\ 25$ psi for $i = 1, 2, 3, 4$; $l_i = 300,\ 400,\ 200,\ 100$ ft for $i = 1, 2, 3, 4$; $d_i = 5,\ 2.5,\ 3,\ 3$ in for $i = 1, 2, 3, 4$.

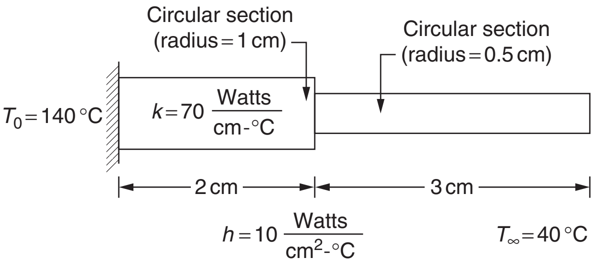


FIGURE 1.50 A stepped fin.

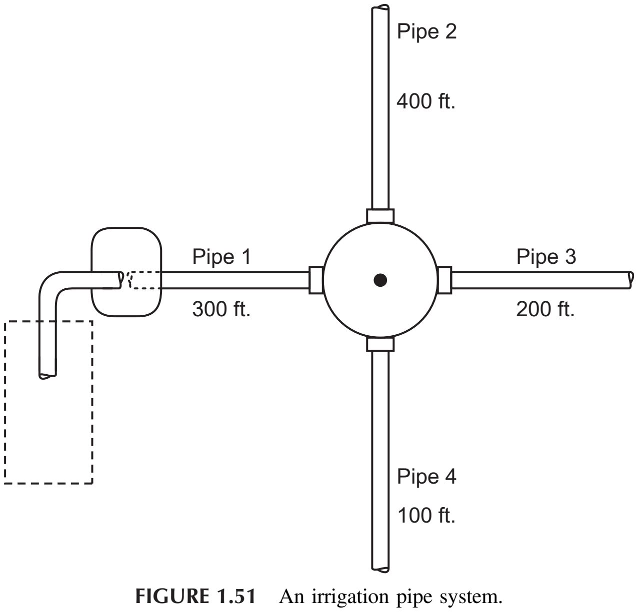


FIGURE 1.51 An irrigation pipe system.

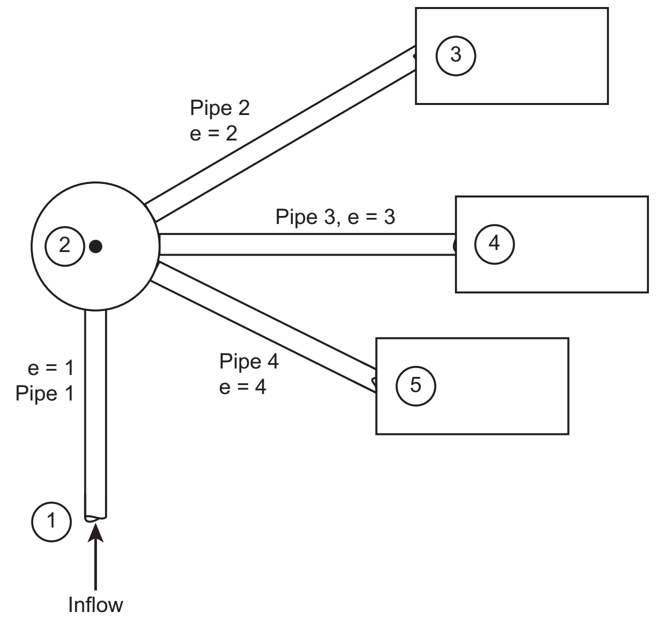


FIGURE 1.52 Flow of crude oil in a refinery.

If the viscosity and density of water are 2.034 lb-s/ft^2 and 62.3 lb/ft^3, respectively, determine the flow rates of water through the various pipe segments.

1.41 Crude oil flows through a pipeline to a refinery, where the crude oil flows through three pipes to different refining stations to produce three grades of gasoline as shown in Fig. 1.52. The pressures at various points and the lengths and diameters of the different pipes are:
$p_i = 50, 25, 30, 20$ psi for $i = 1, 2, 3, 4$; $l_i = 1000, 250, 200, 300$ ft for $i = 1, 2, 3, 4$; $d_i = 4, 2, 2.5, 3$ in for $i = 1, 2, 3, 4$.
If the viscosity and density of crude oil are 2.5×10^{-6}-lb-s/in^2 and 2.5 slug/ft^3, respectively, determine the flow rates of crude oil through the various pipe segments.

1.42 Assuming a cubic polynomial for the transverse deflection of a beam element shown in Fig. 1.9 as $w(x) = a + bx + cx^2 + dx^3$, evaluate the constants a, b, c, and d in terms of the nodal dof shown at the two nodes, $W_i^{(e)}$, $i = 1, 2, 3, 4$, and derive the expression shown on the right-hand side of Eq. (1.17).

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Part II

Basic Procedure

Chapter 2

Discretization of the Domain

Chapter Outline

2.1 INTRODUCTION

In most engineering problems, we need to find the values of a field variable such as displacement, stress, temperature, pressure, and velocity as a function of spatial coordinates (x, y, z). In the case of transient or unsteady-state problems, the field variable has to be found as a function of not only the spatial coordinates (x, y, z) but also time (t). The geometry (domain or solution region) of the problem is often irregular. The first step of the finite element analysis involves the discretization of the irregular domain into smaller and regular subdomains, known as elements. This is equivalent to replacing the domain having an infinite number of degrees of freedom (dof) by a system having a finite number of dof.

A variety of methods can be used to model a domain with finite elements. Different methods of dividing the domain into finite elements involve varying amounts of computational time and often lead to different approximations to the solution of the physical problem. The process of discretization is essentially an exercise of engineering judgment. Efficient methods of finite element idealization require some experience and knowledge of simple guidelines. For large problems involving complex geometries, finite element idealization based on manual procedures requires considerable effort and time on the part of the analyst. Some automatic mesh generation programs have been developed for the efficient idealization of complex domains requiring minimal interface with the analyst.

2.2 BASIC ELEMENT SHAPES

The shapes, sizes, number, and configurations of the elements have to be chosen carefully such that the original body or domain is simulated as closely as possible without increasing the computational effort needed for the solution. Mostly the choice of the type of element is dictated by the geometry of the body and the number of independent coordinates necessary to describe the system. If the geometry, material properties, and field variable of the problem can be described in terms of a single spatial coordinate, we can use the one-dimensional or line elements shown in Fig. 2.1A. The temperature distribution in a rod (or fin), the pressure distribution in a pipe flow, and the deformation of a bar under axial load, for example, can be determined using these elements. Although these elements have a cross-sectional area, they are generally shown schematically as a line element (Fig. 2.1B). In some cases, the cross-sectional area of the element may be nonuniform.

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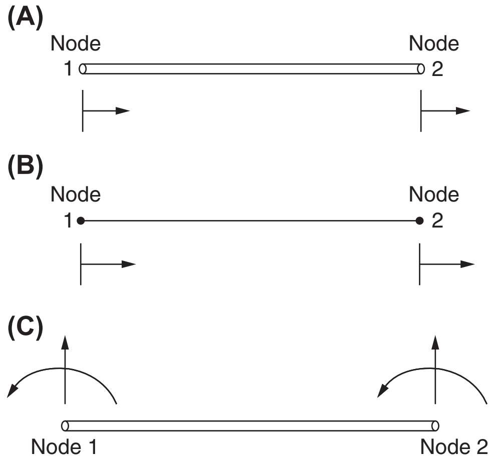

FIGURE 2.1 One-dimensional elements.

For a simple analysis, one-dimensional elements are assumed to have two nodes, one at each end, with the corresponding value of the field variable chosen as the unknown (dof). However, for the analysis of beams, the values of the field variable (transverse displacement) and its derivative (slope) are chosen as the unknowns (dof) at each node as shown in Fig. 2.1C.

When the configuration and other details of the problem can be described in terms of two independent spatial coordinates, we can use the two-dimensional elements shown in Fig. 2.2. The basic element useful for two-dimensional analysis is the triangular element. Although a quadrilateral element (or its special forms, the rectangle and parallelogram) can be obtained by assembling two or four triangular elements, as shown in Fig. 2.3; in some cases the use of quadrilateral (or rectangle or parallelogram) elements proves to be advantageous. For the bending analysis of plates, multiple dof (transverse displacement and its derivatives) are used at each node.

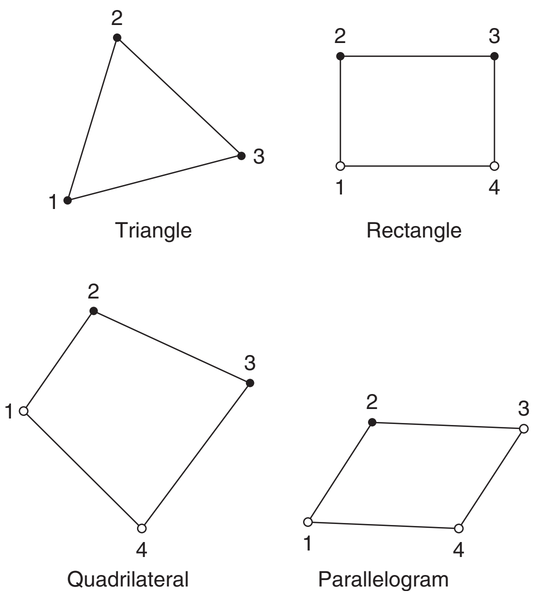

FIGURE 2.2 Two-dimensional elements.

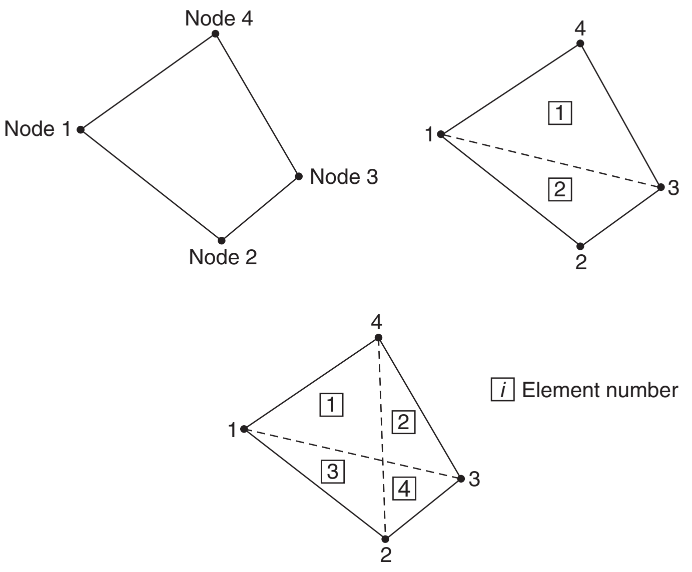

FIGURE 2.3 A quadrilateral element as an assemblage of two or four triangular elements.

If the geometry, material properties, and other parameters of the body can be described by three independent spatial coordinates, we can idealize the body by using the three-dimensional elements shown in Fig. 2.4. The basic three-dimensional element, analogous to the triangular element in the case of two-dimensional problems, is the tetrahedron element. In some cases the hexahedron element, which can be obtained by assembling five tetrahedrons as indicated in Fig. 2.5, can be used advantageously. Some problems, which are actually three-dimensional, can be described by only one or two independent coordinates. Such problems can be idealized by using an axisymmetric or ring type of elements shown in Fig. 2.6. The problems that possess axial symmetry, such as pistons, storage tanks, valves, rocket nozzles, and reentry vehicle heat shields, fall into this category.

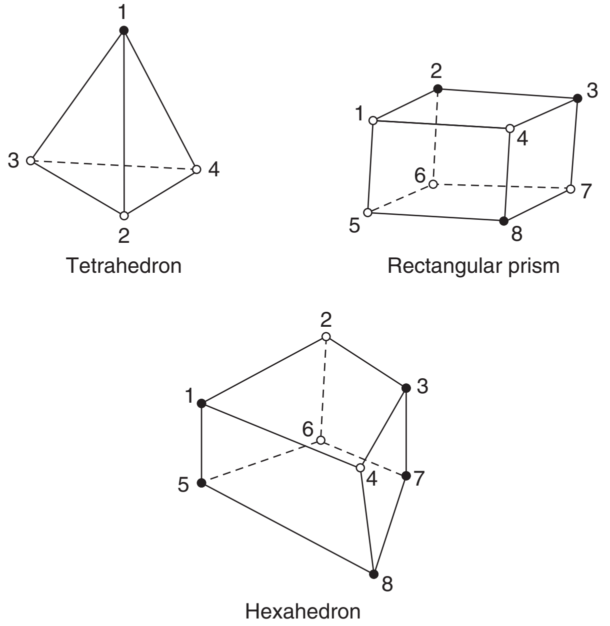

FIGURE 2.4 Three-dimensional finite elements.

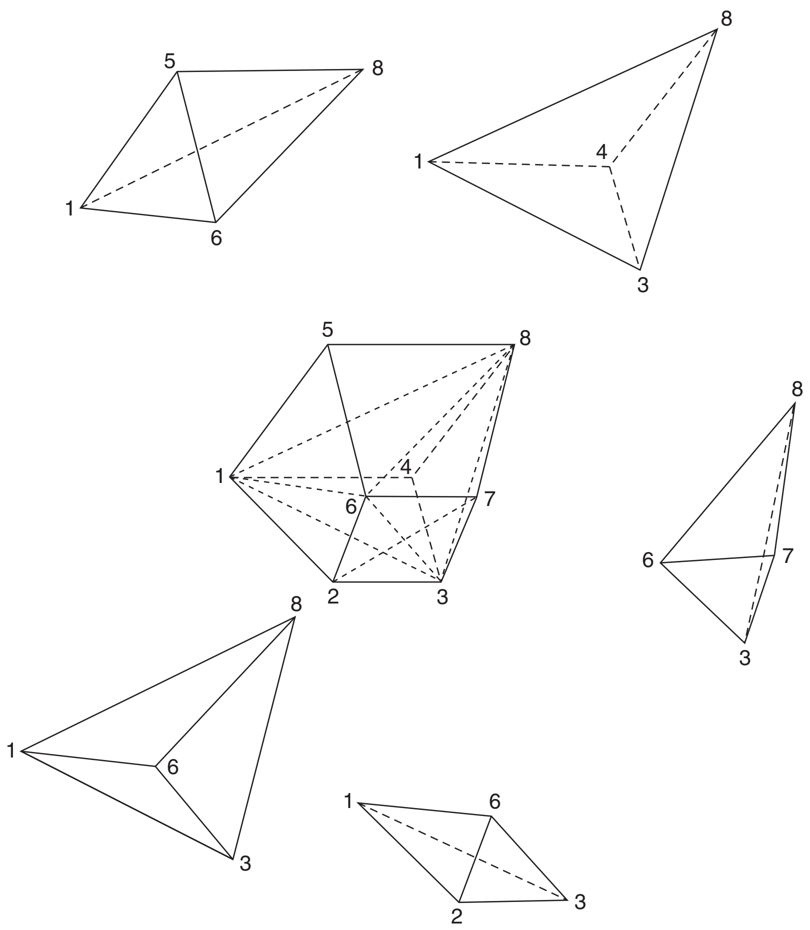

FIGURE 2.5 A hexahedron element as an assemblage of five tetrahedron elements.

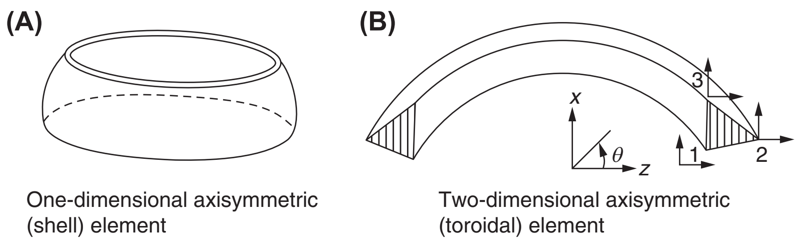

FIGURE 2.6 Axisymmetric elements.

For the discretization of problems involving curved geometries, finite elements with curved sides are useful. Typical elements having curved boundaries are shown in Fig. 2.7. The ability to model curved boundaries has been made possible by the addition of mid-side nodes. Finite elements with straight sides are known as linear elements, whereas those with curved sides are called higher-order elements.

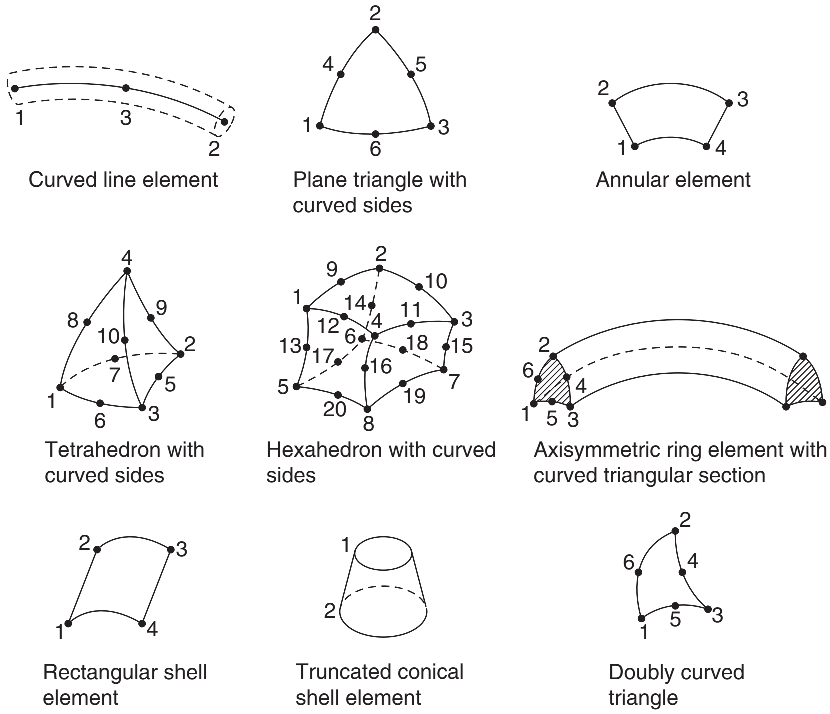

FIGURE 2.7 Finite elements with curved boundaries.

2.3 DISCRETIZATION PROCESS

Various considerations to be taken in the discretization process [2.1] are discussed in the following sections.

2.3.1 Type of Elements

Often, the type of elements to be used will be evident from the physical problem. For example, if the problem involves the analysis of a truss structure under a given set of load conditions (Fig. 2.8A), the type of elements to be used for idealization is obviously the bar or line elements as shown in Fig. 2.8B. Similarly, in the case of stress analysis of the short beam shown in Fig. 2.9A, the finite element idealization can be done using three-dimensional solid elements as shown in Fig. 2.9B.

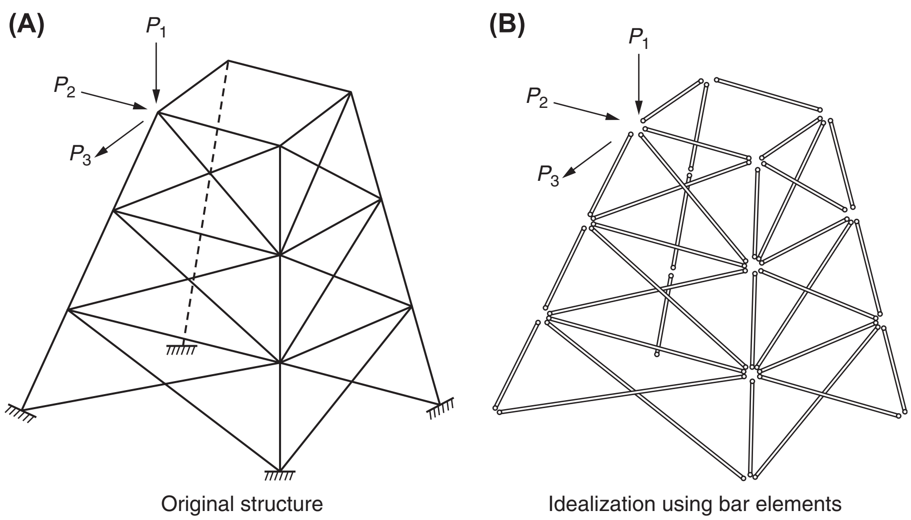

FIGURE 2.8 A truss structure.

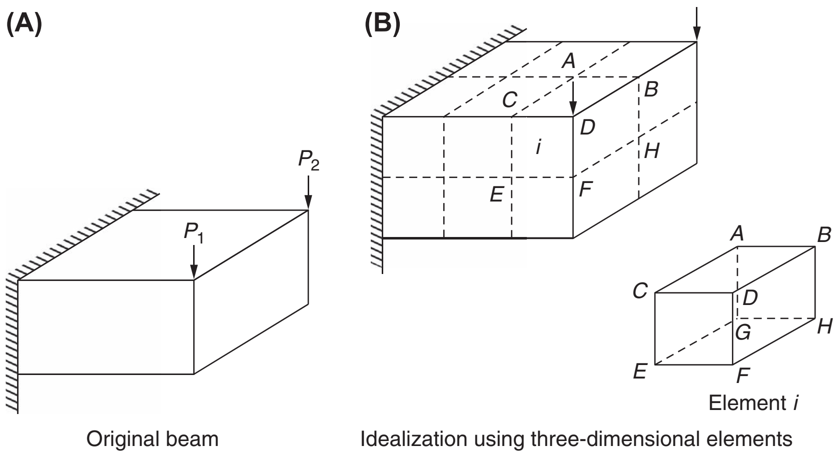

FIGURE 2.9 A short beam.

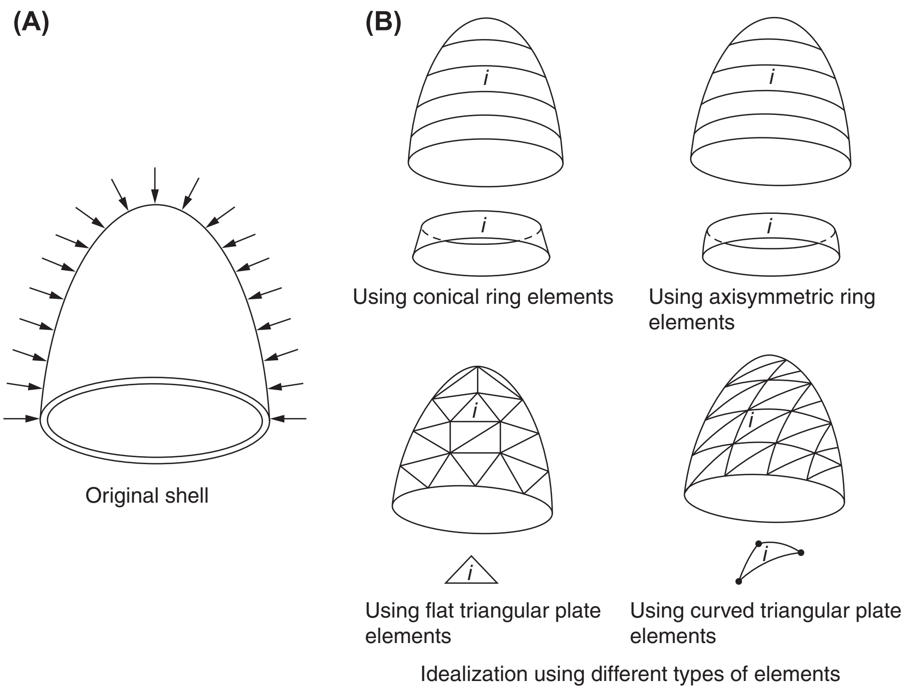

FIGURE 2.10 A thin-walled shell under pressure.

However, the type of elements to be used for idealization may not be apparent, and in such cases we have to choose the type of elements judicially. As an example, consider the problem of analysis of the thin-walled shell shown in Fig. 2.10A. In this case, the shell can be idealized by several types of elements as shown in Fig. 2.10B. Here, the number of dof needed, the expected accuracy, the ease with which the necessary equations can be derived, and the degree to which the physical structure can be modeled without approximation will dictate the choice of the element type to be used for idealization. In certain problems, the given body cannot be represented as an assemblage of only one type of elements. In such cases, we may have to use two or more types of elements for idealization. An example of this would be the analysis of an aircraft wing. Since the wing consists of top and bottom covers, stiffening webs, and flanges, three types of element—triangular plate elements (for covers), rectangular shear panels (for webs), and frame elements (for flanges)—have been used in the idealization shown in Fig. 2.11.

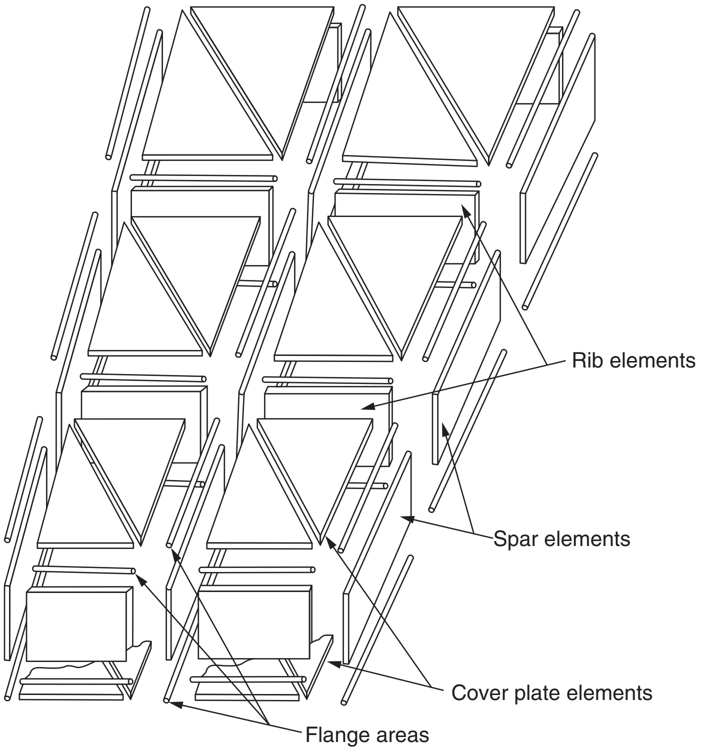

FIGURE 2.11 Idealization of an aircraft wing using different types of elements.

EXAMPLE 2.1

A helical spring is subjected to a compressive load as shown in Fig. 2.12A. Suggest different methods of modeling the spring using one-dimensional elements.

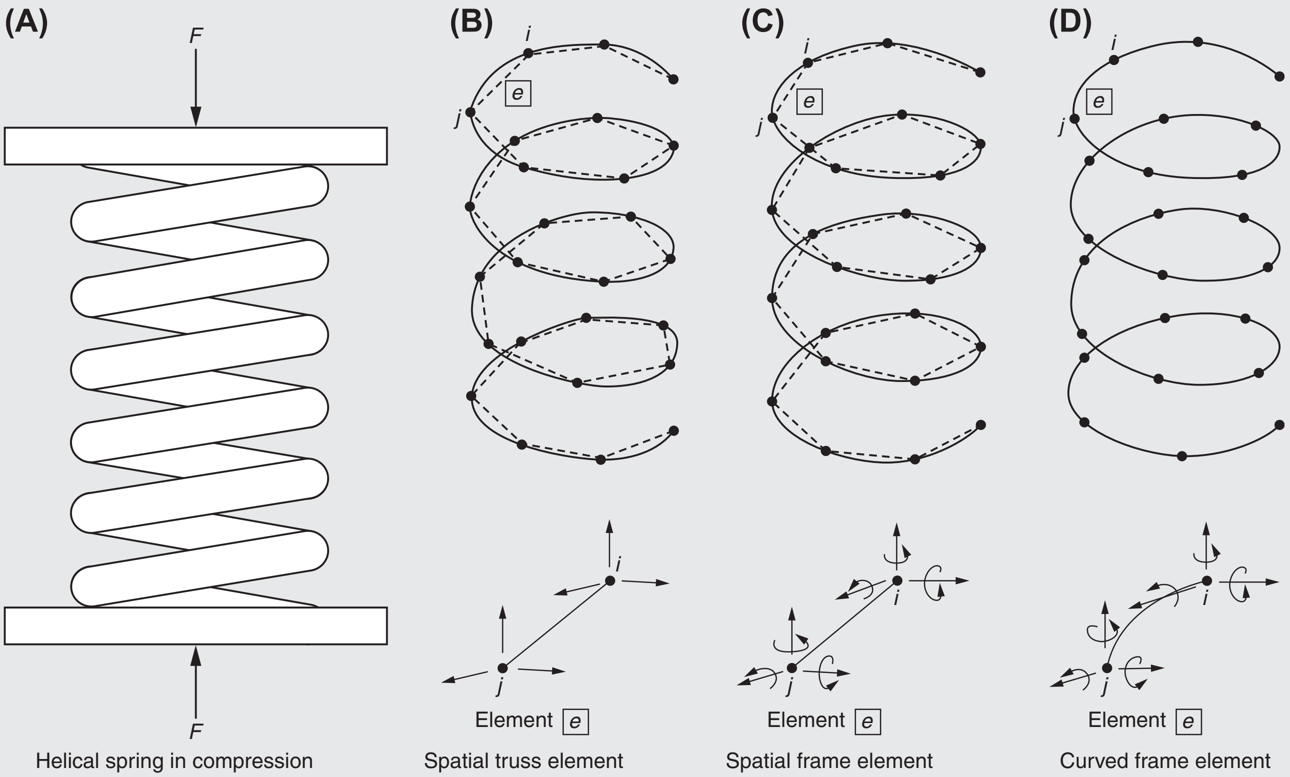

FIGURE 2.12 Modeling of a helical spring.

Continued

EXAMPLE 2.1—cont'd

Solution

Approach: Use various one-dimensional or line elements.

The helical spring (in the form of curved wire) can be divided into several lines or one-dimensional segments. These segments can be straight or curved. Each of the straight line segments (or elements) can be assumed to be a spatial truss element with each of its endpoints (or nodes) having three displacement dof (parallel to the x, y, and z axes) as shown in Fig. 2.12B. Since this element has only translational dof (with no rotational dof), it will not be able to carry any moment. As such, the element may not be able to represent the behavior of the helical spring accurately.

Alternately, each of the straight line segments (or elements) can be assumed to be a spatial frame element with each of its endpoints (or nodes) having three displacement dof (parallel to the x, y, and z axes) and three rotational dof (about the x, y, and z axes) as shown in Fig. 2.12C. In the case of the curved line segments (elements), each element can be treated as a curved frame element with three displacement dof (parallel to the x, y, and z axes) and three rotational dof (about the x, y, and z axes) at each end as shown in Fig. 2.12D. Because of the inclusion of rotational dof, the models shown in Fig. 2.12C and D will be able to simulate the behavior of the helical spring more accurately.

2.3.2 Size of Elements

The size of elements influences the convergence of the solution directly, and hence it has to be chosen with care. If the size of the elements is small, the final solution is expected to be more accurate. However, we have to remember that the use of smaller-sized elements will also mean more computation time. Sometimes, we may have to use elements of different sizes in the same body. For example, in the case of stress analysis of the box beam shown in Fig. 2.13A, the size of all the elements can be approximately the same, as shown in Fig. 2.13B.

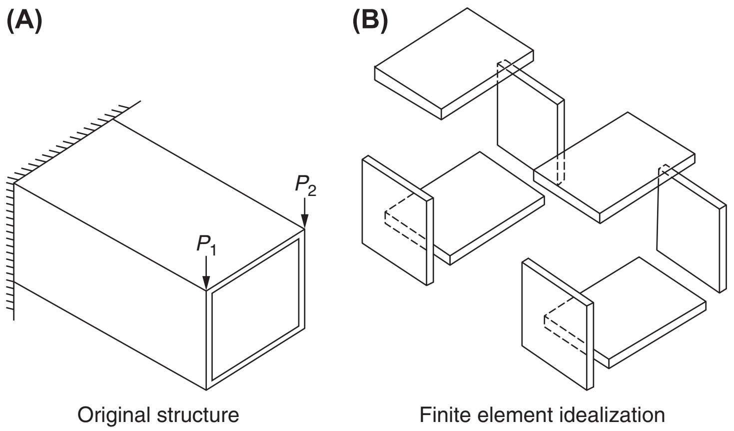

FIGURE 2.13 A box beam.

However, in the case of stress analysis of a plate with a hole shown in Fig. 2.14A, elements of different sizes have to be used, as shown in Fig. 2.14B. The size of elements has to be very small near the hole (where stress concentration is expected) compared to distant places. In general, whenever steep gradients of the field variable are expected, we have to

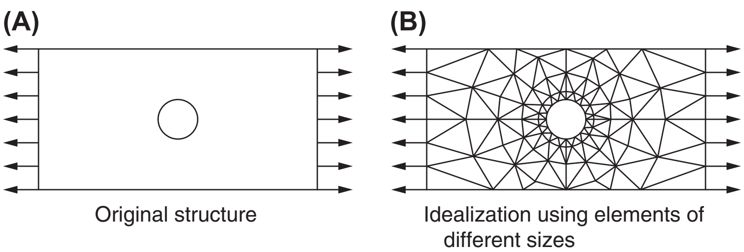

FIGURE 2.14 A plate with a hole.

use a finer mesh in those regions. Another characteristic related to the size of elements that affects the finite element solution is the aspect ratio of the elements. The aspect ratio describes the shape of the element in the assemblage of elements. For two-dimensional elements, the aspect ratio is taken as the ratio of the largest dimension of the element to the smallest dimension. Elements with an aspect ratio of nearly unity generally yield the best results [2.2].

2.3.3 Location of Nodes

If the body has no abrupt changes in geometry, material properties, and external conditions (e.g., load and temperature), the body can be divided into equal subdivisions and hence the spacing of the nodes can be uniform. On the other hand, if there are any discontinuities in the problem, nodes have to be introduced at these discontinuities, as shown in Fig. 2.15.

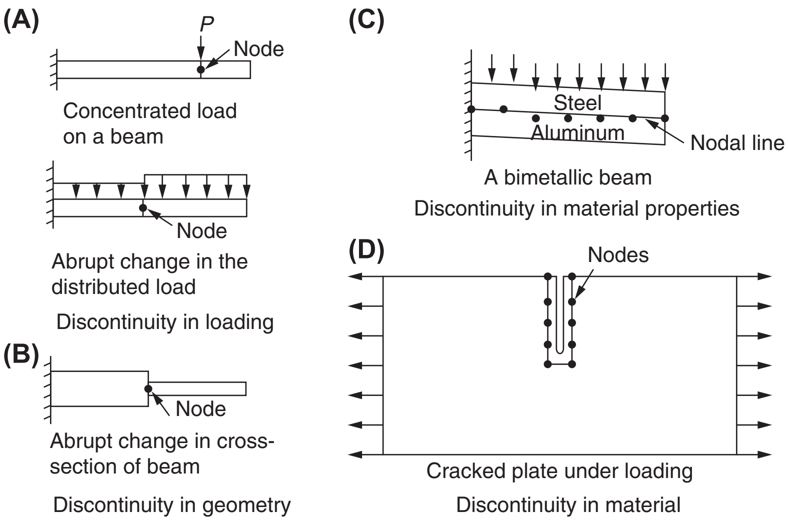

FIGURE 2.15 Location of nodes at discontinuities.

2.3.4 Number of Elements

The number of elements to be chosen for idealization is related to the accuracy desired, size of elements, and the number of dof involved. Although an increase in the number of elements generally means more accurate results, for any given problem, there will be a certain number of elements beyond which the accuracy cannot be significantly improved. This behavior is shown graphically in Fig. 2.16. Moreover, since the use of a large number of elements involves a large number of dof, we may not be able to store the resulting matrices in the available computer memory.

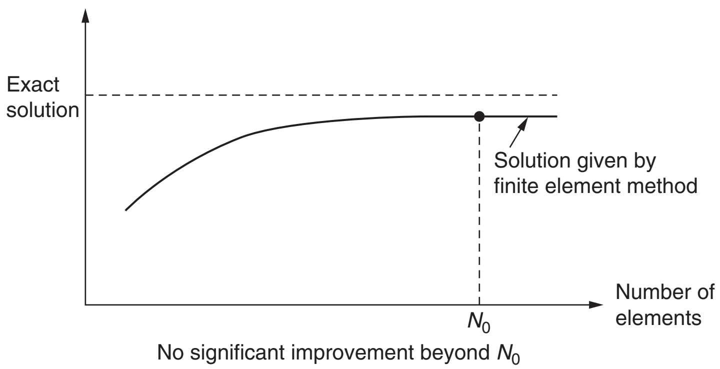

FIGURE 2.16 Effect of varying the number of elements.

2.3.5 Simplifications Afforded by the Physical Configuration of the Body

If the configuration of the body as well as the external conditions are symmetric, we may consider only half of the body for finite element idealization. The symmetry conditions, however, have to be incorporated in the solution procedure. This is illustrated in Fig. 2.17, where only half of the plate with a hole, having symmetry in both geometry and loading, is considered for analysis.[1] Since there cannot be a horizontal displacement along the line of symmetry AA, the condition that $u = 0$ has to be incorporated while finding the solution.

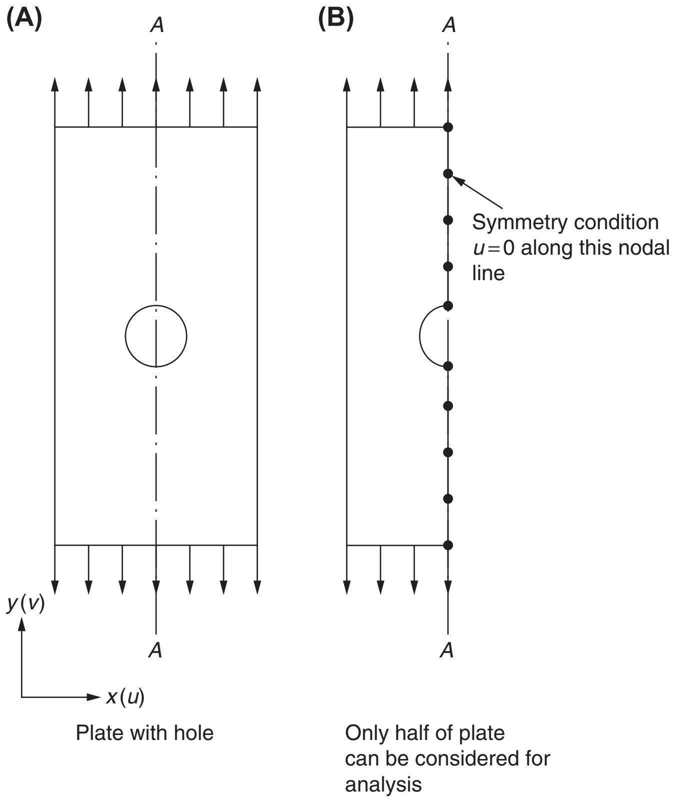

FIGURE 2.17 A plate with a hole with symmetric geometry and loading.

2.3.6 Finite Representation of Infinite Bodies

In most of the problems, like in the analysis of beams, plates, and shells, the boundaries of the body or continuum are clearly defined. Hence, the entire body can be considered for element idealization. However, in some cases, as in the analysis of dams, foundations, and semi-infinite bodies, the boundaries are not clearly defined. In the case of dams (Fig. 2.18), since the geometry is uniform and the loading does not change in the length direction, a unit slice of the dam can be considered for idealization and analyzed as a plane strain problem.

However, in the case of the foundation problem shown in Fig. 2.19A, we cannot idealize the complete semi-infinite soil by finite elements. Fortunately, it is not really necessary to idealize the infinite body. Since the effect of loading decreases gradually with increasing distance from the point of loading, we can consider only that much of the continuum in which the loading is expected to have a significant effect, as shown in Fig. 2.19B. Once the significant extent of the infinite body is identified as shown in Fig. 2.19B, the boundary conditions for this finite body have to be incorporated in the solution. For example, if the horizontal movement only has to be restrained for sides AB and CD (i.e., $u = 0$), these sides are supposed to be on rollers as shown in Fig. 2.19B. In this case, the bottom boundary can be either completely fixed ($u = v = 0$) or constrained only against vertical movement ($v = 0$). The fixed conditions ($u = v = 0$ along BC) are often used if the lower boundary is taken at the known location of a bedrock surface.

1. In this example, even one-fourth of the plate can be considered for analysis due to symmetry about both horizontal and vertical center lines.

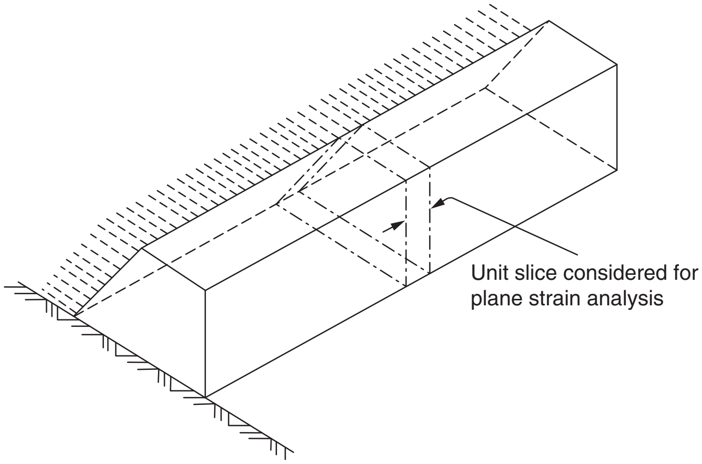

FIGURE 2.18 A dam with uniform geometry and loading.

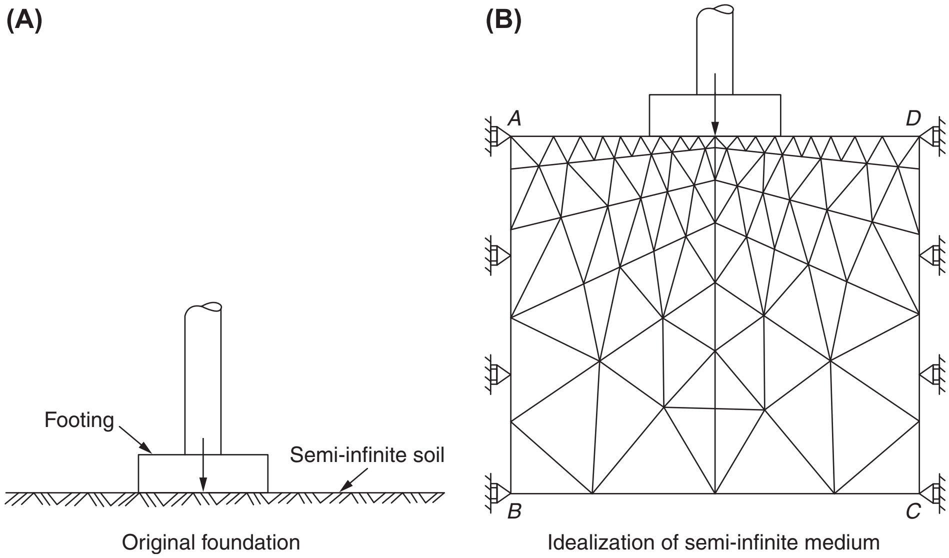

FIGURE 2.19 A foundation under concentrated load.

In Fig. 2.19, the semi-infinite soil has been simulated by considering only a finite portion of the soil. In some applications, the determination of the size of the finite domain may pose a problem. In such cases, we can use infinite elements for modeling [2.3–2.5]. As an example, Fig. 2.20 shows a four-node element that is infinitely long in the x direction. The coordinates of the nodes of this infinite element can be transformed to the natural coordinate system (s, t) as follows:

$$s = 1 - 2\left\{\frac{1}{x}\cdot\frac{(y_3 - y)x_1 + (y - y_1)x_4}{(y_3 - y_1)}\right\}^m; \quad m \geq 1$$

$$t = 1 - 2\left\{\frac{y_3 - y}{y_3 - y_1}\right\}$$

See Section 4.3.3 for the definition of the natural coordinate system.

FIGURE 2.20 A four-node infinite element.

2.4 NODE NUMBERING SCHEME

As seen in Chapter 1, the finite element analysis of practical problems often leads to matrix equations in which the matrices involved will be banded. The advances in the finite element analysis of large practical systems have been made possible largely due to the banded nature of the matrices. Furthermore, since most of the matrices involved (e.g., stiffness matrices) are symmetric, the demands on the computer storage can be substantially reduced by storing only the elements involved in half bandwidth instead of storing the entire matrix.

The bandwidth of the overall or global characteristic matrix depends on the node numbering scheme and the number of dof considered per node [2.6]. If we can minimize the bandwidth, the storage requirements as well as solution time can also be minimized. Since the number of dof per node is generally fixed for any given type of problem, the bandwidth can be minimized by using a proper node numbering scheme. As an example, consider a three-bay frame with rigid joints, 20 stories high, shown in Fig. 2.21. Assuming that there are three dof per node, there are 252 unknowns in the final equations

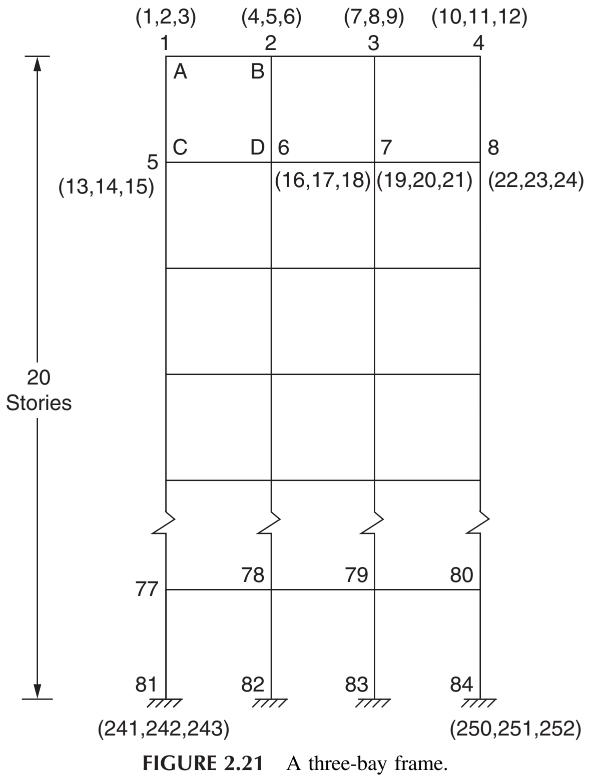

FIGURE 2.21 A three-bay frame.

(including the dof corresponding to the fixed nodes), and if the entire stiffness matrix is stored in the computer, it will require $252^2 = 63{,}504$ locations. The bandwidth (strictly speaking, half-bandwidth) of the overall stiffness matrix can be shown to be 15, and thus the storage required for the upper half-band is only $15 \times 252 = 3780$ locations.

Before we attempt to minimize the bandwidth, we discuss the method of calculating the bandwidth. For this, we consider again the rigid jointed frame shown in Fig. 2.21. By applying constraints to all the nodal dof except number 1 at node 1 (joint A), it is clear that an imposed unit displacement in the direction of 1 will require constraining forces at the nodes directly connected to node A—that is, B and C. These constraining forces are nothing but the cross-stiffnesses appearing in the stiffness matrix, and these forces are confined to the nodes B and C. Thus, the nonzero terms in the first row of the global stiffness matrix (Fig. 2.22) will be confined to the first 15 positions. This defines the bandwidth (B) as

$$\text{Bandwidth } (B) = (\text{maximum difference between the numbered dof at the ends of any member} + 1)$$

This definition can be generalized so as to be applicable for any type of finite element as

$$\text{Bandwidth } (B) = (D+1)\cdot f \tag{2.1}$$

where D is the maximum largest difference in the node numbers occurring for all elements of the assemblage, and f is the number of dof at each node.

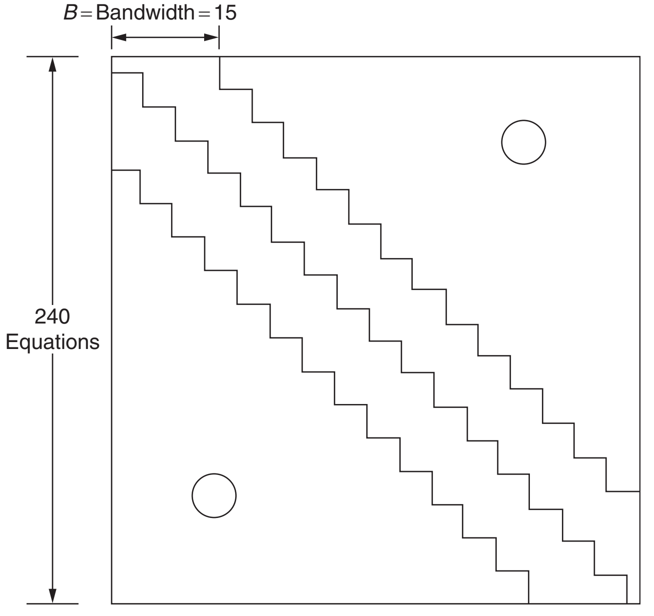

FIGURE 2.22 Banded nature of the stiffness matrix for the frame of Fig. 2.21.

The previous equation indicates that D has to be minimized in order to minimize the bandwidth. Thus, a shorter bandwidth can be obtained simply by numbering the nodes across the shortest dimension of the body. This is clear from Fig. 2.23 also, where the numbering of nodes along the shorter dimension produces a bandwidth of $B = 15$ ($D = 4$), whereas the numbering along the longer dimension produces a bandwidth of $B = 66$ ($D = 21$).

As observed previously, the bandwidth of the overall system matrix depends on the manner in which the nodes are numbered. For simple systems or regions, it is easy to label the nodes so as to minimize the bandwidth. But for large systems, the procedure becomes nearly impossible. Hence, automatic mesh generation algorithms, capable of discretizing any geometry into an efficient finite element mesh without user intervention, have been developed [2.7,2.8]. Most commercial finite element software has built-in automatic mesh generation codes. An automatic mesh generation program generates the locations of the node points and elements, labels the nodes and elements, and provides the element–node connectivity relationships.

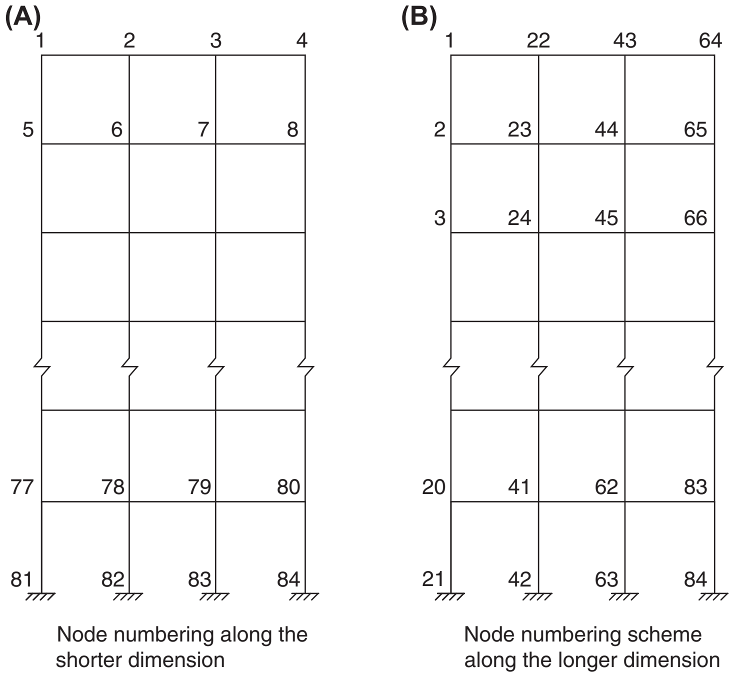

FIGURE 2.23 Different node numbering schemes.

EXAMPLE 2.2

A drilling machine is modeled using one-dimensional beam elements as shown in Fig. 2.24A. If two dof are associated with each node, label the node numbers for minimizing the bandwidth of the stiffness matrix of the system.

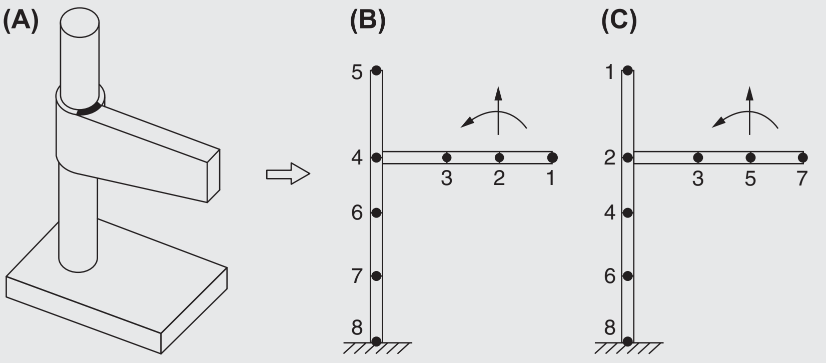

FIGURE 2.24 A drilling machine.

Solution

Approach: Number the nodes along the shorter side of the machine first.

Because the column (vertical member) of the machine has five nodes and the arm (horizontal member) has only four nodes, we number the nodes along the shorter side as shown in Fig. 2.24B. Noting that the maximum difference between the numbers of the end nodes among all the elements is 2, the bandwidth of the resulting stiffness matrix of the system is given by

$$B = (D+1)f = (2+1)2 = 6$$

Note that the nodes can also be numbered as shown in Fig. 2.24C, which also yields the same bandwidth of $B = 6$.

2.5 AUTOMATIC MESH GENERATION[2]

Mesh generation is the process of dividing a physical domain into smaller subdomains (called elements) to facilitate an approximate solution of the governing ordinary or partial differential equation. For this, one-dimensional domains (straight or curved lines) are subdivided into smaller line segments, two-dimensional domains (planes or surfaces) are subdivided into triangle or quadrilateral shapes, and three-dimensional domains (volumes) are subdivided into tetrahedron and hexahedron shapes. If the physical domain is simple and the number of elements used is small, mesh generation can be done manually. However, most practical problems, such as those encountered in aerospace, automobile, and construction industries have complex geometries that require the use of thousands and sometimes millions of elements. In such cases, the manual process of mesh generation is impossible and we have to use automatic mesh generation schemes based on the use of a CAD or solid modeling package.

Automatic mesh generation involves the subdivision of a given domain, which may be in the form of a curve, surface, or solid (described by a CAD or solid modeling package) into a set of nodes (or vertices) and elements (subdomains) to represent the domain as closely as possible subject to the specified element shape and size restrictions. Many automatic mesh generation schemes use a bottom-up approach in that nodes (or vertices or corners of the domain) are meshed first, followed by curves (boundaries), then surfaces, and finally solids. Thus, for a given geometric domain of the problem, nodes are first placed at the corner points of the domain, and then nodes are distributed along the geometric curves that define the boundaries. Next, the boundary nodes are used to develop nodes in the surface(s), and finally the nodes on the various surfaces are used to develop nodes within the given volume (or domain). The nodes or mesh points are used to define line elements if the domain is one-dimensional; triangular or quadrilateral elements if the domain is two-dimensional; and tetrahedral or hexahedral elements if the domain is three-dimensional.

The automatic mesh generation schemes are usually tied to solid modeling and computer-aided design schemes. When the user supplies information on the surfaces and volumes of the material domains that make up the object or system, an automatic mesh generator generates the nodes and elements in the object. The user can also specify minimum permissible element sizes for different regions of the object. Many mesh generation schemes first create all the nodes and then produce a mesh of triangles by connecting the nodes to form triangles (in a plane region). In a particular scheme, known as Delaunay triangulation, the triangular elements are generated by maximizing the sum of the smallest angles of the triangles; thus the procedure avoids generation of thin elements.

The most common methods used in the development of automatic mesh generators are the tesselation and octree methods [2.9,2.10]. In the tesselation method, the user gives a collection of node points and also an arbitrary starting node. The method then creates the first simplex element using the neighboring nodes. Then a subsequent or neighboring element is generated by selecting the node point that gives the least distorted element shape. The procedure is continued until all the elements are generated. The step-by-step procedure involved in this method is illustrated in Fig. 2.25 for a

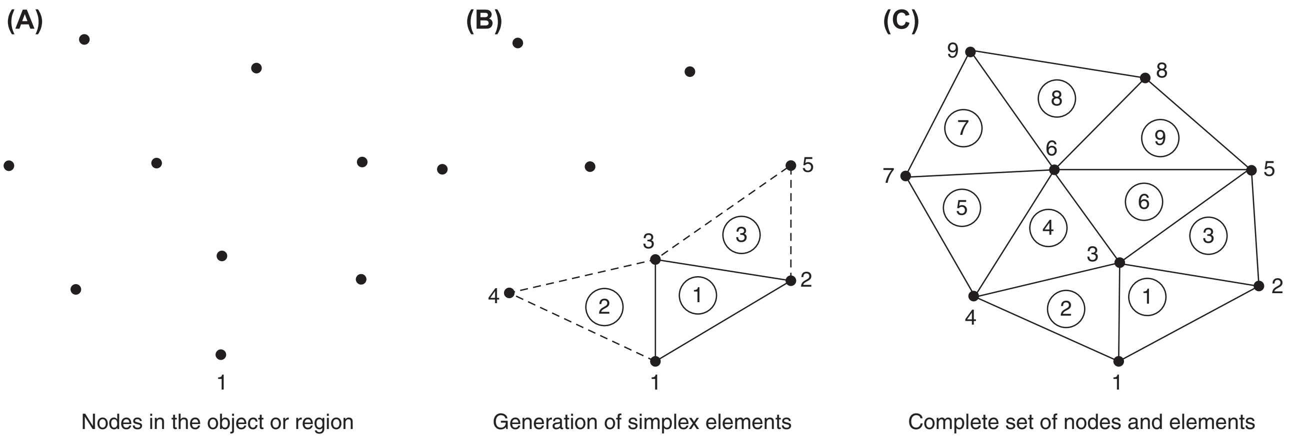

FIGURE 2.25 Mesh generation using the tesselation method.

2. This section may be omitted without loss of continuity in the text material.

two-dimensional example. Alternately, the user can define the boundary of the object by a series of nodes. Then the tesselation method connects selected boundary nodes to generate simplex elements. The stepwise procedure used[3] in this approach is shown in Fig. 2.26.

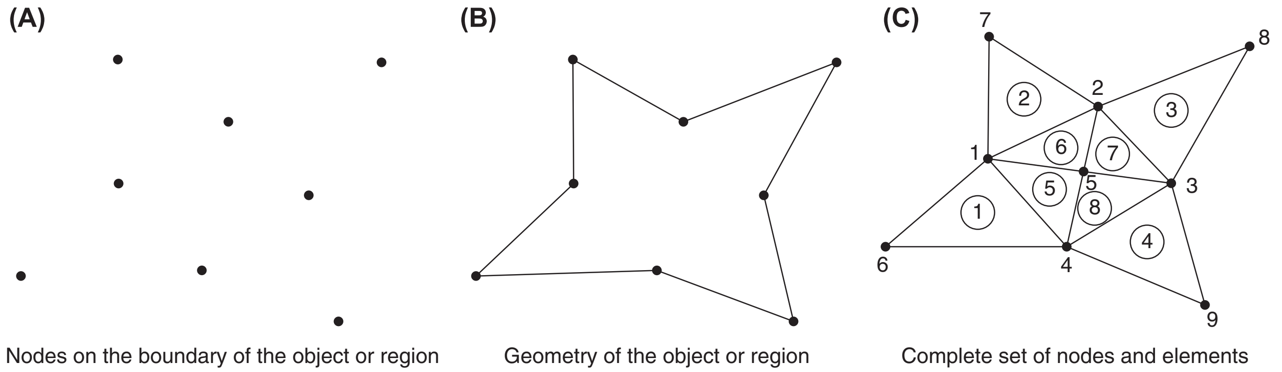

FIGURE 2.26 The tesselation method with nodes defined on the boundary.

The octree methods belong to a class of mesh generation schemes known as tree structure methods, which are extensively used in solid modeling and computer graphics display methods. In the octree method, the object is first considered enclosed in a three-dimensional cube. If the object does not completely (uniformly) cover the cube, the cube is subdivided into eight equal parts. In the two-dimensional analog of the octree method, known as the quadtree method, the object is first considered enclosed in a square region. If the object does not completely cover the square, the square is subdivided into four equal quadrants. If any one of the resulting quadrants is full (completely occupied by the object) or empty (not occupied by the object), then it is not subdivided further. On the other hand, if any one of the resulting quadrants is partially full (partially occupied by the object), it is subdivided into four quadrants. This procedure of subdividing partially full quadrants is continued until all the resulting regions are either full or empty, or until some predetermined level of resolution is achieved. At the final stage, the partially full quadrants are assumed to be either full or empty arbitrarily based on a prespecified criterion.

The approaches indicated in this section can be extended naturally to three and higher dimensional spaces.

EXAMPLE 2.3

Generate the finite element mesh for the two-dimensional object (region) shown by the crossed lines in Fig. 2.27A using the quadtree method.

Solution

Approach: Use the quadtree method.

First, the object is enclosed in a square region as shown by the dotted lines in Fig. 2.27A. Since the object does not occupy the complete square, the square is divided into four parts as shown in Fig. 2.27B. Since none of these parts are fully occupied by the object, each part is subdivided into four parts as shown in Fig. 2.27C. It can be seen that parts 1, 3, and 4 of A, part 3 of B, parts 2−4 of C, and parts 1−3 of D are completely occupied by the object, whereas parts 1, 2, and 4 of B and part 1 of C are empty (not occupied by the object). In addition, part 2 of A and part 4 of D are partially occupied by the object; hence, they are further subdivided into four parts each as shown in Fig. 2.27D. It can be noted that parts α and γ of part 2 (of A) and parts α and β of part 4 (of D) are completely occupied while the remaining parts, namely β and δ of part 2 (of A) and γ and δ of part 4 (of D), are empty. Since all the parts at this stage are either completely occupied or completely empty, no further subdivision is necessary. The corresponding quadtree representation is shown in Fig. 2.27E. Note that the shape of the finite elements is assumed to be square in this example.

3. A simplex in an n-dimensional space is defined as a geometric figure having $n+1$ nodes or corners. Thus, the simplex will be a triangle in two-dimensional space and a tetrahedron in three-dimensional space.

EXAMPLE 2.3—cont'd

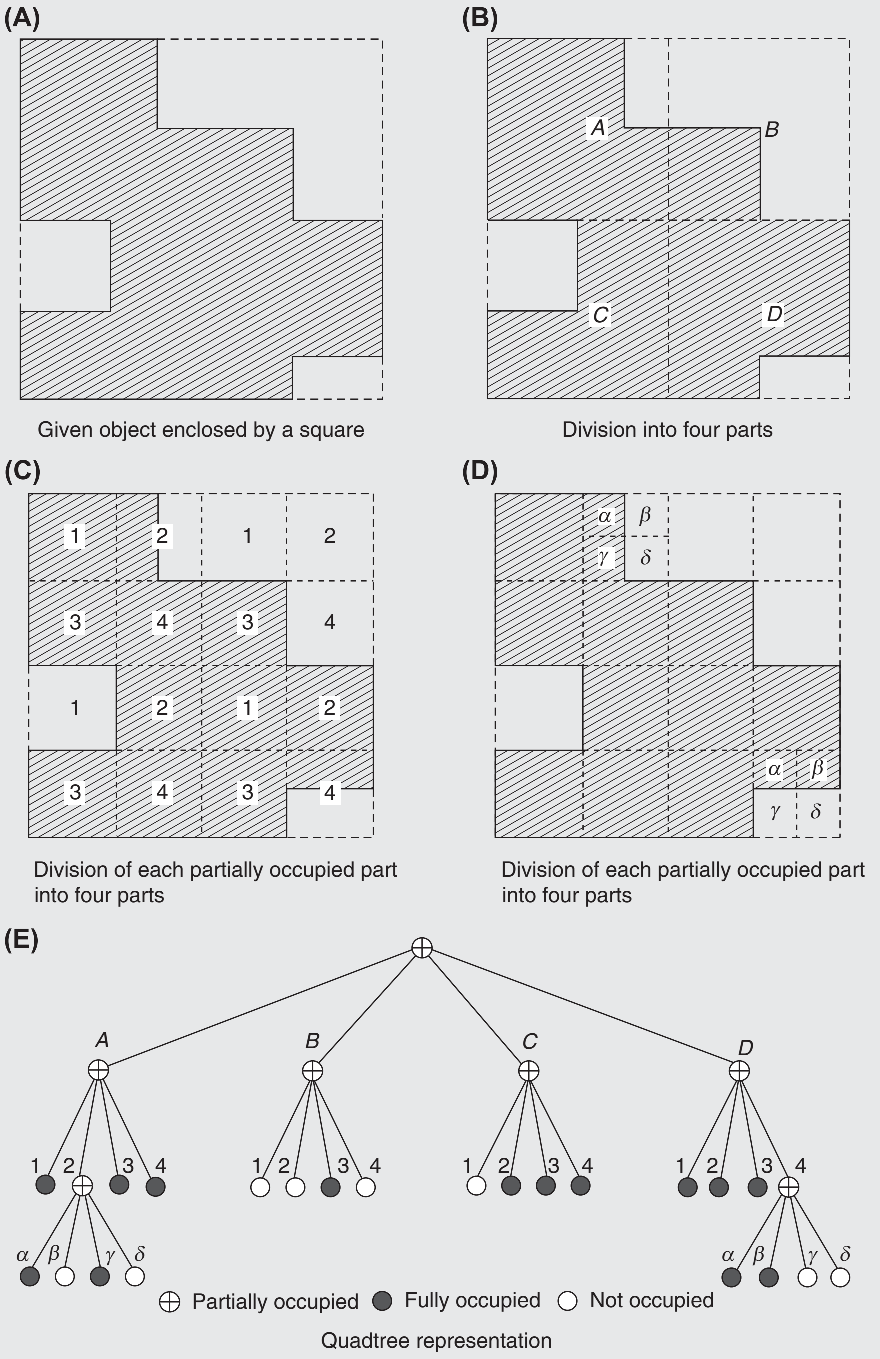

FIGURE 2.27 Mesh generation using the quadtree method.

REVIEW QUESTIONS

2.1 Give brief answers to the following questions.

1. State two factors that influence the selection of a specific finite element in the modeling process.
2. When can we use a one-dimensional element for modeling?
3. Give two examples of physical problems that can be modeled using one-dimensional or line elements.
4. What is the reason for using two degrees of freedom at each node of a one-dimensional beam element?
5. Give an example of a practical problem where the type of element to be used is unique.
6. Give two practical problems that require different types of elements for modeling.
7. Define the aspect ratio of an element.
8. Can we always expect improved accuracy of the finite element solution by increasing the number of elements?
9. How do you model the semi-infinite soil under a column subjected to a large compressive load?
10. Define the bandwidth of a matrix.
11. What is the purpose of the tessellation method?
12. Indicate three practical problems that can be modeled using axisymmetric elements.

2.2 Fill in the blank(s) with suitable word(s).

1. Several methods can be used to ________ any domain in the finite element method.
2. The finite element mesh generation of complex domains, with minimal interface with the analyst, can be done using an ________ program or software.
3. The basic shape of the element for three-dimensional problems is ________.
4. A hexahedron element can be formed using ________ tetrahedron elements.
5. Elements with curved sides are considered ________ order elements.
6. A ________ element can be obtained using two triangular elements.
7. An aspect ratio of ________ is desirable for best results.
8. ________ are to be located at points of abrupt change in geometry, material properties, or loads.
9. If the geometry and load conditions have double symmetry in a problem, the finite element model needs to consider only for ________ of the problem.
10. For modeling a long dam, we need to consider only a ________ of the dam in the axial direction.
11. ________ elements are available for modeling infinite regions.
12. It is desirable to ________ the bandwidth of the stiffness matrix for efficient solution.
13. A shorter bandwidth can be obtained simply by numbering the nodes across the ________ dimension of the body.
14. The octree method can be used for automatic ________.

2.3 Indicate whether the following statement is true or false.

1. The finite element method is used to find an approximate solution of an infinite number of degrees of freedom system using only a finite number of degrees of freedom.

2.4 Select the most appropriate answer from the multiple choices given.

1. The basic shape of the element for two-dimensional problems is:
   (a) Rectangle (b) Triangle (c) Quadrilateral

PROBLEMS

2.1 A thick-walled pressure vessel is subjected to an internal pressure as shown in Fig. 2.28. Model the cross section of the pressure vessel by taking advantage of the symmetry of the geometry and load condition.

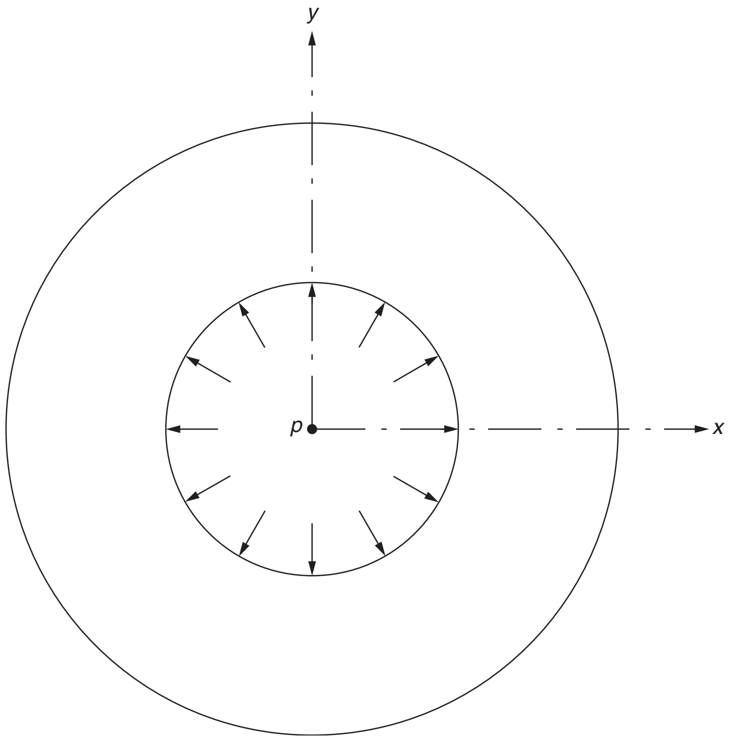

FIGURE 2.28 A thick-walled pressure vessel.

2.2 A rectangular plate with a V-notch is shown in Fig. 2.29. Model the plate using triangular elements by taking advantage of the symmetry of the system.

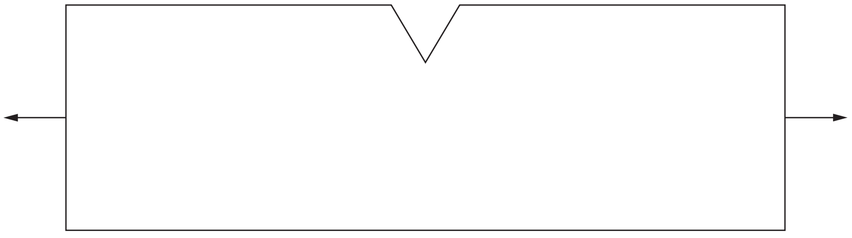

FIGURE 2.29 A rectangular plate with a notch.

2.3 The plate shown in Fig. 2.30 is modeled using 13 triangular and 2 quadrilateral elements. Label the nodes such that the bandwidth of the system matrix is minimal. Compute the resulting bandwidth assuming 1 dof at each node.

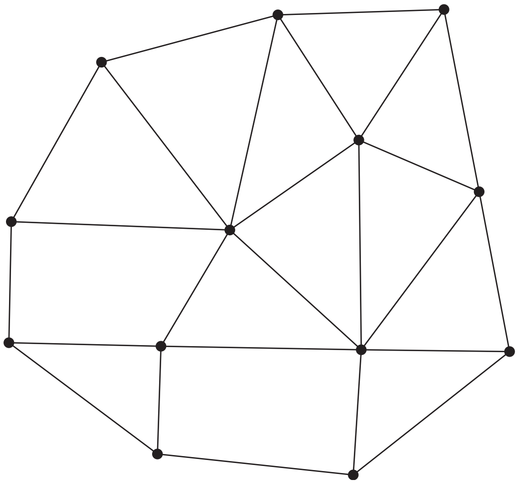

FIGURE 2.30 A plate modeled with triangular and quadrilateral elements.

2.4–2.8 Label the elements and nodes for each of the systems shown in Figs. 2.31 through 2.35 to produce a minimum bandwidth. In addition, find the resulting bandwidth in each case.

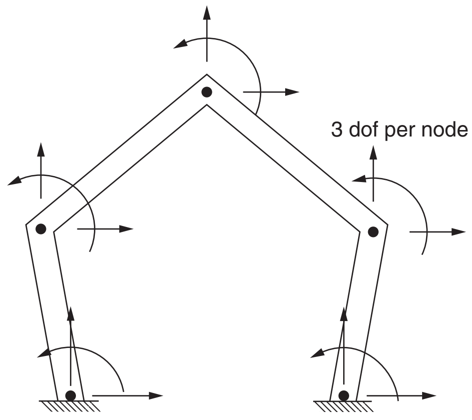

FIGURE 2.31 A planar frame.

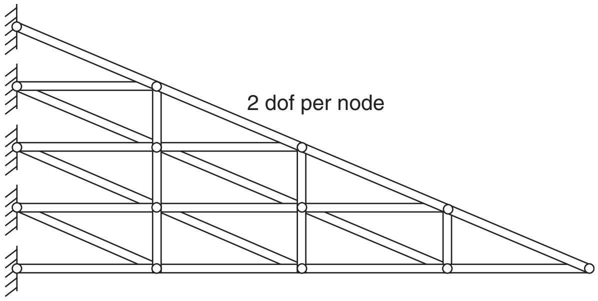

FIGURE 2.32 A planar truss.

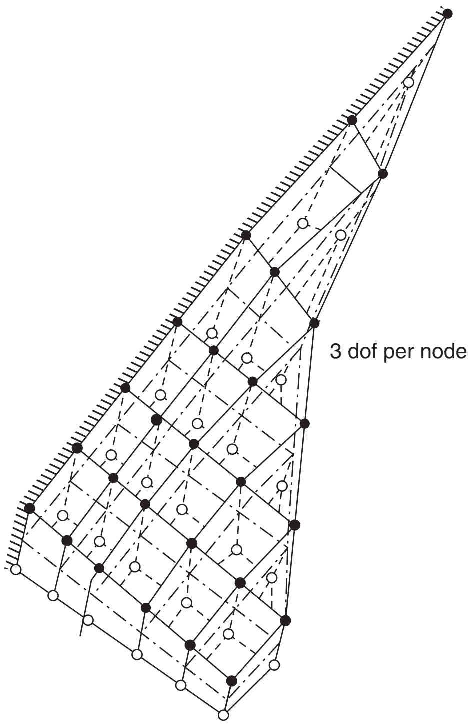

FIGURE 2.33 An aircraft wing.

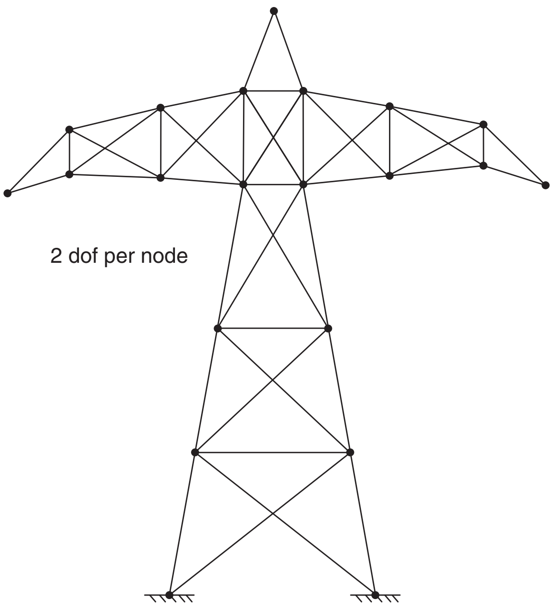

FIGURE 2.34 A planar truss.

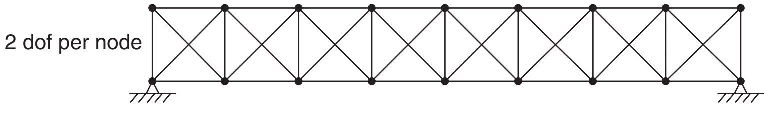

FIGURE 2.35 A planar truss.

2.9 Consider the collection of node points shown in Fig. 2.36 for a two-dimensional object. Generate the finite element mesh using the tesselation method.

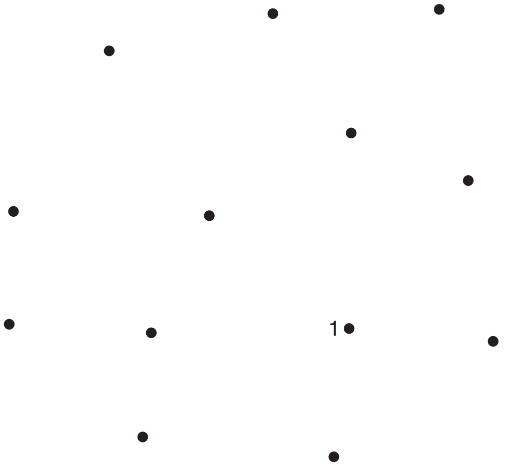

FIGURE 2.36 Node points for a two-dimensional object.

2.10 Generate the finite element mesh for the two-dimensional object shown in Fig. 2.37 using the quadtree method.

2.11 State the reasons for the desirability of elements with an aspect ratio close to 1 in finite element modeling.

2.12 Give two practical examples, each of which can be modeled using one-, two-, and three-dimensional finite elements to achieve different levels of accuracy.

2.13 The transmission lines (wires) carrying electricity are supported by electric transmission towers and are subjected to axial tension and gravity, and wind and snow loads. Discuss possible types of finite elements/models that can be used for the stress analysis of transmission lines.

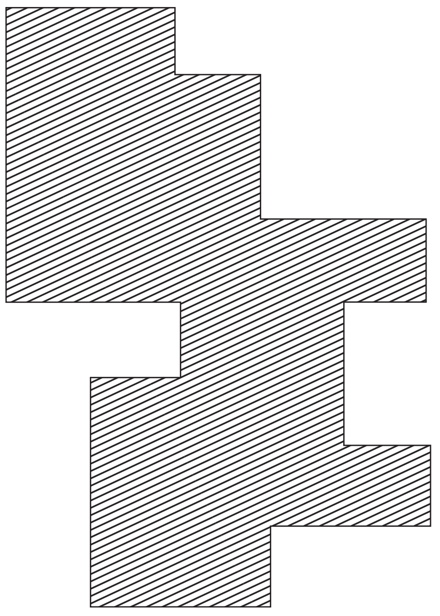

FIGURE 2.37 A two-dimensional object.

2.14 Water at pressure p_i flows through an underground cast iron pipe. The outer surface of the pipe is subjected to a uniform axisymmetric pressure along its length by the surrounding soil (Fig. 2.38). The pipe has an inner diameter of 0.7 m, outer diameter of 1.0 m, and a length of 1000 m. Indicate a suitable finite element idealization for the deformation and stress analysis of the water pipe.

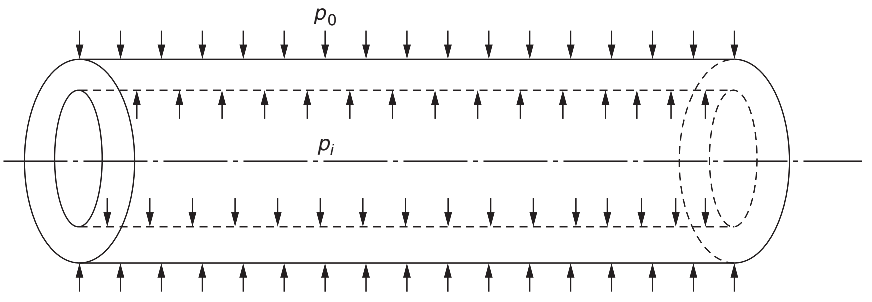

FIGURE 2.38 An underground pipe.

2.15 Label the nodes of the planar truss shown in Fig. 2.39 to minimize the bandwidth of the resulting stiffness matrix. Assume that each node has two dof (components of the displacement of the node parallel to the x and y axes). Also determine the resulting bandwidth of the stiffness matrix.

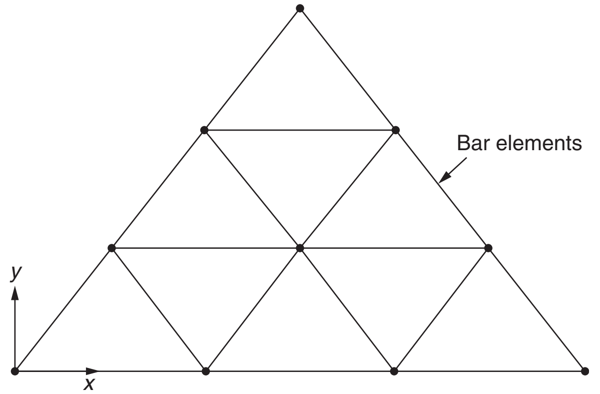

FIGURE 2.39 A planar truss.

2.16 Label the nodes of the truncated conical shell shown in Fig. 2.40. Assume that each node has three dof (components of the displacement of the node parallel to the x, y, and z axes). Also determine the resulting bandwidth of the stiffness matrix.

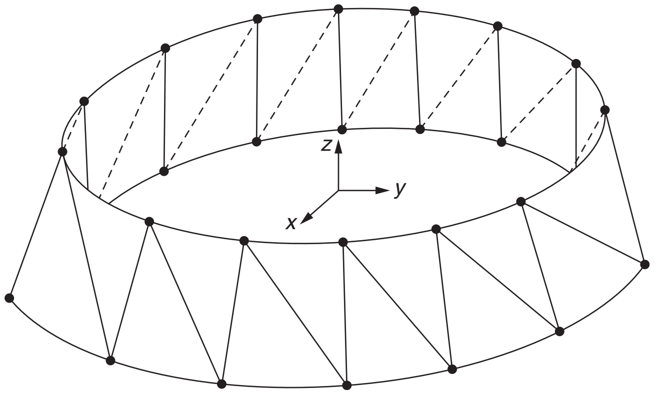

FIGURE 2.40 A truncated conical shell.

2.17 A semicircular plate is subjected to loads P_1 and P_2 as shown in Fig. 2.41. Using straight-sided triangular elements to model the plate is proposed. Show a sequence of three finite element meshes with increasing number of elements ensuring that each finer mesh includes the previous coarse mesh(es).

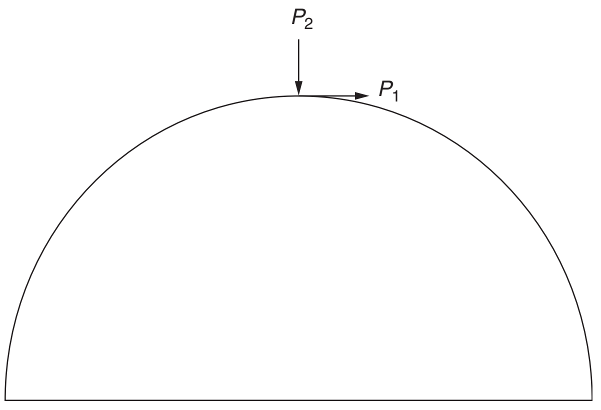

FIGURE 2.41 A semicircular plate.

2.18 The data for the uniform beam shown in Fig. 2.42 are given by $L = 100$ cm, $P = 1000$ N, $E = 70 \times 10^9$ Pa, and $I = 2$ cm^4. Determine the deflection of the beam at the center using a single beam element.
Hint: Use the symmetry of the geometry and load of the beam.

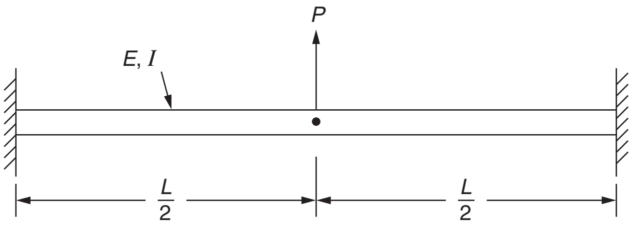

FIGURE 2.42 A uniform fixed–fixed beam.

2.19 A mechanical link is subjected to a symmetric distributed load p and concentrated loads P as shown in Fig. 2.43. Indicate the boundary conditions to be incorporated if only a quarter of the link is to be modeled using suitable finite elements to determine the stresses induced in the link.

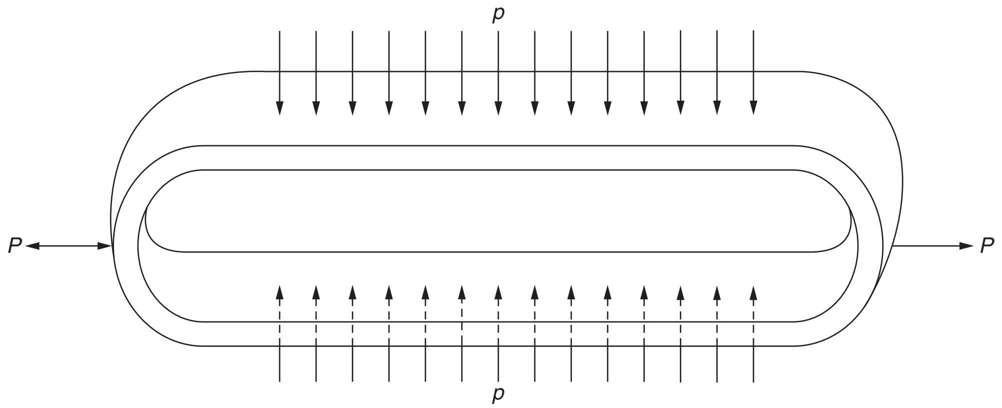

FIGURE 2.43 A mechanical link.

2.20 Generate the finite element mesh for the two-dimensional plate shown in Fig. 2.44 using the quadtree method.

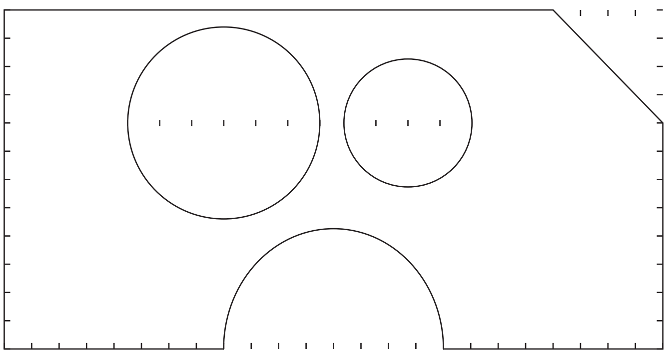

FIGURE 2.44 A two-dimensional plate.

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FURTHER READING

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Chapter 3

Interpolation Models

Chapter Outline

3.1 INTRODUCTION

As stated earlier, the basic idea of the finite element method is piecewise approximation—that is, the solution of a complicated problem is obtained by dividing the region of interest into small regions (finite elements) and approximating the solution over each subregion by a simple function. Thus, a necessary and important step is that of choosing a simple function for the solution in each element. The functions used to represent the behavior of the solution within an element are called interpolation functions, approximating functions, or interpolation models. Polynomial-type interpolation functions have been most widely used in the literature due to the following reasons:

1. It is easier to formulate and computerize the finite element equations with polynomial-type interpolation functions. Specifically, it is easier to perform differentiation or integration with polynomials.
2. It is possible to improve the accuracy of the results by increasing the order of the polynomial, as shown in Fig. 3.1. Theoretically, a polynomial of infinite order corresponds to the exact solution. But in practice we use polynomials of finite order only as an approximation.

Although trigonometric functions also possess some of these properties, they are seldom used in the finite element analysis [3.1]. We shall consider only polynomial-type interpolation functions in this book.

When the interpolation polynomial is of order one, the element is termed a linear element. A linear element is called a simplex element if the number of nodes in the element is 2, 3, and 4 in one, two, and three dimensions, respectively. If the interpolation polynomial is of order two or more, the element is known as a higher order element. In higher order elements, some secondary (mid-side and/or interior) nodes are introduced in addition to the primary (corner) nodes in order to match the number of nodal degrees of freedom with the number of constants (generalized coordinates) in the interpolation polynomial.

In general, fewer higher order elements are needed to achieve the same degree of accuracy in the final results. Although it does not reduce the computational time, the reduction in the number of elements generally reduces the effort needed in the preparation of data and hence the chances of errors in the input data. The higher order elements are especially useful in cases in which the gradient of the field variable is expected to vary rapidly. In these cases the simplex elements, which approximate the gradient by a set of constant values, do not yield good results. The combination of greater accuracy and a

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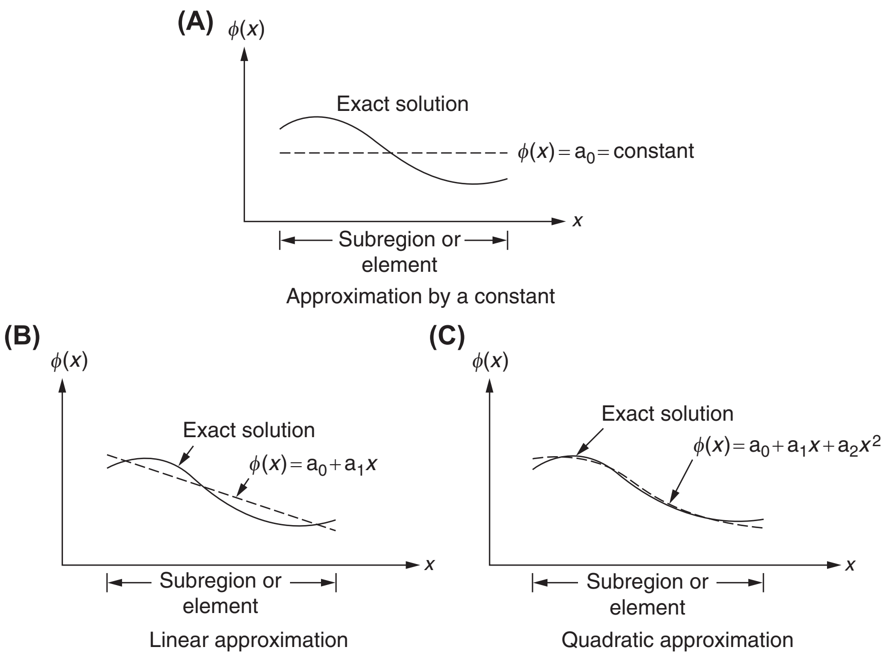

FIGURE 3.1 Polynomial approximation in one dimension.

reduction in the data preparation effort has resulted in the widespread use of higher order elements in several practical applications. We shall consider mostly linear elements in this chapter.

If the order of the interpolation polynomial is fixed, the discretization of the region (or domain) can be improved by two methods. In the first method, known as the *r-method*, the locations of the nodes are altered without changing the total number of elements. In the second method, known as the *h-method*, the number of elements is increased. On the other hand, if improvement in accuracy is sought by increasing the order of the interpolation of polynomial, the method is known as the *p-method*.

Problems involving curved boundaries cannot be modeled satisfactorily by using straight-sided elements. The family of elements known as isoparametric elements has been developed for this purpose. The basic idea underlying the isoparametric elements is to use the same interpolation functions to define the element shape or geometry as well as the variation of the field variable within the element. To derive the isoparametric element equations, we first introduce a local or natural coordinate system for each element shape. Then the interpolation or shape functions are expressed in terms of the natural coordinates. The representation of geometry in terms of (nonlinear) shape functions can be considered a mapping procedure that transforms a regular shape, such as a straight-sided triangle or rectangle in the local coordinate system, into a distorted shape, such as a curved-sided triangle or rectangle in the global Cartesian coordinate system. This concept can be used in representing problems with curved boundaries with the help of curved-sided isoparametric elements. Today, isoparametric elements are extensively used in three-dimensional and shell-analysis problems. The formulation of isoparametric elements, along with the aspect of numerical integration that is essential for computations with isoparametric elements, is considered in the next chapter.

3.2 POLYNOMIAL FORM OF INTERPOLATION FUNCTIONS

If a polynomial type of variation is assumed for the field variable $\phi(x)$ in a one-dimensional element, $\phi(x)$ can be expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots + \alpha_m x^n \tag{3.1}$$

Similarly, in two- and three-dimensional finite elements the polynomial form of interpolation functions can be expressed as

$$\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \cdots + \alpha_m y^n \tag{3.2}$$

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx + \cdots + \alpha_m z^n \tag{3.3}$$

where α_1, α_2, …, α_m are the coefficients of the polynomial, also known as generalized coordinates; n is the degree of the polynomial; and the number of polynomial coefficients m is given by

$$m = n + 1 \text{ for one-dimensional elements (Eq. 3.1)} \tag{3.4}$$

$$m = \sum_{j=1}^{n+1} j \text{ for two-dimensional elements (Eq. 3.2)} \tag{3.5}$$

$$m = \sum_{j=1}^{n+1} j(n+2-j) \text{ for three-dimensional elements (Eq. 3.3)} \tag{3.6}$$

In most practical applications, the order of the polynomial in the interpolation functions is taken as one, two, or three. Thus, Eqs. (3.1) to (3.3) reduce to the following equations for various cases of practical interest.

For $n = 1$ (linear model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x \tag{3.7}$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{3.8}$$

Three-dimensional case:

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \tag{3.9}$$

For $n = 2$ (quadratic model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{3.10}$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy \tag{3.11}$$

Three-dimensional case:

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz \tag{3.12}$$

For $n = 3$ (cubic model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{3.13}$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_8 y^3 + \alpha_9 x^2 y + \alpha_{10} xy^2 \tag{3.14}$$

Three-dimensional case:

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz + \alpha_{11} x^3 + \alpha_{12} y^3 + \alpha_{13} z^3 + \alpha_{14} x^2 y + \alpha_{15} x^2 z + \alpha_{16} y^2 z + \alpha_{17} xy^2 + \alpha_{18} xz^2 + \alpha_{19} yz^2 + \alpha_{20} xyz \tag{3.15}$$

3.3 SIMPLEX, COMPLEX, AND MULTIPLEX ELEMENTS

Finite elements can be classified into three categories—simplex, complex, and multiplex elements—depending on the geometry of the element and the order of the polynomial used in the interpolation function [3.2]. The simplex elements are those for which the approximating polynomial consists of constant and linear terms. Thus, the polynomials given by

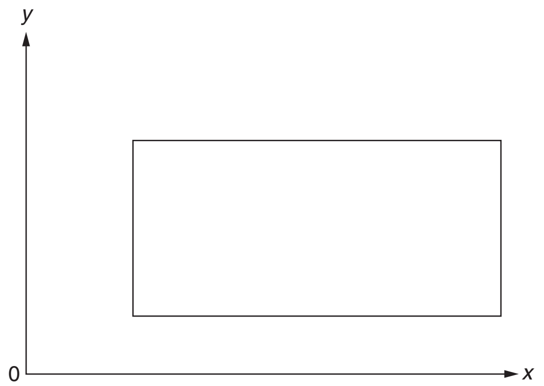

FIGURE 3.2 Example of a multiplex element.

Eqs. (3.7) to (3.9) represent the simplex functions for one-, two-, and three-dimensional elements. Noting that a simplex is defined as a geometric figure obtained by joining $n+1$ joints (nodes) in an n-dimensional space, we can consider the corners of the elements as nodes in simplex elements. For example, the simplex element in two dimensions is a triangle with three nodes (corners). The three polynomial coefficients α_1, α_2, and α_3 of Eq. (3.8) can thus be expressed in terms of the nodal values of the field variable ϕ. The complex elements are those for which the approximating polynomial consists of quadratic, cubic, and higher order terms, according to the need, in addition to the constant and linear terms. Thus, the polynomials given by Eqs. (3.10) to (3.15) denote complex functions.

The complex elements may have the same shapes as the simplex elements but will have additional boundary nodes and, sometimes, internal nodes. For example, the interpolating polynomial for a two-dimensional complex element (including terms up to quadratic terms) is given by Eq. (3.11). Since this equation has six unknown coefficients α_i, the corresponding complex element must have six nodes. Thus, a triangular element with three corner nodes and three mid-side nodes satisfies this requirement. The multiplex elements are those whose boundaries are parallel to the coordinate axes to achieve interelement continuity, and whose approximating polynomials contain higher order terms. The rectangular element shown in Fig. 3.2 is an example of a multiplex element in two dimensions. Note that the boundaries of the simplex and complex elements need not be parallel to the coordinate axes.

3.4 INTERPOLATION POLYNOMIAL IN TERMS OF NODAL DEGREES OF FREEDOM

The basic idea of the finite element method is to consider a body as composed of several elements (or subdivisions) that are connected at specified node points. The unknown solution or the field variable (e.g., displacement, pressure, or temperature) inside any finite element is assumed to be given by a simple function in terms of the nodal values of that element. The nodal values of the solution, also known as nodal degrees of freedom, are treated as unknowns in formulating the system or overall equations. The solution of the system equations (e.g., force equilibrium equations or thermal equilibrium equations or continuity equations) gives the values of the unknown nodal degrees of freedom. Once the nodal degrees of freedom are known, the solution within any finite element (and hence within the complete body) will also be known to us.

Thus, we need to express the approximating polynomial in terms of the nodal degrees of freedom of a typical finite element e. For this, let the finite element have M nodes. We can evaluate the values of the field variable at the nodes by substituting the nodal coordinates into the polynomial equation given by Eqs. (3.1) to (3.3). For example, Eq. (3.1) can be expressed as

$$\phi(x) = \vec{\eta}^T \vec{\alpha} \tag{3.16}$$

where

$$\vec{\eta}^{\mathrm{T}} = \begin{Bmatrix} 1 & x & x^2 & \ldots & x^n \end{Bmatrix},$$

and

$$\vec{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n+1} \end{Bmatrix}$$

The evaluation of Eq. (3.16) at the various nodes of element e gives

$$\begin{Bmatrix} \phi(\text{at node } 1) \\ \phi(\text{at node } 2) \\ \vdots \\ \phi(\text{at node } M) \end{Bmatrix}^{(e)} = \vec{\Phi}^{(e)} = \begin{bmatrix} \vec{\eta}^{\mathrm{T}}(\text{at node } 1) \\ \vec{\eta}^{\mathrm{T}}(\text{at node } 2) \\ \vdots \\ \vec{\eta}^{\mathrm{T}}(\text{at node } M) \end{bmatrix} \vec{\alpha} \equiv \left[\underset{\sim}{\eta}\right] \vec{\alpha} \tag{3.17}$$

where $\vec{\Phi}^{(e)}$ is the vector of nodal values of the field variable corresponding to element e, and the square matrix $\left[\underset{\sim}{\eta}\right]$ can be identified from Eq. (3.17). By inverting Eq. (3.17), we obtain

$$\vec{\alpha} = \left[\underset{\sim}{\eta}\right]^{-1} \vec{\Phi}^{(e)} \tag{3.18}$$

Substitution of Eq. (3.18) into Eqs. (3.1) to (3.3) gives

$$\phi = \vec{\eta}^{\mathrm{T}} \vec{\alpha} = \vec{\eta}^{\mathrm{T}} \left[\underset{\sim}{\eta}\right]^{-1} \vec{\Phi}^{(e)} = [N] \vec{\Phi}^{(e)} \tag{3.19}$$

where

$$[N] = \vec{\eta}^{T} \left[\underset{\sim}{\eta}\right]^{-1} \tag{3.20}$$

Eq. (3.19) now expresses the interpolating polynomial inside any finite element in terms of the nodal unknowns of that element, $\vec{\Phi}^{(e)}$.

Note

A major limitation of polynomial-type interpolation functions is that we have to invert the matrix $\left[\underset{\sim}{\eta}\right]$ to find ϕ, and $\left[\underset{\sim}{\eta}\right]^{-1}$ may become singular in some cases [3.3]. The latter difficulty can be avoided by using other types of interpolation functions discussed in Chapter 4.

The following example illustrates the use of Eq. (3.20) in finding the shape functions of an element.

EXAMPLE 3.1

The nodes of a one-dimensional element are located at $x_1 = 20$ in and $x_2 = 25$ in. The values of the field variable at the two nodes (nodal unknowns) of the element are denoted Φ_1 and Φ_2. Assuming a linear interpolation model for the field variable as

$$\phi(x) = \{1 \quad x\} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \equiv \vec{\eta}^{T} \vec{\alpha} \tag{E.1}$$

where

$$\vec{\eta}^{T} = \{1 \quad x\} \quad \text{and} \quad \vec{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

determine the matrix of shape functions $[N(x)]$ using Eq. (3.20).

Solution

Approach: Use a linear interpolation model in the form of Eq. (E.1).

The nodal values of the field variable can be expressed, using Eq. (E.1), as

$$\Phi_1 = \phi(x = x_1) = \{1 \quad 20\}\vec{\alpha}$$
$$\Phi_2 = \phi(x = x_2) = \{1 \quad 25\}\vec{\alpha}$$

so that

$$\Phi^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix} = \begin{bmatrix} 1 & 20 \\ 1 & 25 \end{bmatrix} \vec{\alpha} \equiv \left[\underset{\sim}{\eta}\right] \vec{\alpha}$$

Continued

EXAMPLE 3.1 —cont'd

The inverse of the matrix $\left[\underset{\sim}{\eta}\right]$ is given by

$$\left[\underset{\sim}{\eta}\right]^{-1} = \begin{bmatrix} 1 & 20 \\ 1 & 25 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -4 \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Thus, the matrix of shape functions of the element is given by Eq. (3.20):

$$\left[N(x)\right] = \vec{\eta}^{\mathrm{T}}\left[\underset{\sim}{\eta}\right]^{-1} = \{1 \quad x\}\begin{bmatrix} 5 & -4 \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix} = \left\{\left(5-\frac{x}{5}\right) \quad \left(-4+\frac{x}{5}\right)\right\} \equiv \{N_1(x) \quad N_2(x)\}$$

where the shape functions associated with nodes 1 and 2, $N_1(x)$ and $N_2(x)$, can be identified as

$$N_1(x) = 5-\frac{x}{5}, \quad N_2(x) = -4+\frac{x}{5}$$

3.5 SELECTION OF THE ORDER OF THE INTERPOLATION POLYNOMIAL

While choosing the order of the polynomial in a polynomial-type interpolation function, the following considerations have to be taken into account:

1. The interpolation polynomial should satisfy, as far as possible, the convergence requirements stated in Section 3.6.
2. The pattern of variation of the field variable resulting from the polynomial model should be independent of the local coordinate system.
3. The number of generalized coordinates (α_i) should be equal to the number of nodal degrees of freedom of the element (Φ_i).

A discussion on the first consideration, namely, the convergence requirements to be satisfied by the interpolation polynomial, is given in the next section. According to the second consideration, as can be felt intuitively also, it is undesirable to have a preferential coordinate direction. That is, the field variable representation within an element, and hence the polynomial, should not change with a change in the local coordinate system (when a linear transformation is made from one Cartesian coordinate system to another). This property is called *geometric isotropy* or *geometric invariance* or *spatial isotropy* [3.4]. In order to achieve geometric isotropy, the polynomial should contain terms that do not violate symmetry in Fig. 3.3, which is known as *Pascal triangle* in the case of two dimensions, and *Pascal tetrahedron* or *pyramid* in the case of three dimensions.

Thus, in the case of a two-dimensional simplex element (triangle), the interpolation polynomial should include terms containing both x and y, but not only one of them, in addition to the constant term. In the case of a two-dimensional complex element (triangle), if we neglect the term x^3 (or x^2y) for any reason, we should not include y^3 (or xy^2) also in order to maintain geometric isotropy of the model. Similarly, in the case of a three-dimensional simplex element (tetrahedron), the approximating polynomial should contain terms involving x, y, and z in addition to the constant term.

The final consideration in selecting the order of the interpolation polynomial is to make the total number of terms involved in the polynomial equal to the number of nodal degrees of freedom of the element. The satisfaction of this requirement enables us to express the polynomial coefficients in terms of the nodal unknowns of the element as indicated in Section 3.4.

3.6 CONVERGENCE REQUIREMENTS

Since the finite element method is a numerical technique, we obtain a sequence of approximate solutions as the element size is reduced successively. This sequence will converge to the exact solution if the interpolation polynomial satisfies the following convergence requirements [3.5−3.8]:

1. The field variable must be continuous within the elements. This requirement is easily satisfied by choosing continuous functions as interpolation models. Since polynomials are inherently continuous, the polynomial type of interpolation models discussed in Section 3.2 satisfy this requirement.

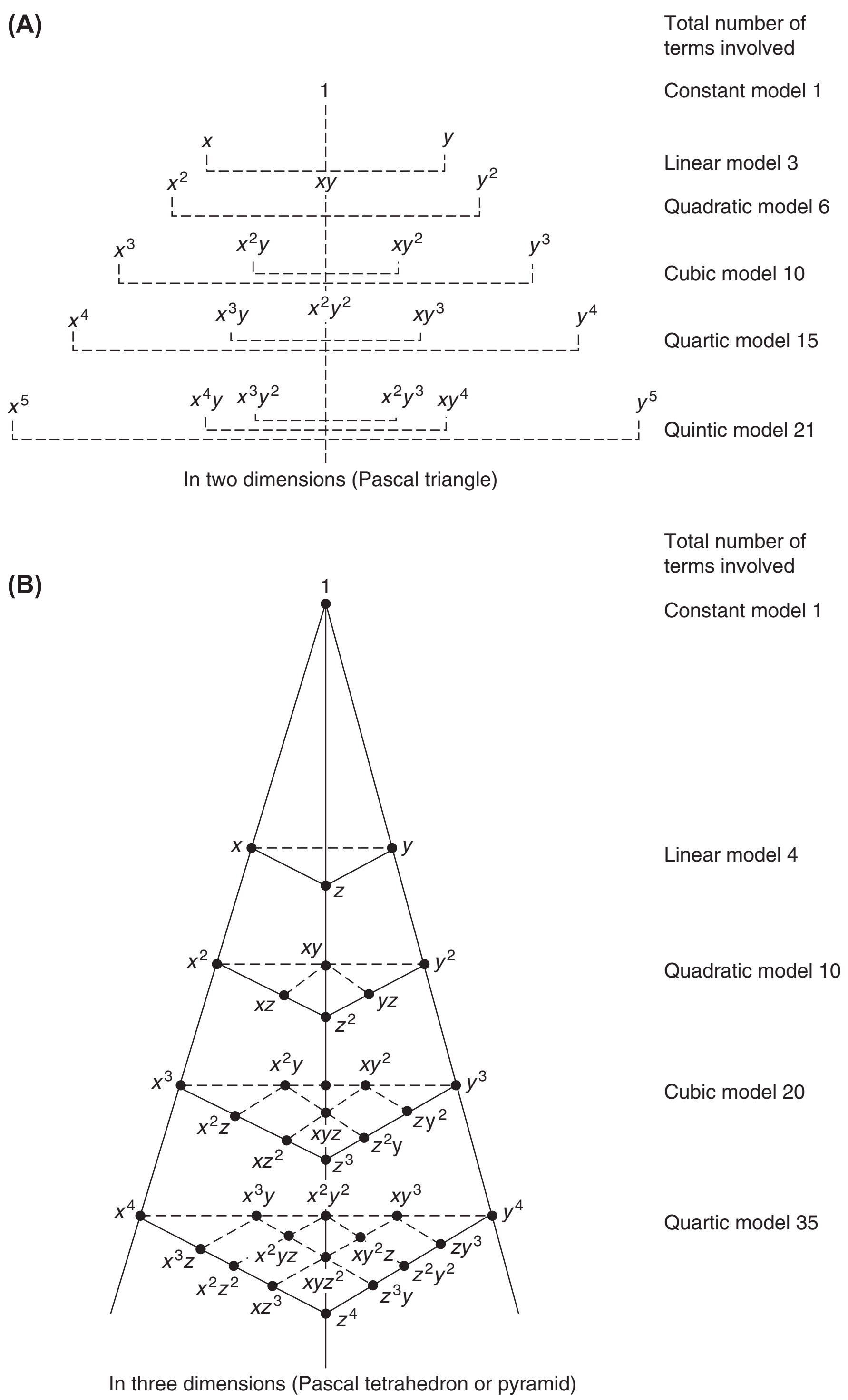

FIGURE 3.3 Array of terms in complete polynomials of various orders.

2. All uniform states of the field variable ϕ and its partial derivatives up to the highest order appearing in the functional $I(\phi)$ must have representation in the interpolation polynomial when, in the limit, the element size reduces to zero. The necessity of this requirement can be explained physically. The uniform or constant value of the field variable is the most elementary type of variation. Thus, the interpolation polynomial must be able to give a constant value of the field variable within the element when the nodal values are numerically identical. Similarly, when the body is subdivided

into smaller and smaller elements, the partial derivatives of the field variable up to the highest order appearing in the functional[1] $I(\phi)$ approach a constant value within each element. Thus, we cannot hope to obtain convergence to the exact solution unless the interpolation polynomial permits this constant derivative state.
In the case of solid mechanics and structural problems, this requirement states that the assumed displacement model must permit the rigid body (zero strain) and the constant strain states of the element.
3. The field variable ϕ and its partial derivatives up to one order less than the highest order derivative appearing in the functional $I(\phi)$ must be continuous at element boundaries or interfaces.

We know that in the finite element method the discrete model for the continuous function ϕ is taken as a set of piecewise continuous functions, each defined over a single element. As seen in Examples 1.2 to 1.4, we need to evaluate integrals of the form

$$\int \frac{\mathrm{d}^r\phi}{\mathrm{d}x^r}\,\mathrm{d}x$$

to derive the element characteristic matrices and vectors. We know that the integral of a stepwise continuous function, say $f(x)$, is defined if $f(x)$ remains bounded in the interval of integration. Thus, for the integral

$$\int \frac{\mathrm{d}^r\phi}{\mathrm{d}x^r}\,\mathrm{d}x$$

to be defined, ϕ must be continuous to the order $(r-1)$ to ensure that only finite jump discontinuities occur in the r-th derivative of ϕ. This is precisely the requirement stated previously.

The elements whose interpolation polynomials satisfy the requirements (1) and (3) are called *compatible* or *conforming* elements and those satisfying condition (2) are called *complete* elements. If r-th derivative of the field variable ϕ is continuous, then ϕ is said to have C^r continuity. In terms of this notation, the completeness requirement implies that ϕ must have C^r continuity within an element, whereas the compatibility requirement implies that ϕ must have C^{r-1} continuity at element interfaces.[2]

In the case of general solid and structural mechanics problems, this requirement implies that the element must deform without causing openings, overlaps, or discontinuities between adjacent elements. In the case of beam, plate, and shell elements, the first derivative of the displacement (slope) across interelement boundaries also must be continuous.

Although it is desirable to satisfy all the convergence requirements, several interpolation polynomials that do not meet all the requirements have been used in the finite element literature. In some cases, acceptable convergence or convergence to an incorrect solution has been obtained. In particular, the interpolation polynomials that are complete but not conforming have been found to give satisfactory results.

If the interpolation polynomial satisfies all three requirements, the approximate solution converges to the correct solution when we refine the mesh and use an increasing number of smaller elements. In order to prove the convergence mathematically, the mesh refinement has to be made in a regular fashion so as to satisfy the following conditions:

1. All previous (coarse) meshes must be contained in the refined meshes.
2. The elements must be made smaller in such a way that every point of the solution region can always be within an element.
3. The form of the interpolation polynomial must remain unchanged during the process of mesh refinement.

Conditions (1) and (2) are illustrated in Fig. 3.4, in which a two-dimensional region (in the form of a parallelogram) is discretized with an increasing number of triangular elements. From Fig. 3.5, in which the solution region is assumed to have a curved boundary, it can be seen that conditions (1) and (2) are not satisfied if we use elements with straight boundaries. In structural problems, interpolation polynomials satisfying all the convergence requirements always lead to the convergence of the displacement solution from below while nonconforming elements may converge either from below or from above.

---

1. Finite element method can be considered as an approximate method of minimizing a functional $I(\phi)$ in the form of an integral of the type

$$I(\phi) = I\left(\phi, \frac{\mathrm{d}\phi}{\mathrm{d}x}, \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}, \cdots, \frac{\mathrm{d}^r\phi}{\mathrm{d}x^r}\right)$$

The functionals for simple one-dimensional problems were given in Examples 1.2–1.4.
2. This statement assumes that the functional (I) corresponding to the problem contains derivatives of ϕ up to the r-th order.

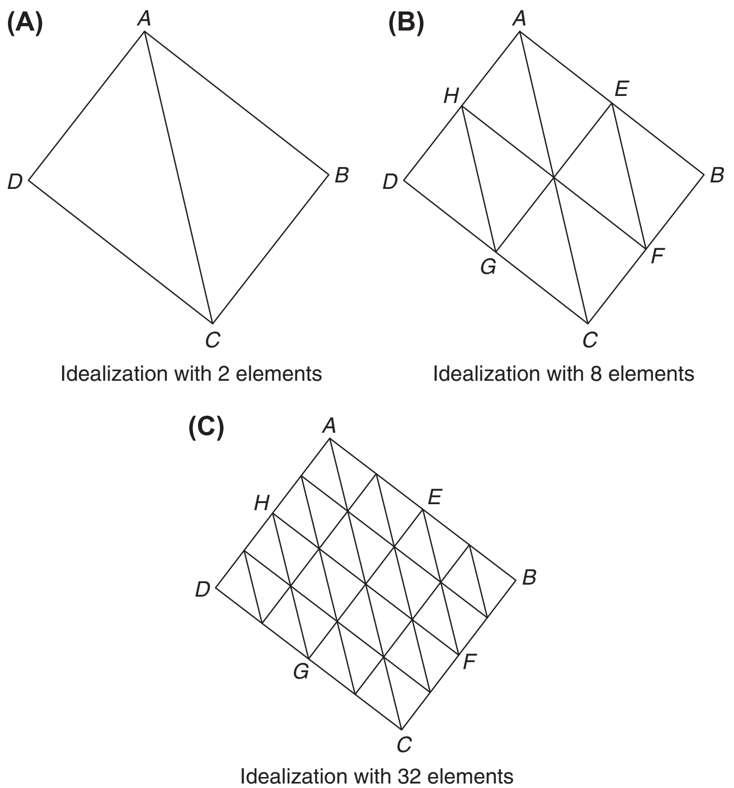

FIGURE 3.4 All previous meshes contained in refined meshes.

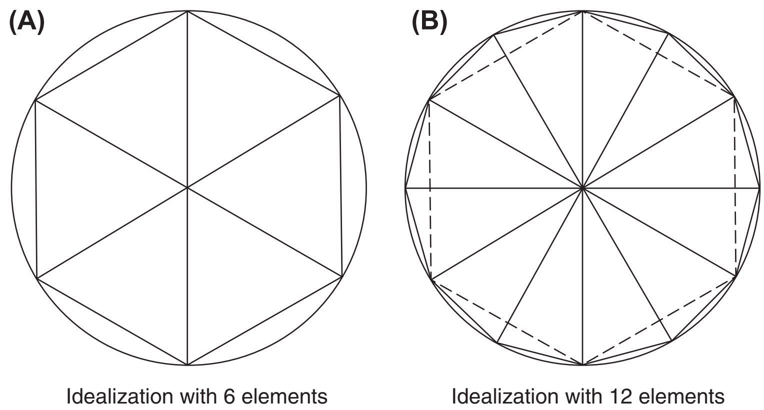

FIGURE 3.5 Previous mesh is not contained in the refined mesh.

Notes

1. For any physical problem, the selection of finite elements and interpolation polynomials to achieve C^0 continuity is not very difficult. However, the difficulty increases rapidly when higher order continuity is required. In general, the construction of finite elements to achieve specified continuity of order C^0, C^1, C^2, …, requires skill, ingenuity, and experience. Fortunately, most of the time, we would be able to use the elements already developed in an established area such as stress analysis for solving new problems.
2. The construction of an efficient finite element model involves (a) representing the geometry of the problem accurately, (b) developing a finite element mesh to reduce the bandwidth, and (c) choosing a proper interpolation model to obtain the desired accuracy in the solution. Unfortunately, there is no a priori method of creating a reasonably efficient finite element

Continued

Notes—cont'd

model that can ensure a specified degree of accuracy. Several numerical tests are available for assessing the convergence of a finite element model [3.9,3.10].

Some adaptive finite element methods have been developed to employ the results from previous meshes to estimate the magnitude and distribution of solution errors and to adaptively improve the finite element model [3.11–3.15]. There are four basic approaches to adaptively improve a finite element model:

a. Subdivide selected elements (called h-method).
b. Increase the order of the polynomial of selected elements (called p-refinement).
c. Move node points in fixed element topology (called r-refinement).
d. Define a new mesh having a better distribution of elements.

Various combinations of these approaches are also possible. Determining which of these approaches is the best for a particular class of problems is a complex problem that must consider the cost of the entire solution process.

3.7 LINEAR INTERPOLATION POLYNOMIALS IN TERMS OF GLOBAL COORDINATES

The linear interpolation polynomials correspond to simplex elements. In this section, we derive the linear interpolation polynomials for the basic one-, two-, and three-dimensional elements in terms of the global coordinates that are defined for the entire domain or body.

3.7.1 One-Dimensional Simplex Element

Consider a one-dimensional element (line segment) of length l with two nodes, one at each end, as shown in Fig. 3.6. Let the nodes be denoted as i and j and the nodal values of the field variable ϕ as Φ_i and Φ_j. The variation of ϕ inside the element is assumed to be linear as

$$\phi(x) = \alpha_1 + \alpha_2 x \tag{3.21}$$

where α_1 and α_2 are the unknown coefficients. By using the nodal conditions

$$\phi(x) = \Phi_i \quad \text{at} \quad x = x_i$$

$$\phi(x) = \Phi_j \quad \text{at} \quad x = x_j$$

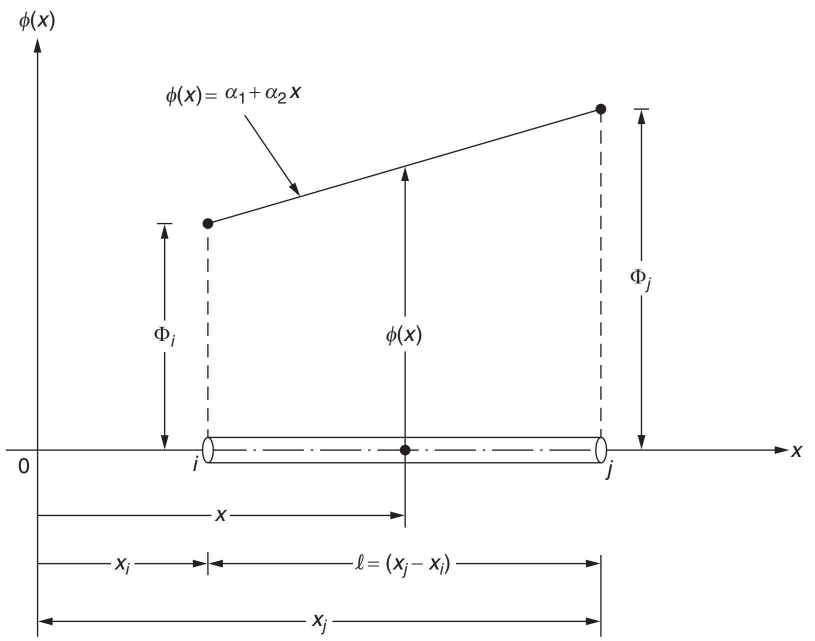

FIGURE 3.6 One-dimensional simplex element.

and Eq. (3.21), we obtain

$$\Phi_i = \alpha_1 + \alpha_2 x_i$$

$$\Phi_j = \alpha_1 + \alpha_2 x_j$$

The solution of these equations gives

$$\left.\begin{aligned}\alpha_1 &= \frac{\Phi_i x_j - \Phi_j x_i}{l}\\ \alpha_2 &= \frac{\Phi_j - \Phi_i}{l}\end{aligned}\right\} \tag{3.22}$$

where x_i and x_j denote the global coordinates of nodes i and j, respectively. By substituting Eq. (3.22) into Eq. (3.21), we obtain

$$\phi(x) = \left(\frac{\Phi_i x_j - \Phi_j x_i}{l}\right) + \left(\frac{\Phi_j - \Phi_i}{l}\right)x \tag{3.23}$$

This equation can be written, after rearrangement of terms, as

$$\phi(x) = N_i(x)\Phi_i + N_j(x)\Phi_j = [N(x)]\overrightarrow{\Phi}^{(e)} \tag{3.24}$$

where

$$[N(x)] = [N_i(x) \;\; N_j(x)] \tag{3.25}$$

$$\left.\begin{aligned}N_i(x) &= \frac{x_j - x}{l}\\ N_j(x) &= \frac{x - x_i}{l}\end{aligned}\right\} \tag{3.26}$$

and

$$\overrightarrow{\Phi}^{(e)} = \begin{Bmatrix}\Phi_i\\ \Phi_j\end{Bmatrix} = \text{vector of nodal unknowns of elements } e \tag{3.27}$$

Note that the superscript e is not used for Φ_i and Φ_j for simplicity.

The linear functions of x defined in Eq. (3.26) are called *interpolation* or *shape functions*.[3] Note that each interpolation function has a subscript to denote the node to which it is associated. Furthermore, the value of $N_i(x)$ can be seen to be 1 at node i ($x = x_i$) and 0 at node j ($x = x_j$). Likewise, the value of $N_j(x)$ will be 0 at node i and 1 at node j. These represent the common characteristics of interpolation functions. They will be equal to 1 at one node and 0 at each of the other nodes of the element.

EXAMPLE 3.2

The nodal temperatures of nodes i and j (same as local nodes 1 and 2) of an element in a one-dimensional fin are known to be $T_i = 120°\text{C}$ and $T_j = 80°\text{C}$ with the x-coordinates $x_i = 30$ cm and $x_j = 50$ cm. Find the following:

a. Shape functions associated with the nodal values T_i and T_j.
b. Interpolation model for the temperature inside the element, $T(x)$.
c. Temperature in the element at $x = 45$ cm.

Continued

3. The original polynomial type of interpolation model $\phi = \overrightarrow{\eta}^T\overrightarrow{\alpha}$ (which is often called the *interpolation polynomial* or interpolation model of the element) should not be confused with the *interpolation functions* N_i associated with the nodal degrees of freedom. There is a clear difference between the two. The expression $\overrightarrow{\eta}^T\overrightarrow{\alpha}$ denotes an interpolation polynomial that applies to the entire element and expresses the variation of the field variable inside the element in terms of the generalized coordinates α_i. The interpolation function N_i corresponds to the i-th nodal degree of freedom $\Phi_i^{(e)}$ and only the sum $\sum N_i \Phi i^{(e)}$ represents the variation of the field variable inside the element in terms of the nodal degrees of freedom $\Phi_i^{(e)}$. In fact, the interpolation function corresponding to the i-th nodal degree of freedom (N_i) assumes a value of 1 at node i, and 0 at all the other nodes of the element.

EXAMPLE 3.2 —cont'd

Solution

1. The shape functions $N_i(x)$ and $N_j(x)$ are given by Eq. (3.26):

$$N_i(x) = \frac{x_j - x}{l} = \frac{50 - x}{50 - 30} = 2.5 - 0.05\,x \tag{E.1}$$

$$N_j(x) = \frac{x - x_i}{l} = \frac{x - 30}{50 - 30} = 0.05\,x - 1.5 \tag{E.2}$$

2. The interpolation model for the temperature inside the element can be expressed, using Eq. (3.24), as

$$T(x) = N_i(x)T_i + N_j(x)T_j = (2.5 - 0.05\,x)120 + (0.05\,x - 1.5)80°\text{C} \tag{E.3}$$

3. The temperature at $x = 45$ cm can be determined from Eq. (E.3) as

$$T(x = 45) = (2.5 - 0.05(45))120 + (0.05(45) - 1.5)\,80 = 90°\text{C} \tag{E.4}$$

EXAMPLE 3.3

A one-dimensional tapered fin element has the nodal coordinates $x_i = 20$ mm and $x_j = 60$ mm with the area of cross section changing linearly from a value of $A_i = 20$ mm^2 at x_i to a value of $A_j = 10$ mm^2 at x_j as shown in Fig. 3.7. (1) Determine the matrix of shape functions, and (2) express the area of cross section of the fin element in terms of the shape functions.

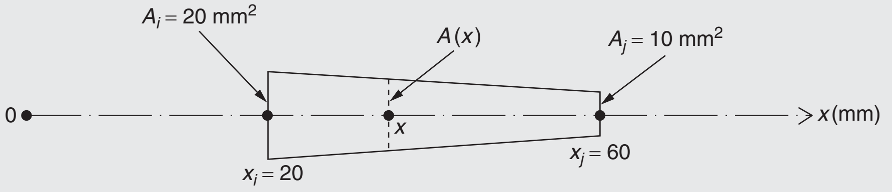

FIGURE 3.7 A tapered fin element.

Approach: (1) Use the shape functions corresponding to linear variation of the field variable. (2) Express linear variation of cross-sectional area in terms of shape functions similar to the variation of the field variable.

Solution

1. The linear variation of the field variable $\phi(x)$ can be expressed by Eq. (3.21) or, equivalently, by Eq. (3.24):

$$\phi(x) = \alpha_1 + \alpha_2 x = N_i(x)\Phi_i + N_j(x)\Phi_j = [N(x)]\vec{\Phi}^{(e)} \tag{E.1}$$

where the matrix of shape functions $[N(x)]$ is given by Eq. (3.25):

$$[N(x)] = \left[N_i(x)N_j(x)\right] \equiv \left[\frac{x_j - x}{x_j - x_i} \quad \frac{x - x_i}{x_j - x_i}\right]$$
$$= \left[\frac{60 - x}{60 - 20} \quad \frac{x - 20}{60 - 20}\right] = \left[\frac{60 - x}{40} \quad \frac{x - 20}{40}\right] \tag{E.2}$$

2. The linear variation of the cross-sectional area of the element can be expressed as

$$A(x) = \beta_1 + \beta_2 x \tag{E.3}$$

where the values of the constants β_1 and β_2 can be found by using the known areas of cross section at the two nodes:

$$A(x = x_i = 20) = A_i = 20\text{ mm}^2 \text{ and } A(x = x_j = 60) = A_j = 10\text{ mm}^2$$

This gives

$$\beta_1 = \frac{A_i x_j - A_j x_i}{x_j - x_i} = \frac{20(60) - 10(20)}{60 - 20} = 25, \quad \beta_2 = \frac{A_j - A_i}{x_j - x_i} = \frac{10 - 20}{60 - 20} = -0.25 \tag{E.4}$$

EXAMPLE 3.3 —cont'd

Using Eq. (E.4), Eq. (E.3) can be expressed in terms of the shape functions as

$$A(x) = A_i N_i(x) + A_j N_j(x) \equiv [N(x)] \vec{A}^{(e)} \tag{E.5}$$

where the matrix of shape functions, $[N(x)]$, is given by Eq. (E.2) and $\vec{A}^{(e)}$ is the vector of nodal areas of cross section of the element:

$$\vec{A}^{(e)} = \begin{Bmatrix} A_i \\ A_j \end{Bmatrix} = \begin{Bmatrix} 20 \\ 10 \end{Bmatrix} \text{mm}^2 \tag{E.6}$$

Note

Eq. (E.4) gives $\beta_1 = 25$ and $\beta_2 = -0.25$ so that the variation of $A(x)$ can also be expressed as

$$A(x) = 25 - 0.25x \tag{E.7}$$

3.7.2 Two-Dimensional Simplex Element

The two-dimensional simplex element is a straight-sided triangle with three nodes, one at each corner, as indicated in Fig. 3.8. Let the nodes be labeled as i, j, and k by proceeding counterclockwise from node i, which is arbitrarily specified. Let the global coordinates of the nodes i, j, and k be given by (x_i, y_i), (x_j, y_j), and (x_k, y_k), and the nodal values of the field variable $\phi\,(x, y)$ by Φ_i, Φ_j, and Φ_k, respectively. The variation of ϕ inside the element is assumed to be linear as

$$\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{3.28}$$

The nodal conditions

$$\phi(x, y) = \Phi_i \quad \text{at} \quad (x = x_i, y = y_i)$$
$$\phi(x, y) = \Phi_j \quad \text{at} \quad (x = x_j, y = y_j)$$
$$\phi(x, y) = \Phi_k \quad \text{at} \quad (x = x_k, y = y_k)$$

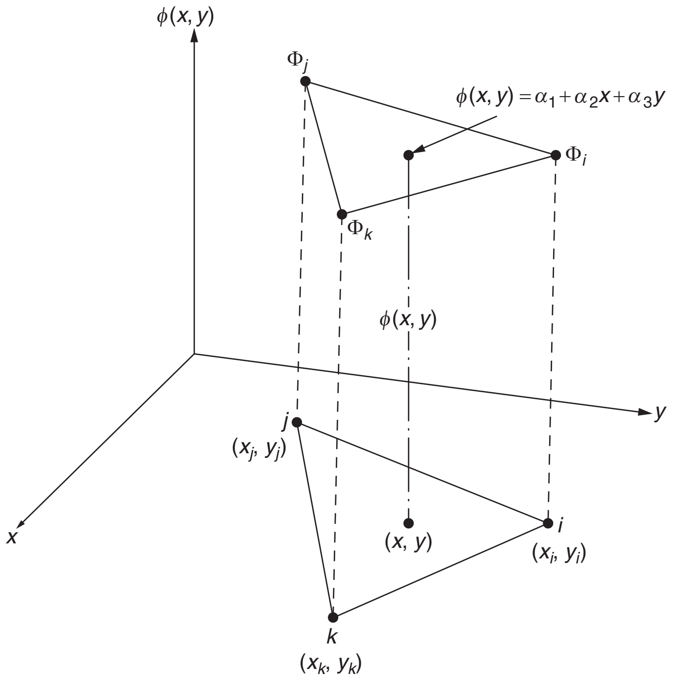

FIGURE 3.8 Two-dimensional simplex element.

lead to the system of equations

$$\begin{aligned} \Phi_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ \Phi_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ \Phi_k &= \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \end{aligned} \tag{3.29}$$

The solution of Eq. (3.29) yields

$$\begin{aligned} \alpha_1 &= \frac{1}{2A}(a_i\Phi_i + a_j\Phi_j + a_k\Phi_k) \\ \alpha_2 &= \frac{1}{2A}(b_i\Phi_i + b_j\Phi_j + b_k\Phi_k) \\ \alpha_3 &= \frac{1}{2A}(c_i\Phi_i + c_j\Phi_j + c_k\Phi_k) \end{aligned} \tag{3.30}$$

where A is the area of the triangle ijk given by

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - x_i y_k - x_j y_i - x_k y_j) \tag{3.31}$$

$$\begin{aligned} a_i &= x_j y_k - x_k y_j \\ a_j &= x_k y_i - x_i y_k \\ a_k &= x_i y_j - x_j y_i \\ b_i &= y_j - y_k \\ b_j &= y_k - y_i \\ b_k &= y_i - y_j \\ c_i &= x_k - x_j \\ c_j &= x_i - x_k \\ c_k &= x_j - x_i \end{aligned} \tag{3.32}$$

Substitution of Eq. (3.30) into Eq. (3.28) and rearrangement yields the equation

$$\phi(x,y) = N_i(x,y)\Phi_i + N_j(x,y)\Phi_j + N_k(x,y)\Phi_k = [N(x,y)]\vec{\Phi}^{(e)} \tag{3.33}$$

where

$$[N(x,y)] = [N_i(x,y) \;\; N_j(x,y) \;\; N_k(x,y)] \tag{3.34}$$

$$\begin{aligned} N_i(x,y) &= \frac{1}{2A}(a_i + b_i x + c_i y) \\ N_j(x,y) &= \frac{1}{2A}(a_j + b_j x + c_j y) \\ N_k(x,y) &= \frac{1}{2A}(a_k + b_k x + c_k y) \end{aligned} \tag{3.35}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{Bmatrix} = \text{vector of nodal unknowns of element } e \tag{3.36}$$

Notes

1. The shape function $N_i(x, y)$ when evaluated at node i (x_i, y_i) gives

$$\begin{aligned} N_i(x_i, y_i) &= \frac{1}{2A}(a_i + b_i x_i + c_i y_i) \\ &= \frac{1}{2A}(x_j y_k - x_k y_j + x_i y_j - x_i y_k + x_k y_i - x_j y_i) = 1 \end{aligned} \tag{3.37}$$

   It can be shown that $N_i(x, y) = 0$ at nodes j and k, and at all points on the line passing through these nodes. Similarly, the shape functions N_j and N_k have a value of 1 at nodes j and k, respectively, and 0 at other nodes.
2. Since the interpolation functions are linear in x and y, the gradient of the field variable in x or y direction will be a constant. For example,

$$\frac{\partial \phi(x, y)}{\partial x} = \frac{\partial}{\partial x}[N(x, y)]\vec{\Phi}^{(e)} = (b_i \Phi_i + b_j \Phi_j + b_k \Phi_k)/2A \tag{3.38}$$

   Since Φ_i, Φ_j, and Φ_k are the nodal values of ϕ (independent of x and y), and b_i, b_j, and b_k are constants whose values are fixed once the nodal coordinates are specified, $(\partial\phi/\partial x)$ will be a constant. A constant value of the gradient of ϕ within an element means that many small elements have to be used in locations where rapid changes are expected in the value of ϕ.

EXAMPLE 3.4

The temperatures at the nodes of a triangular element are given by $T_i = 210$ °F, $T_j = 270$ °F, and $T_k = 250$ °F. If the nodal coordinates are $(x_i, y_i) = (50, 30)$ in, $(x_j, y_j) = (70, 50)$ in, and $(x_k, y_k) = (55, 60)$ in, determine (a) the shape functions of the element and (b) temperature at the point $(x, y) = (60, 40)$ in the element.

Solution

1. From the known nodal coordinates, the area of the triangular element and the constants a_i, b_i, c_i, … involved in the shape functions can be determined as

$$A = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - x_i y_k - x_j y_i - x_k y_j)$$

$$= \frac{1}{2}(50 \times 50 + 70 \times 60 + 55 \times 30 - 50 \times 60 - 70 \times 30 - 55 \times 50) = 250 \text{ in}^2$$

$$a_i = x_j y_k - x_k y_j = 70 \times 60 - 55 \times 50 = 1450$$

$$a_j = x_k y_i - x_i y_k = 55 \times 30 - 50 \times 60 = -1350$$

$$a_k = x_i y_j - x_j y_i = 50 \times 50 - 70 \times 30 = 400$$

$$b_i = y_j - y_k = 50 - 60 = -10$$

$$b_j = y_k - y_i = 60 - 30 = 30$$

$$b_k = y_i - y_j = 30 - 50 = -20$$

$$c_i = x_k - x_j = 55 - 70 = -15$$

$$c_j = x_i - x_k = 50 - 55 = -5$$

$$c_k = x_j - x_i = 70 - 50 = 20$$

Continued

EXAMPLE 3.4 —cont'd

The shape functions can be found as

$$N_i(x,y) = \frac{1}{2A}(a_i + b_i x + c_i y) = \frac{1}{500}(1450 - 10x - 15y) = 2.9 - 0.02x - 0.03y$$

$$N_j(x,y) = \frac{1}{2A}(a_j + b_j x + c_j y) = \frac{1}{500}(-1350 + 30x - 5y) = -2.7 + 0.06x - 0.01y$$

$$N_k(x,y) = \frac{1}{2A}(a_k + b_k x + c_k y) = \frac{1}{500}(400 - 20x + 20y) = 0.8 - 0.04x + 0.04y$$

2. The temperature distribution in the element can be expressed as

$$T(x,y) = N_i(x,y)T_i + N_j(x,y)T_j + N_k(x,y)T_k$$

$$= 210(2.9 - 0.02x - 0.03y) + 270(-2.7 + 0.06x - 0.01y) + 250(0.8 - 0.04x + 0.04y)$$

The temperature at the point $(x, y) = (60, 40)$ in can be found as

$$T(60,40) = 210(2.9 - 1.2 - 1.2) + 270(-2.7 + 3.6 - 0.4) + 250(0.8 - 2.4 + 1.6) = 240°\text{F}$$

3.7.3 Three-Dimensional Simplex Element

The three-dimensional simplex element is a flat-faced tetrahedron with four nodes, one at each corner, as shown in Fig. 3.9. Let the nodes be labeled as i, j, k, and l, where i, j, and k are labeled in a counterclockwise sequence on any face as viewed from the vertex opposite this face, which is labeled as l. Let the values of the field variable be Φ_i, Φ_j, Φ_k, and Φ_l and the global coordinates be (x_i, y_i, z_i), (x_j, y_j, z_j), (x_k, y_k, z_k), and (x_l, y_l, z_l) at nodes i, j, k, and l, respectively. If the variation of $\phi(x, y, z)$ is assumed to be linear,

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \tag{3.39}$$

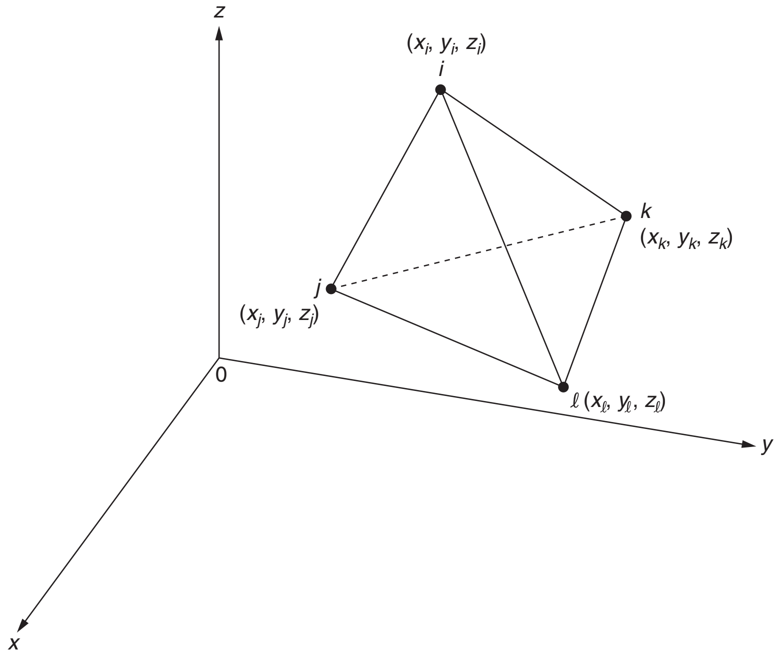

FIGURE 3.9 A three-dimensional simplex element.

the nodal conditions $\phi = \Phi_i$ at (x_i, y_i, z_i), $\phi = \Phi_j$ at (x_j, y_j, z_j), $\phi = \Phi_k$ at (x_k, y_k, z_k), and $\phi = \Phi_l$ at (x_l, y_l, z_l) produce the system of equations

$$\begin{aligned}\Phi_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i \\ \Phi_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j + \alpha_4 z_j \\ \Phi_k &= \alpha_1 + \alpha_2 x_k + \alpha_3 y_k + \alpha_4 z_k \\ \Phi_l &= \alpha_1 + \alpha_2 x_l + \alpha_3 y_l + \alpha_4 z_l\end{aligned} \tag{3.40}$$

Eq. (3.40) can be solved and the coefficients α_1, α_2, α_3, and α_4 can be expressed as

$$\begin{aligned}\alpha_1 &= \frac{1}{6V}(a_i\Phi_i + a_j\Phi_j + a_k\Phi_k + a_l\Phi_l) \\ \alpha_2 &= \frac{1}{6V}(b_i\Phi_i + b_j\Phi_j + b_k\Phi_k + b_l\Phi_l) \\ \alpha_3 &= \frac{1}{6V}(c_i\Phi_i + c_j\Phi_j + c_k\Phi_k + c_l\Phi_l) \\ \alpha_4 &= \frac{1}{6V}(d_i\Phi_i + d_j\Phi_j + d_k\Phi_k + d_l\Phi_l)\end{aligned} \tag{3.41}$$

where V is the volume of the tetrahedron $i\ j\ k\ l$ given by

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix} \tag{3.42}$$

$$a_i = \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} \tag{3.43}$$

$$b_i = -\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} \tag{3.44}$$

$$c_i = -\begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{vmatrix} \tag{3.45}$$

and

$$d_i = -\begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{vmatrix} \tag{3.46}$$

with the other constants defined by cyclic interchange of the subscripts in the order l, i, j, and k. The signs in front of the determinants in Eqs. (3.43) to (3.46) are to be reversed when generating a_j, b_j, c_j, d_j and a_l, b_l, c_l, d_l. By substituting Eq. (3.41) into Eq. (3.39), we obtain

$$\begin{aligned}\phi(x,y,z) &= N_i(x,y,z)\Phi_i + N_j(x,y,z)\Phi_j + N_k(x,y,z)\Phi_k + N_l(x,y,z)\Phi_l \\ &= [N(x,y,z)]\vec{\Phi}^{(e)}\end{aligned} \tag{3.47}$$

where

$$[N(x,y,z)] = [N_i(x,y,z)\ N_j(x,y,z)\ N_k(x,y,z)\ N_l(x,y,z)]$$

$$\begin{aligned}
N_i(x,y,z) &= \frac{1}{6V}(a_i + b_i x + c_i y + d_i z)\\
N_j(x,y,z) &= \frac{1}{6V}(a_j + b_j x + c_j y + d_j z)\\
N_k(x,y,z) &= \frac{1}{6V}(a_k + b_k x + c_k y + d_k z)\\
N_l(x,y,z) &= \frac{1}{6V}(a_l + b_l x + c_l y + d_l z)
\end{aligned} \tag{3.48}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \\ \Phi_l \end{Bmatrix} \tag{3.49}$$

EXAMPLE 3.5

Consider a tetrahedron element with node numbers i, j, k, and l as shown in Fig. 3.9. Noting that any of the nodes can be considered as the first (local) node, and the next three (local) nodes must follow a counterclockwise direction as viewed from the first node, enumerate the 12 different ways in which the node (local) numbers of the element can be assigned.

Solution

If node i is labeled as the first (local) node, the other nodes j, k, and l can be numbered as j, k, l, or k, l, j or l, j, k to satisfy the counterclockwise requirement (as viewed from node i). Similar considerations when node j (or k or l) is labeled as the first (local) node lead to the permissible numbering schemes as indicated in Table 3.1.

TABLE 3.1 Proper Node Numbering Schemes for Tetrahedron Element

Local Node 1	Local Node 2	Local Node 3	Local Node 4
i	j	k	l
i	k	l	j
i	l	j	k
j	k	i	l
j	i	l	k
j	l	k	i
k	i	j	l
k	j	l	i
k	l	i	j
l	i	k	j
l	k	j	i
l	j	i	k

EXAMPLE 3.6

A tetrahedron element with global node numbers 7, 8, 12, and 17 is shown in Fig. 3.10. Determine which of the following (local) numbering sequences satisfy the node numbering convention.

$$8, 12, 7, 17; 17, 7, 8, 12; 12, 7, 8, 17$$

Solution

The numbering scheme 8, 12, 7, 17 (also the scheme 12, 7, 8, 17) does not satisfy the node numbering convention because the nodes 12, 7, and 17 (7, 8, and 17) correspond to a clockwise order as seen from node 8 (12). Only the numbering scheme 17, 7, 8, 12 satisfies the node numbering convention because the nodes 7, 8, and 12 correspond to a counterclockwise order as viewed from node 17.

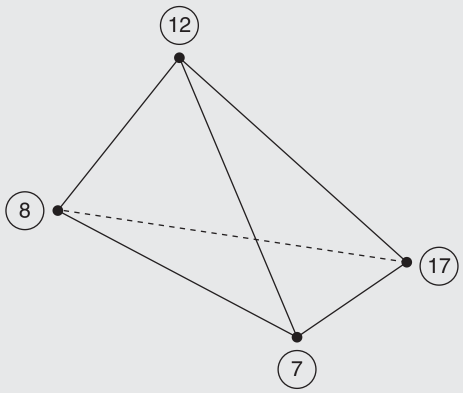

FIGURE 3.10 A tetrahedron element.

EXAMPLE 3.7

The nodal coordinates and nodal temperatures of a tetrahedron simplex element are given by

Node i: $(x_i, y_i, z_i) = (0,0,0)$ mm, $T_i = 100°C$

Node j: $(x_j, y_j, z_j) = (20,0,0)$ mm, $T_j = 80°C$

Node k: $(x_k, y_k, z_k) = (0,30,0)$ mm, $T_k = 120°C$

Node l: $(x_l, y_l, z_l) = (0,0,40)$ mm, $T_l = 50°C$

Express the temperature variation in the element $T(x, y, z)$, in terms of the shape functions.

Approach: Express the linear variation of temperature in the element using Eq. (3.47).

Solution

The volume of the tetrahedron element is given by Eq. (3.42):

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 20 & 0 & 0 \\ 1 & 0 & 30 & 0 \\ 1 & 0 & 0 & 40 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 20 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{vmatrix} = 4000 \text{ mm}^3$$

Continued

EXAMPLE 3.7 —cont'd

The constants defined by Eqs. (3.43)–(3.46) can be computed as

$$a_i = \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} = \begin{vmatrix} 20 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 40 \end{vmatrix} = 24,000$$

$$a_j = -\begin{vmatrix} x_k & y_k & z_k \\ x_l & y_l & z_l \\ x_i & y_i & z_i \end{vmatrix} = -\begin{vmatrix} 0 & 30 & 0 \\ 0 & 0 & 40 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$a_k = \begin{vmatrix} x_l & y_l & z_l \\ x_i & y_i & z_i \\ x_j & y_j & z_j \end{vmatrix} = \begin{vmatrix} 0 & 0 & 40 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{vmatrix} = 0$$

$$a_l = -\begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 0 \\ 20 & 0 & 0 \\ 0 & 30 & 0 \end{vmatrix} = 0$$

$$b_i = -\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 30 & 0 \\ 1 & 0 & 40 \end{vmatrix} = -1(1200) = -1200$$

$$b_j = \begin{vmatrix} 1 & y_k & z_k \\ 1 & y_l & z_l \\ 1 & y_i & z_i \end{vmatrix} = \begin{vmatrix} 1 & 30 & 0 \\ 1 & 0 & 40 \\ 1 & 0 & 0 \end{vmatrix} = 1200$$

$$b_k = -\begin{vmatrix} 1 & y_l & z_l \\ 1 & y_i & z_i \\ 1 & y_j & z_j \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 40 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$b_l = \begin{vmatrix} 1 & y_i & z_i \\ 1 & y_j & z_j \\ 1 & y_k & z_k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 30 & 0 \end{vmatrix} = 0$$

EXAMPLE 3.7 —cont'd

$$c_i = -\begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{vmatrix} = -\begin{vmatrix} 20 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 40 \end{vmatrix} = -\{20(40)\} = -800$$

$$c_j = \begin{vmatrix} x_k & 1 & z_k \\ x_l & 1 & z_l \\ x_i & 1 & z_i \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 40 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$c_k = -\begin{vmatrix} x_l & 1 & z_l \\ x_i & 1 & z_i \\ x_j & 1 & z_j \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 40 \\ 0 & 1 & 0 \\ 20 & 1 & 0 \end{vmatrix} = -\{40(-20)\} = 800$$

$$c_l = \begin{vmatrix} x_i & 1 & z_i \\ x_j & 1 & z_j \\ x_k & 1 & z_k \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 20 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$d_i = -\begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{vmatrix} = -\begin{vmatrix} 20 & 0 & 1 \\ 0 & 30 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\{20(30)\} = -600$$

$$d_j = \begin{vmatrix} x_k & y_k & 1 \\ x_l & y_l & 1 \\ x_i & y_i & 1 \end{vmatrix} = \begin{vmatrix} 0 & 20 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$d_k = -\begin{vmatrix} x_l & y_l & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 20 & 0 & 1 \end{vmatrix} = 0$$

$$d_l = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 20 & 0 & 1 \\ 0 & 30 & 1 \end{vmatrix} = 600$$

Thus, the shape functions of the element can be expressed, using Eq. (3.48), as

$$[N(x,y,z)] = [\,N_i(x,y,z) \quad N_j(x,y,z) \quad N_k(x,y,z) \quad N_l(x,y,z)\,] \tag{E.1}$$

Continued

EXAMPLE 3.7 —cont'd

where

$$N_i(x,y,z) = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) = \frac{1}{6(4000)}(24,000 - 1200x - 800y - 600z)$$
$$= 1 - 0.05x - 0.0333y - 0.025z$$

$$N_j(x,y,z) = \frac{1}{6V}(a_j + b_j x + c_j y + d_j z) = \frac{1}{6(4000)}(1200x) = 0.05x$$

$$N_k(x,y,z) = \frac{1}{6V}(a_k + b_k x + c_k y + d_k z) = \frac{1}{6(4000)}(800y) = 0.0333y$$

$$N_l(x,y,z) = \frac{1}{6V}(a_l + b_l x + c_l y + d_l z) = \frac{1}{6(4000)}(600z) = 0.025z$$

Thus, the temperature distribution inside the element is given by Eq. (3.47):

$$T(x,y,z) = N_i T_i + N_j T_j + N_k T_k + N_l T_l$$
$$= (1 - 0.05x - 0.0333y - 0.025z)100 + (0.05x)80 + (0.0333y)120 + (0.025z)50$$
$$= 100 - x + 0.6667y - 1.25z°\text{C}$$

3.7.4 C^0 Continuity

The one-, two-, and three-dimensional simplex elements considered in this section satisfy the following two properties that imply C^0 continuity:

1. The shape function corresponding to any specific node, such as node i, varies linearly from a value of 1 at that node i to a value of 0 at each of the remaining nodes of the element. Thus, the shape function N_i will have a value of 1 at node i and a value of 0 at each of the remaining nodes of the element.
2. The sum of all the shape functions at any point within the element, including its boundaries, will be equal to 1.

EXAMPLE 3.8

Show that the sum of the shape functions of a three-noded triangular element is equal to one at any point in the element.

Approach: Find the sum of the expressions of the shape functions.

Solution

Using Eq. (3.35), the sum of the shape functions of the triangular element ijk can be expressed as

$$\begin{aligned} N_i(x,y) + N_j(x,y) + N_k(x,y) &= \frac{1}{2A}(a_i + b_i x + c_i y) + \frac{1}{2A}(a_j + b_j x + c_j y) + \frac{1}{2A}(a_k + b_k x + c_k y) \\ &= \frac{1}{2A}(a_i + a_j + a_k + \{b_i + b_j + b_k\}x + \{c_i + c_j + c_k\}y) \end{aligned} \tag{E.1}$$

Using the expressions given in Eq. (3.32), Eq. (E.1) can be rewritten as

$$N_i(x,y) + N_j(x,y) + N_k(x,y) = \frac{1}{2A}\left\{\begin{array}{l} (x_j y_k - x_k y_j + x_k y_i - x_i y_k + x_i y_j - x_j y_i) \\ +(y_j - y_k + y_k - y_i + y_i - y_j)x \\ +(x_k - x_j + x_i - x_k + x_j - x_i)y \end{array}\right\} \tag{E.2}$$

EXAMPLE 3.8 —cont'd

$$N_i(x,y) + N_j(x,y) + N_k(x,y) = \frac{1}{2A}\{(x_j y_k - x_k y_j + x_k y_i - x_i y_k + x_i y_j - x_j y_i)\} \tag{E.3}$$

Noting that the expression in the parenthesis is equal to $2A$ (see Eq. 3.31), the sum of the shape functions can be seen to be equal to 1:

$$N_i(x,y) + N_j(x,y) + N_k(x,y) = \frac{1}{2A}\{2A\} = 1 \tag{E.4}$$

3.8 INTERPOLATION POLYNOMIALS FOR VECTOR QUANTITIES

In Eqs. (3.21), (3.28), and (3.39), the field variable ϕ has been assumed to be a scalar quantity. In some problems the field variable may be a vector quantity having both magnitude and direction (e.g., displacement in solid mechanics problems). In such cases, the usual procedure is to resolve the vector into components parallel to the coordinate axes and treat these components as the unknown quantities. Thus, there will be more than one unknown (degree of freedom) at a node in such problems. The number of degrees of freedom at a node will be one, two, or three depending on whether the problem is one-, two-, or three-dimensional. The notation used in this book for the vector components is shown in Fig. 3.11. All the

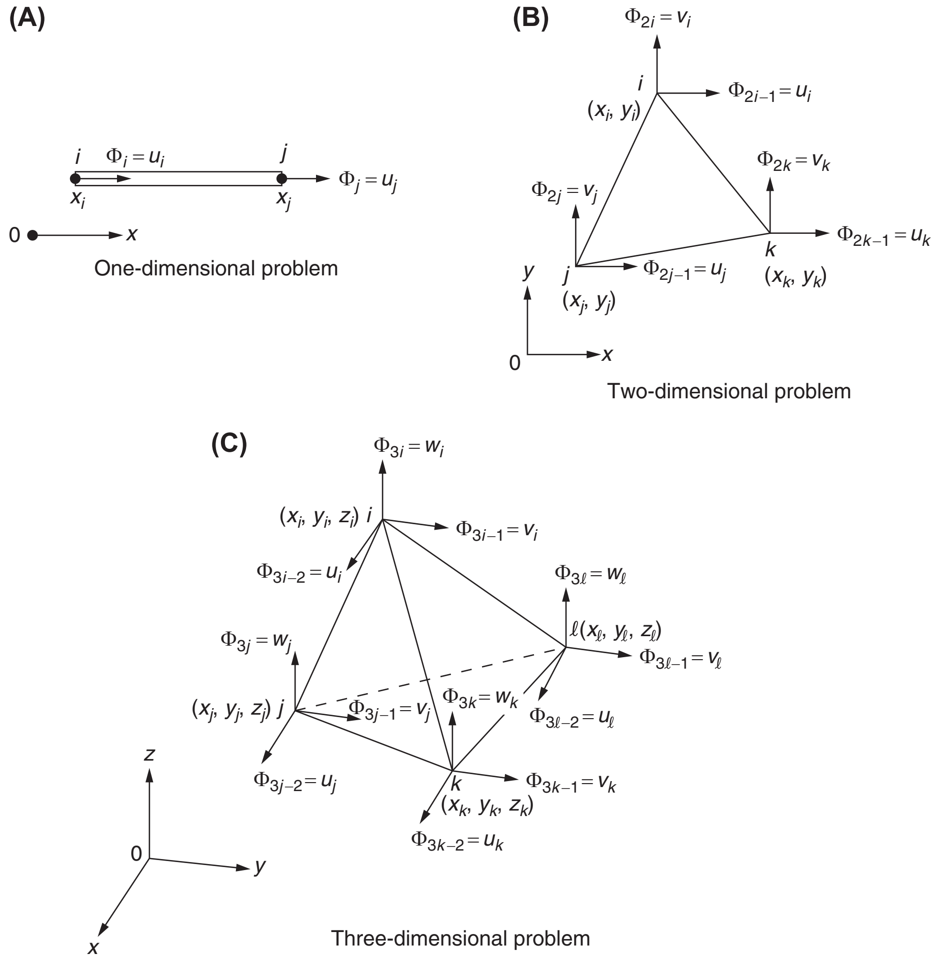

FIGURE 3.11 Nodal degrees of freedom when the field variable is a vector.

components are designated by the same symbol, Φ, with a subscript denoting the individual components. The subscripts, at any node, are ordered in the sequence x, y, z starting with the x component. The x, y, and z components of the vector quantity (field variable) ϕ are denoted by u, v, and w, respectively.

The interpolation function for a vector quantity in a one-dimensional element will be the same as that of a scalar quantity since there is only one unknown at each node. Thus,

$$u(x) = N_i(x)\Phi_i + N_j(x)\Phi_j = [N(x)]\vec{\Phi}^{(e)} \tag{3.50}$$

where

$$[N(x)] = [N_i(x) N_j(x)],$$

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \end{Bmatrix},$$

and u is the component of ϕ (e.g., displacement) parallel to the axis of the element that is assumed to coincide with the x axis. The shape functions $N_i(x)$ and $N_j(x)$ are the same as those given in Eq. (3.26).

For a two-dimensional triangular (simplex) element, the linear interpolation model of Eq. (3.33) will be valid for each of the components of ϕ, namely, u and v. Thus,

$$u(x,y) = N_i(x,y)\Phi_{2i-1} + N_j(x,y)\Phi_{2j-1} + N_k(x,y)\Phi_{2k-1} \tag{3.51}$$

and

$$v(x,y) = N_i(x,y)\Phi_{2i} + N_j(x,y)\Phi_{2j} + N_k(x,y)\Phi_{2k} \tag{3.52}$$

where N_i, N_j, and N_k are the same as those defined in Eq. (3.35); Φ_{2i-1}, Φ_{2j-1}, and Φ_{2k-1} are the nodal values of u (component of ϕ parallel to the x axis); and Φ_{2i}, Φ_{2j}, and Φ_{2k} are the nodal values of v (component of ϕ parallel to the y axis). Eqs. (3.51) and (3.52) can be written in matrix form as

$$\vec{\phi}(x,y) = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N(x,y)]\vec{\Phi}^{(e)} \tag{3.53}$$

where

$$[N(x,y)] = \begin{bmatrix} N_i(x,y) & 0 & N_j(x,y) & 0 & N_k(x,y) & 0 \\ 0 & N_i(x,y) & 0 & N_j(x,y) & 0 & N_k(x,y) \end{bmatrix} \tag{3.54}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_{2i-1} \\ \Phi_{2i} \\ \Phi_{2j-1} \\ \Phi_{2j} \\ \Phi_{2k-1} \\ \Phi_{2k} \end{Bmatrix} = \text{vector of nodal degrees of freedom} \tag{3.55}$$

Extending this procedure to three dimensions, we obtain for a tetrahedron (simplex) element,

$$\vec{\phi}(x,y,z) = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = [N(x,y,z)]\vec{\Phi}^{(e)} \tag{3.56}$$

where

$$
[N(x,y,z)] = \left[\begin{array}{cccc}
N_i(x,y,z) & 0 & 0 & N_j(x,y,z) \\
0 & N_i(x,y,z) & 0 & 0 \\
0 & 0 & N_i(x,y,z) & 0 \\
0 & 0 & N_k(x,y,z) & 0 \\
N_j(x,y,z) & 0 & 0 & N_k(x,y,z) \\
0 & N_j(x,y,z) & 0 & 0 \\
0 & N_l(x,y,z) & 0 & 0 \\
0 & 0 & N_l(x,y,z) & 0 \\
N_k(x,y,z) & 0 & 0 & N_l(x,y,z)
\end{array}\right] \tag{3.57}
$$

$$
\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_{3i-2} \\ \Phi_{3i-1} \\ \Phi_{3i} \\ \Phi_{3j-2} \\ \Phi_{3j-1} \\ \Phi_{3j} \\ \Phi_{3k-2} \\ \Phi_{3k-1} \\ \Phi_{3k} \\ \Phi_{3l-2} \\ \Phi_{3l-1} \\ \Phi_{3l} \end{Bmatrix} \tag{3.58}
$$

and the shape functions N_i, N_j, N_k, and N_l are the same as those defined in Eq. (3.48).

EXAMPLE 3.9

The nodal coordinates of a triangular plate element subjected to in-plane loads follow:

Node i: $(x_i, y_i) = (10, 10)$ mm

Node j: $(x_j, y_j) = (20, 50)$ mm

Node k: $(x_k, y_k) = (-10, 30)$ mm

The in-plane displacement components of the nodes are given by

$$(u_i v_i) = (1, -1)\text{ mm}, (u_j, v_j) = (-2, 1)\text{ mm}, (u_k, v_k) = (0.5, -0.5)\text{ mm}$$

Express the variations of $u(x, y)$ and $v(x, y)$ in the element in terms of the shape functions.

Approach: Express the linear variations of $u(x, y)$ and $v(x, y)$ in the element using Eqs. (3.51) and (3.52).

Solution

The shape functions of the element can be derived as (see Problem 3.56)

$$N_i(x,y) = 1.1 + 0.02x - 0.03y \tag{E.1}$$

$$N_j(x,y) = -0.4 + 0.02x + 0.02y \tag{E.2}$$

Continued

EXAMPLE 3.9 —cont'd

$$N_k(x,y) = 0.3 - 0.04x + 0.01y \tag{E.3}$$

The variation of the u-displacement inside the element can be expressed, using Eq. (3.51), as:

$$\begin{aligned} u(x,y) &= (1.1 + 0.02x - 0.03y)u_i + (-0.4 + 0.02x + 0.02y)u_j + (0.3 - 0.04x + 0.01y)u_k \\ &= (1.1 + 0.02x - 0.03y)(1) + (-0.4 + 0.02x + 0.2y)(-2) + (0.3 - 0.04x + 0.01y)(0.5) \\ &= 2.05 - 0.04x - 0.065y \end{aligned} \tag{E.4}$$

Similarly, the variation of the v-displacement inside the element can be expressed, using Eq. (3.52), as:

$$\begin{aligned} v(x,y) &= (1.1 + 0.02x - 0.03y)v_i + (-0.4 + 0.02x + 0.02y)v_j + (0.3 - 0.04x + 0.01y)v_k \\ &= (1.1 + 0.02x - 0.03y)(-1) + (-0.4 + 0.02x + 0.02y)(1) + (0.3 - 0.04x + 0.01y)(-0.5) \\ &= -1.65 + 0.02x + 0.045y \end{aligned} \tag{E.5}$$

EXAMPLE 3.10

The global nodal coordinates of a triangular plate element used in the stress analysis of a plate subjected to in-plane loads are given by $(x_i, y_i) = (50, 30)$ in, $(x_j, y_j) = (70, 50)$ in, and $(x_k, y_k) = (55, 60)$ in. If $u(x, y)$ and $v(x, y)$ are the field variables with nodal unknowns given by $(\Phi_{2i-1}, \Phi_{2i}) = (u_i, v_i)$, $(\Phi_{2j-1}, \Phi_{2j}) = (u_j, v_j)$ and $(\Phi_{2k-1}, \Phi_{2k}) = (u_k, v_k)$, find the matrix of shape functions of the element.

Solution

The matrix of shape functions, $[N(x, y)]$, is given by Eq. (3.54) with $N_i(x, y)$, $N_j(x, y)$, and $N_k(x, y)$ defined by Eq. (3.35). Using the given nodal coordinates, the shape functions are given by (see Example 3.4):

$$\begin{aligned} N_i(x,y) &= 2.9 - 0.02x - 0.03y \\ N_j(x,y) &= -2.7 + 0.06x - 0.01y \\ N_k(x,y) &= 0.8 - 0.04x + 0.04y \end{aligned}$$

3.9 LINEAR INTERPOLATION POLYNOMIALS IN TERMS OF LOCAL COORDINATES

The derivation of element characteristic matrices and vectors involves the integration of the shape functions or their derivatives or both over the element. These integrals can be evaluated easily if the interpolation functions are written in terms of a local coordinate system that is defined separately for each element.

In this section, we derive the interpolation functions of simplex elements in terms of a particular type of local coordinate systems, known as natural coordinate systems. A *natural coordinate system* is a local coordinate system that permits the specification of any point inside the element by a set of nondimensional numbers whose magnitude lies between 0 and 1. Usually, natural coordinate systems are chosen such that some of the natural coordinates will have unit magnitude at primary[4] or corner nodes of the element.

3.9.1 One-Dimensional Element

The natural coordinates for a one-dimensional (line) element are shown in Fig. 3.12. Any point P inside the element is identified by two natural coordinates, L_1 and L_2, which are defined as

4. The nodes located at places other than at corners (e.g., mid-side nodes and interior nodes) are called secondary nodes.

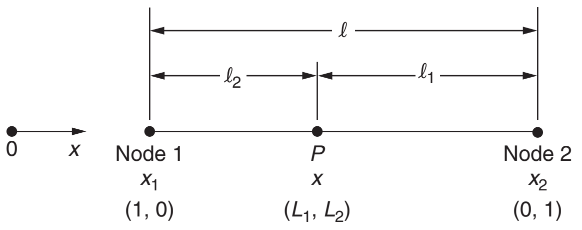

FIGURE 3.12 Natural coordinates for a line element.

$$L_1 = \frac{l_1}{l} = \frac{x_2 - x}{x_2 - x_1}$$
$$L_2 = \frac{l_2}{l} = \frac{x - x_1}{x_2 - x_1} \tag{3.59}$$

where l_1 and l_2 are the distances shown in Fig. 3.12, and l is the length of the element. Since it is a one-dimensional element, there should be only one independent coordinate to define any point P. This is true even with natural coordinates because the two natural coordinates L_1 and L_2 are not independent but are related as

$$L_1 + L_2 = \frac{l_1}{l} + \frac{l_2}{l} = 1 \tag{3.60}$$

A study of the properties of L_1 and L_2 reveals something quite interesting. The natural coordinates L_1 and L_2 are also the shape functions for the line element (compare Eq. (3.59) with Eq. (3.26)). Thus,

$$N_i = L_1, \quad N_j = L_2 \tag{3.61}$$

Any point x within the element can be expressed as a linear combination of the nodal coordinates of nodes 1 and 2 as

$$x = x_1 L_1 + x_2 L_2 \tag{3.62}$$

where L_1 and L_2 may be interpreted as weighting functions. Thus, the relationship between the natural and the Cartesian coordinates of any point P can be written in matrix form as

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} \tag{3.63}$$

or

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{l} \begin{Bmatrix} x_2 & -1 \\ -x_1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \tag{3.64}$$

If f is a function of L_1 and L_2, differentiation of f with respect to x can be performed, using the chain rule, as

$$\frac{df}{dx} = \frac{\partial f}{\partial L_1} \frac{\partial L_1}{\partial x} + \frac{\partial f}{\partial L_2} \frac{\partial L_2}{\partial x} \tag{3.65}$$

where, from Eq. (3.59),

$$\frac{\partial L_1}{\partial x} = -\frac{1}{x_2 - x_1} \quad \text{and} \quad \frac{\partial L_2}{\partial x} = \frac{1}{x_2 - x_1} \tag{3.66}$$

Integration of polynomial terms in natural coordinates can be performed by using the simple formula

$$\int_{x_1}^{x_2} L_1^{\alpha} L_2^{\beta} \, \mathrm{d}x = \frac{\alpha! \beta!}{(\alpha + \beta + 1)!} l \tag{3.67}$$

where $\alpha!$ is the factorial of α given by $\alpha! = \alpha\ (\alpha - 1)\ (\alpha - 2) \ldots$ [1]. The value of the integral in Eq. (3.67) is given for certain combinations of α and β in Table 3.2.

TABLE 3.2 Value of the Integral in Eq. (3.67)

Value of		Value of the Integral in Eq. (3.67)/*l*
α	β	
0	0	1
1	0	1/2
1	1	1/6
2	0	1/3
1	2	1/12
3	0	1/4
4	0	1/5
2	2	1/30
3	1	1/20
1	4	1/30
3	2	1/60
5	0	1/6

3.9.1.1 Two-Dimensional (Triangular) Element

A natural coordinate system for a triangular element (also known as the *triangular coordinate system*) is shown in Fig. 3.13A. Although three coordinates L_1, L_2, and L_3 are used to define a point P, only two of them are independent. The natural coordinates are defined as

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A} \tag{3.68}$$

where A_1 is the area of the triangle formed by the points P, 2 and 3; A_2 is the area of the triangle formed by the points P, 1 and 3; A_3 is the area of the triangle formed by the points P, 1 and 2; and A is the area of the triangle 123 in Fig. 3.13. Because L_i are defined in terms of areas, they are also known as area coordinates. Since

$$A_1 + A_2 + A_3 = A$$

we have

$$\frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = L_1 + L_2 + L_3 = 1 \tag{3.69}$$

A study of the properties of L_1, L_2, and L_3 shows that they are also the shape functions for the two-dimensional simplex (triangular) element:

$$N_i = L_1, \quad N_j = L_2, \quad N_K = L_3 \tag{3.70}$$

The relation between the natural and Cartesian coordinates is given by (see Problem 3.8)

$$\left.\begin{aligned} x &= x_1 L_1 + x_2 L_2 + x_3 L_3 \\ y &= y_1 L_1 + y_2 L_2 + y_3 L_3 \end{aligned}\right\} \tag{3.71}$$

To every set of natural coordinates (L_1, L_2, L_3) (which are not independent but are related by Eq. 3.69), there corresponds a unique set of Cartesian coordinates (x, y). At node 1, $L_1 = 1$ and $L_2 = L_3 = 0$, and so on. The linear relationship between L_i ($i = 1, 2, 3$) and (x, y) implies that the contours of L_1 are equally placed straight lines parallel to the side 2, 3 of the triangle (on which $L_1 = 0$), and so on as shown in Fig. 3.13B.

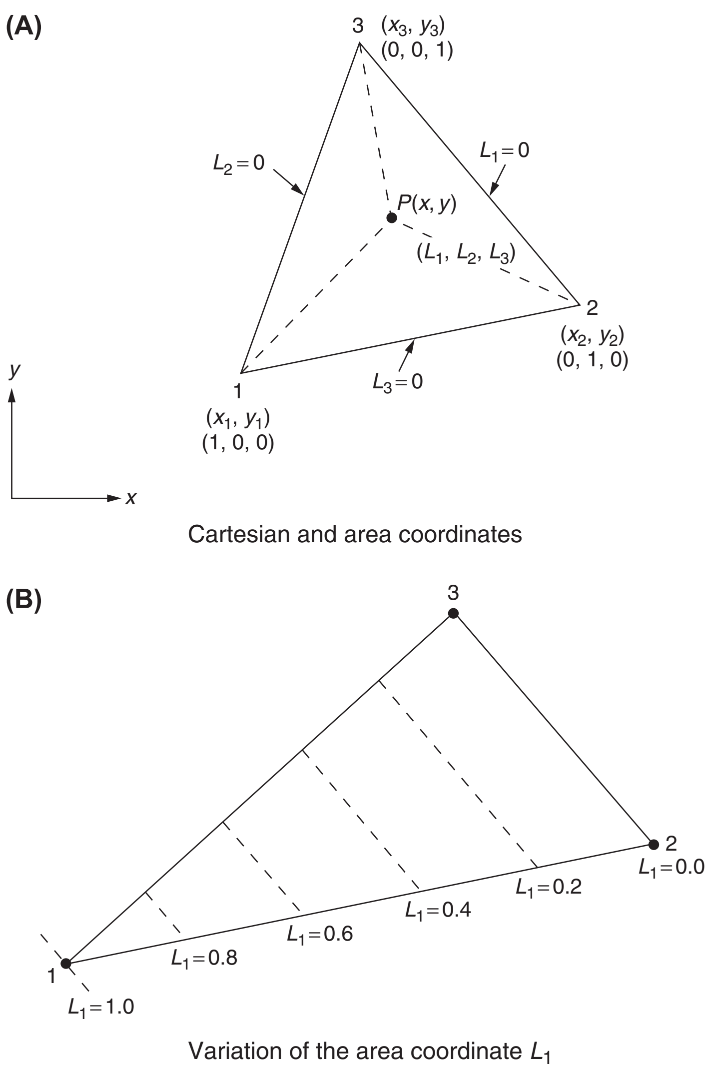

FIGURE 3.13 Area coordinates for a triangular element.

Eqs. (3.69) and (3.71) can be expressed in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \tag{3.72}$$

Eq. (3.72) can be inverted to obtain

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3y_1 - x_1y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1y_2 - x_2y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \tag{3.73}$$

where A is the area of the triangle 1, 2, 3 given by

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{3.74}$$

Note that Eq. (3.73) is identical to Eq. (3.35).

If f is a function of L_1, L_2, and L_3, the differentiation with respect to x and y can be performed as

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x}\\ \frac{\partial f}{\partial y} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y}\end{aligned}\right\} \tag{3.75}$$

where

$$\left.\begin{aligned}\frac{\partial L_1}{\partial x} &= \frac{y_2 - y_3}{2A}, & \frac{\partial L_1}{\partial y} &= \frac{x_3 - x_2}{2A}\\ \frac{\partial L_2}{\partial x} &= \frac{y_3 - y_1}{2A}, & \frac{\partial L_2}{\partial y} &= \frac{x_1 - x_3}{2A}\\ \frac{\partial L_3}{\partial x} &= \frac{y_1 - y_2}{2A}, & \frac{\partial L_3}{\partial y} &= \frac{x_2 - x_1}{2A}\end{aligned}\right\} \tag{3.76}$$

For integrating polynomial terms in natural coordinates, we can use the relations

$$\int_L L_1^{\alpha} L_2^{\beta} \cdot d\mathscr{L} = \frac{\alpha!\beta!}{(\alpha+\beta+1)!}\mathscr{L} \tag{3.77}$$

and

$$\iint_A L_1^{\alpha} L_2^{\beta} L_3^{\gamma} \cdot \mathrm{d}A = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}2A \tag{3.78}$$

Eq. (3.77) is used to evaluate an integral that is a function of the length along an edge of the element. Thus, the quantity $\mathscr{L}$ denotes the distance between the two nodes that define the edge under consideration. Eq. (3.78) is used to evaluate area integrals. Table 3.3 gives the values of the integral for various combinations of α, β, and γ.

TABLE 3.3 Values of the Integrals in Eqs. (3.77) and (3.78)

Value of			Value of the Integral in Eq. (3.77)/$\mathscr{L}$	Value of the Integral in Eq. (3.78)/A
α	β	γ		
0	0	0	1	1
1	0	0	1/2	1/3
2	0	0	1/3	1/6
1	1	0	1/6	1/12
3	0	0	1/4	1/10
2	1	0	1/12	1/30
1	1	1	—	1/60
4	0	0	1/5	1/15
3	1	0	1/20	1/60
2	2	0	1/30	1/90
2	1	1	—	1/180
5	0	0	1/6	1/21
4	1	0	1/30	1/105
3	2	0	1/60	1/210
3	1	1	—	1/420
2	2	1	—	1/630

EXAMPLE 3.11

Show that the natural (area) coordinate L_i ($i = 1, 2, 3$) is the same as the shape function N_i given by Eq. (3.35).

Solution

The area coordinate L_1, defined as the ratio of the area of the shaded triangle to the total area of the triangle *ijk* shown in Fig. 3.14, can be expressed as

$$L_1 = \frac{A_1}{A} = \frac{\frac{1}{2}b\,d}{\frac{1}{2}b\,h} = \frac{d}{h} \tag{E.1}$$

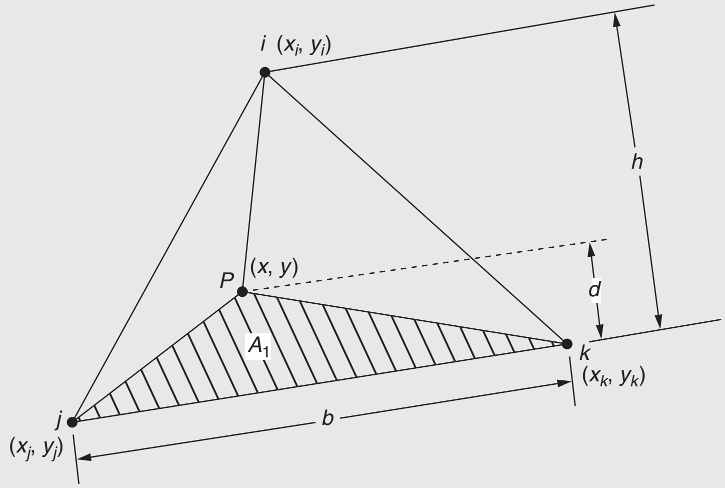

FIGURE 3.14 Area coordinates.

where d and h denote the distances of the perpendiculars from the points P and i to the base jk of the triangle. The area A_1 of the triangle Pjk can be determined in terms of the coordinates of P, j, and k as

$$2A_1 = \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = x_j y_k - x_k y_j + x(y_j - y_k) + y(x_k - x_j) \tag{E.2}$$

Eqs. (E.1) and (E.2) lead to

$$L_1 = \frac{2A_1}{2A} = \frac{1}{2A}\{x_j y_k - x_k y_j + x(y_j - y_k) + y(x_k - x_j)\} \tag{E.3}$$

which can be seen to be identical to the shape function N_i given by Eq. (3.35). This shows that the area coordinates of a linear triangular element are identical to the shape functions.

3.9.3 Three-Dimensional (Tetrahedron) Element

The natural coordinates for a tetrahedron element can be defined analogous to those of a triangular element. Thus, four coordinates L_1, L_2, L_3, and L_4 will be used to define a point P, although only three of them are independent. These natural coordinates are defined as

$$L_1 = \frac{V_1}{V}, \quad L_2 = \frac{V_2}{V}, \quad L_3 = \frac{V_3}{V}, \quad L_4 = \frac{V_4}{V} \tag{3.79}$$

where V_i is the volume of the tetrahedron formed by the points P and the vertices other than the vertex i ($i = 1, 2, 3, 4$), and V is the volume of the tetrahedron element defined by the vertices 1, 2, 3, and 4 (Fig. 3.15). Because the natural coordinates are defined in terms of volumes, they are also known as *volume* or *tetrahedral coordinates*. Since

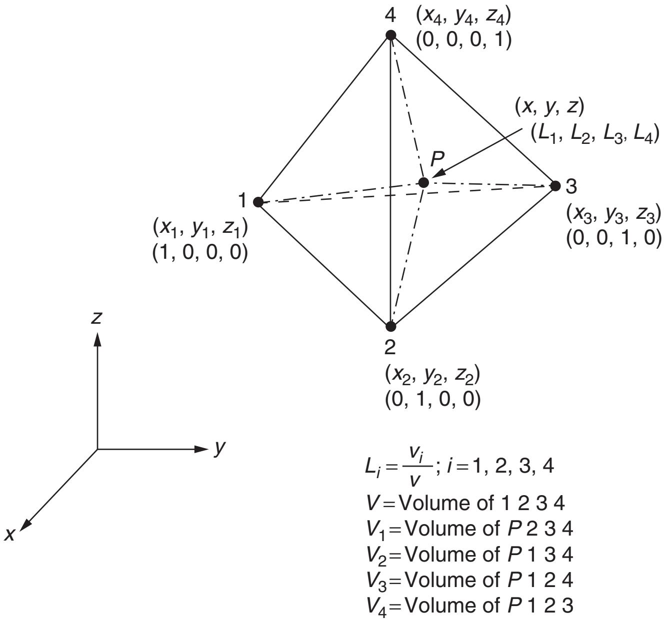

FIGURE 3.15 Volume coordinates for a tetrahedron element.

$$V_1 + V_2 + V_3 + V_4 = V$$

we obtain

$$\frac{V_1}{V} + \frac{V_2}{V} + \frac{V_3}{V} + \frac{V_4}{V} = L_1 + L_2 + L_3 + L_4 = 1 \tag{3.80}$$

The volume coordinates L_1, L_2, L_3, and L_4 are also the shape functions for a three-dimensional simplex element:

$$N_i = L_1, \quad Nj = L_2, \quad N_k = L_3, \quad N_l = L_4 \tag{3.81}$$

The Cartesian and natural coordinates are related as

$$\left.\begin{aligned} x &= L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4 \\ y &= L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4 \\ z &= L_1z_1 + L_2z_2 + L_3z_3 + L_4z_4 \end{aligned}\right\} \tag{3.82}$$

Eqs. (3.80) and (3.82) can be expressed in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \tag{3.83}$$

The inverse relations can be expressed as

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \tag{3.84}$$

where

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \text{volume of the tetrahedron } 1,2,3,4 \tag{3.85}$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \tag{3.86}$$

$$b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix} \tag{3.87}$$

$$c_1 = -\begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix} \tag{3.88}$$

$$d_1 = -\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \tag{3.89}$$

and the other constants are obtained through a cyclic permutation of subscripts 1, 2, 3, and 4. These constants are the cofactors of the terms in the determinant of Eq. (3.85) and hence it is necessary to give proper signs to them. If the tetrahedron element is defined in a right-handed Cartesian coordinate system as shown in Fig. 3.15, Eqs. (3.86) to (3.89) are valid only when the nodes 1, 2, and 3 are numbered in a counterclockwise manner when viewed from node 4.

An alternate method of showing the equivalency of L_i and N_i is indicated in the following example.

EXAMPLE 3.12

Show that the natural coordinate L_i is same as the shape function N_i given by Eq. (3.48).

Approach: Use the definition of the volume of a tetrahedron.

Solution

Using the definition of L_i given in Eq. (3.79), we have

$$L_i = \frac{\text{Volume of tetrahedron } pjkl}{\text{Volume of tetrahedron } ijkl} = \frac{V_i}{V} \tag{E.1}$$

where the volume V_i can be expressed as (from Fig. 3.16):

$$V_i = \frac{1}{6}\begin{vmatrix} 1 & x & y & z \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix}$$

$$= \frac{1}{6}\left\{ 1\begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} - x\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} + y\begin{vmatrix} 1 & x_j & z_j \\ 1 & x_k & z_k \\ 1 & x_l & z_l \end{vmatrix} - z\begin{vmatrix} 1 & x_j & y_j \\ 1 & x_k & y_k \\ 1 & x_l & y_l \end{vmatrix} \right\} \tag{E.2}$$

Continued

EXAMPLE 3.12 —cont'd

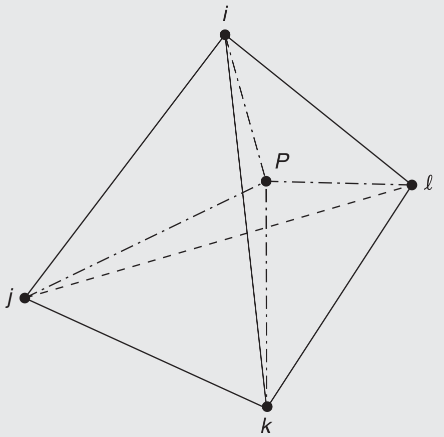

FIGURE 3.16 Tetrahedron element.

Using Eqs. (3.43) to (3.46), Eq. (E.2) can be rewritten as

$$V_i = \frac{1}{6}(a_i + b_i x + c_i y + d_i z) \tag{E.3}$$

Thus, L_i, defined by Eq. (E.1), can be expressed as

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z) \tag{E.4}$$

which can be seen to be identical to the expression of N_i given in Eq. (3.48).

If f is a function of the natural coordinates, it can be differentiated with respect to Cartesian coordinates as

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x} \\ \frac{\partial f}{\partial y} &= \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y} \\ \frac{\partial f}{\partial z} &= \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial z} \end{aligned}\right\} \tag{3.90}$$

where

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{6V}, \quad \frac{\partial L_i}{\partial y} = \frac{c_i}{6V}, \quad \frac{\partial L_i}{\partial z} = \frac{d_i}{6V} \tag{3.91}$$

The integration of polynomial terms in natural coordinates can be performed using the relation

$$\iiint_V L_1^{\alpha} L_2^{\beta} L_3^{\gamma} L_4^{\delta} \mathrm{d}V = \frac{\alpha!\beta!\gamma!\delta!}{(\alpha+\beta+\gamma+\delta+3)!} 6V \tag{3.92}$$

The values of this integral for different values of α, β, γ, and δ are given in Table 3.4.

TABLE 3.4 Value of the Integral in Eq. (3.92)

Value of				
α	β	γ	δ	Value of the Integral in Eq. (3.92)/V
0	0	0	0	1
1	0	0	0	1/4
2	0	0	0	1/10
1	1	0	0	1/20
3	0	0	0	1/20
2	1	0	0	1/60
1	1	1	0	1/120
4	0	0	0	1/35
3	1	0	0	1/140
2	2	0	0	1/210
2	1	1	0	1/420
1	1	1	1	1/840
5	0	0	0	1/56
4	1	0	0	1/280
3	2	0	0	1/560
3	1	1	0	1/1120
2	2	1	0	1/1680
2	1	1	1	1/3360

3.10 INTEGRATION OF FUNCTIONS OF NATURAL COORDINATES

The integration of polynomial terms defined in terms of natural coordinates is considered in this section. For illustration, the formula for a one-dimensional element is presented. Let the integral to be evaluated be given by

$$I = \int_{x_1}^{x_2} L_1^{\alpha}(x) L_2^{\beta}(x) \mathrm{d}x \tag{3.93}$$

where L_1 and L_2 can be expressed in terms of s as (see Fig. 3.17):

$$L_1 = \frac{l-s}{l} = 1 - \frac{s}{l}, \quad L_2 = \frac{s}{l} \tag{3.94}$$

Using

$$L_1 = 1 - \frac{s}{l} = 1 - L_2 \tag{3.95}$$

$$s = lL_2 \tag{3.96}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}L_2} = l \quad \text{or} \quad \mathrm{d}s = l\,\mathrm{d}L_2 \tag{3.97}$$

Eq. (3.93) can be rewritten as

$$I = \int_{s=0}^{l} L_1^{\alpha}(s) L_2^{\beta}(s) \mathrm{d}s = \int_{L_2=0}^{1} L_1^{\alpha} L_2^{\beta} l\, \mathrm{d}L_2 \tag{3.98}$$

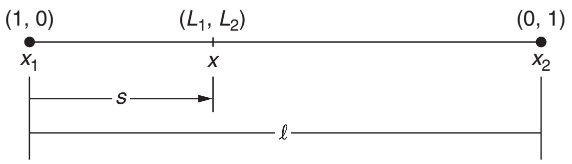

FIGURE 3.17 One-dimensional element.

Using $L_1 = 1 - L_2$, Eq. (3.98) can be written as

$$I = l\int_0^1 (1 - L_2)^\alpha L_2^\beta \mathrm{d}L_2 \tag{3.99}$$

Noting that the integral of Eq. (3.99) is in the form of the known integral [3.16]

$$\int_0^1 (1-t)^{a-1} t^{b-1}\, \mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{3.100}$$

where $\Gamma(i+1) = i!$. Thus, the integral of Eq. (3.99) becomes

$$I = l\int_0^1 (1-L_2)^\alpha L_2^\beta\, \mathrm{d}L_2 = l\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1+1)} = l\frac{\alpha!\beta!}{(\alpha+\beta+1)!} \tag{3.101}$$

Eq. (3.101) can be seen to be same as Eq. (3.67).

3.11 PATCH TEST

When a given problem is solved repeatedly using a finer mesh (smaller-size elements) each time, the convergence of the results to the exact solution (such as displacements, strains, and/or stresses) is not guaranteed unless the elements pass a test known as the *patch test*. In a standard patch test, a small domain of the problem is modeled with a number of finite elements, known as patch of elements, such that there is at least one interior node and just enough supports to prevent rigid body motion of the patch. A typical patch of quadrilateral elements is shown in Fig. 3.18. Note that the shape of the elements and the mesh is not regular or uniform because some errors in the formulation of elements may not be evident with a regular mesh. The following example illustrates the patch test for bar elements undergoing axial deformation.

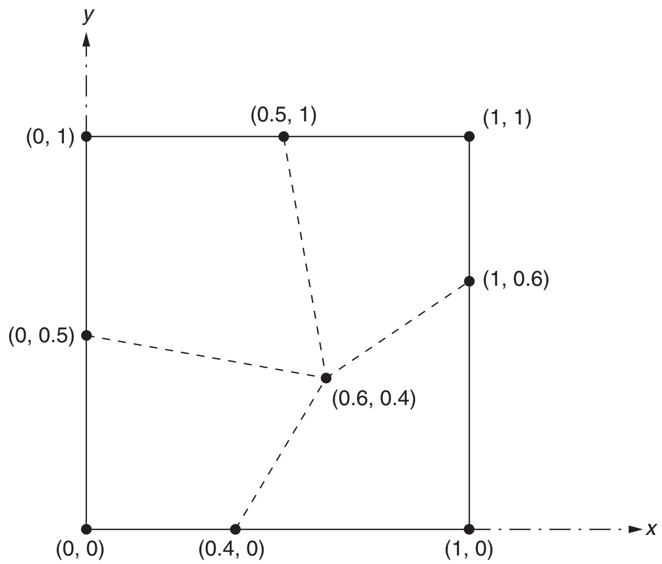

FIGURE 3.18 A patch of quadrilateral elements.

EXAMPLE 3.13

Consider a patch of two bar elements as shown in Fig. 3.19. The stiffness matrices of the elements are given by (see Eq. (E.10) of Example 1.2):

$$[K^{(1)}] = 10^6\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}, \quad [K^{(2)}] = 10^6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The system or assembled equilibrium equations can be expressed as

$$10^6\begin{bmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \tag{E.1}$$

Determine whether the element, with the shape functions $N_1(x) = 1 - \frac{x}{l} = \frac{x_{i+1}-x}{x_{i+1}-x_i}$ and $N_2(x) = \frac{x}{l} = \frac{x - x_i}{x_{i+1} - x_i}$ passes the patch test for rigid body motion and constant strain.

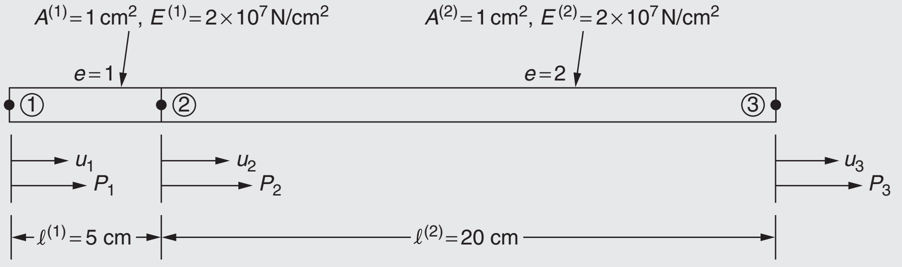

FIGURE 3.19 A patch of two bar elements.

Solution

To test the bar element for rigid body motion, first the displacements of the end nodes are specified to have the same value and then solve Eq. (E.1) and see whether the middle node has the same value of displacement. Let the end nodal displacements be specified as $u_1 = u_3 = 1$. Then Eq. (E.1) takes the form

$$10^6\begin{bmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ u_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ R_3 \end{Bmatrix} \tag{E.2}$$

where R_1 and R_3 denote the reactions (unknown) at nodes 1 and 3, respectively. The solution of the second equation of (E.2) yields

$$10^6(-4 + 5u_2 - 1) = 0 \text{ or } u_2 = 1 \tag{E.3}$$

Because $u_2 = 1$, the element passes the patch test for rigid body motion. To test the bar element for constant strain, the bar elements must undergo a linear displacement. For this, we specify the values of the displacements of end nodes as $u_1 = 0$ and $u_3 = 5$. If the displacement variation is linear (or if the strain is constant), the displacement of node 2 must be equal to 1. When the values $u_1 = 0$ and $u_3 = 5$ are specified, Eq. (E.1) takes the form

$$10^6\begin{bmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ R_3 \end{Bmatrix} \tag{E.4}$$

where R_1 and R_3 denote the reactions (unknown) at nodes 1 and 3, respectively. The solution of the second equation of (E.4) yields

$$10^6(0 + 5u_2 - 5) = 0 \text{ or } u_2 = 1 \tag{E.5}$$

Because $u_2 = 1$, the element passes the patch test for constant strain state.

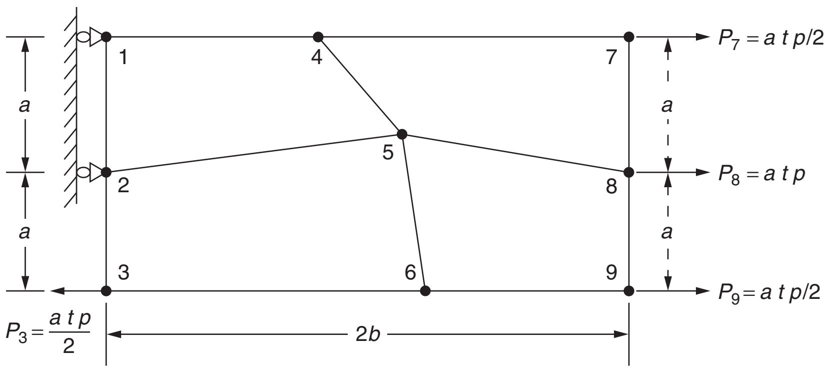

FIGURE 3.20 A patch of two-dimensional elements.

Patch Test for Two-Dimensional Elements

To illustrate the patch test for two-dimensional elements, consider the patch shown in Fig. 3.20 with four quadrilateral-shaped elements. If the patch is subjected to a uniform stress in the x-direction, p, on the right edge, the equivalent concentrated loads applied at the nodes can be taken as

$$P_3 = P_9 = \frac{atp}{2}, \quad P_6 = atp$$

where t is the thickness of the patch (in the z-direction). The nodes 1 and 2 are assumed to be pin-connected or simply supported to prevent the patch from having rigid body motion in the xy plane. If the patch test is successful, the finite element solution of the patch should give the stresses as $\sigma_x = p$, $\sigma_y = 0$, and $\tau_{xy} = 0$ at node 5 as well as at all other points where the stresses are computed.

Note that we just have described the patch test for the stress state σ_x. If the patch is subjected to a general plane stress condition (σ_x, σ_y, and τ_{xy} nonzero), we need to conduct patch tests for constant values of each of the stresses σ_x, σ_y, and τ_{xy} separately. If an element passes the required patch tests, we can be assured that the finite element model that uses this type of element will yield exact results as the mesh size is refined repeatedly.

REVIEW QUESTIONS

3.1 Give brief answers to the following questions.

1. What is an interpolation function?
2. What are primary nodes of a finite element?
3. What is p-method?
4. What is the name of the element that can be used to model curved boundaries?
5. What type of coordinate system is usually used to derive isoparametric element equations?
6. State the polynomial type of interpolation function for a one-dimensional problem.
7. What is a nodal degree of freedom?
8. What is the field variable in a thermal problem?
9. What is the field variable in a stress analysis problem?
10. What type of equations are used for finding the nodal displacements in a stress analysis problem?
11. What is Pascal triangle?
12. What is C^2 continuity within an element?
13. What is the primary purpose of using local coordinates in defining the interpolation functions?
14. Why do we need two local coordinates, $L_1 = \frac{x_2 - x}{x_2 - x_1}$ and $L_2 = \frac{x - x_1}{x_2 - x_1}$, for a one-dimensional (line) element?
15. Which local coordinates of a triangular element among L_1, L_2, and L_3 are zero along the edge 23?

3.2 Fill in the blank with a suitable word.

1. Theoretically a polynomial of order ________________ corresponds to the exact solution of any problem.
2. A linear element is one whose interpolation polynomial is of order ________.

3. The number of secondary nodes introduced plus the number of primary nodes must match the number of ______________ degrees of freedom.
4. When the order of interpolation polynomial is fixed, the discretization of the domain can be improved using the r-method ______________ h-method.
5. When the locations of the nodes are altered without changing the total number of elements, the method is known as ______ method.
6. Curved boundaries cannot be modeled using straight-sided elements for good ______________.
7. The basic idea of isoparametric elements is to use the same interpolation to ______________ the element shape as well as the variation of the field variable.
8. The quadratic interpolation function for a one-dimensional problem is ______________.
9. A ______________ element is one for which the interpolation polynomial consists of a constant and linear terms.
10. A multiplex element is one whose boundaries are ______________ to the coordinate axes.
11. Geometric isotropy is the same as ______________ invariance.
12. The requirement of continuity of the field variable within the element is automatically ______________ by polynomial interpolation functions.
13. The field variable ϕ and its partial derivatives up to the highest order appearing in the functional $I(\phi)$ must have ______________ in the interpolation polynomial.
14. The field variable ϕ and its partial derivatives up to one order less than the highest derivative appearing in the functional $I(\phi)$ must be ______________ at element boundaries.
15. For a systematic mesh refinement, all coarse meshes must be ______________ in the finer meshes.
16. The shape function $N_i(x) = \frac{x_j - x}{l}$ corresponds to a one-dimensional ______________ element.
17. Natural coordinates permit representation of the coordinates of any point within an element between 0 and ______________.
18. The sum of all the local coordinates at any point in a simplex element must be equal to ______________.
19. The natural coordinates of a tetrahedron element are also known as ______________ coordinates.
20. The convergence of a finite element to exact solution is guaranteed only if the element passes the ______________ test.

3.3 Indicate whether each of the following statements is true or false.

1. Trigonometric functions are more frequently used as interpolation functions.
2. In general, fewer higher order elements achieve the same accuracy compared to larger numbers of linear elements for the same computational effort.
3. The quartic type of polynomial interpolation function is commonly used.
4. The complex element is one that has a complex boundary.
5. The shapes of simplex elements and complex elements can be the same.
6. The boundaries of complex elements will not be parallel to the coordinate axes.
7. The number of generalized coordinates used in the interpolation polynomial can be different from the number of degrees of freedom of the element.
8. The field variable representation within an element can change with the local coordinates used.
9. Geometric isotropy and spatial isotropy are the same.
10. The interpolation function $\phi(x,y) = a_1x + a_2y + a_3xy$ satisfies geometric isotropy.
11. The interpolation function $\phi(x,y) = a_0 + a_1x + a_2x^2 + a_3y$ satisfies geometric isotropy.
12. The interpolation function $\phi(x,y) = a_0 + a_1x^2 + a_2y^2 + a_3xy)$ satisfies geometric isotropy.
13. If the interpolation polynomial does not satisfy all the convergence requirements, the results may converge to an incorrect solution.
14. Interpolation polynomials that satisfy completeness but not compatibility requirement cannot give satisfactory results.
15. To prove convergence of an interpolation polynomial, the mesh refinement can be done any convenient way.
16. The form of the interpolation polynomial can be changed during the mesh refinement process.
17. The shape function $N_i(x,y) = \frac{1}{2A}(a_i = b_ix + c_iy)$ corresponds to a two-dimensional complex element.
18. Linear interpolation polynomials can be stated using global or local coordinates.
19. The natural coordinates are local coordinates.
20. The natural coordinates of a triangular element can be called area coordinates.

3.4 Select the most appropriate answer from the multiple choices given.

1. Polynomial type of interpolation functions are commonly used because they permit:
(a) More accurate solutions
(b) Easier computerization of finite element equations
(c) Quicker solutions

2. An element with interpolation polynomial of order 2 or more is called:
(a) Quadratic element (b) More accurate element (c) Higher order element

3. Secondary nodes are those whose locations are:
(a) Along the sides of the element
(b) At the corner points of the element
(c) Mid-side and/or interior of the element

4. Higher order elements are desirable when:
(a) Gradients of the field variable varies rapidly
(b) Use of linear elements is not possible
(c) Reduced computational effort is required

5. When locations of nodes are changed by fixing the number of elements, the method is known as:
(a) h-method (b) p-method (c) mixed method

6. Isoparametric elements are extensively used for the analysis of:
(a) Plates (b) Shells (c) Beams

7. The shape of a simplex element in two dimensions is:
(a) Rectangle (b) Triangle (c) Quadrilateral

8. The number of nodes of a complex element compared to ac simplex element is:
(a) Same (b) Less (c) More

9. Pascal pyramid can be defined for:
(a) One-dimensional problems (b) Two-dimensional problems (c) Three-dimensional problems

10. If ϕ has C^r continuity within the element and C^{r-1} continuity at element interfaces, the element is considered to satisfy the following requirement:
(a) Compatibility (b) Continuity (c) Both compatibility and continuity

3.5 Match the following.

1. h-method	(a) move node points in fixed element topology
2. p-method	(b) subdivide selected elements
3. r-method	(c) change order of polynomial

3.6 Match the function to the corresponding element type.

1. $\phi(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$	(a) complex element
2. $\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$	(b) simplex element
3. $\phi(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z$	(c) multiplex element

3.7 Match the following for a three-dimensional tetrahedron element with nodes 1, 2, 3, and 4.

1. $L_1 = L_3 = L_4 = 0,\ L_2 = 1$	(a) on face 124
2. $L_3 = 0$	(b) along edge 24
3. $L_1 = L_3 = 0,\ L_2, L_4 \neq 0$	(c) along edge 13
4. $L_1 + L_3 = 1,\ L_2 = L_4 = 0$	(d) node 2

PROBLEMS

3.1 What kind of interpolation model would you propose for the field variable ϕ for the six-node rectangular element shown in Fig. 3.21? Discuss how the various considerations given in Section 3.5 are satisfied.

3.2 A one-dimensional simplex element has been used to find the temperature distribution in a straight fin. It is found that the nodal temperatures of the element are 140°C and 100°C at nodes i and j, respectively. If the nodes i and j are located 2 cm and 8 cm from the origin, find the temperature at a point 5 cm from the origin. Also find the temperature gradient inside the element.

3.3 Two-dimensional simplex elements have been used for modeling a heated flat plate. The (x, y) coordinates of nodes i, j, and k of an interior element are given by (5, 4) cm, (8, 6) cm, and (4, 8) cm, respectively. If the nodal temperatures are found to be $T_i = 100°\text{C}$, $T_j = 80°\text{C}$, and $T_k = 110°\text{C}$, find (a) the temperature gradients inside the element and (b) the temperature at point P located at $(x_p, y_p) = (6, 5)$ cm.

3.4 Three-dimensional simplex elements are used to find the pressure distribution in a fluid medium. The (x, y, z) coordinates of nodes i, j, k, and l of an element are given by (2, 4, 2) in, (0, 0, 0) in, (4, 0, 0) in, and (2, 0, 6) in. Find the shape functions N_i, N_j, N_k, and N_l of the element.

3.5 Show that the condition to be satisfied for constant value of the field variable is $\sum_{i=1}^{r} N_i = 1$, where N_i denotes the shape function corresponding to node i and r represents the number of nodes in the element.

3.6 Triangular elements are used for the stress analysis of a plate subjected to in-plane loads. The components of displacement parallel to (x, y) axes at the nodes i, j, and k of an element are found to be (−0.001, 0.01) cm, (−0.002, 0.01) cm, and (−0.002, 0.02) cm, respectively. If the (x,y) coordinates of the nodes shown in Fig. 3.22 are in centimeters, find (a) the distribution of the (x,y) displacement components inside the element and (b) the components of displacement of the point $(x_p, y_p) = (30, 25)$ cm.

3.7 The temperatures at the corner nodes of a rectangular element, in degrees centigrade, are $T_i = 90$, $T_j = 84$, $T_k = 75$, and $T_l = 85$. If the length and width of the element are $x_{ij} = 15$ mm, and $y_{il} = 10$ mm, and the conduction coefficient of the material is $k = 42.5$ W/m-°C, determine the following:

a. Temperature distribution in the element

b. Heat flow rates in x and y directions (q_x and q_y) using the relation

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -k \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix}$$

3.8 Derive the relationship between the natural (area) and Cartesian coordinates of a triangular element (Eq. 3.71).

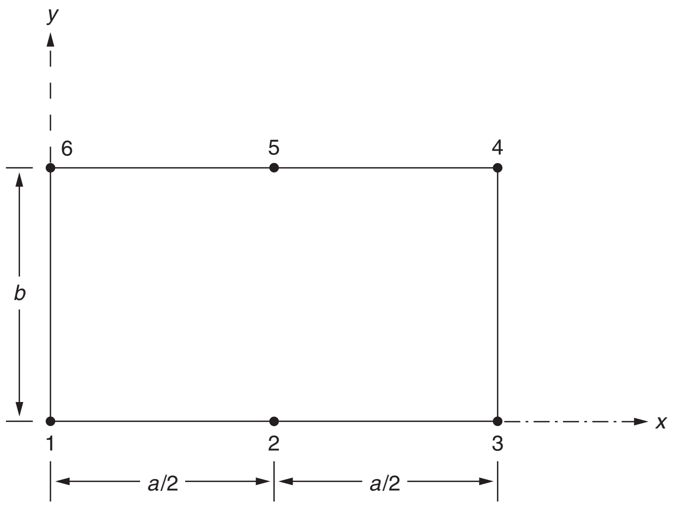

FIGURE 3.21 A six-node rectangular element.

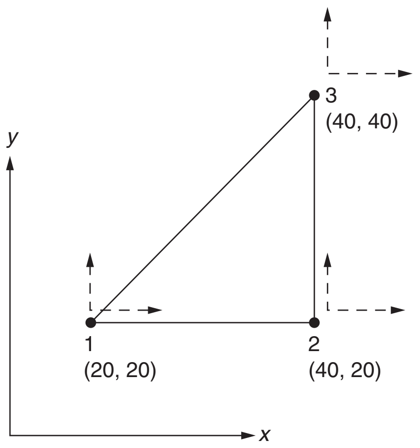

FIGURE 3.22 A triangular element.

3.9 The quadratic interpolation function of a one-dimensional element with three nodes is given by

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

If the x coordinates of nodes 1, 2, and 3 are given by 1, 3, and 5 in, respectively, determine the matrices $\left[\underset{\sim}{\eta}\right]$, $\left[\underset{\sim}{\eta}\right]^{-1}$, and $[N]$ of Eqs. (3.17), (3.18), and (3.20).

3.10 The cubic interpolation function for the displacement of a beam element is expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

with the nodal degrees of freedom defined as $\phi_1 = \phi(x = x_1)$, $\phi_2 = (\mathrm{d}\phi/\mathrm{d}x)\,(x = x_1)$, $\phi_3 = \phi(x = x_2)$, and $\phi_4 = (\mathrm{d}\phi/\mathrm{d}x)$ $(x = x_2)$, where x_1 and x_2 denote the x coordinates of nodes 1 and 2 of the element. If $x_1 = 1.0$ in and $x_2 = 6$ in, determine the matrices $\left[\underset{\sim}{\eta}\right]$, $\left[\underset{\sim}{\eta}\right]^{-1}$, and $[N]$ of Eqs. (3.17), (3.18), and (3.20).

3.11 The transverse displacement of a triangular bending element (w) is expressed as

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 \left(x^2 y + xy^2\right) + \alpha_9 y^3$$

The nodal degrees of freedom are defined as $\phi_i = w(x_i, y_i)$, $\phi_{i+3} = (\phi w/\phi y)\,(x_i, y_i)$, $\phi_{i+6} = (\phi w/\phi x)\,(x_i, y_i)$; $i = 1, 2, 3$, where (x_i, y_i) denote the coordinates of node i. If the (x, y) coordinates of nodes 1, 2, and 3 are given by (0, 0), (0, 5), and (10, 0), respectively, determine the matrices $\left[\underset{\sim}{\eta}\right]$, $\left[\underset{\sim}{\eta}\right]^{-1}$ and $[N]$ of Eqs. (3.17), (3.18), and (3.20).

3.12 Consider the displacement model of a triangular bending element given in Problem 3.11. Determine whether the convergence requirements of Section 3.6 are satisfied by this model.

Note

The expression for the functional I (potential energy) of a plate in bending is given by

$$I = \frac{1}{2}\iint_A D\left\{\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 - 2(1-v)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left[\frac{\partial^2 w}{\partial x \partial y}\right]^2\right]\right\}\mathrm{d}x\,\mathrm{d}y - \int_A (pw)\,\mathrm{d}x\,\mathrm{d}y$$

where p is the distributed transverse load per unit area, D is the flexural rigidity, v is the Poisson's ratio, and A is the surface area of the plate.

3.13 The coordinates of the nodes of a three-dimensional simplex element are given below:

Coordinates of the Node

Node Number	x	y	Z
i	0	0	0
j	10	0	0
k	0	15	0
l	0	0	20

Determine the shape functions of the element.

3.14 The shape function matrix of a uniform one-dimensional simplex element is given by $[N] = [N_i \ N_j]$, with $N_i = 1 - (x/l)$ and $N_j = (x/l)$. Evaluate the integral $\iiint_V [N]^T [N] \mathrm{d}V$, where $V = A \, \mathrm{d}x$, A is the cross-sectional area, and l is the length of the element.

3.15 Evaluate the integral $\int_{\mathscr{L}_{ij}} \overrightarrow{N} \overrightarrow{N}^T \mathrm{d}\mathscr{L}$ along the edge ij of a simplex triangle, where $\mathscr{L}_{ij}$ denotes the distance between the nodes i and j, and the vector of shape functions $\overrightarrow{N}$ is given by $\overrightarrow{N}^T = (N_i N_j N_k)$.

3.16 Evaluate the integral $\int_{S_{ijk}} \overrightarrow{N} \, \overrightarrow{N}^T \mathrm{d}S$ on the face ijk of a simplex tetrahedron, where S_{ijk} denotes the surface area bounded by the nodes i, j, and k, and the vector of shape functions is given by $\overrightarrow{N}^T = (N_i N_j N_k N_l)$.

3.17 The nodes of a one-dimensional simplex element are located at $x_1 = 30$ mm and $x_2 = 50$ mm. The values of the field variable at the two nodes are U_1 and U_2. Determine the matrix of shape functions, $[N(x)]$, using Eq. (3.20) assuming a linear interpolation model for the field variable as

$$\phi(x) = \overrightarrow{\eta}^T \overrightarrow{\alpha} \quad \text{where} \quad \overrightarrow{\eta} = \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad \text{and} \quad \overrightarrow{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

3.18 The temperatures of a one-dimensional element at nodes i and j are given by 130 °F and 80 °F, respectively. If the nodes i and j are located at a distance of 30 in and 70 in, respectively, from the origin, express the matrix of shape functions using Eq. (3.20).

3.19 The u-components of displacement at nodes i, j, and k of a triangular element are denoted U_i, U_j, and U_k, respectively. The nodal coordinates are given by $(x_i, y_i) = (5, 0)$ in, $(x_j, y_j) = (0, 4)$ in, and $(x_k, y_k) = (0, 0)$ in. Assuming a linear variation of $u(x, y)$ inside the element, determine the shape functions of the element using Eq. (3.20).

3.20 The (x, y) coordinates of the nodes of a triangular element in the temperature analysis of a rectangular plate are given by $(x_i, y_i) = (30, 40)$ mm, $(x_j, y_j) = (60, 80)$ mm, and $(x_k, y_k) = (50, 70)$ mm. If the nodal temperatures are given by $T_i = 130°\mathrm{C}$, $T_j = 70°\mathrm{C}$, and $T_k = 150°\mathrm{C}$, find the following:

a. Matrix of shape functions using Eq. (3.20)
b. Expression for the temperature distribution inside the element
c. Temperature gradient inside the element

3.21 The horizontal component of fluid velocity, $u(x)$, in a tube is known to be $U_i = 3$ m/s and $U_j = 2$ m/s at nodes i and j located at $x_i = 10$ cm and $x_j = 15$ cm, respectively, in a finite element. Determine the following:

a. Matrix of shape functions using Eq. (3.20)
b. Expression for the velocity distribution, $u(x)$, in the element
c. Velocity gradient in the element

3.22 The temperatures at the nodes of a triangular simplex element are given by
Node i: $T = 110°\mathrm{C}$, $(x, y) = (2, 2)$ cm
Node j: $T = 38°\mathrm{C}$, $(x, y) = (5, 3)$ cm
Node k: $T = 125°\mathrm{C}$, $(x, y) = (-1, 4)$ cm
Find the shape functions associated with nodes i, j, and k using Eq. (3.20). Also, determine the temperature variation, $T(x, y)$, in the element.

3.23 The (x, y, z) coordinates of the nodes i, j, k, and l of a tetrahedron element are given by (30, 20, 50), (10, 10, 0), (50, 40, 0), and (20, 60, 0) mm, respectively. Determine the shape functions corresponding to the various nodes using Eq. (3.20).

3.24 The temperatures at the nodes of a tetrahedron simplex element are given by:
Node i: $T = 150°$ F, $(x, y, z) = (4, 5, 7)$ in
Node j: $T = 125$ °F, $(x, y, z) = (2, 2, 2)$ in
Node k: $T = 85$ F, $(x, y, z) = (8, 5, 3)$ in
Node l: $T = 115$ °F, $(x, y, z) = (3, 6, 4)$ in
Find the shape functions associated with the nodes using Eq. (3.20). Also, determine the temperature variation, $T(x, y, z)$, in the element.

3.25 The displacement components parallel to the x axis of the nodes of a tetrahedron element are:
Node i: $u = 0.005$ mm, $(x, y, z) = (30, 40, 20)$ mm
Node j: $u = -0.002$ mm, $(x, y, z) = (10, 60, -20)$ mm
Node k: $u = 0.003$ mm, $(x, y, z) = (30, 80, -40)$ mm
Node l: $u = -0.004$ mm, $(x, y, z) = (50, 50, -30)$ mm
Find the shape functions associated with the nodes using Eq. (3.20). Also, determine the displacement variation, $u(x, y, z)$, in the element.

3.26 The nodal coordinates of a tetrahedron element, in mm, are given by (0, 0, 0), (10, 0, 30), (20, 0, 0), and (10, 20, 10). Find the shape functions of the element using Eq. (3.20).

3.27 A two-dimensional simplex element has the nodal coordinates (10, 10) mm, (40, 30) mm, and (10, 30) mm. The element undergoes in-plane displacement with components u and v. Determine the shape functions of the element and express the variations $u(x, y)$ and $v(x, y)$ in terms of the nodal components of displacement, (U_α, V_α), $\alpha = i, j, k$.

3.28 Solve Problem 3.1 using the relations of Section 3.7.1.

3.29 Solve Problem 3.2 using the relations of Section 3.7.1.

3.30 Solve Problem 3.3 using the relations of Section 3.7.2.

3.31 Solve Problem 3.4 using the relations of Section 3.7.2.

3.32 Solve Problem 3.5 using the relations of Section 3.7.1.

3.33 Solve Problem 3.6 using the relations of Section 3.7.2.

3.34 Solve Problem 3.7 using the relations of Section 3.7.3.

3.35 Solve Problem 3.8 using the relations of Section 3.7.3.

3.36 Solve Problem 3.9 using the relations of Section 3.7.3.

3.37 Solve Problem 3.10 using the relations of Section 3.7.3.

3.38 Consider the nodal coordinates of a tetrahedron element as given in Problem 3.7. If the (u, v, w) displacements at the nodes i, j, k, and l are given by (0.001, −0.002, 0.003) mm, (−0.002, 0.001, 0.002) mm, (0.0015, 0.002, −0.002) mm, and (0.003, −0.001, 0.001) mm, respectively, determine the variations of $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$ in the element.

3.39 The nodal coordinates of a triangular plate element subject to in-plane loads are $(x_i, y_i) = (10, 20)$ mm, $(x_j, y_j) = (50, 30)$ mm, and $(x_k, y_k) = (40, 60)$ mm.
The in-plane displacement components, (u, v), of the nodes i, j, and k are given by (1.5, −0.5) mm, (−1.0, 2.0) mm, and (1.5, −1.0) mm, respectively. Express the variations of $u(x, y)$ and $v(x, y)$ in the element.

3.40 Evaluate the line integral $\int_L [N]^T \, dL$ along the side ij of a linear triangular element ijk using natural coordinates.

3.41 Evaluate the line integral $\int_L [N]^T dL$ along the side jk of a linear triangular element ijk using natural coordinates.

3.42 Evaluate the line integral $\int_L [N]^T dL$ along the side ki of a linear triangular element ijk using natural coordinates.

3.43 Show that the area coordinates L_1, L_2, and L_3 along the edge ij of the triangular element ijk shown in Fig. 3.23 can be expressed as

$$L_1 = 1 - \frac{s}{l_{ij}}, \quad L_2 = \frac{s}{l_{ij}}, \quad L_3 = 0$$

where l_{ij} is the length of the side ij.

3.44 Show that the area coordinates L_1, L_2, and L_3 along the edge jk of the triangular element ijk shown in Fig. 3.23 can be expressed as

$$L_1 = 0, \ L_2 = 1 - \frac{s}{l_{jk}}, \ L_3 = \frac{s}{l_{jk}}$$

where l_{ik} is the length of the side jk.

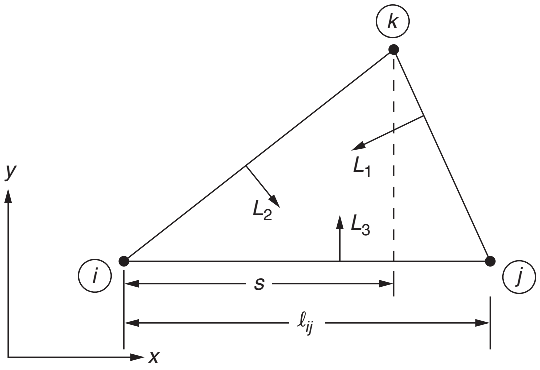

FIGURE 3.23 Area coordinates of a triangular element.

3.45 Show that the area coordinates L_1, L_2, and L_3 along the edge ki of the triangular element ijk shown in Fig. 3.23 can be expressed as

$$L_1 = \frac{s}{l_{ki}}, \quad L_2 = 0, \quad L_3 = 1 - \frac{s}{l_{ki}}$$

where l_{ki} is the length of the side ki.

3.46 Evaluate the integral $\int_L L_1^\alpha L_2^\beta L_3^\gamma \, dL$ along the side ij of a triangular element ijk.

3.47 Show that the shape functions of a triangular simplex element (with three nodes) satisfy the two properties of C^0 continuity stated in Section 3.7.4.

3.48 Show that the value of the shape function N_i is equal to 1 at node i and 0 at the other nodes j, k, and l for a four-noded tetrahedron element.

3.49 The nodal temperatures of a triangular element, T_i, T_j, and T_k, are assumed to be the unknowns in the heat transfer analysis of a two-dimensional (plate) fin. Assuming a linear temperature model for the temperature inside the element is

$$\phi(x, y) = \{1\ x\ y\} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \equiv \vec{\eta}^{\,T} \vec{\alpha} \tag{P1}$$

determine the shape function matrix $[N(x, y)]$ and the shape functions associated with the three nodes of the element. Assume the nodal coordinates as $(x_i, y_i) = (-5, 10)$ in, $(x_j, y_j) = (2, 5)$ in, and $(x_k, y_k) = (0, 11)$ in.

3.50 Consider a tetrahedron element with global nodes 3, 7, 11, and 25 as shown in Fig. 3.24. Indicate the 12 acceptable ways in which the four nodes can be labeled (as local 1, 2, 3, 4) to satisfy the ordering convention.

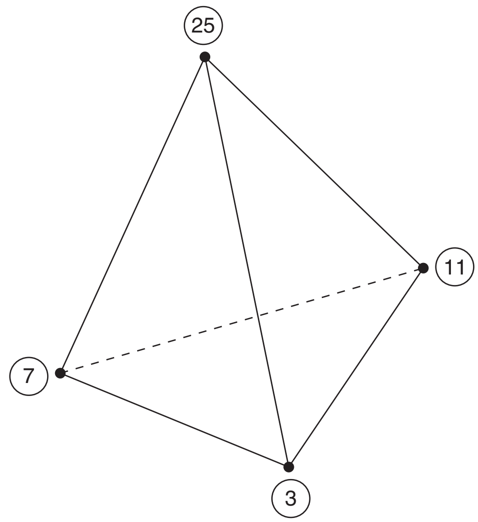

FIGURE 3.24 A tetrahedron element.

3.51 The nodal coordinates x_1 and x_2 of an element in a heat transfer problem are given by 10 in and 15 in with the corresponding nodal unknowns (temperatures) as $T_1^{(e)}$ and $T_2^{(e)}$, respectively. Express the linear interpolation model of the element, $T(x) = \alpha_1 + \alpha_2 x$, in terms of the nodal unknowns using Eq. (3.19).

3.52 The axial displacements at the two end nodes (nodes 1 and 2) of a bar element are known to be $u_1 = 0.2$ mm and $u_2 = 0.4$ mm. If the x-coordinates of nodes 1 and 2 are $x_1 = 100$ mm and $x_2 = 150$ mm, express the linear displacement model of the element, $u(x) = \alpha_1 + \alpha_2 x$, in terms of the nodal values u_1 and u_2 using Eq. (3.19).

3.53 The nodal temperatures of nodes i and j of a line element in a one-dimensional heat transfer problem are given by $T_i = 230°C$ and $T_j = 180°C$ with the x-coordinates of the nodes i and j given by $x_i = 25$ cm and $x_j = 45$ cm. Determine the following:

a. Shape functions associated with the nodal temperatures T_i and T_j

b. Interpolation model for the temperature in the element

c. Temperature in the element at $x = 40$ cm

3.54 The axial displacements of the end nodes i and j of a bar element are known to be $u_i = 0.22$ in and $u_j = 0.27$ in. If the x-coordinates of the nodes are $x_i = 8.0$ in and $x_j = 10.0$ in, find the following:

a. Shape functions associated with the nodal displacements u_i and u_j

b. Interpolation model for the axial displacement in the element

c. Axial displacement at $x = 8.5$ in in the element

3.55 The global nodal coordinates of a triangular plate element used in the stress analysis of a plate subjected to in-plane loads are given by $(x_i, y_i) = (50, 30)$ in, $(x_j, y_j) = (70, 50)$ in, and $(x_k, y_k) = (55, 60)$ in. If $u(x, y)$ and $v(x, y)$ are the field variables with nodal unknowns given by

$$(\Phi_{2i-1}, \Phi_{2i}) = (u_i, v_i), (\Phi_{2j-1}, \Phi_{2j}) = (u_j, v_j), \text{ and } (\Phi_{2k-1}, \Phi_{2k}) = (u_k, v_k)$$

find the shape function matrix of the element.

3.56 The nodal coordinates and nodal temperatures of a triangular simplex element follow:

Node i: $(x_i, y_i) = (10, 10)$ mm, $T_i = 100°C$

Node j: $(x_j, y_j) = (20, 50)$ mm, $T_j = 80°C$

Node k: $(x_k, y_k) = (-10, 30)$ mm, $T_k = 130°C$

Express the temperature variation in the element, $T(x, y)$, in terms of the shape functions.

3.57 The global X-coordinates of the nodes of a one-dimensional element e, shown in Fig. 3.3, are

$$X_1^{(e)} = 12 \text{ cm and } X_2^{(e)} = 18 \text{ cm}$$

If the nodal values of the field variable are denoted as $\Phi_1^{(e)}$ and $\Phi_2^{(e)}$, express the linear interpolation model of the element in terms of the nodal unknowns $\Phi_1^{(e)}$ and $\Phi_2^{(e)}$.

3.58 Check whether the patch tests for rigid body motion and constant strain state are satisfied for a bar element with the shape functions $N_1(x) = \dfrac{x_{i+1} - x}{x_{i+1} - x_i}$ and $N_2(x) = \dfrac{x - x_i}{x_{i+1} - x_i}$ by considering the patch shown in Fig. 3.25.

3.59 Check whether the patch test for rigid body motion is satisfied for the element with the shape functions $N_i(x)$ given in Eq. (1.17) by considering the patch of elements shown in Fig. 3.26.

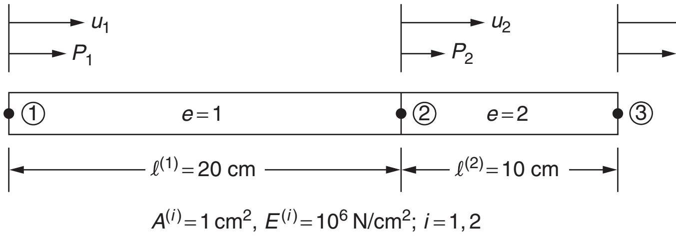

FIGURE 3.25 A patch of two bar elements.

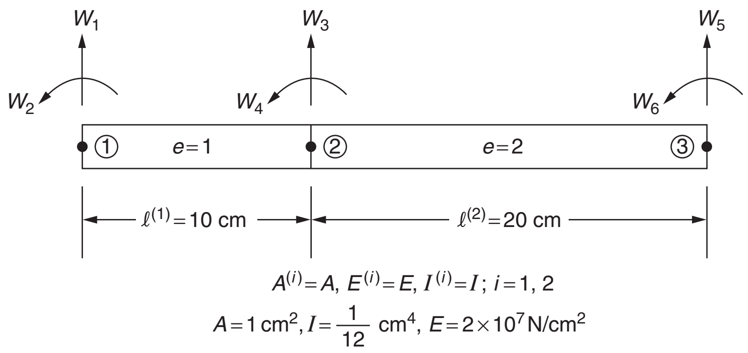

FIGURE 3.26 A patch of two beam elements.

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Chapter 4

Higher Order and Isoparametric Elements

Chapter Outline

4.1 INTRODUCTION

As stated earlier, if the interpolation polynomial is of order two or more, the element is known as a *higher order element*. A higher order element can be either complex or multiplex. In higher order elements, some secondary (mid-side and/or interior) nodes are introduced in addition to the primary (corner) nodes in order to match the number of nodal degrees of freedom with the number of constants (also known as generalized coordinates) in the interpolation polynomial.

For problems involving curved boundaries, a family of elements known as *isoparametric elements* can be used. In isoparametric elements, the same interpolation functions used to define the element geometry are also used to describe the variation of the field variable within the element. Both higher order and isoparametric elements are considered in this chapter.

4.2 HIGHER ORDER ONE-DIMENSIONAL ELEMENTS

4.2.1 Quadratic Element

The quadratic interpolation model for a one-dimensional element can be expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{4.1}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00004-3

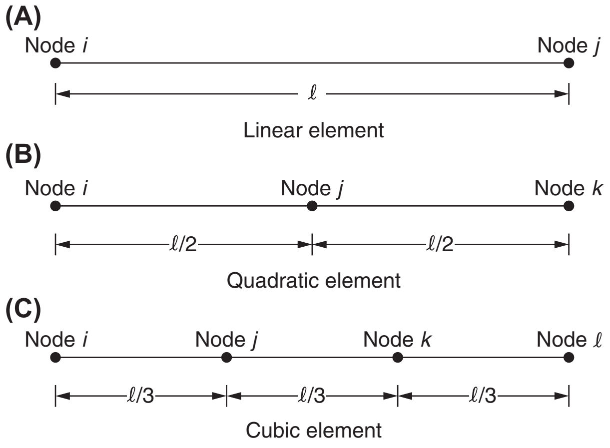


FIGURE 4.1 Location of nodes in one-dimensional element (global node numbers indicated).

Since there are three constants (α_1, α_2, and α_3) in Eq. (4.1), the element is assumed to have three degrees of freedom, one at each of the ends and one at the middle point as shown in Fig. 4.1B. By requiring that

$$\begin{aligned} \phi(x) &= \Phi_i \quad \text{at} \quad x = 0 \\ \phi(x) &= \Phi_j \quad \text{at} \quad x = l/2 \\ \phi(x) &= \Phi_k \quad \text{at} \quad x = l \end{aligned} \tag{4.2}$$

we can evaluate the constants α_1, α_2, and α_3 as

$$\begin{aligned} \alpha_1 &= \Phi_i \\ \alpha_2 &= (4\Phi_j - 3\Phi_i - \Phi_k)/l \\ \alpha_3 &= 2(\Phi_i - 2\Phi_j + \Phi_k)/l^2 \end{aligned} \tag{4.3}$$

With the help of Eq. (4.2), Eq. (4.1) can be expressed after rearrangement as

$$\phi(x) = [N(x)]\vec{\Phi}^{(e)} \tag{4.4}$$

where

$$\begin{aligned} [N(x)] &= [N_i(x) \quad N_j(x) \quad N_k(x)] \\ N_i(x) &= \left(1 - 2\frac{x}{l}\right)\left(1 - \frac{x}{l}\right) \\ N_j(x) &= 4\frac{x}{l}\left(1 - \frac{x}{l}\right) \\ N_k(x) &= -\frac{x}{l}\left(1 - 2\frac{x}{l}\right) \end{aligned} \tag{4.5}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{Bmatrix} \tag{4.6}$$

4.2.2 Cubic Element

The cubic interpolation model can be expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{4.7}$$

Because there are four unknown coefficients α_1, α_2, α_3, and α_4, the element is assumed to have four degrees of freedom, one at each of the four nodes shown in Fig. 4.1C. By requiring that

$$\begin{aligned} \phi(x) &= \Phi_i \quad \text{at} \quad x = 0 \\ \phi(x) &= \Phi_j \quad \text{at} \quad x = l/3 \\ \phi(x) &= \Phi_k \quad \text{at} \quad x = 2l/3 \\ \phi(x) &= \Phi_l \quad \text{at} \quad x = l \end{aligned} \tag{4.8}$$

the constants α_1, α_2, α_3, and α_4 can be evaluated. The substitution of values of these constants into Eq. (4.7) leads to

$$\phi(x) = [N(x)]\vec{\Phi}^{(e)} \tag{4.9}$$

where

$$[N(x)] = [N_i(x) \quad N_j(x) \quad N_k(x) \quad N_l(x)]$$

$$\begin{aligned} N_i(x) &= \left(1 - \frac{3x}{l}\right)\left(1 - \frac{3x}{2l}\right)\left(1 - \frac{x}{l}\right) \\ N_j(x) &= 9\frac{x}{l}\left(1 - \frac{3x}{2l}\right)\left(1 - \frac{x}{l}\right) \\ N_k(x) &= -\frac{9}{2}\frac{x}{l}\left(1 - \frac{3x}{l}\right)\left(1 - \frac{x}{l}\right) \\ N_l(x) &= \frac{x}{l}\left(1 - \frac{3x}{l}\right)\left(1 - \frac{3x}{2l}\right) \end{aligned} \tag{4.10}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \\ \Phi_l \end{Bmatrix} \tag{4.11}$$

It can be observed that the application of the previous procedure for determining the coefficients α_i and the nodal interpolation functions $N_i(x)$ becomes more tedious as the order of the interpolation polynomial increases. The nodal interpolation functions $N_i(x)$ can be constructed in a simpler manner by employing either natural coordinates or classical interpolation polynomials.

4.3 HIGHER ORDER ELEMENTS IN TERMS OF NATURAL COORDINATES

4.3.1 One-Dimensional Element

QUADRATIC ELEMENT

The normalized or natural coordinates L_1 and L_2 for a one-dimensional element were shown in Fig. 3.12. If the values of ϕ at three stations, x_1,$(x_1 + x_2)/2$, and x_2 are taken as nodal unknowns, the quadratic model for $\phi(x)$ can be expressed as

$$\phi(x) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3]\vec{\Phi}^{(e)} \tag{4.12}$$

where

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(\text{at } L_1 = 1, L_2 = 0) \\ \phi\left(\text{at } L_1 = \frac{1}{2}, L_2 = \frac{1}{2}\right) \\ \phi(\text{at } L_1 = 0, L_2 = 1) \end{Bmatrix}^{(e)} \tag{4.13}$$

and the quadratic nodal interpolation functions N_i can be expressed in general form as

$$N_i = \alpha_1^{(i)} L_1 + \alpha_2^{(i)} L_2 + \alpha_3^{(i)} L_1 L_2; \quad i = 1, 2, 3 \tag{4.14}$$

where $\alpha_1^{(i)}, \alpha_2^{(i)}$, and $\alpha_3^{(i)}$ are constants (to be determined) with the superscript (i) identifying the ith interpolation function, N_i. By imposing on N_i, the three requirements

$$N_1 = \begin{cases} 1 \text{ at node } 1 & (L_1 = 1, L_2 = 0) \\ 0 \text{ at node } 2 & \left(L_1 = L_2 = \frac{1}{2}\right) \\ 0 \text{ at node } 3 & (L_1 = 0, L_2 = 1) \end{cases}$$

and find the values of the constants $\alpha_1^{(1)}, \alpha_2^{(1)}$, and $\alpha_3^{(1)}$ as

$$a_1^{(1)} = 1, \quad a_2^{(1)} = 0, \quad a_3^{(1)} = -2$$

so that Eq. (4.14) becomes

$$N_1 = L_1 - 2L_1 L_2$$

By using the condition $L_1 + L_2 = 1$, we obtain

$$N_1 = L_1(2L_1 - 1) \tag{4.15}$$

Similarly, the other two nodal interpolation functions can be derived as

$$N_2 = 4L_1 L_2 \tag{4.16}$$

and

$$N_3 = L_2(2L_2 - 1) \tag{4.17}$$

The nodal interpolation functions N_i appearing in Eqs. (4.15) to (4.17) are shown in Fig. 4.2.

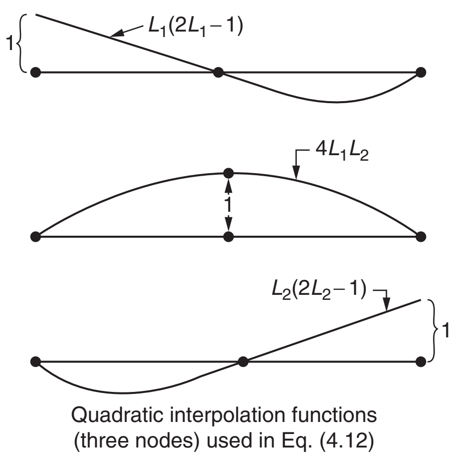


FIGURE 4.2 Nodal interpolation (or shape) functions for a line element.

CUBIC ELEMENT

For a cubic element, we consider four nodal degrees of freedom, one at each of the nodes shown in Fig. 4.1C. The cubic interpolation model can be written as

$$\phi(x) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3 \quad N_4]\vec{\Phi}^{(e)} \tag{4.18}$$

where

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(\text{at } L_1 = 1,\ L_2 = 0) \\ \phi(\text{at } L_1 = 2/3,\ L_2 = 1/3) \\ \phi(\text{at } L_1 = 1/3,\ L_2 = 2/3) \\ \phi(\text{at } L_1 = 0,\ L_2 = 1) \end{Bmatrix}^{(e)}$$

and the nodal interpolation functions N_i appearing in Eq. (4.18) can be expressed in terms of the natural coordinates as

$$N_i = a_1^{(i)} L_1 + a_2^{(i)} L_2 + a_3^{(i)} L_1 L_2 + a_4^{(i)} L_1^2 L_2 \tag{4.19}$$

By requiring that N_i be equal to 1 at node i and 0 at each of the other nodes, we find that

$$N_1 = L_1\left(1 - \frac{9}{2} L_1 L_2\right) \tag{4.20}$$

$$N_2 = -\frac{9}{2} L_1 L_2 (1 - 3L_1) \tag{4.21}$$

$$N_3 = 9 L_1 L_2 \left(1 - \frac{3}{2} L_1\right) \tag{4.22}$$

$$N_4 = L_2 - \frac{9}{2} L_1 L_2 (1 - L_1) \tag{4.23}$$

EXAMPLE 4.1

A bar, subjected to an axial force, is divided into a number of quadratic elements. For a particular element, the nodes i, j, and k are located at 15 mm, 18 mm, and 21 mm, respectively, from the origin. If the axial displacements of the three nodes are given by $u_i = 0.0015$ mm, $u_j = 0.0024$ mm, and $u_k = 0.0033$ mm, determine the following:

a. Shape functions
b. Variation of the displacement, $u(x)$, in the element
c. Axial strain, ε_{xx}, in the element

Solution

Approach: Use Eqs. (4.5) and (4.4) and the relation $\varepsilon_{xx} = \frac{du}{dx}$.

a. The length of the element is $l = x_k - x_i = 21-15 = 6$ mm, and the shape functions are given by Eq. (4.5):

$$N_i(x) = \left(1 - \frac{2x}{6}\right)\left(1 - \frac{x}{6}\right) = \frac{1}{18}(3 - x)(6 - x) \tag{E.1}$$

$$N_j(x) = \frac{4x}{6}\left(1 - \frac{x}{6}\right) = \frac{1}{9} x(6 - x) \tag{E.2}$$

$$N_k(x) = -\frac{x}{6}\left(1 - \frac{2x}{6}\right) = -\frac{x}{18}(3 - x) \tag{E.3}$$

b. The variation of the axial displacement can be expressed using Eq. (4.4):

$$\begin{aligned} u(x) &= N_i(x) u_i + N_j(x) u_j + N_k(x) u_k \\ &= \frac{1}{18}(3 - x)(6 - x)(0.0015) + \frac{1}{9} x(6 - x)(0.0024) - \frac{x}{18}(3 - x)(0.0033) \\ &= \left(18 - 9x + x^2\right)(0.00008333) + \left(6x - x^2\right)(0.00026667) + \left(x^2 - 3x\right)(0.00018333) \end{aligned}$$

c. The axial strain in the element is given by

$$\varepsilon_{xx} = \frac{du}{dx} = (-9 + 2x)(0.00008333) + (6 - 2x)(0.00026667) + (2x - 3)(0.00018333); \quad 0 \le x \le 6 \text{ mm}$$

EXAMPLE 4.2

The nodal temperatures in a one-dimensional cubic element used in heat transfer analysis are given by $T_i = 80\ °F$, $T_j = 90\ °F$, $T_k = 110\ °F$, and $T_l = 140\ °F$. If the length of the element is 6 in, determine (a) the temperature variation and (b) the rate of change of temperature along the axial distance of the element.

Solution

Approach: Use Eqs. (4.9) and (4.10).

a. Here the length of the element is $l = 6$ in, and hence the shape functions associated with the nodes are as follows:

$$N_i(x) = \left(1-\frac{3x}{6}\right)\left(1-\frac{3x}{12}\right)\left(1-\frac{x}{6}\right) = \frac{1}{48}(2-x)(4-x)(6-x)$$

$$N_j(x) = 9\frac{x}{6}\left(1-\frac{3x}{12}\right)\left(1-\frac{x}{6}\right) = \frac{x}{16}(4-x)(6-x)$$

$$N_k(x) = -\frac{9}{2}\frac{x}{6}\left(1-\frac{3x}{6}\right)\left(1-\frac{x}{6}\right) = -\frac{3x}{48}(2-x)(6-x)$$

$$N_l(x) = \frac{x}{6}\left(1-\frac{3x}{6}\right)\left(1-\frac{3x}{12}\right) = \frac{x}{48}(2-x)(4-x)$$

Variation of temperature in the element:

$$\begin{aligned} T(x) &= N_i(x)T_i + N_j(x)T_j + N_k(x)T_k + N_l(x)T_l \\ &= \frac{1}{48}(2-x)(4-x)(6-x)(80) + \frac{x}{16}(4-x)(6-x)(90) - \frac{3x}{48}(2-x)(6-x)(110) + \frac{x}{48}(2-x)(4-x)(140) \\ &= \frac{5}{3}(48-44x+12x^2-x^3) + \frac{45}{8}(24x-10x^2+x^3) - \frac{55}{8}(12x-8x^2+x^3) + \frac{35}{12}(8x-6x^2+x^3)\ °F \end{aligned}$$

b. Rate of change of temperature in the element:

$$\frac{dT}{dx} = \frac{5}{3}(-44+24x-3x^2) + \frac{45}{8}(24-20x+3x^2) - \frac{55}{8}(12-16x+3x^2) + \frac{35}{12}(8-12x+3x^2)°F/in$$

4.3.2 Two-Dimensional (Triangular) Element

QUADRATIC ELEMENT

The natural or triangular coordinates L_1, L_2, and L_3 of a triangular element were shown in Fig. 3.13. For a quadratic interpolation model, the values of the field variable at three corner nodes and three mid-side nodes (Fig. 4.3A) are taken as the nodal unknowns and $\phi\ (x, y)$ is expressed as

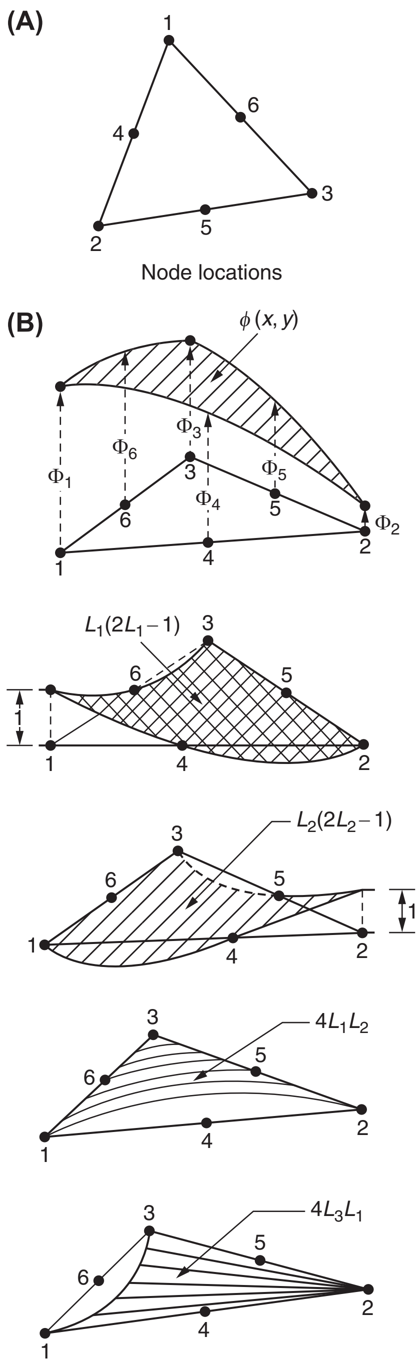
$$\phi(x,y) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad \cdots \quad N_6]\vec{\Phi}^{(e)} \tag{4.24}$$

where N_i can be derived from the general quadratic relationship

$$N_i = a_1^{(i)}L_1 + a_2^{(i)}L_2 + a_3^{(i)}L_3 + a_4^{(i)}L_1L_2 + a_5^{(i)}L_2L_3 + a_6^{(i)}L_1L_3 \tag{4.25}$$

as

$$\begin{aligned} N_i &= L_i(2\,L_i - 1), \quad i = 1,2,3 \\ N_4 &= 4\,L_1L_2 \\ N_5 &= 4\,L_2L_3 \\ N_6 &= 4\,L_1L_3 \end{aligned} \tag{4.26}$$



Shape functions

FIGURE 4.3 Nodes and shape functions of a quadratic triangular element.

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_6 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \vdots \\ \phi(x_6, y_6) \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(\text{at } L_1 = 1,\ L_2 = L_3 = 0) \\ \phi(\text{at } L_2 = 1,\ L_1 = L_3 = 0) \\ \vdots \\ \phi\left(\text{at } L_1 = L_3 = \frac{1}{2},\ L_2 = 0\right) \end{Bmatrix}^{(e)} \tag{4.27}$$

The nodal interpolation or shape functions of Eq. (4.26) are shown in Fig. 4.3B.

CUBIC ELEMENT

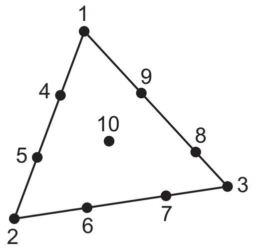


FIGURE 4.4 Location of nodes of a cubic triangular element.

If a cubic interpolation model is used, 10 nodal unknowns are required. The location of the nodes is shown in Fig. 4.4, in which the nodes 4 and 5 are located at one-third points along the edge 12 with similar locations for the nodes 6 and 7, and 8 and 9 along the edges 23 and 31, respectively. The node 10 is located at the centroid of the triangle 123. In this case, the interpolation model is given by

$$\phi(x, y) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad \cdots \quad N_{10}]\vec{\Phi}^{(e)} \tag{4.28}$$

where the general form of the nodal interpolation function can be assumed as

$$N_i = a_1^{(i)}L_1 + a_2^{(i)}L_2 + a_3^{(i)}L_3 + a_4^{(i)}L_1L_2 + a_5^{(i)}L_2L_3 + a_6^{(i)}L_1L_3 + a_7^{(i)}L_1^2L_2 + a_8^{(i)}L_2^2L_3 + a_9^{(i)}L_3^2L_1 + a_{10}^{(i)}L_1L_2L_3 \tag{4.29}$$

By imposing the conditions that N_i be equal to 1 at node i and 0 at each of the remaining nine nodes, we can obtain

$$\begin{aligned}
N_i &= \frac{1}{2}L_i(3L_i - 1)(3L_i - 2), \quad i = 1,\ 2,\ 3 \\
N_4 &= \frac{9}{2}L_1L_2(3L_1 - 1) \\
N_5 &= \frac{9}{2}L_1L_2(3L_2 - 1) \\
N_6 &= \frac{9}{2}L_2L_3(3L_2 - 1) \\
N_7 &= \frac{9}{2}L_2L_3(3L_3 - 1) \\
N_8 &= \frac{9}{2}L_1L_3(3L_3 - 1) \\
N_9 &= \frac{9}{2}L_1L_3(3L_1 - 1) \\
N_{10} &= 27\,L_1L_2L_3
\end{aligned} \tag{4.30}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{10} \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \vdots \\ \phi(x_{10}, y_{10}) \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(\text{at } L_1 = 1,\ L_2 = L_3 = 0) \\ \phi(\text{at } L_2 = 1,\ L_1 = L_3 = 0) \\ \vdots \\ \phi\left(\text{at } L_1 = L_2 = L_3 = \frac{1}{3}\right) \end{Bmatrix}^{(e)} \tag{4.31}$$

4.3.3 Two-Dimensional (Quadrilateral) Element

NATURAL COORDINATES

A different type of natural coordinate system can be established for a quadrilateral element in two dimensions as shown in Fig. 4.5. For the local r, s (natural) coordinate system, the origin is taken as the intersection of lines joining the midpoints of opposite sides and the sides are defined by $r = \pm 1$ and $s = \pm 1$. The natural and Cartesian coordinates are related by the following equation:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{4.32}$$

where (x_i, y_i) are the (x, y) coordinates of node i $(i = 1, 2, 3, 4)$,

$$N_i = \frac{1}{4}(1 + r r_i)(1 + s s_i), \quad i = 1, 2, 3, 4 \tag{4.33}$$

and the natural coordinates of the four nodes of the quadrilateral are given by

$$\begin{aligned} (r_1, s_1) &= (-1, -1), \quad (r_2, s_2) = (1, -1), \\ (r_3, s_3) &= (1, 1), \text{ and } \quad (r_4, s_4) = (-1, 1) \end{aligned} \tag{4.34}$$

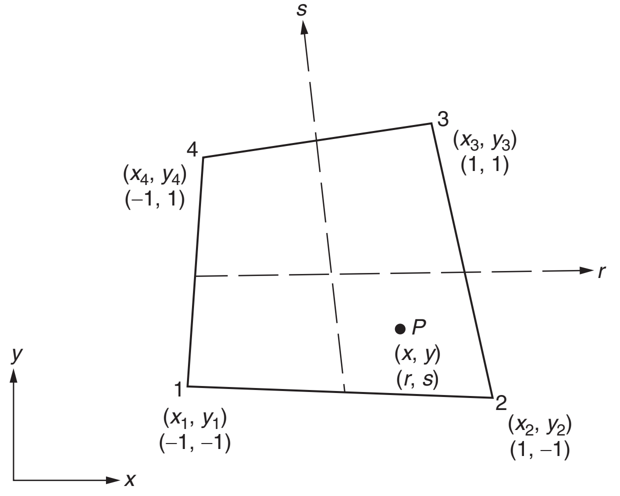


FIGURE 4.5 Natural coordinates for a quadrilateral element.

The Jacobian matrix, $[J]$, is defined as a matrix containing the derivatives of the global coordinates with respect to the natural coordinates of the element. Thus, the 2×2 Jacobian matrix of the element can be expressed as

$$[J] = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial s & \partial y/\partial s \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(1-s) & (1-s) & (1+s) & -(1+s) \\ -(1-r) & -(1+r) & (1+r) & (1-r) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{4.35}$$

If ϕ is a function of the natural coordinates r and s, its derivatives with respect to x and y can be obtained as

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial s}$$

or

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial r} \\ \dfrac{\partial \phi}{\partial s} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} \equiv [J] \begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix}$$

This equation can be inverted to obtain

$$\begin{Bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \phi}{\partial r} \\ \dfrac{\partial \phi}{\partial s} \end{Bmatrix} \tag{4.36}$$

The integration of functions of r and s has to be performed numerically with

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \det[J] \cdot \mathrm{d}r\mathrm{d}s \tag{4.37}$$

and the limits of both r and s will be -1 and 1.

LINEAR ELEMENT

For a quadrilateral element, it is not possible to have linear variation of the field variable (in terms of two independent coordinates) if one degree of freedom is chosen at each of the four corner nodes. Hence, we take the interpolation model as

$$\phi(x,y) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3 \quad N_4]\vec{\Phi}^{(e)} \tag{4.38}$$

where

$$N_i = (1 + r r_i)(1 + s s_i)/4, \quad i = 1,2,3,4 \tag{4.39}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1,y_1) \\ \phi(x_2,y_2) \\ \phi(x_3,y_3) \\ \phi(x_4,y_4) \end{Bmatrix}^{(e)} \equiv \begin{Bmatrix} \phi(\text{at } r=-1,\ s=-1) \\ \phi(\text{at } r=1,\ s=-1) \\ \phi(\text{at } r=1,\ s=1) \\ \phi(\text{at } r=-1,\ s=1) \end{Bmatrix}^{(e)} \tag{4.40}$$

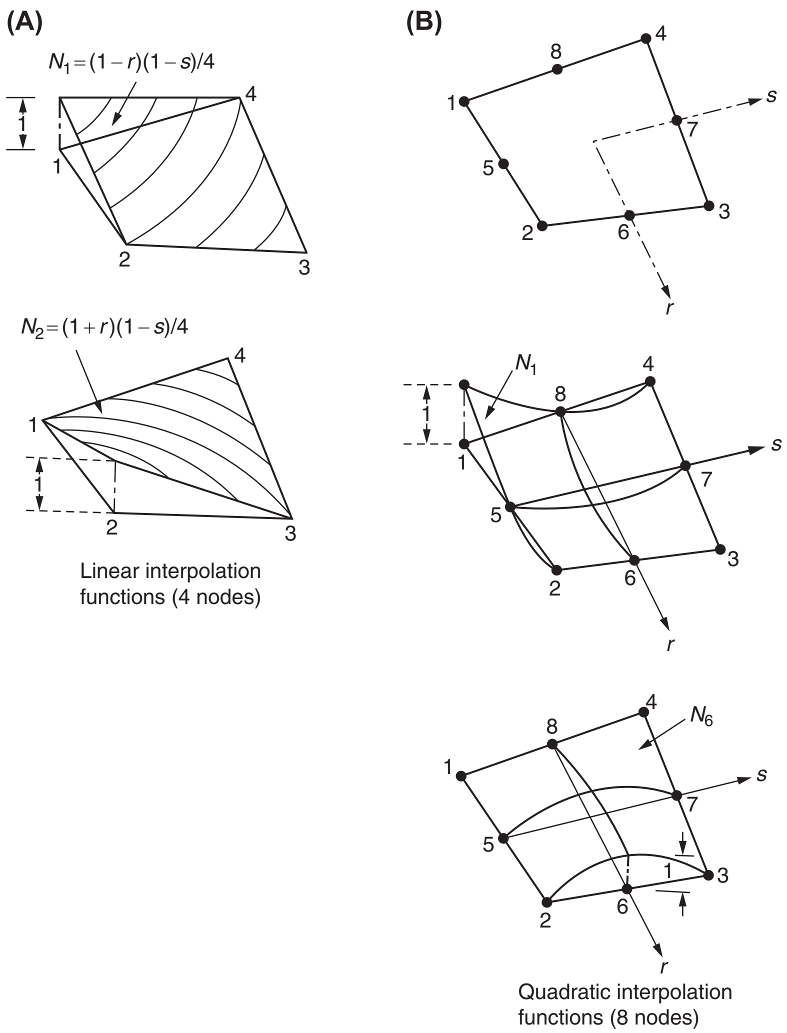


FIGURE 4.6 Interpolation functions for a quadrilateral element.

The nodal shape functions represented by Eq. (4.39) are shown in Fig. 4.6A. It can be seen that the variation of the field variable along the edges of the quadrilateral is linear. Hence, this element is often called a linear element.

EXAMPLE 4.3

The global coordinates of the four corners of a linear quadratic element are given by

$$(x_1, y_1) = (6, 9) \text{ in}, (x_2, y_2) = (2, 7) \text{ in}, (x_3, y_3) = (3, 10) \text{ in}, \text{and} (x_4, y_4) = (10, 6) \text{ in} \tag{E.1}$$

Find the global coordinates corresponding to the natural coordinates $r = -0.75$ and $s = 0.5$.

Solution

Eq. (4.32) gives

$$x = \sum_{i=1}^{4} N_i x_i \tag{E.2}$$

$$y = \sum_{i=1}^{4} N_i y_i \tag{E.3}$$

Continued

EXAMPLE 4.3 —cont'd

where N_i can be expressed using Eqs. (4.33) and (4.34) as

$$N_1 = \frac{1}{4}(1 + rr_1)(1 + ss_1) = \frac{1}{4}(1 - r)(1 - s) \tag{E.4}$$

$$N_2 = \frac{1}{4}(1 + rr_2)(1 + ss_2) = \frac{1}{4}(1 + r)(1 - s) \tag{E.5}$$

$$N_3 = \frac{1}{4}(1 + rr_3)(1 + ss_3) = \frac{1}{4}(1 + r)(1 + s) \tag{E.6}$$

$$N_4 = \frac{1}{4}(1 + rr_4)(1 + ss_4) = \frac{1}{4}(1 - r)(1 + s) \tag{E.7}$$

Substitution of Eqs. (E.4)—(E.7), along with the data in Eq. (E.1), into Eqs. (E.2) and (E.3) gives

$$x = \frac{1}{4}[(1 - r)(1 - s)6 + (1 + r)(1 - s)2 + (1 + r)(1 + s)3 + (1 - r)(1 + s)10] \tag{E.8}$$

$$y = \frac{1}{4}[(1 - r)(1 - s)9 + (1 + r)(1 - s)7 + (1 + r)(1 + s)10 + (1 - r)(1 + s)6] \tag{E.9}$$

For the given values of $r = -0.75$ and $s = 0.5$, Eqs. (E.8) and (E.9) yield

$$x = \frac{1}{4}[(1 + 0.75)(1 - 0.5)6 + (1 - 0.75)(1 - 0.5)2 + (1 - 0.75)(1 + 0.5)3 + (1 + 0.75)(1 + 0.5)10)] = 8.21875 \text{ in.}$$

$$y = \frac{1}{4}[(1 + 0.75)(1 - 0.5)9 + (1 - 0.75)(1 - 0.5)7 + (1 - 0.75)(1 + 0.5)10 + (1 + 0.75)(1 + 0.5)6)] = 7.0625 \text{ in.}$$

QUADRATIC ELEMENT

If the values of $\phi(x, y)$ at the four corner nodes and four mid-side nodes are taken as the nodal unknowns, we get a quadratic element for which the variation of the field variable along any edge is given by a quadratic equation. In this case, the interpolation model can be expressed as

$$\phi(x, y) = [N]\overrightarrow{\Phi}^{(e)} = [N_1 \quad N_2 \quad \cdots \quad N_8]\overrightarrow{\Phi}^{(e)} \tag{4.41}$$

where

$$\begin{aligned} N_i &= \frac{1}{4}(1 + rr_i)(1 + ss_i)(rr_i + ss_i - 1), \quad i = 1, 2, 3, 4 \\ N_5 &= \frac{1}{2}\left(1 - r^2\right)(1 + ss_5) \\ N_6 &= \frac{1}{2}(1 + rr_6)\left(1 - s^2\right) \\ N_7 &= \frac{1}{2}\left(1 - r^2\right)(1 + ss_7) \\ N_8 &= \frac{1}{2}(1 + rr_8)\left(1 - s^2\right) \end{aligned} \tag{4.42}$$

(r_i, s_i) are the natural coordinates of node i ($i = 1, 2, \ldots, 8$), and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_8 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \vdots \\ \phi(x_8, y_8) \end{Bmatrix}^{(e)} \equiv \begin{Bmatrix} \phi(\text{at } r = -1,\ s = -1) \\ \phi(\text{at } r = 1,\ s = -1) \\ \vdots \\ \phi(\text{at } r = -1,\ s = 0) \end{Bmatrix}^{(e)} \tag{4.43}$$

Typical quadratic interpolation or shape functions used in Eq. (4.41) are shown in Fig. 4.6B.

4.3.4 Three-Dimensional (Tetrahedron) Element

QUADRATIC ELEMENT

The natural or tetrahedral coordinates L_1, L_2, L_3, and L_4 of a tetrahedron element were shown in Fig. 3.15. For a quadratic interpolation model, there will be 10 nodal unknowns, one at each of the nodes indicated in Fig. 4.7A. Here, the nodes 1, 2, 3, and 4 correspond to the corners, whereas the nodes 5 to 10 are located at the midpoints of the edges of the tetrahedron. The variation of the field variable is given by

$$\phi(x, y, z) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad \cdots \quad N_{10}]\vec{\Phi}^{(e)} \tag{4.44}$$

where N_i can be found as

$$\begin{aligned} N_i &= L_i(2L_i - 1), \quad i = 1, 2, 3, 4 \\ N_5 &= 4L_1L_2 \\ N_6 &= 4L_2L_3 \\ N_7 &= 4L_1L_3 \\ N_8 &= 4L_1L_4 \\ N_9 &= 4L_2L_4 \\ N_{10} &= 4L_3L_4 \end{aligned} \tag{4.45}$$

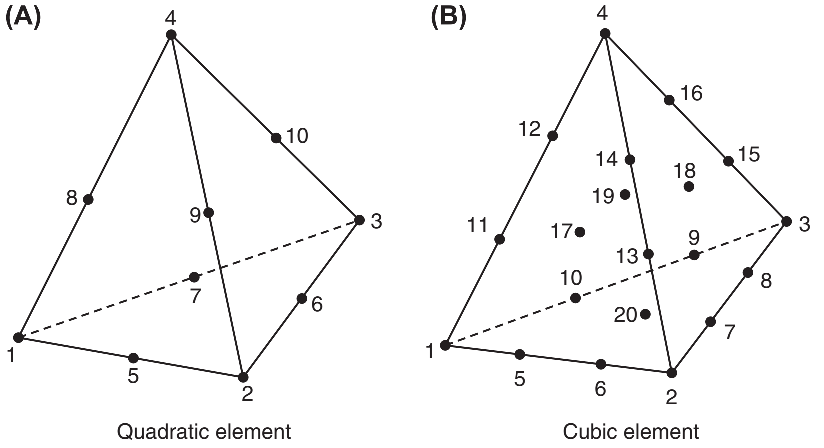


FIGURE 4.7 Location of nodes in tetrahedron element.

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{10} \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1, y_1, z_1) \\ \phi(x_2, y_2, z_2) \\ \vdots \\ \phi(x_{10}, y_{10}, z_{10}) \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(\text{at } L_1 = 1,\ L_2 = L_3 = L_4 = 0) \\ \phi(\text{at } L_2 = 1,\ L_1 = L_3 = L_4 = 0) \\ \vdots \\ \phi\left(\text{at } L_3 = L_4 = \frac{1}{2},\ L_1 = L_2 = 0\right) \end{Bmatrix}^{(e)} \tag{4.46}$$

CUBIC ELEMENT

The cubic interpolation model involves 20 nodal unknowns (nodes are shown in Fig. 4.7B) and can be expressed as

$$\phi(x, y, z) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad \cdots \quad N_{20}]\vec{\Phi}^{(e)} \tag{4.47}$$

where the nodal shape functions can be determined as follows:

$$\text{For corner nodes: } N_i = \frac{1}{2} L_i(3L_i - 1)(3L_i - 2), \quad i = 1, 2, 3, 4 \tag{4.48}$$

$$\begin{aligned} \text{For one-third points of edges:} \quad & N_5 = \frac{9}{2} L_1 L_2 (3L_1 - 1) \\ & N_6 = \frac{9}{2} L_1 L_2 (3L_2 - 1) \\ & N_7 = \frac{9}{2} L_2 L_3 (3L_2 - 1) \\ & N_8 = \frac{9}{2} L_2 L_3 (3L_3 - 1), \text{ and so on} \end{aligned} \tag{4.49}$$

$$\begin{aligned} \text{For midface nodes:} \quad & N_{17} = 27 L_1 L_2 L_4 \\ & N_{18} = 27 L_2 L_3 L_4 \\ & N_{19} = 27 L_1 L_3 L_4 \\ & N_{20} = 27 L_1 L_2 L_3 \end{aligned} \tag{4.50}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{20} \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x_1, y_1, z_1) \\ \phi(x_2, y_2, z_2) \\ \vdots \\ \phi(x_{20}, y_{20}, z_{20}) \end{Bmatrix}^{(e)} \equiv \begin{Bmatrix} \phi(\text{at } L_1 = 1,\ L_2 = 0, L_3 = 0, L_4 = 0) \\ \phi(\text{at } L_1 = 0,\ L_2 = 1, L_3 = 0, L_4 = 0) \\ \vdots \\ \phi\left(\text{at } L_1 = L_2 = L_3 = L_4 = \frac{1}{3}\right) \end{Bmatrix}^{(e)} \tag{4.51}$$

4.4 HIGHER ORDER ELEMENTS IN TERMS OF CLASSICAL INTERPOLATION POLYNOMIALS

It is possible to construct the nodal interpolation functions N_i by employing classical interpolation polynomials (instead of natural coordinates). We consider the use of Lagrange and Hermite interpolation polynomials in this section.

4.4.1 Classical Interpolation Functions

In numerical mathematics, an approximation polynomial that is equal to the function it approximates at a number of specified stations or points is called an *interpolation function*. A generalization of the interpolation function is obtained by requiring agreement with not only the function value $\phi(x)$ but also the first N derivatives of $\phi(x)$ at any number of distinct points x_i, $i = 1, 2, \ldots, n+1$. When $N = 0$—that is, when only the function values are required to match (agree) at each point of interpolation—the (classical) interpolation polynomial is called the *Lagrange interpolation formula*. For the case of $N = 1$—that is, when the function and its first derivative are to be assigned at each point of interpolation—the (classical) interpolation polynomial is called the *Hermite* or *osculatory interpolation formula*. If higher derivatives of $\phi(x)$ are assigned (i.e., when $N > 1$), we obtain the *hyperosculatory interpolation formula* [4.6–4.8].

4.4.2 Lagrange Interpolation Functions for *n* Stations

The Lagrange interpolation polynomials are defined as [4.1]

$$L_k(x) = \prod_{\substack{i=0,\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \tag{4.52}$$

$L_k(x)$ is an n-th degree polynomial because it is given by the product of n linear factors. If $x = x_k$, the numerator would be equal to the denominator in Eq. (4.52), and hence $L_k(x)$ will have a value of 1. On the other hand, if $x = x_i$ and $i \neq k$, the numerator and hence $L_k(x)$ will be 0. This property of $L_k(x)$ can be used to approximately represent any arbitrary function $\phi(x)$ over an interval on the x axis.

For example, if the values of $\phi\ (x)$ are known only at the discrete points x_0, x_1, x_2, and x_3, the approximating polynomial $\underset{\sim}{\phi}\ (x)$ can be written as

$$\phi(x) = \underset{\sim}{\phi}\ (x) = \sum_{i=0}^{3} \Phi_i L_i(x) \tag{4.53}$$

where Φ_i is the value of ϕ at $x = x_i$, $i = 0, 1, 2, 3$. Fig. 4.8 shows the typical shape of $L_i(x)$. Here, the function $\underset{\sim}{\phi}\ (x)$ is called the Lagrange interpolation formula. Thus, Lagrange interpolation functions can be used if the matching of only the function values (not derivatives) is involved for a line element.

4.4.3 General Two-Station Interpolation Functions

We denote a general one-dimensional interpolation polynomial as $H_{ki}^{(N)}(x)$, where N is the number of derivatives to be interpolated, k is an index varying from 0 to N, and i corresponds to the station index (i.e., the i-th point of the discrete set of points of interpolation). For simplicity we consider the case in which there are only two points of interpolation (as in the

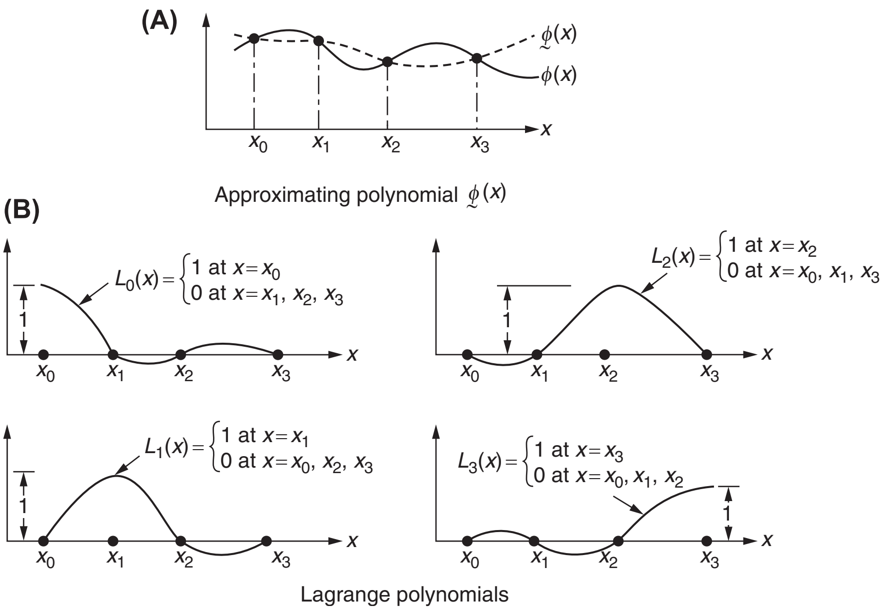


FIGURE 4.8 Lagrange interpolation formula using Lagrange polynomials.

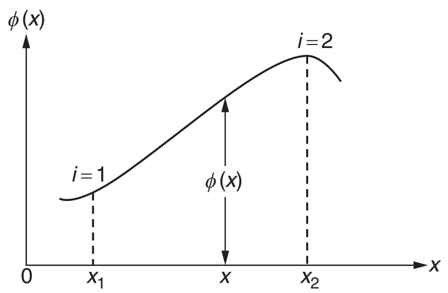


FIGURE 4.9 A one-dimensional function to be interpolated between stations x_1 and x_2.

case of one-dimensional elements). We denote the first point of interpolation as $i = 1(x_1 = 0)$ and the second point as $i = 2(x_2 = l)$, where l is the distance between the two points. Any function $\phi(x)$ shown in Fig. 4.9 can be approximated by using the Hermite functions as

$$\begin{aligned}\phi(x) &= \sum_{i=1}^{2} \sum_{k=0}^{N} H_{ki}^{(N)}(x)\Phi_i^k \\ &= \sum_{i=1}^{2}\left[H_{0i}^{(N)}(x)\Phi_i^{(0)} + H_{1i}^{(N)}(x)\Phi_i^{(1)} + H_{2i}^{(N)}(x)\Phi_i^{(2)} + \cdots + H_{Ni}^{(N)}(x)\Phi_i^{(N)}\right]\end{aligned} \tag{4.54}$$

where $\Phi_i^{(k)}$ are undetermined parameters. The Hermite polynomials have the following property:

$$\left.\begin{aligned}\frac{\mathrm{d}^r H_{ki}^{(N)}}{\mathrm{d}x^r}(x_p) &= \delta_{ip}\delta_{kr} \quad \text{for} \quad i,\ p = 1,\ 2, \text{and} \\ k, r &= 0, 1, 2, \ldots, N\end{aligned}\right\} \tag{4.55}$$

where x_p is the value of x at p-th station, and δ_{mn} is the Kronecker delta having the property

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \tag{4.56}$$

By using the property of Eq. (4.55) the undetermined parameters $\Phi_i^{(k)}$ appearing in Eq. (4.54) can be shown to have certain physical meaning. The r-th derivative of $\phi\ (x)$ at $x = x_p$ can be written as, from Eq. (4.54),

$$\frac{\mathrm{d}^r\phi}{\mathrm{d}x^r}(x_p) = \sum_{i=1}^{2}\sum_{k=0}^{N}\frac{\mathrm{d}^r H_{ki}^{(N)}}{\mathrm{d}x^r}(x_p)\Phi_i^{(k)} \tag{4.57}$$

Using Eq. (4.55), Eq. (4.57) can be reduced to

$$\frac{\mathrm{d}^r\phi}{\mathrm{d}x^r}(x_p) = \sum_{i=1}^{2}\sum_{k=0}^{N}\delta_{ip}\delta_{kr}\Phi_i^{(k)} = \Phi_p^{(r)} \tag{4.58}$$

Thus, $\Phi_p^{(r)}$ indicates the value of r-th derivative of $\phi\ (x)$ at station p. For $r = 0$ and 1, the parameters $\Phi_p^{(r)}$ are shown in Fig. 4.10. From Eqs. (4.58) and (4.54) the function $\phi\ (x)$ can be expressed as

$$\phi(x) = \sum_{i=1}^{2}\sum_{k=0}^{N} H_{ki}^{(N)}(x)\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x_i) \tag{4.59}$$

Hermite interpolation functions find application in certain one- and two-dimensional (structural beam- and plate-bending) problems in which continuity of derivatives across element interfaces is important.

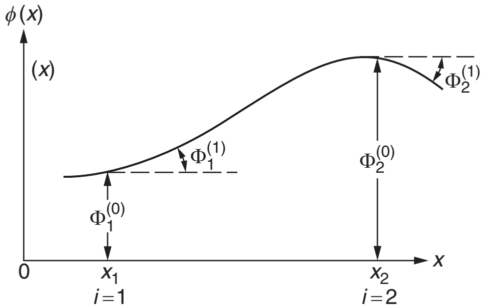


FIGURE 4.10 Physical meaning of the parameter $\Phi_p^{(r)}$.

4.4.4 Zeroth-Order Hermite Interpolation Function

The general expression given in Eq. (4.54) can be specialized to the case of the two-station zeroth-order Hermite (Lagrange) interpolation formula as

$$\phi(x) = \sum_{i=1}^{2} H_{0i}^{(0)} \Phi_i^{(0)} = \sum_{i=1}^{2} H_{0i}^{(0)}(x)\phi(x_i) \tag{4.60}$$

To find the polynomials $H_{01}^{(0)}(x)$ and $H_{02}^{(0)}(x)$, we use the property given by Eq. (4.55). For the polynomial $H_{01}^{(0)}(x)$, we have

$$\frac{d^{(0)} H_{01}^{(0)}(x_p)}{dx^{(0)}} = \delta_{1p}\delta_{00} = \delta_{1p} \equiv H_{01}^{(0)}(x_p)$$

or

$$H_{01}^{(0)}(x_1) = 1 \quad \text{and} \quad H_{01}^{(0)}(x_2) = 0 \tag{4.61}$$

Since two conditions are known (Eq. 4.61), we assume $H_{01}^{(0)}(x)$ as a polynomial involving two unknown coefficients as

$$H_{01}^{(0)}(x) = a_1 + a_2 x \tag{4.62}$$

By using Eq. (4.61), we find that

$$a_1 = 1 \quad \text{and} \quad a_2 = -1/l$$

by assuming that $x_1 = 0$ and $x_2 = l$. Thus, we have

$$H_{01}^{(0)}(x) = 1 - \frac{x}{l} \tag{4.63}$$

Similarly, the polynomial $H_{02}^{(0)}(x)$ can be found by using the conditions

$$H_{02}^{(0)}(x_1) = 0 \quad \text{and} \quad H_{02}^{(0)}(x_2) = 1 \tag{4.64}$$

as

$$H_{02}^{(0)}(x) = \frac{x}{l} \tag{4.65}$$

The shape of the Lagrange polynomials $H_{01}^{(0)}(x)$ and $H_{02}^{(0)}(x)$ and the variation of the function ϕ (x) approximated by Eq. (4.60) between the two stations are shown in Fig. 4.11 and 4.12, respectively. Note that the Lagrange polynomials given by Eqs. (4.63) and (4.65) are special cases (two-station formulas) of the more general (n-station) polynomial given by Eq. (4.52).

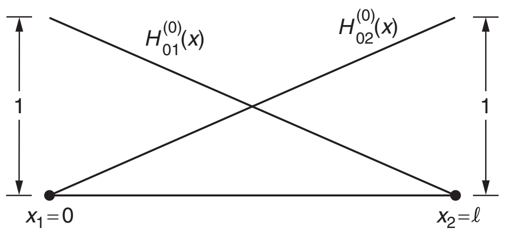


FIGURE 4.11 Variation of Lagrange polynomials between the two stations.

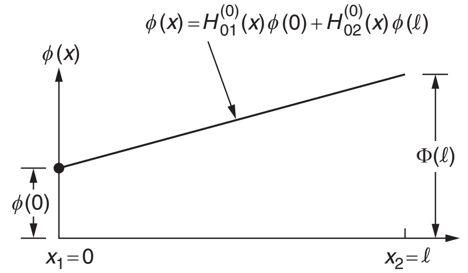


FIGURE 4.12 Variation of ϕ (x) approximated by Lagrange polynomials between the two stations.

4.4.5 First-Order Hermite Interpolation Function

If the function values as well as the first derivatives of the function are required to match with their true values, the two-station interpolation function is known as first-order Hermite (or osculatory) interpolation and is given by

$$\phi(x) = \sum_{i=1}^{2}\sum_{k=0}^{1} H_{ki}^{(1)}(x)\Phi_i^k = \sum_{i=1}^{2}\sum_{k=0}^{1} H_{ki}^{(1)}(x)\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}(x_i) \tag{4.66}$$

To determine the polynomials $H_{01}^{(1)}(x), H_{02}^{(1)}(x), H_{11}^{(1)}(x)$, and $H_{12}^{(1)}(x)$, four conditions are known from Eq. (4.55) for each of the polynomials. Thus, to find $H_{01}^{(1)}(x)$, we have

$$H_{01}^{(1)}(x_1) = 1, \quad H_{01}^{(1)}(x_2) = 0, \quad \frac{\mathrm{d}H_{01}^{(1)}}{\mathrm{d}x}(x_1) = 0, \quad \text{and} \quad \frac{\mathrm{d}H_{01}^{(1)}}{\mathrm{d}x}(x_2) = 0 \tag{4.67}$$

Since four conditions are known, we assume a cubic equation, which involves four unknown coefficients, for $H_{01}^{(1)}(x)$ as

$$H_{01}^{(1)}(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \tag{4.68}$$

By using Eq. (4.67), the constants can be found as

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{3}{l^2}, \quad \text{and} \quad a_4 = \frac{2}{l^3}$$

Thus, the Hermite polynomial $H_{01}^{(1)}(x)$ becomes

$$H_{01}^{(1)}(x) = \frac{1}{l^3}\left(2x^3 - 3lx^2 + l^3\right) \tag{4.69}$$

Similarly, the other first-order Hermite polynomials can be obtained as

$$H_{02}^{(1)}(x) = -\frac{1}{l^3}\left(2x^3 - 3lx^2\right) \tag{4.70}$$

$$H_{11}^{(1)}(x) = \frac{1}{l^2}\left(x^3 - 2lx^2 + l^2x\right) \tag{4.71}$$

$$H_{12}^{(1)}(x) = \frac{1}{l^2}\left(x^3 - lx^2\right) \tag{4.72}$$

The variations of the first-order Hermite polynomials between the two stations are shown in Fig. 4.13. The variation of the function approximated by these polynomials, namely,

$$\phi(x) = H_{01}^{(1)}(x)\phi(0) + H_{02}^{(1)}(x)\phi(l) + H_{11}^{(1)}(x)\frac{d\phi}{dx}(0) + H_{12}^{(1)}(x)\frac{d\phi}{dx}(l) \tag{4.73}$$

is shown in Fig. 4.14.

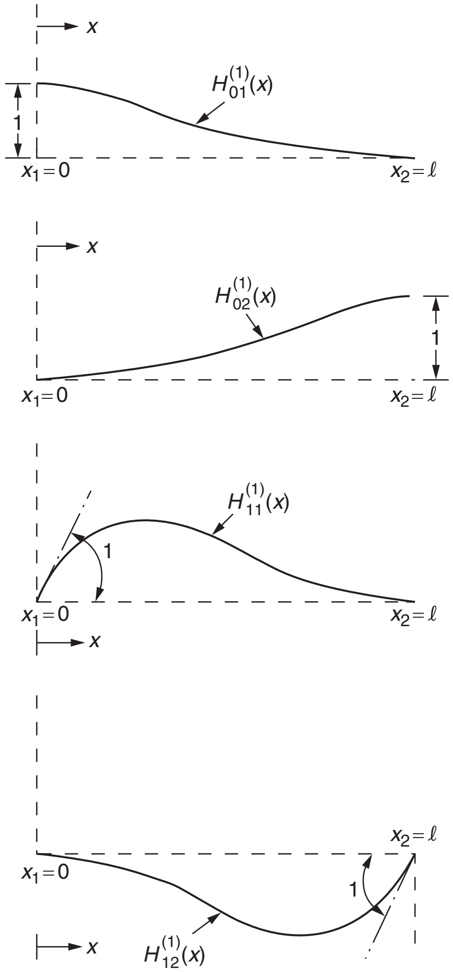


FIGURE 4.13 Variation of first-order Hermite polynomials between the two stations.

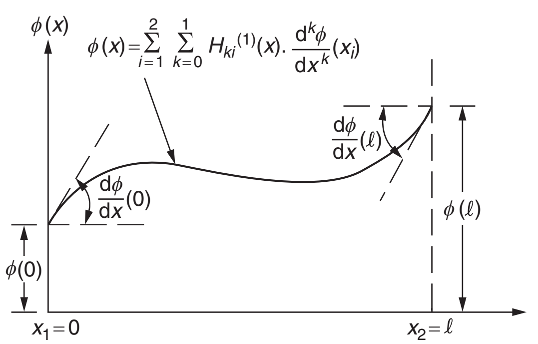


FIGURE 4.14 Variation of ϕ (x) given by Eq. (4.73) between the two stations.

4.5 ONE-DIMENSIONAL ELEMENTS USING CLASSICAL INTERPOLATION POLYNOMIALS

4.5.1 Linear Element

If the field variable $\phi\,(x)$ varies linearly along the length of a one-dimensional element and if the nodal values of the field variable, namely $\Phi_1 = \phi\,(x = x_1 = 0)$ and $\Phi_2 = \phi\,(x = x_2 = l)$, are taken as the nodal unknowns, we can use zeroth-order Hermite polynomials to express $\phi(x)$ as

$$\phi(x) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2]\vec{\Phi}^{(e)} \tag{4.74}$$

where

$$N_1 = H_{01}^{(0)}(x) = 1 - \frac{x}{l}$$

$$N_2 = H_{02}^{(0)}(x) = \frac{x}{l}$$

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix}^{(e)}$$

and l is the length of the element e.

4.5.2 Quadratic Element

If $\phi(x)$ is assumed to vary quadratically along x and the values of $\phi\,(x)$ at three points x_1, x_2, and x_3 are taken as nodal unknowns, $\phi(x)$ can be expressed in terms of three-station Lagrange interpolation polynomials as

$$\phi(x) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3]\vec{\Phi}^{(e)} \tag{4.75}$$

where

$$N_1 = L_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$N_2 = L_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$N_3 = L_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(x = x_1) \\ \phi(x = x_2) \\ \phi(x = x_3) \end{Bmatrix}^{(e)}$$

4.5.3 Cubic Element

If $\phi\,(x)$ is to be taken as a cubic polynomial and if the values of $\phi\,(x)$ and $(d\phi/dx)\,(x)$ at two nodes are taken as nodal unknowns, the first-order Hermite polynomials can be used to express $\phi\,(x)$ as

$$\phi(x) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3 \quad N_4]\vec{\Phi}^{(e)} \tag{4.76}$$

where

$$N_1(x) = H_{01}^{(1)}(x), \quad N_2(x) = H_{11}^{(1)}(x), \quad N_3(x) = H_{02}^{(1)}(x), \quad N_4(x) = H_{12}^{(1)}(x)$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix}^{(e)} \equiv \begin{Bmatrix} \phi(x = x_1) \\ \dfrac{d\phi}{dx}(x = x_1) \\ \phi(x = x_2) \\ \dfrac{d\phi}{dx}(x = x_2) \end{Bmatrix}^{(e)}$$

4.6 TWO-DIMENSIONAL (RECTANGULAR) ELEMENTS USING CLASSICAL INTERPOLATION POLYNOMIALS

4.6.1 Using Lagrange Interpolation Polynomials

The Lagrange interpolation polynomials defined in Eq. (4.52) for one-dimensional problems can be used to construct interpolation functions for two-dimensional or higher dimensional problems. For example, in two dimensions (see Fig. 4.15A), the product of Lagrange interpolation polynomials in x and y directions can be used to represent the interpolation functions of a rectangular element as

$$\phi(r,s) = [N]\vec{\Phi}^{(e)} = [N_1 \quad N_2 \quad N_3 \quad N_4]\vec{\Phi}^{(e)} \tag{4.77}$$

where

$$N_i(r,s) = L_i(r) \cdot L_i(s), \quad i = 1,2,3,4 \tag{4.78}$$

and

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi(r = -1, s = -1) \\ \phi(r = 1, s = -1) \\ \phi(r = 1, s = 1) \\ \phi(r = -1, s = 1) \end{Bmatrix}^{(e)} \tag{4.79}$$

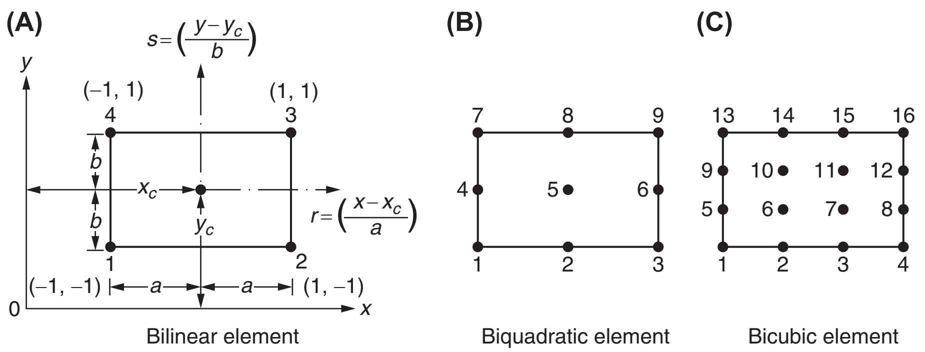


FIGURE 4.15 Location of nodes in rectangular elements.

$L_i(r)$ and $L_i(s)$ denote Lagrange interpolation polynomials in r and s directions corresponding to node i and are defined, with reference to Fig. 4.15A, as

$$L_1(r) = \frac{r-r_2}{r_1-r_2},\quad L_2(r) = \frac{r-r_1}{r_2-r_1},\quad L_3(r) = \frac{r-r_4}{r_3-r_4},\quad L_4(r) = \frac{r-r_3}{r_4-r_3}$$
$$L_1(s) = \frac{s-s_4}{s_1-s_4},\quad L_2(s) = \frac{s-s_3}{s_2-s_3},\quad L_3(s) = \frac{s-s_2}{s_3-s_2},\quad L_4(s) = \frac{s-s_1}{s_4-s_1} \tag{4.80}$$

The nodal interpolation functions N_i given by Eq. (4.78) are called bilinear since they are defined as products of two linear functions.

The higher order elements, such as biquadratic and bicubic elements, can be formulated precisely the same way by taking products of Lagrange interpolation polynomials of degree two and three, respectively, as

$$N_i(r,s) = L_i(r)\cdot L_i(s) \tag{4.81}$$

where $L_i(r)$ and $L_i(s)$ can be obtained with the help of Eq. (4.52) and Fig. 4.15B and C. For example, in the case of the biquadratic element shown in Fig. 4.15B, the Lagrange interpolation polynomials are defined as follows:

$$L_1(r) = \frac{(r-r_2)(r-r_3)}{(r_1-r_2)(r_1-r_3)},\quad L_1(s) = \frac{(s-s_4)(s-s_7)}{(s_1-s_4)(s_1-s_7)} \tag{4.82}$$

$$L_2(r) = \frac{(r-r_1)(r-r_3)}{(r_2-r_1)(r_2-r_3)},\quad L_2(s) = \frac{(s-s_5)(s-s_8)}{(s_2-s_5)(s_2-s_8)} \tag{4.83}$$

and so on. In this case, node 5 represents an interior node. It can be observed that the higher order Lagrangian elements contain a large number of interior nodes and this limits the usefulness of these elements. Of course, a technique known as static condensation can be used to suppress the degrees of freedom associated with the internal nodes in the final computation (see Problem 12.7).

4.6.2 Using Hermite Interpolation Polynomials

Just as we have done with Lagrange interpolation polynomials, we can form products of one-dimensional Hermite polynomials and derive the nodal interpolation functions N_i for rectangular elements. If we use first-order Hermite polynomials for this purpose, we have to take the values of ϕ, $(\partial\phi/\partial x)$, $(\partial\phi/\partial y)$, and $(\partial^2\phi/\partial x\partial y)$ as nodal degrees of freedom at each of the four corner nodes. Thus, by using a two-number scheme for identifying the nodes of the rectangle as shown in Fig. 4.16, the interpolation model for $\phi(x,y)$ can be expressed as

$$\begin{aligned}\phi(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg[&H_{0i}^{(1)}(x)\cdot H_{0j}^{(1)}(y)\cdot\phi_{ij} + H_{1i}^{(1)}(x)\cdot H_{0j}^{(1)}(y)\cdot\left(\frac{\partial\phi}{\partial x}\right)_{ij} + H_{0i}^{(1)}(x)\cdot H_{1j}^{(1)}(y)\cdot\left(\frac{\partial\phi}{\partial y}\right)_{ij}\\ &+ H_{1i}^{(1)}(x)\cdot H_{1j}^{(1)}(y)\cdot\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_{ij}\Bigg]\end{aligned} \tag{4.84}$$

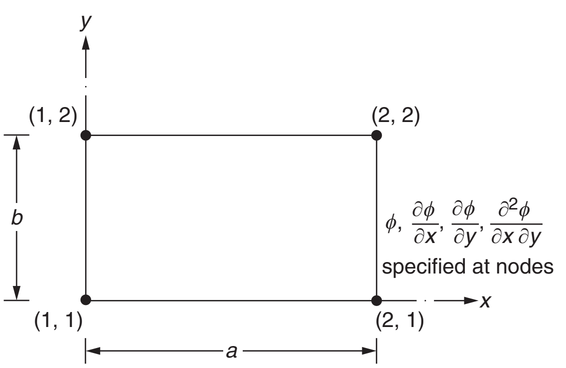


FIGURE 4.16 Rectangular element with 16 degrees of freedom.

where ϕ_{ij}, $(\partial\phi/\partial x)_{ij}$, $(\partial\phi/\partial y)_{ij}$, and $\left(\partial^2\phi/\partial x\partial y\right)_{ij}$ denote the values of ϕ, $(\partial\phi/\partial x)$, $(\partial\phi/\partial y)$, and $(\partial^2\phi/\partial x\partial y)$, respectively, at node (i, j). Eq. (4.84) can be rewritten in the familiar form as

$$\phi(x,y) = [N(x,y)]\overrightarrow{\Phi}^{(e)} = [N_1(x,y)\ldots N_{16}(x,y)]\overrightarrow{\Phi}^{(e)} \tag{4.85}$$

where

$$\begin{aligned} N_1(x,y) &= H_{01}^{(1)}(x)H_{01}^{(1)}(y) \\ N_2(x,y) &= H_{11}^{(1)}(x)H_{01}^{(1)}(y) \\ N_3(x,y) &= H_{01}^{(1)}(x)H_{11}^{(1)}(y) \\ N_4(x,y) &= H_{11}^{(1)}(x)H_{11}^{(1)}(y) \\ N_5(x,y) &= H_{02}^{(1)}(x)H_{01}^{(1)}(y) \\ &\vdots \\ N_{16}(x,y) &= H_{12}^{(1)}(x)H_{12}^{(1)}(y) \end{aligned} \tag{4.86}$$

and

$$\overrightarrow{\Phi}^{(e)} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \vdots \\ \Phi_{16} \end{Bmatrix}^{(e)} = \begin{Bmatrix} \phi_{11} \\ \left(\dfrac{\partial\phi}{\partial x}\right)_{11} \\ \left(\dfrac{\partial\phi}{\partial y}\right)_{11} \\ \left(\dfrac{\partial^2\phi}{\partial x\partial y}\right)_{11} \\ \phi_{21} \\ \vdots \\ \left(\dfrac{\partial^2\phi}{\partial x\partial y}\right)_{22} \end{Bmatrix}^{(e)} \tag{4.87}$$

4.7 CONTINUITY CONDITIONS

As seen in Section 3.6, the interpolation model assumed for the field variable ϕ has to satisfy the following conditions:

1. It has to be continuous inside and between the elements up to order $r-1$, where r is the order of the highest derivative in the functional I. For example, if the governing differential equation is quasi-harmonic as in the case of Example 1.3, ϕ have to be continuous (i.e., C^0 continuity is required). On the other hand, if the governing differential equation is biharmonic ($\nabla^4\phi = 0$), ϕ as well as its derivative $(\partial\phi/\partial n)$ have to be continuous inside and between elements (i.e., C^1 continuity is required). The continuity of the higher order derivatives associated with the free or natural boundary conditions need not be imposed because their eventual satisfaction is implied in the variational statement of the problem.
2. As the size of the elements decreases, the derivatives appearing in the functional of the variational statement will tend to have constant values. Thus, it is necessary to include terms that represent these conditions in the interpolation model of ϕ.

For elements requiring C^0 continuity (i.e., continuity of only the field variable ϕ at element interfaces), we usually take the nodal values of ϕ only as the degrees of freedom. To satisfy the interelement continuity condition, we have to take the number of nodes along a side of the element (and hence the number of nodal values of ϕ) to be sufficient to determine the variation of ϕ along that side uniquely. Thus, if a cubic interpolation model is assumed within the element and retains its cubic behavior along the element sides, then we have to take four nodes (and hence four nodal values of ϕ) along each side.

It can be observed that the number of elements (of a given shape) capable of satisfying C^0 continuity is infinite. This is because we can continue to add nodes and degrees of freedom to the elements to form ever-increasing higher order elements. All such elements will satisfy the C^0 continuity. In general, higher order elements can be derived by increasing the number of nodes, and hence the nodal degrees of freedom, and assuming a higher order interpolation model for the field variable ϕ. As stated earlier, in general, smaller numbers of higher order elements yield more accurate results compared to larger numbers of simpler elements for the same overall effort. But this does not mean that we should always favor elements of very high order. Although there are no general guidelines available for choosing the order of the element for a given problem, elements that require polynomials of order greater than three have seldom been used for problems requiring C^0 continuity. The main reason for this is that the computational effort saved with fewer numbers of higher order elements will become overshadowed by the increased effort required in formulating and evaluating the element characteristic matrices and vectors.

4.7.1 Elements With C^0 Continuity

All simplex elements considered in Section 3.7 satisfy C^0 continuity because their interpolation models are linear. Furthermore, all higher order one-, two-, and three-dimensional elements considered in this chapter also satisfy the C^0 continuity. For example, each of the triangular elements shown in Fig. 4.3 has a sufficient number of nodes (and hence the nodal degrees of freedom) to uniquely specify a complete polynomial of the order necessary to give C^0 continuity. Thus, the corresponding interpolation models satisfy the requirements of compatibility, completeness, and geometric isotropy. In general, for a triangular element, a complete polynomial of order n requires $(1/2)\,(n+1)\,(n+2)$ nodes for its specification. Similarly, a tetrahedron element requires $(1/6)\,(n+1)\,(n+2)\,(n+3)$ nodes in order to have the interpolation model in the form of a complete polynomial of order n. For such elements, if the nodal values of ϕ only are taken as degrees of freedom, the conditions of compatibility, completeness, and geometric isotropy will be satisfied.

The quadrilateral element discussed in Section 4.3.3 considers only the nodal values of ϕ as the degrees of freedom and satisfies C^0 continuity. For rectangular elements, if the nodal interpolation functions are defined by-products of Lagrange interpolation polynomials (Fig. 4.15), then the C^0 continuity is satisfied.

4.7.2 Elements With C^1 Continuity

The construction of elements that satisfy C^1 continuity of the field variable ϕ is much more difficult than constructing elements for C^0 continuity. To satisfy the C^1 continuity, we have to ensure continuity of ϕ as well as its normal derivative $\partial\phi/\partial n$ along the element boundaries. The one-dimensional cubic element considered in Section 4.5.3 guarantees the continuity of both ϕ and $d\phi/dn$ at the nodes and hence it satisfies the C^1 continuity.

For two-dimensional elements, we have to ensure that ϕ and $\partial\phi/\partial n$ are specified uniquely along an element boundary by the nodal degrees of freedom associated with the nodes of that particular boundary. The rectangular element considered in Fig. 4.16 (Eq. 4.84) considers ϕ, $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial^2\phi/\partial x\partial y$ as nodal degrees of freedom and satisfies the C^1 continuity. In the case of a triangular element, some authors have treated the values of $\phi, (\partial\phi/\partial x), (\partial\phi/\partial y), \left(\partial^2\phi/\partial x\partial y\right), \left(\partial^2\phi/\partial x^2\right)$, and $(\partial^2\phi/\partial y^2)$ at the three corner nodes and the values of $(\partial\phi/\partial n)$ at the three mid-side nodes (Fig. 4.17) as degrees of freedom, and represented the interpolation model of ϕ by a complete quintic polynomial.

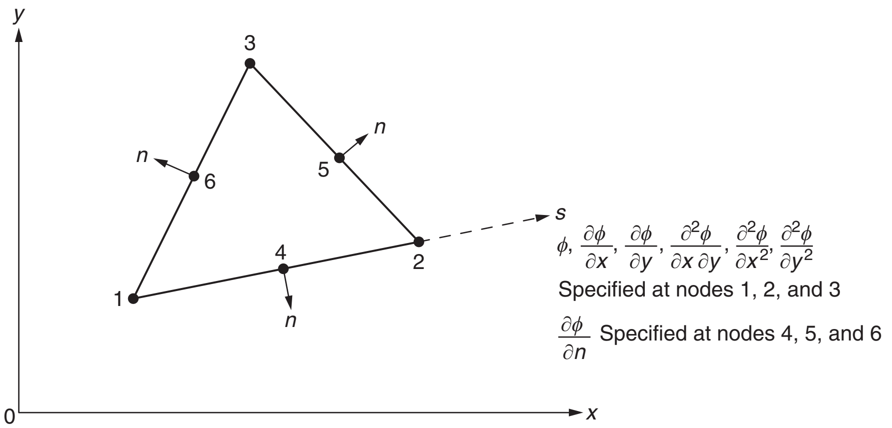


FIGURE 4.17 Triangular element with C^1 continuity.

If s denotes the linear coordinate along any boundary of the element, then ϕ varies along s as a fifth-degree polynomial. This fifth-degree polynomial is uniquely determined by the six nodal degrees of freedom, namely ϕ, $(\partial\phi/\partial s)$, and $(\partial^2\phi/\partial s^2)$ at each of the two end nodes. Hence, ϕ will be continuous along the element boundaries. Similarly, the normal slope $(\partial\phi/\partial n)$ can be seen to vary as a fourth-degree polynomial in s along the element boundary. There are five nodal degrees of freedom to determine this quartic polynomial uniquely. These are the values of $(\partial\phi/\partial n)$ and $(\partial^2\phi/\partial n^2)$ at each of the end nodes and $(\partial\phi/\partial n)$ at the mid-side node. Hence, the normal slope $(\partial\phi/\partial n)$ will also be continuous along the element boundaries. In the case of three-dimensional elements, the satisfaction of C^1 continuity is quite difficult and practically no such element has been used in the literature.

Note

Since the satisfaction of C^1 continuity is difficult to achieve, many investigators have used finite elements that satisfy slope continuity at the nodes and other requirements but violate slope continuity along the element boundaries. Such elements are known as incompatible or nonconforming elements and have been used with surprising success in plate-bending (two-dimensional structural) problems.

4.8 COMPARATIVE STUDY OF ELEMENTS

The relative accuracy of the results obtained by using interpolation polynomials of different orders was studied by Emery and Carson [4.2]. They considered the solution of a one-dimensional steady-state diffusion equation as a test case. The governing equation is

$$\frac{d^2\phi}{dx^2} = \psi(x), \quad 0 \le x \le 1 \tag{4.88}$$

with $\psi(x) = x^5$; the boundary conditions are

$$\phi(x=0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(x=1) = 0 \tag{4.89}$$

By dividing the region ($x = 0$ to 1) into different numbers of finite elements, they obtained the results using linear, quadratic, and cubic interpolation models. The results are shown in Fig. 4.18 along with those given by the finite difference method. The ordinate in Fig. 4.18 denotes the error in the temperature (ϕ) at the point $x = 1$. The exact solution obtained by integrating Eq. (4.88) with the boundary conditions, Eq. (4.89), gives the value of ϕ at $x = 1$ as 0.1429. The results indicate

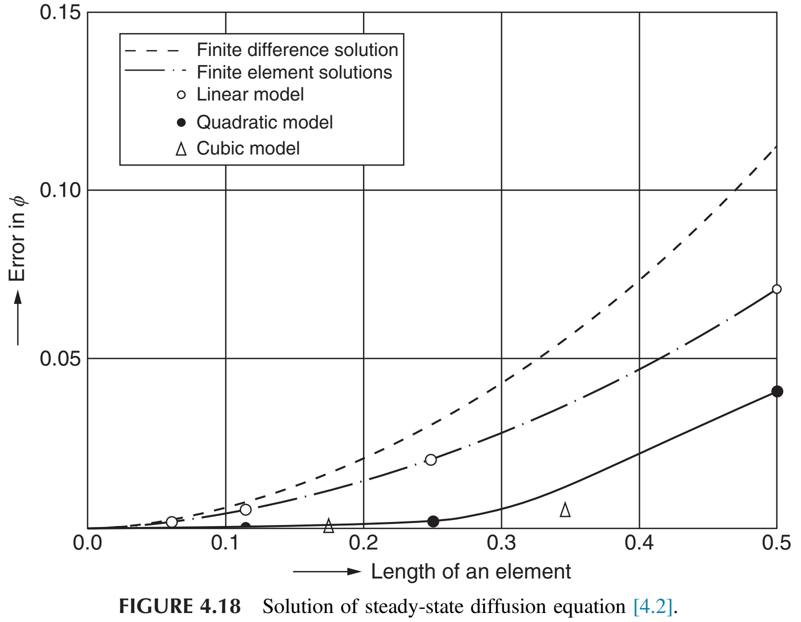


FIGURE 4.18 Solution of steady-state diffusion equation [4.2].

that the higher order models yield better results in this case. This characteristic has been found to be true even for higher dimensional problems. If the overall computational effort involved and the accuracy achieved are compared, we might find the quadratic model to be the most efficient one for use in complex practical problems.

4.9 ISOPARAMETRIC ELEMENTS

4.9.1 Definitions

In the case of one-dimensional elements, Eqs. (3.62) and (3.24) give

$$x = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{4.90}$$

and

$$\phi = [N_1 \quad N_2]\begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix}^{(e)} \tag{4.91}$$

where $N_1 = L_1$ and $N_2 = L_2$. In the case of a triangular element, if we consider ϕ as a vector quantity with components $u(x, y)$ and $v(x, y)$, Eqs. (3.71) and (3.33) give

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{4.92}$$

and

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{4.93}$$

where $N_1 = L_1$, $N_2 = L_2$, $N_3 = L_3$, and (x_i, y_i) are the Cartesian coordinates of node i, and u_i and v_i are the values of u and v, respectively, at node i ($i = 1, 2, 3$). Similarly, for a quadrilateral element, the geometry and field variable are given by Eqs. (4.32) and (4.38) as (assuming ϕ to be a vector with components u and v)

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{4.94}$$

and

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \tag{4.95}$$

where N_i ($i = 1, 2, 3, 4$) are given by Eq. (4.33), (x_i, y_i) are the Cartesian coordinates of node i, and (u_i, v_i) are the components of $\phi(u, v)$ at node i. A comparison of Eqs. (4.90) and (4.91), or (4.92) and (4.93), or (4.94) and (4.95) shows that the geometry and field variables of these elements are described in terms of the same parameters and of the same order.

Such elements whose shape (or geometry) and field variables are described by the same interpolation functions of the same order are known as isoparametric elements. These elements have been used with great success in solving two- and three-dimensional elasticity problems, including those involving plates and shells [4.3]. These elements have become popular for the following reasons:

1. If one element is understood, the same concepts can be extended for understanding all isoparametric elements.
2. Although linear elements have straight sides, quadratic and higher order isoparametric elements may have either straight or curved sides. Hence, these elements can be used for idealizing regions having curved boundaries.

It is not necessary to use interpolation functions of the same order for describing both geometry and the field variable of an element. If geometry is described by a lower order model compared to the field variable, the element is called a subparametric element. On the other hand, if the geometry is described by a higher order interpolation model than the field variable, the element is termed a superparametric element.

4.9.2 Shape Functions in Coordinate Transformation

The equations that describe the geometry of the element, namely,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 \ldots N_p & 0 & 0 \ldots 0 & 0 & 0 \ldots 0 \\ 0 & 0 \ldots 0 & N_1 & N_2 \ldots N_p & 0 & 0 \ldots 0 \\ 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & N_1 & N_2 \ldots N_p \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \\ y_1 \\ y_2 \\ \vdots \\ y_p \\ z_1 \\ z_2 \\ \vdots \\ z_p \end{Bmatrix} \tag{4.96}$$

(p = number of nodes of the element) can be considered a transformation relation between the Cartesian (x, y, z) coordinates and curvilinear (r, s, t or L_1, L_2, L_3, L_4) coordinates if the shape functions N_i are nonlinear in terms of the natural coordinates of the element. Eq. (4.96) can also be considered as the mapping of a straight-sided element in local coordinates into a curved-sided element in the global Cartesian coordinate system. Thus, for any set of coordinates L_1, L_2, L_3, and L_4, or r, s, and t, there corresponds a set of x, y, and z. Such mapping permits elements of one-, two-, and three-dimensional types to be mapped into distorted forms in the manner illustrated in Fig. 4.19.

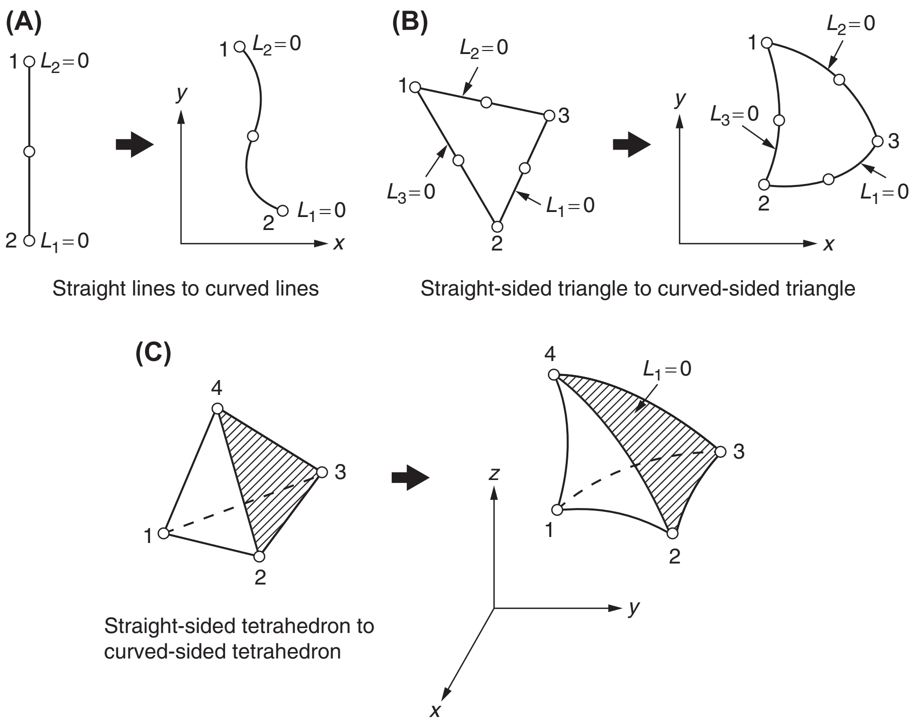


FIGURE 4.19 Mapping of elements.

To each set of local coordinates, there will be, in general, only one set of Cartesian coordinates. However, in some cases, a nonuniqueness may arise with violent distortion. In order to have unique mapping of elements, the number of coordinates (L_1, L_2, L_3, L_4) or (r, s, t) and (x, y, z) must be identical and the Jacobian, defined by

$$|[J]| = \frac{\partial(x, y, \ldots)}{\partial(L_1, L_2, \ldots)} = \left| \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial x}{\partial L_2} & \cdots \\ \dfrac{\partial y}{\partial L_1} & \dfrac{\partial y}{\partial L_2} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix} \right| \tag{4.97}$$

must not change sign in the domain.

4.9.3 Curved-Sided Elements

The main idea underlying the development of curved-sided elements centers on mapping or transforming simple geometric shapes (with straight edges or flat surfaces) in some local coordinate system into distorted shapes (with curved edges or surfaces) in the global Cartesian coordinate system and then evaluating the element equations for the resulting curved-sided elements.

To clarify the idea, we shall consider a two-dimensional example. The extension of the idea to one- and three-dimensional problems will be straightforward. Let the problem to be analyzed in a two-dimensional (x, y) space be as shown in Fig. 4.20A and let the finite element mesh consist of curved-sided quadrilateral elements as indicated in Fig. 4.20A. Let the field variable ϕ (e.g., displacement) be taken to vary quadratically within each element. According to the discussion of Section 4.3.3, if we want to take only the nodal values of ϕ (but not the derivatives of ϕ) as degrees of freedom of the element, we need to take three nodes on each side of the quadrilateral element. In order to derive the finite element equations, we consider one typical element in the assemblage of Fig. 4.20A and focus our attention on the simpler

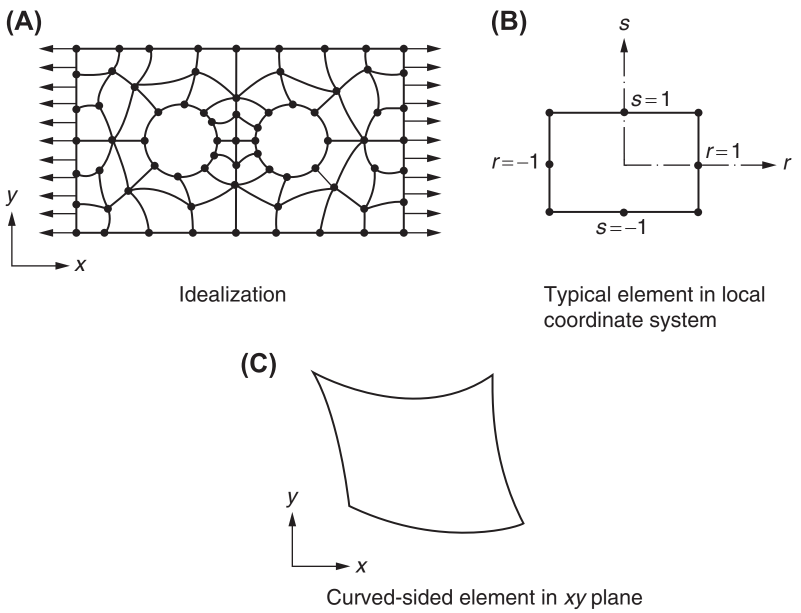


FIGURE 4.20 Curved-sided elements.

parent element in the local (r, s) coordinate system as shown in Fig. 4.20B. We find from Section 4.3.3 that the quadratic variation of ϕ within this parent element can be expressed as

$$\phi(r,s) = \sum_{i=1}^{8} N_i(r,s)\Phi_i \tag{4.98}$$

where N_i are the quadratic shape or interpolation functions used in Eq. (4.41). The eight nodes in the (r, s) plane may be mapped into corresponding nodes in the (x, y) plane by defining the relations

$$\left.\begin{aligned} x &= \sum_{i=1}^{8} f_i(r,s)x_i \\ y &= \sum_{i=1}^{8} f_i(r,s)y_i \end{aligned}\right\} \tag{4.99}$$

where $f_i(r, s)$ are the mapping functions. These functions, in this case, must be at least quadratic since the curved boundaries of the element in the (x, y) plane need at least three points for their specification and the f_i should take the proper values of 0 and 1 when evaluated at the corner nodes in the (r, s) plane.

If we take the quadratic shape functions N_i given in Eq. (4.41) for this purpose, we can write

$$\left.\begin{aligned} x &= \sum_{i=1}^{8} N_i(r,s)x_i \\ y &= \sum_{i=1}^{8} N_i(r,s)y_i \end{aligned}\right\} \tag{4.100}$$

The mapping defined by Eq. (4.100) results in a curved-sided quadrilateral element as shown in Fig. 4.20C. Thus, for this element, the functional description of the field variable ϕ as well as its curved boundaries are expressed by interpolation functions of the same order. According to the definition, this element is an isoparametric element. Similarly, the element is called subparametric or superparametric if the functional representation of geometry of the element [$f_i(r, s)$] is expressed in terms of a lower or a higher order polynomial than the one used for representing the field variable ϕ.

4.9.4 Continuity and Compatibility

We need to preserve the continuity and compatibility conditions in the global (x, y) coordinate system while constructing isoparametric elements using the following observations [4.4]:

1. If the interpolation functions in natural (local) coordinates satisfy continuity of geometry and field variable both within the element and between adjacent elements, the compatibility requirement will be satisfied in the global coordinates. The polynomial interpolation models discussed in Section 4.3 are inherently continuous within the element. Furthermore, we can notice that the field variable along any edge of the element depends only on the nodal degrees of freedom occurring on that edge when interpolation functions in natural coordinates are used. This can also be seen from Figs. 4.4 and 4.6, where, for example, the field variable along the edge 2-6-3 of Fig. 4.6B depends only on the values of the field variable at nodes 2, 6, and 3.
2. If the interpolation model provides constant values of ϕ in the local coordinate system, the conditions of both constant values of ϕ and its derivatives will be satisfied in the global coordinates.

Let the functional relation for the components of the vector-valued field variable in an isoparametric element be given by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N] \begin{Bmatrix} u_1 \\ \vdots \\ u_p \\ v_1 \\ \vdots \\ v_p \\ w_1 \\ \vdots \\ w_p \end{Bmatrix} \tag{4.101}$$

where

$$[N] = \begin{bmatrix} N_1 \ldots N_p & 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & N_1 \ldots N_p & 0 \ldots 0 \\ 0 \ldots 0 & 0 \ldots 0 & N_1 \ldots N_p \end{bmatrix} \tag{4.102}$$

and p is the number of nodes in the element. Thus, the u component of ϕ is given by

$$u = \sum_{i=1}^{p} N_i u_i \tag{4.103}$$

Let the geometry be given by

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [N] \begin{Bmatrix} x_1 \\ \vdots \\ x_p \\ y_1 \\ \vdots \\ y_p \\ z_1 \\ \vdots \\ z_p \end{Bmatrix} \tag{4.104}$$

where (x_i, y_i, z_i) are the coordinates of node $i (i = 1, 2, \ldots)$. For constant u, all points on the element must have the same value of u—for example, u_0. Hence, Eq. (4.103) becomes

$$u_0 = \left(\sum_{i=1}^{p} N_i \right) u_0 \tag{4.105}$$

Thus, we obtain the following necessary condition to be satisfied for constant values of ϕ in local coordinates:

$$\sum_{i=1}^{p} N_i = 1 \tag{4.106}$$

4.9.5 Derivation of Element Equations

In the case of structural and solid mechanics problems, the element characteristic (stiffness) matrix is given by[1]

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \cdot \mathrm{d}V \tag{4.107}$$

and the element characteristic (load) vector by

$$\vec{p}^{(e)} = \iiint_{V^{(e)}} [B]^T [D] \vec{\varepsilon}_0 \cdot \mathrm{d}V + \iiint_{V^{(e)}} [N]^T \vec{\overline{\phi}} \cdot \mathrm{d}V + \iint_{S_1^{(e)}} [N]^T \vec{\overline{\Phi}} \cdot \mathrm{d}S_1 \tag{4.108}$$

where $[B]$ is the matrix relating strains and nodal displacements, $[D]$ is the elasticity matrix, $\vec{\overline{\Phi}}$ is the vector of distributed surface forces, $\vec{\overline{\phi}}$ is the vector of body forces, and $\vec{\varepsilon}_0$ is the initial strain vector.

For a plane stress or plane strain problem, we have

$$\begin{Bmatrix} u(x,y) \\ u(x,y) \end{Bmatrix} = [N]\vec{\Phi}^{(e)} \equiv \begin{bmatrix} N_1 & N_2 \ldots N_p & 0 & 0 \ldots 0 \\ 0 & 0 \ldots 0 & N_1 & N_2 \ldots N_p \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \\ v_1 \\ v_2 \\ \vdots \\ v_p \end{Bmatrix} \tag{4.109}$$

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = [B]\,\vec{Q}^{(e)} \tag{4.110}$$

and

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} \cdots \dfrac{\partial N_p}{\partial x} & 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} \cdots \dfrac{\partial N_p}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} \cdots \dfrac{\partial N_p}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} \cdots \dfrac{\partial N_p}{\partial x} \end{bmatrix} \tag{4.111}$$

where p denotes the number of nodes of the element, (u_i, v_i) denote the values of (u, v) at node i, and N_i is the shape function associated with node i expressed in terms of natural coordinates (r, s) or (L_1, L_2, L_3). Thus, in order to evaluate $[K^{(e)}]$ and $\vec{p}^{(e)}$, two transformations are necessary. First, the shape functions N_i are defined in terms of local curvilinear coordinates (e.g., r and s), and hence the derivatives of N_i with respect to the global coordinates x and y must be expressed in terms of derivatives of N_i with respect to the local coordinates. Second, the volume and surface integrals needed in Eqs. (4.107) and (4.108) have to be expressed in terms of local coordinates with appropriate change of limits of integration.

For the first transformation, let us consider the differential of N_i with respect to the local coordinate r. Then, by the chain rule of differentiation, we have

1. Eqs. (4.107) and (4.108) are derived in Chapter 8.

$$\frac{\partial N_i}{\partial r} = \frac{\partial N_i}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial y}{\partial r} \tag{4.112}$$

Similarly,

$$\frac{\partial N_i}{\partial s} = \frac{\partial N_i}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Thus, we can express

$$\begin{Bmatrix} \partial N_i/\partial r \\ \partial N_i/\partial s \end{Bmatrix} = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial s & \partial y/\partial s \end{bmatrix} \begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} = [J] \begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} \tag{4.113}$$

where the matrix $[J]$, called the Jacobian matrix, is given by

$$[J] = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial s & \partial y/\partial s \end{bmatrix} \tag{4.114}$$

Since x and y (geometry) are expressed as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = [N] \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \\ y_1 \\ y_2 \\ \vdots \\ y_p \end{Bmatrix} \tag{4.115}$$

we can obtain the derivatives of x and y with respect to the local coordinates directly, and hence the Jacobian matrix can be expressed as

$$[J] = \begin{bmatrix} \sum_{i=1}^{p} \left(\frac{\partial N_i}{\partial r}\cdot x_i \right) & \sum_{i=1}^{p} \left(\frac{\partial N_i}{\partial r}\cdot y_i \right) \\ \sum_{i=1}^{p} \left(\frac{\partial N_i}{\partial s}\cdot x_i \right) & \sum_{i=1}^{p} \left(\frac{\partial N_i}{\partial s}\cdot y_i \right) \end{bmatrix} \tag{4.116}$$

Thus, we can find the global derivatives needed in Eq. (4.111) as

$$\begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \partial N_i/\partial r \\ \partial N_i/\partial s \end{Bmatrix} \tag{4.117}$$

For the second transformation, we use the relation $dV = t\,dx\,dy = t\det[J]\,dr\,ds$ (for plane problems), where t is the thickness of the plate element, and $dV = dx\,dy\,dz = \det[J]\,dr\,ds\,dt$[2] (for three-dimensional problems). Assuming that the inverse of $[J]$ can be found, the volume integration implied in Eq. (4.107) can be performed as

$$\left[K^{(e)}\right] = t \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] \cdot \det[J]\,dr\,ds \tag{4.118}$$

and a similar expression can be written for Eq. (4.108).

2. For carrying out the volume integration, we assume that r, s, and t are the local coordinates and x, y, and z are the global coordinates so that the Jacobian matrix is given by $[J] = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \\ \partial x/\partial s & \partial y/\partial s & \partial z/\partial s \\ \partial x/\partial t & \partial y/\partial t & \partial z/\partial t \end{bmatrix}$

Notes

1. Although a two-dimensional (plane) problem is considered for explanation, a similar procedure can be adopted in the case of one- and three-dimensional isoparametric elements.
2. If the order of the shape functions used is different in describing the geometry of the element compared to the displacements (i.e., for subparametric or superparametric elements), the shape functions used for describing the geometry would be used in Eqs. (4.115) and (4.116), whereas the shape functions used for describing the displacements would be used in Eq. (4.112).
3. Although the limits of integration in Eq. (4.118) appear to be very simple, unfortunately, the explicit form of the matrix product $[B]^T[D][B]$ is not very easy to express in closed form. Hence, it is necessary to resort to numerical integration. However, this is not a severe restriction because general computer programs, not tied to a particular element, can be written for carrying out the numerical integration.

4.10 NUMERICAL INTEGRATION

4.10.1 In One Dimension

There are several schemes available for the numerical evaluation of definite integrals. Because the Gauss quadrature method has proven to be most useful in finite element applications, we shall consider only this method in this section.

Let the one-dimensional integral to be evaluated be

$$I = \int_{-1}^{1} f(r)\,\mathrm{d}r \tag{4.119}$$

The simplest and crudest way of evaluating I is to sample (evaluate) f at the middle point and multiply by the length of the interval as shown in Fig. 4.21A to obtain

$$I = 2f_1 \tag{4.120}$$

This result would be exact only if the curve happens to be a straight line. Generalization of this relation gives

$$I = \int_{-1}^{1} f(r)\,\mathrm{d}r \simeq \sum_{i=1}^{n} w_i f_i = \sum_{i=1}^{n} w_i f_i(r_i) \tag{4.121}$$

where w_i is called the weight associated with the i-th point, and n is the number of sampling points. This means that in order to evaluate the integral, we evaluate the function at several sampling points, multiply each value f_i by an appropriate weight w_i, and add. Fig. 4.21 illustrates sampling at one, two, and three points.

In the Gauss method, the location of sampling points is such that for a given number of points, greatest accuracy is obtained. The sampling points are located symmetrically about the center of the interval. The weight would be the same for

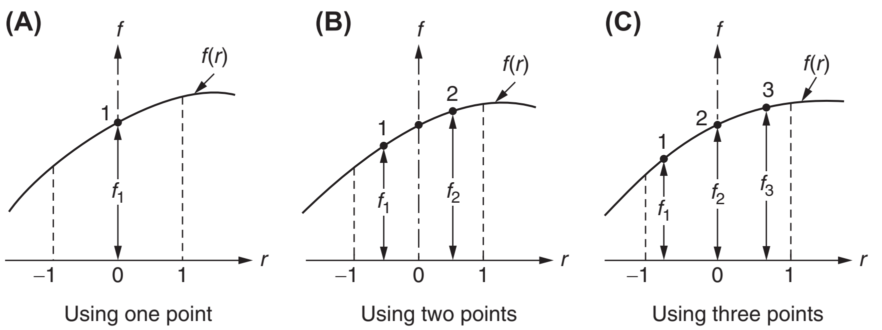


FIGURE 4.21 Gauss integration.

TABLE 4.1 Locations (r_i) and Weights (w_i) in Gaussian Integration (Eq. 4.121)

Number of Points (n)	Location (r_i)	Weight (w_i)
1	$r_1 = 0.00000\ 00000\ 00000$	2.00000 00000 00000
2	$r_1,\ r_2 = \pm 0.57735\ 02691\ 89626$	1.00000 00000 00000
3	$r_1,\ r_3 = \pm 0.77459\ 66692\ 41483$	0.55555 55555 55555
	$r_2 = 0.00000\ 00000\ 00000$	0.88888 88888 88889
4	$r_1,\ r_4 = \pm 0.86113\ 63115\ 94053$	0.34785 48451 47454
	$r_2,\ r_3 = \pm 0.33998\ 10435\ 84856$	0.65214 51548 62546
5	$r_1,\ r_5 = \pm 0.90617\ 98459\ 38664$	0.23692 68850 56189
	$r_2,\ r_4 = \pm 0.53846\ 93101\ 05683$	0.47862 86704 99366
	$r_3 = 0.00000\ 00000\ 00000$	0.56888 88888 88889
6	$r_1,\ r_6 = \pm 0.93246\ 95142\ 03152$	0.17132 44923 79170
	$r_2,\ r_5 = \pm 0.66120\ 93864\ 66265$	0.36076 15730 48139
	$r_3,\ r_4 = \pm 0.23861\ 91860\ 83197$	0.46791 39345 72691

symmetrically located points. Table 4.1 shows the locations and weights for Gaussian integration up to six points. Thus, for example, if we use the three-point Gaussian formula, we get

$$I \simeq 0.555556\, f_1 + 0.888889\, f_2 + 0.555556\, f_3$$

which is the exact result if $f(r)$ is a polynomial of order less than or equal to 5. In general, Gaussian quadrature using n points is exact if the integrand is a polynomial of degree $2n-1$ or less.

The principle involved in deriving the Gauss quadrature formula can be illustrated by considering a simple function,

$$f(r) = a_1 + a_2 r + a_3 r^2 + a_4 r^3$$

If $f(r)$ is integrated between -1 and 1, the area under the curve $f(r)$ is

$$I = 2a_1 + \frac{2}{3} a_3$$

By using two symmetrically located points $r = \pm r_i$, we propose to calculate the area as

$$\underset{\sim}{I} = w \cdot f(-r_i) + w \cdot f(r_i) = 2w\left(a_1 + a_3 r_i^2\right)$$

If we want to minimize the error $e = I - \underset{\sim}{I}$ for any values of a_1 and a_3, we must have

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial a_3} = 0$$

These equations give

$$w = 1$$

and

$$r_i = \frac{1}{\sqrt{3}} = 0.577350\ \ldots$$

4.10.2 In Two Dimensions

IN RECTANGULAR REGIONS

In two-dimensional (rectangular) regions, we obtain the Gauss quadrature formula by integrating first with respect to one coordinate and then with respect to the second as

$$\begin{aligned} I &= \int_{-1}^{1}\int_{-1}^{1} f(r,s)\,\mathrm{d}r\,\mathrm{d}s = \int_{-1}^{1}\left[\sum_{i=1}^{n} w_i f(r_i,s)\right]\mathrm{d}s \\ &= \sum_{j=1}^{n} w_j\left[\sum_{i=1}^{n} w_i f(r_i,s_j)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(r_i,s_j) \end{aligned} \tag{4.122}$$

Thus, for example, a four-point Gaussian rule (Fig. 4.22) gives

$$I \simeq (1.000000)(1.000000)[f(r_1,s_1) + f(r_2,s_2) + f(r_3,s_3) + f(r_4,s_4)] \tag{4.123}$$

where the four sampling points are located at r_i, $s_i = \pm 0.577350$. In Eq. (4.122), the number of integration points in each direction was assumed to be the same. Clearly, it is not necessary and sometimes it may be advantageous to use different numbers in each direction.

IN TRIANGULAR REGIONS

The integrals involved for triangular elements would be in terms of triangular or area coordinates. The following Gauss-type formula has been developed by Hammer and Stroud [4.5]:

$$I = \iint_A f(L_1,L_2,L_3)\,\mathrm{d}A \simeq \sum_{i=1}^{n} w_i f\left(L_1^{(i)},L_2^{(i)},L_3^{(i)}\right) \tag{4.124}$$

where for $n = 1$ (linear triangle):

$$w_1 = 1;\quad L_1^{(1)} = L_2^{(1)} = L_3^{(1)} = \frac{1}{3}$$

for $n = 3$ (quadratic triangle):

$$w_1 = \frac{1}{3};\quad L_1^{(1)} = L_2^{(1)} = \frac{1}{2}, L_3^{(1)} = 0$$

$$w_2 = \frac{1}{3};\quad L_1^{(2)} = 0, L_2^{(2)} = L_3^{(2)} = \frac{1}{2}$$

$$w_3 = \frac{1}{3}; L_1^{(3)} = L_3^{(3)} = \frac{1}{2}, L_2^{(3)} = 0$$

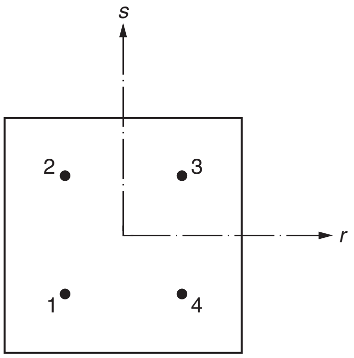


FIGURE 4.22 Four-point Gaussian quadrature rule.

for $n = 7$ (cubic triangle):

$$w_1 = \frac{27}{60}; \quad L_1^{(1)} = L_2^{(1)} = L_3^{(1)} = \frac{1}{3}$$

$$w_2 = \frac{8}{60}; \quad L_1^{(2)} = L_2^{(2)} = \frac{1}{2}, L_3^{(2)} = 0$$

$$w_3 = \frac{8}{60}; \quad L_1^{(3)} = 0, L_2^{(3)} = L_3^{(3)} = \frac{1}{2}$$

$$w_4 = \frac{8}{60}; \quad L_1^{(4)} = L_3^{(4)} = \frac{1}{2}, L_2^{(4)} = 0$$

$$w_5 = \frac{3}{60}; \quad L_1^{(5)} = 1, L_2^{(5)} = L_3^{(5)} = 0$$

$$w_6 = \frac{3}{60}; \quad L_1^{(6)} = L_3^{(6)} = 0, L_2^{(6)} = 1$$

$$w_7 = \frac{3}{60}; \quad L_1^{(7)} = L_2^{(7)} = 0, L_3^{(7)} = 1$$

The locations of the integration points are shown in Fig. 4.23.

4.10.3 In Three Dimensions

IN RECTANGULAR PRISM-TYPE REGIONS

For a right prism, we can obtain an integration formula similar to Eq. (4.122) as

$$\begin{aligned} I &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(r,s,t)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_i w_j w_k f(r_i, s_j, t_k) \end{aligned} \tag{4.125}$$

where an equal number of integration points (n) in each direction is taken only for convenience.

IN TETRAHEDRAL REGIONS

For tetrahedral regions, four volume coordinates are involved and the integral can be evaluated as in Eq. (4.124):

$$I \simeq \sum_{i=1}^{n} w_i f\left(L_1^{(i)}, L_2^{(i)}, L_3^{(i)}, L_4^{(i)}\right) \tag{4.126}$$

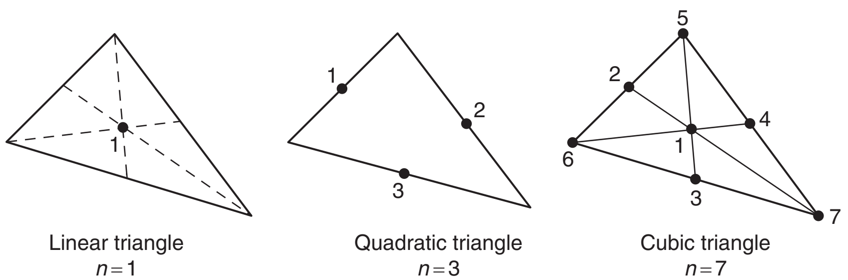


FIGURE 4.23 Integration points within a triangular region according to Eq. (4.124).

where for $n = 1$ (linear tetrahedron):

$$w_1 = 1; \quad L_1^{(1)} = L_2^{(1)} = L_3^{(1)} = L_4^{(1)} = \frac{1}{4}$$

for $n = 4$ (quadratic tetrahedron):

$$w_1 = \frac{1}{4}; \quad L_1^{(1)} = a, \; L_2^{(1)} = L_3^{(1)} = L_4^{(1)} = b$$

$$w_2 = \frac{1}{4}; \quad L_2^{(2)} = a, \; L_1^{(2)} = L_3^{(2)} = L_4^{(2)} = b$$

$$w_3 = \frac{1}{4}; \quad L_3^{(3)} = a, \; L_1^{(3)} = L_2^{(3)} = L_4^{(3)} = b$$

$$w_4 = \frac{1}{4}; \quad L_4^{(4)} = a, \; L_1^{(4)} = L_2^{(4)} = L_3^{(4)} = b$$

with

$$a = 0.58541020$$

and

$$b = 0.13819660$$

for $n = 4$ (cubic tetrahedron):

$$w_1 = -\frac{4}{5}; \quad L_1^{(1)} = L_2^{(1)} = L_3^{(1)} = L_4^{(1)} = \frac{1}{4}$$

$$w_2 = \frac{9}{20}; \quad L_1^{(2)} = \frac{1}{3}, L_2^{(2)} = L_3^{(2)} = L_4^{(2)} = \frac{1}{6}$$

$$w_3 = \frac{9}{20}; \quad L_2^{(3)} = \frac{1}{3}, L_1^{(3)} = L_3^{(3)} = L_4^{(3)} = \frac{1}{6}$$

$$w_4 = \frac{9}{20}; \quad L_3^{(4)} = \frac{1}{3}, L_1^{(4)} = L_2^{(4)} = L_4^{(4)} = \frac{1}{6}$$

$$w_5 = \frac{9}{20}; \quad L_4^{(5)} = \frac{1}{3}, L_1^{(5)} = L_2^{(5)} = L_3^{(5)} = \frac{1}{6}$$

The locations of the integration points used in Eq. (4.126) are shown in Fig. 4.24.

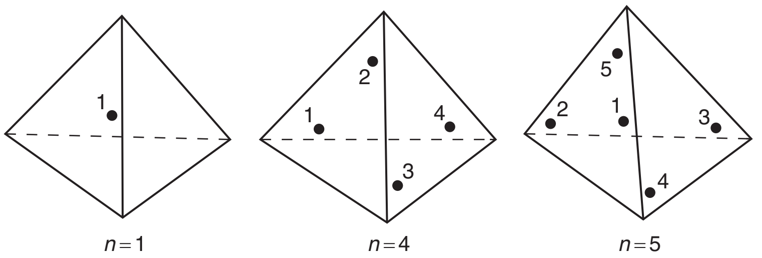


FIGURE 4.24 Location of integration points within a tetrahedron according to Eq. (4.126).

REVIEW QUESTIONS

4.1 Give brief answers to the following questions.

1. What is a higher order element?
2. State the cubic interpolation model for a one-dimensional element.
3. Show the locations of nodes for a one-dimensional element with a cubic interpolation model.
4. Show the locations of nodes of a triangular element with a quadratic interpolation model.
5. Show the natural coordinates for a quadrilateral element.
6. How many primary nodes are there for a tetrahedral quadratic element?
7. Name two types of classical interpolation polynomials.
8. Show the locations of nodes in a biquadratic rectangular element.

4.2 Fill in the blank with a suitable word.

1. A higher order element can be either a complex or a ________________ element.
2. Higher order elements will have ________________ nodes in addition to primary nodes.
3. Isoparametric elements can be used for problems involving ________________ boundaries.
4. Elements whose shape and field variables are described by the same interpolation functions of the same order are known as ________________ elements.
5. ________________ quadrature is most useful in finite element applications.

4.3 Indicate whether each of the following statements is true or false.

1. C^1 continuity means that the field variable ϕ is continuous inside as well as between the elements.
2. The number of elements of a given shape capable of satisfying C^0 continuity is infinite.
3. In general, a smaller number of higher order elements yields somewhat less accurate results compared to a larger number of simpler elements for the same computational effort.
4. An incompatible element can be called a nonconforming element.
5. Satisfying C^1 continuity is more difficult for a rectangular element compared to a triangular element.
6. Linear isoparametric elements can have curved edges.
7. Quadratic isoparametric elements can have straight or curved edges.

4.4 Match the following two-station first-order interpolation functions with their corresponding polynomials.

1. $H_{01}^{(1)}(x)$ (a) $-\frac{1}{l^3}(2x^3 - 3lx^2)$
2. $H_{02}^{(1)}(x)$ (b) $\frac{1}{l}(x^3 - 2lx^2 + l^2x)$
3. $H_{11}^{(1)}(x)$ (c) $\frac{1}{2}(x^3 - lx^2)$
4. $H_{12}^{(1)}(x)$ (d) $\frac{1}{l}(2x^3 - 3lx^2 + l^3)$

4.5 Match the following.

1. Quadratic interpolation model of a 1D element in terms of natural coordinates	(a) $\phi(x) = \sum_{i=0}^{2} L_i(x)\Phi_i$
2. Quadratic interpolation model of a 1D element in terms of Cartesian coordinates	(b) $\phi(x) = a_1 + a_2x + a_3x^2$
3. Quadratic interpolation model of a 1D element in terms of Lagrange interpolation functions	(c) $\phi(x) = a_1 + a_2x + a_3y$

PROBLEMS

4.1 Consider the shape functions, $N_i(x)$, $N_j(x)$, and $N_k(x)$, corresponding to the nodes i, j, and k, of the one-dimensional quadratic element described in Section 4.2.1. Show that the shape function corresponding to a particular node i (j or k) has a value of 1 at node i (j or k) and 0 at the other two nodes j (k or i) and k (i or j).

4.2 Consider the shape functions described in Eq. (4.10) for a one-dimensional cubic element. Show that the shape function corresponding to a particular node i, $N_i(x)$, has a value of 1 at node i and 0 at the other three nodes j, k, and l. Repeat the procedure for the shape functions $N_j(x)$, $N_k(x)$, and $N_l(x)$.

4.3 The Cartesian (global) coordinates of the corner nodes of a quadrilateral element are given by (0, −1), (−2, 3), (2, 4), and (5, 3). Find the coordinate transformation between the global and local (natural) coordinates. Using this, determine the Cartesian coordinates of the point defined by $(r, s) = (0.5, 0.5)$ in the global coordinate system.

4.4 Determine the Jacobian matrix for the quadrilateral element defined in Problem 4.3. Evaluate the Jacobian matrix at the point $(r, s) = (0.5, 0.5)$.

4.5 The Cartesian (global) coordinates of the corners of a triangular element are given by $(-2, -1)$, $(2, 4)$, and $(4, 1)$. Find expressions for the natural (triangular) coordinates L_1, L_2, and L_3. Determine the values of L_1, L_2, and L_3 at the point $(x, y) = (0, 0)$.

4.6 Consider a triangular element with the corner nodes defined by the Cartesian coordinates $(-2, -1)$, $(2, 4)$, and $(4, 1)$. Using the expressions derived in Problem 4.5 for L_1, L_2, and L_3, evaluate the following in terms of the Cartesian coordinates x and y:

a. Shape functions N_1, N_2, and N_3 corresponding to a linear interpolation model
b. Shape functions $N_1, N_2, \ldots, N_6$ corresponding to a quadratic interpolation model
c. Shape functions $N_1, N_2, \ldots, N_{10}$ corresponding to a cubic interpolation model

4.7 The interpolation functions corresponding to node i of a triangular element can be expressed in terms of natural coordinates L_1, L_2, and L_3 using the relationship

$$N_i = f^{(i)}(L_1) f^{(i)}(L_2) f^{(i)}(L_3) \tag{P.1}$$

where

$$f^{(i)}(L_j) = \begin{cases} \prod_{k=1}^{p} \frac{1}{k}(mL_j - k + 1) & \text{if } p \geq 1 \\ 1 & \text{if } p = 0 \end{cases} \tag{P.2}$$

with $i = 1, 2, \ldots, n$; n = total number of nodes in the element; $p = mL_j^{(i)}$; m = order of the interpolation model (2 for quadratic, 3 for cubic, etc.); and $L_j^{(i)}$ = value of the coordinate L_j at node i.

Using Eq. (P.1), find the interpolation function corresponding to node 1 of a quadratic triangular element.

4.8 Using Eq. (P.1) given in Problem 4.7, find the interpolation function corresponding to node 4 of a cubic triangular element.

4.9 Using Eq. (P.1) given in Problem 4.7, find the interpolation function corresponding to node 10 of a cubic triangular element.

4.10 Using Eq. (P.1) given in Problem 4.7, find the interpolation function corresponding to node 4 of a quadratic triangular element.

4.11 Using Eq. (P.1) given in Problem 4.7, find the interpolation function corresponding to node 1 of a cubic triangular element.

4.12 The interpolation functions corresponding to node i of a tetrahedron element can be expressed in terms of the natural coordinates L_1, L_2, L_3, and L_4 using the relationship

$$N_i = f^{(i)}(L_1) f^{(i)}(L_2) f^{(i)}(L_3) f^{(i)}(L_4) \tag{P.3}$$

where $f^{(i)}(L_j)$ is defined by Eq. (P.2) of Problem 4.7. Using this relation, find the interpolation function corresponding to node 1 of a quadratic tetrahedron element.

4.13 Using Eq. (P.3) given in Problem 4.12, find the interpolation function corresponding to node 5 of a quadratic tetrahedron element.

4.14 Using Eq. (P.3) given in Problem 4.12, find the interpolation function corresponding to node 1 of a cubic tetrahedron element.

4.15 Using Eq. (P.3) given in Problem 4.12, find the interpolation function corresponding to node 5 of a cubic tetrahedron element.

4.16 Using Eq. (P.3) given in Problem 4.12, find the interpolation function corresponding to node 17 of a cubic tetrahedron element.

4.17 The Cartesian (global) coordinates of the corners of a tetrahedron element are given by $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Find expressions for the natural (tetrahedral) coordinates, L_1, L_2, L_3, and L_4. Determine the values of L_1, L_2, L_3, and L_4 at the point $(x, y, z) = (0.25, 0.25, 0.25)$.

4.18 The nodes of a quadratic one-dimensional element are located at $x = 0$, $x = l/2$, and $x = l$. Express the shape functions using Lagrange interpolation polynomials.

4.19 Derive expressions for the shape functions of the rectangular element shown in Fig. 4.25 using Lagrange interpolation polynomials.

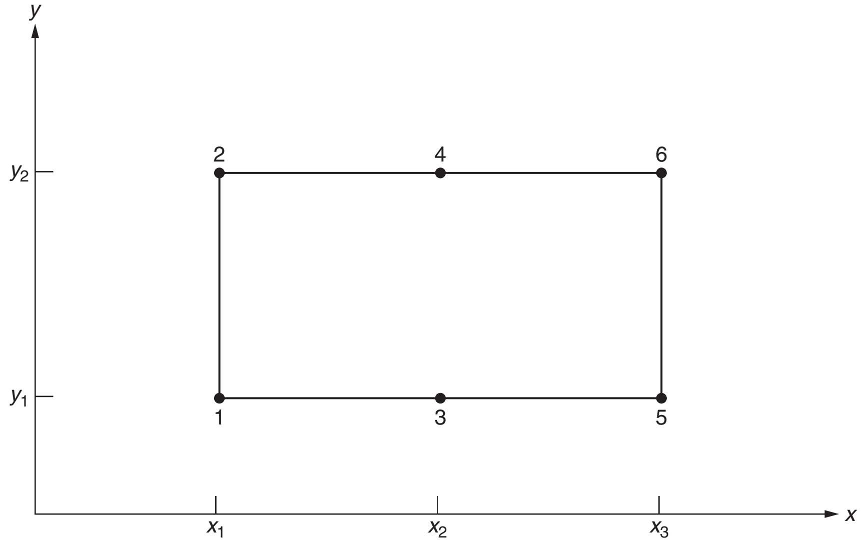


FIGURE 4.25 Rectangular element.

4.20 The Cartesian coordinates of the nodes of a quadratic quadrilateral isoparametric element are shown in Fig. 4.26. Determine the coordinate transformation relation between the local and global coordinates. Using this relation, find the global coordinates corresponding to the point $(r, s) = (0, 0)$.

4.21 A boundary value problem, governed by the Laplace equation, is stated as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ in A}$$

$$\phi = \phi_0 \text{ on C}$$

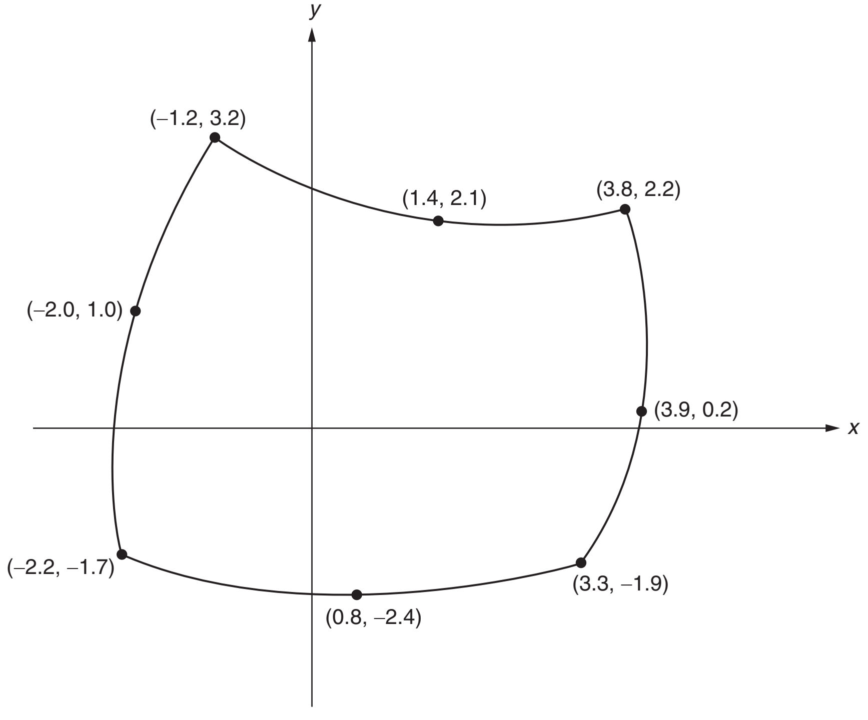


FIGURE 4.26 Quadrilateral isoparametric element.

The characteristic (stiffness) matrix of an element corresponding to this problem can be expressed as

$$[K^{(e)}] = \int\int_{A^{(e)}} [B]^T [D][B] \, dA$$

where

$$[D] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_p}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_p}{\partial y} \end{bmatrix}$$

and $A^{(e)}$ is the area of the element. Derive the matrix $[B]$ for a quadratic quadrilateral isoparametric element whose nodal coordinates are shown in Fig. 4.26.

4.22 Evaluate the integral

$$I = \int_{-1}^{1} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right) dx$$

using the following methods and compare the results:

a. Two-point Gauss integration

b. Analytical integration

4.23 Evaluate the integral

$$I = \int_{-1}^{1} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right) dx$$

using the following methods and compare the results:

a. Three-point Gauss integration

b. Analytical integration

4.24 Evaluate the integral

$$I = \int_{-1}^{1} \int_{-1}^{1} \left(r^2 s^3 + r s^4\right) dr \, ds$$

using the following methods and compare the results:

a. Gauss integration

b. Analytical integration

4.25 Determine the Jacobian matrix for the quadratic isoparametric triangular element shown in Fig. 4.27.

4.26 How do you generate an isoparametric quadrilateral element for C^1 continuity? (Hint: In this case we need to transform the second-order partial derivatives and the Jacobian will be a variable matrix.)

4.27 Consider a ring element with triangular cross section as shown in Fig. 4.28. If the field variable ϕ does not change with respect to θ, propose linear, quadratic, and cubic interpolation models for C^0 continuity. Develop the necessary element equations for the linear case for solving the Laplace's equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

4.28 Evaluate $\partial N_4/\partial x$ and $\partial N_4/\partial y$ at the point (1.5, 2.0) for the quadratic triangular element shown in Fig. 4.29. Hint: Since the sides of the element are straight, define the geometry of the element using Eq. (3.71). Evaluate the Jacobian matrix,

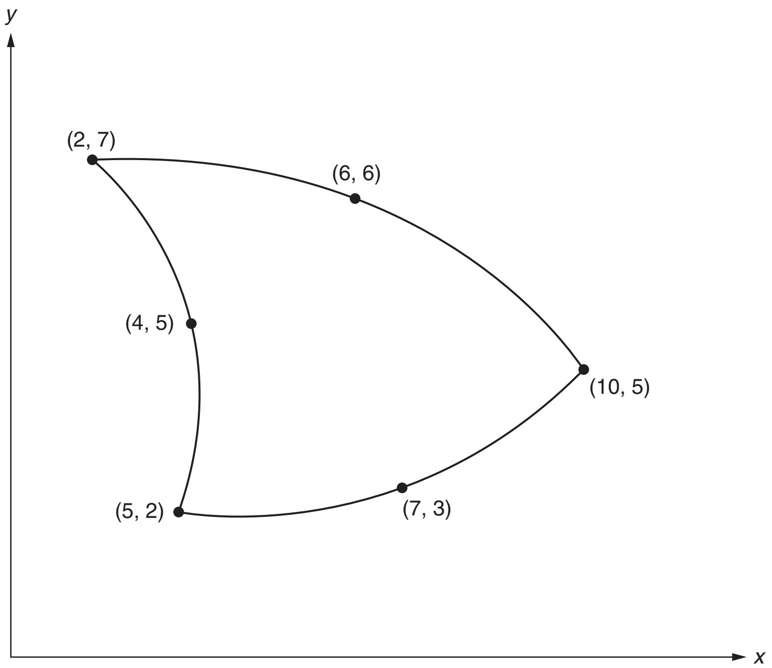


FIGURE 4.27 Isoparametric triangular element.

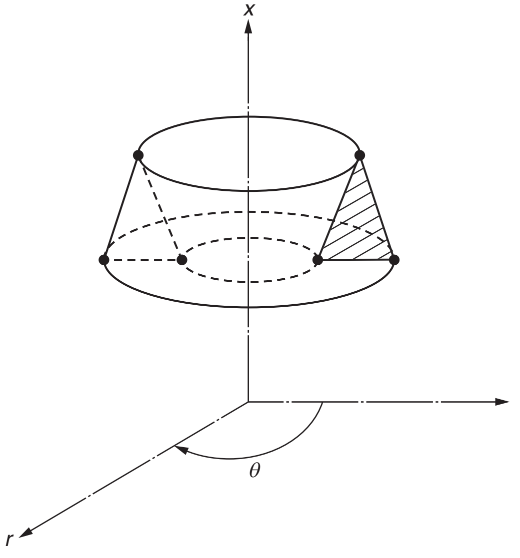


FIGURE 4.28 Ring element.

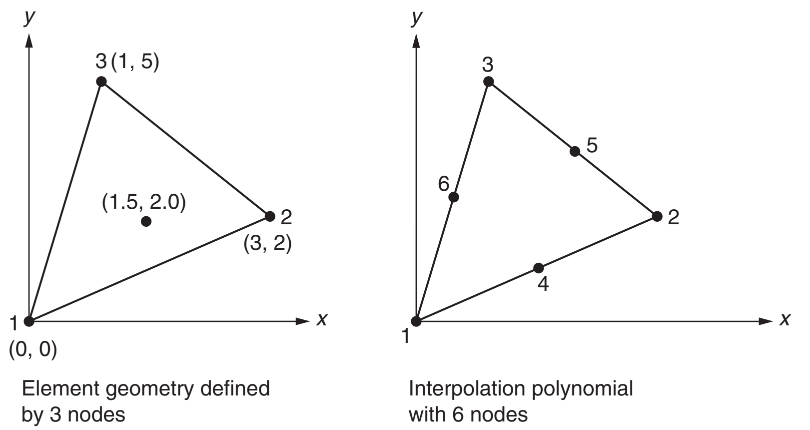


FIGURE 4.29 Quadratic triangular element.

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix}$$

Differentiate the quadratic shape function N_4 given by Eq. (4.26) with respect to x and y. Use the relation

$$\begin{Bmatrix} \dfrac{\partial N_4}{\partial x} \\ \dfrac{\partial N_4}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_4}{\partial L_1} \\ \dfrac{\partial N_4}{\partial L_2} \end{Bmatrix}$$

to obtain the desired result.

4.29 Evaluate the partial derivatives $(\partial N_1/\partial x)$ and $(\partial N_1/\partial y)$ of the quadrilateral element shown in Fig. 4.30 at the point $(r = 1/2,\ s = 1/2)$, assuming that the scalar field variable ϕ is approximated by a quadratic interpolation model.

4.30 Derive Eqs. (4.16) and (4.17) for a one-dimensional quadratic element.

4.31 Derive Eqs. (4.20) to (4.23) for a one-dimensional cubic element.

4.32 Derive Eq. (4.26) for a quadratic triangular element.

4.33 Derive Eq. (4.30) for a cubic triangular element.

4.34 Derive Eq. (4.36) for a quadrilateral element.

4.35 Derive the Hermite polynomials indicated in Eqs. (4.70) to (4.72).

4.36 A column, subjected to an axial load, is modeled with quadratic elements. The nodes 1, 2, and 3 of one of the elements are located at $x = 5$ cm, 8 cm, and 11 cm. If the axial displacements of nodes 1, 2, and 3 are 0.005 cm, 0.009 cm, and 0.016 cm, respectively, determine the shape functions of the element, strain in the element, and the stress in the element. Assume the Young's modulus of the material as 207 GPa.

4.37 In the heat transfer analysis of a one-dimensional fin, the rod (or fin) is modeled with quadratic finite elements. For an interior element, the nodes 1, 2, and 3 are located at $x = 12$ cm, 15 cm, and 18 cm with the nodal temperatures as 180 °F, 170 °F, and 155 °F, respectively. Determine the shape functions of the element and the temperature gradient $\left(\frac{dT}{dx}\right)$ in the element.

4.38 The global coordinates of the four corners of a linear quadratic element are given by $(x_1, y_1) = (4, 3)$ cm, $(x_2, y_2) = (10, 1)$ cm, $(x_3, y_3) = (8, 8)$ cm, and $(x_4, y_4) = (2, 6)$ cm. Find the global coordinates corresponding to the natural coordinates $r = 0.5$ and $s = 1.0$.

4.39 The global coordinates of the four corners of a linear quadratic element are given by $(x_1, y_1) = (2, 4)$ in, $(x_2, y_2) = (4, 1)$ in, $(x_3, y_3) = (10, 3)$ in, and $(x_4, y_4) = (6, 8)$ in. Find the global coordinates corresponding to the natural coordinates $r = 1.0$ and $s = -0.5$.

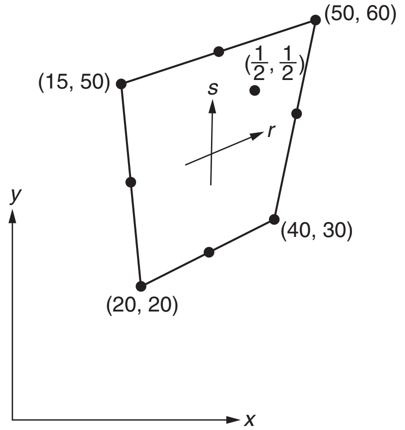


FIGURE 4.30 Quadrilateral element.

4.40 The global coordinates of the four corners of a linear quadratic element are given by $(x_1, y_1) = (13, 3)$ cm, $(x_2, y_2) = (6, 9)$ cm, $(x_3, y_3) = (1, 4)$ cm, and $(x_4, y_4) = (4, -3)$ cm. Find the global coordinates corresponding to the natural coordinates $r = 0.5$ and $s = 1.0$.

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Chapter 5

Derivation of Element Matrices and Vectors

Chapter Outline

5.1 INTRODUCTION

The characteristic matrices and characteristic vectors (also termed vectors of nodal actions) of finite elements can be derived by using any of the following approaches.

1. Direct approach
   The direct method was presented along with several examples from different areas of engineering in Section 1.6. As can be seen from these examples, the direct method is based on using direct physical reasoning to establish the element properties (i.e., the characteristic matrices and vectors) in terms of pertinent variables. Because the approach uses the basic principles of engineering science, it aids in understanding the physical basis of the finite element method [5.1,5.2]. However, the method is applicable only for simple problems, and insurmountable difficulties arise when we try to apply the method to complex problems involving two- and three-dimensional finite elements. As such, the direct method is not used in the finite element analysis of most practical problems.
2. Variational approach
   In this method, the finite element analysis is interpreted as an approximate means for solving variational problems. Since most physical and engineering problems can be formulated in variational form, the finite element method can be readily applied for finding their approximate solutions. The variational approach has been most widely used in

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the literature in formulating finite element equations. A major limitation of the method is that it requires the physical or engineering problem to be stated in variational form, which may not be possible in all cases.

3. Weighted residual approach
In this method, the element matrices and vectors are derived directly from the governing differential equations of the problem without reliance on the variational statement of the problem. This method offers the most general procedure for deriving finite element equations and can be applied to almost all practical problems of science and engineering. Again, within the weighted residual approach, different procedures, such as the Galerkin method and the least squares method, can be used in deriving the element equations.

The variational and weighted residual approaches are presented for deriving element characteristic matrices and vectors in this chapter.

5.2 VARIATIONAL APPROACH

The variational approach is based on the application of variational calculus, which deals with the extremization of functionals in the form of integrals. In this section, we interpret the finite element method as an approximate method of solving variational problems. Most of the solutions reported in the literature for physical and engineering problems have been based on this approach. Finite element equations are derived using the variational approach.

5.2.1 Specification of Continuum Problems

Most continuum problems can be specified in one of two ways. In the first, a variational principle valid over the entire domain of the problem is postulated and an integral I is defined in terms of the unknown parameters and their derivatives. The correct solution of the problem is one that minimizes the integral I subject to specified boundary conditions. In the second, differential equations governing the behavior of a typical infinitesimal domain are given along with the boundary conditions. These two approaches are mathematically equivalent, an exact solution of one being the solution of the other. The final equations of the finite element method can be derived by proceeding either from the differential equations or from the variational principle of the problem.

Although the differential equation approach is more popular, the variational approach will be of special interest in studying the finite element method. This is due to the fact that the consideration of the finite element method as a variational approach has contributed significantly in formulating and solving problems of different branches of engineering in a unified manner. Thus, a knowledge of the basic concepts of calculus of variations is useful in understanding the general finite element method.

5.2.2 Approximate Methods of Solving Continuum Problems

If the physical problem is specified as a variational problem and the exact solution, which minimizes the integral I, cannot be found easily, we would like to find an approximate solution that approximately minimizes the integral I. Similarly, if the problem is specified in terms of differential equations and boundary conditions, and if the correct solution, which satisfies all the equations exactly, cannot be obtained easily, we would like to find an approximate solution that satisfies the boundary conditions exactly but not the governing differential equations. Of the various approximate methods available, the methods using trial functions have been more popular. Depending on the manner in which the problem is specified, two types of approximate methods, namely variational methods (e.g., Rayleigh–Ritz method) and weighted residual methods (e.g., Galerkin method), are available. The finite element method can be considered as a variational (Rayleigh–Ritz) method and also as a weighted residual (Galerkin) method. The consideration of the finite element method as a variational approach (which minimizes the integral I approximately) is discussed in this section. The consideration of the finite element method as a weighted residual approach (which satisfies the governing differential equations approximately) is discussed in the next section.

5.2.3 Calculus of Variations

The calculus of variations is concerned with the determination of extrema (maxima and minima) or stationary values of functionals. A functional can be defined as a function of several other functions. The basic problem in variational calculus is to find the function $\phi(x)$ that makes the functional (integral)

$$I = \int_{x_1}^{x_2} F(x, \phi, \phi_x, \phi_{xx}) \cdot \mathrm{d}x \tag{5.1}$$

stationary. Here, x is the independent variable, $\phi_x = \mathrm{d}\phi/\mathrm{d}x$, $\phi_{xx} = \mathrm{d}^2\phi/\mathrm{d}x^2$, and I and F can be called functionals. The functional I usually possesses a clear physical meaning in most applications. For example, in structural and solid mechanics, the potential energy (π) plays the role of the functional (π is a function of the displacement vector $\vec{\phi}$, whose components are u, v, and w, which is a function of the coordinates x, y, and z). The integral in Eq. (5.1) is defined in the region or domain [x_1, x_2]. Let the value of ϕ be prescribed on the boundaries as $\phi\,(x_1) = \phi_1$ and $\phi\,(x_2) = \phi_2$. These are called the boundary conditions of the problem.

One of the procedures that can be used to solve the problem in Eq. (5.1) is as follows:

1. Select a series of trial or tentative solutions $\phi\,(x)$ for the given problem and express the functional I in terms of each of the tentative solutions.
2. Compare the values of I given by the different tentative solutions.
3. Find the correct solution to the problem as that particular tentative solution that makes the functional I assume an extreme or stationary value.

The mathematical procedure used to select the correct solution from a number of tentative solutions is called the calculus of variations.

STATIONARY VALUES OF FUNCTIONALS

Any tentative solution $\overline{\phi}(x)$ in the neighborhood of the exact solution $\phi\,(x)$ may be expressed as

$$\underset{\text{tentative solution}}{\overline{\phi}(x)} = \underset{\text{exact solution}}{\phi(x)} + \underset{\text{variation of }\phi}{\delta\,\phi(x)} \tag{5.2}$$

(see Fig. 5.1). The variation in ϕ (i.e., $\delta\phi$) is defined as an infinitesimal, arbitrary change in ϕ for a fixed value of the variable x (i.e., for $\delta x = 0$). Here, δ is called the variational operator (similar to the differential operator d). The operation of variation is commutative with respect to both integration and differentiation:

$$\delta\left(\int F \cdot \mathrm{d}x\right) = \int (\delta F)\,\mathrm{d}x \tag{5.3}$$

and

$$\delta\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\delta\phi) \tag{5.4}$$

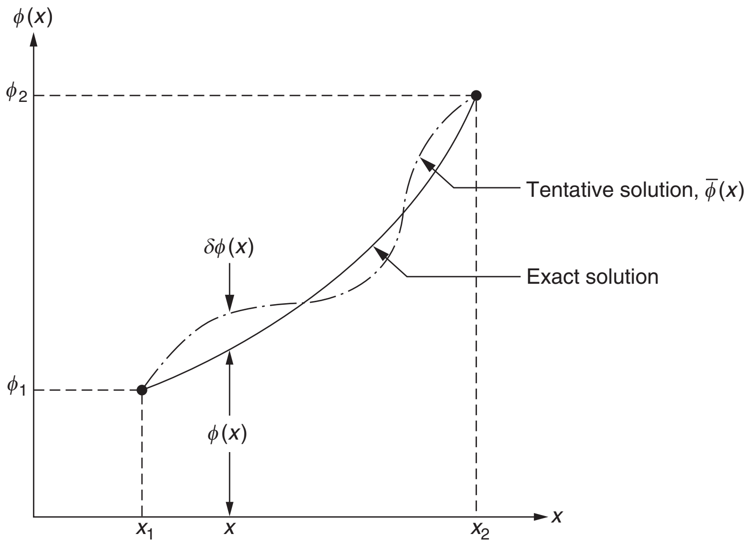

FIGURE 5.1 Tentative and exact solutions.

Also, we define the variation of a functional or a function of several variables in a manner similar to the calculus definition of a total differential as

$$\delta F = \frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi_x}\delta\phi_x + \frac{\partial F}{\partial \phi_{xx}}\delta\phi_{xx} + \frac{\partial F}{\partial x}\cancelto{0}{\delta x} \tag{5.5}$$

(since we are finding variation of ϕ for a fixed value of x, $\delta x = 0$).

Now, let us consider the variation in $I(\delta I)$ corresponding to variations in the solution ($\delta\phi$). If we want the condition for the stationariness of I, we take the necessary condition as the vanishing of the first derivative of I (similar to maximization or minimization of simple functions in calculus):

$$\delta I = \int_{x_1}^{x_2} \delta F \, \mathrm{d}x = 0 \tag{5.6}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi_x}\delta\phi_x + \frac{\partial F}{\partial \phi_{xx}}\delta\phi_{xx} \right) \mathrm{d}x = 0 \tag{5.7}$$

Integrate the second and third terms by parts to obtain

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \phi_x}\delta\phi_x \mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial \phi_x}\delta\left(\frac{\partial \phi}{\partial x}\right)\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial \phi_x}\frac{\partial}{\partial x}(\delta\phi)\,\mathrm{d}x \tag{5.8}$$

$$= \frac{\partial F}{\partial \phi_x}\delta\phi\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_x}\right)\delta\phi\,\mathrm{d}x \tag{5.9}$$

and

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \phi_{xx}}\delta\phi_{xx}\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial \phi_{xx}}\frac{\partial}{\partial x}(\delta\phi_x)\,\mathrm{d}x \tag{5.10}$$

$$= \frac{\partial F}{\partial \phi_{xx}}\delta\phi_x\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\delta\phi_x\,\mathrm{d}x \tag{5.11}$$

$$= \frac{\partial F}{\partial \phi_{xx}}\delta\phi_x\bigg|_{x_1}^{x_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\delta\phi\bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\delta\phi\,\mathrm{d}x \tag{5.12}$$

$$\therefore \delta I = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\right]\delta\phi\,\mathrm{d}x + \left[\frac{\partial F}{\partial \phi_x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\right]\delta\phi\bigg|_{x_1}^{x_2} + \left[\left(\frac{\partial F}{\partial \phi_{xx}}\right)\delta\phi_x\right]\bigg|_{x_1}^{x_2} = 0 \tag{5.13}$$

Since $\delta\phi$ is arbitrary, each term must vanish individually so that

$$\frac{\partial F}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_x}\right) + \frac{\mathrm{d}}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial \phi_{xx}}\right) = 0 \tag{5.14}$$

$$\left[\frac{\partial F}{\partial \phi_x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\right]\delta\phi\bigg|_{x_1}^{x_2} = 0 \tag{5.15}$$

$$\frac{\partial F}{\partial \phi_{xx}}\delta\phi_x\bigg|_{x_1}^{x_2} = 0 \tag{5.16}$$

Eq. (5.14) will be the governing differential equation for the given problem and is called the *Euler equation* or *Euler–Lagrange equation*. Eqs. (5.15) and (5.16) give the boundary conditions. The conditions

$$\left[\frac{\partial F}{\partial \phi_x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \phi_{xx}}\right)\right]\bigg|_{x_1}^{x_2} = 0 \tag{5.17}$$

and

$$\frac{\partial F}{\partial \phi_{xx}}\bigg|_{x_1}^{x_2} = 0 \tag{5.18}$$

are called *natural boundary conditions* (if they are satisfied, they are called *free boundary conditions*). If the natural boundary conditions are not satisfied, we should have

$$\delta\phi(x_1) = 0 \tag{5.19}$$

$$\delta\phi(x_2) = 0 \tag{5.20}$$

and

$$\delta\phi_x(x_1) = 0 \tag{5.21}$$

$$\delta\phi_x(x_2) = 0 \tag{5.22}$$

in order to satisfy Eqs. (5.15) and (5.16). These are called *geometric* or *essential* or *forced boundary conditions*. Thus, the boundary conditions, Eqs. (5.15) and (5.16), can be satisfied by any combination of free and forced boundary conditions. If the finite element equations are derived on the basis of a variational principle, the natural boundary conditions will be automatically incorporated in the formulation; hence, only the geometric boundary conditions are to be enforced on the solution.

5.2.4 Several Dependent Variables and One Independent Variable

Although Eqs. (5.14) to (5.16) were derived for a single dependent variable, the method can be extended to the case of several dependent variables $\phi_i(x)$ to obtain the set of Euler–Lagrange equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\frac{\partial F}{\partial(\phi_i)_{xx}}\right\} - \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\partial F}{\partial(\phi_i)_x}\right\} + \frac{\partial F}{\partial(\phi_i)_i} = 0, \quad i = 1, 2, \ldots, n \tag{5.23}$$

In general, the integrand F will involve derivatives of higher order than the second order so that

$$I = \int_{x_1}^{x_2} F\left[x, \phi_i, \phi_i^{(1)}, \phi_i^{(2)}, \ldots, \phi_i^{(j)}\right] \mathrm{d}x, \quad i = 1, 2, \ldots, n \tag{5.24}$$

where $\phi_i^{(j)}$ indicates the j-th derivative of ϕ_i with respect to x. The corresponding Euler–Lagrange equations can be expressed as [5.3]

$$\sum_{j=0}^{n} (-1)^{n-j} \frac{\mathrm{d}^{n-j}}{\mathrm{d}x^{n-j}}\left\{\frac{\partial F}{\partial \phi_i^{(n-j)}}\right\} = 0, \quad i = 1, 2, \ldots, n \tag{5.25}$$

5.2.5 Several Independent Variables and One Dependent Variable

Consider the following functional with three independent variables:

$$I = \int_V F\left(x, y, z, \phi, \phi_x, \phi_y, \phi_z\right) \mathrm{d}V \tag{5.26}$$

where $\phi_x = (\partial\phi/\partial x)$, $\phi_y = (\partial\phi/\partial y)$, and $\phi_z = (\partial\phi/\partial z)$. The variation of I due to an arbitrary small change in the solution ϕ can be expressed as

$$\delta I = \int_V \left(\frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi_x}\delta\phi_x + \frac{\partial F}{\partial \phi_y}\delta\phi_y + \frac{\partial F}{\partial \phi_z}\delta\phi_z\right) \mathrm{d}V \tag{5.27}$$

$$= \int_V \left[\frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi_x}\frac{\partial}{\partial x}(\delta\phi) + \frac{\partial F}{\partial \phi_y}\frac{\partial}{\partial y}(\delta\phi) + \frac{\partial F}{\partial \phi_z}\frac{\partial}{\partial z}(\delta\phi)\right] \mathrm{d}V \tag{5.28}$$

since $\delta\phi_x = \delta\,(\partial\phi/\partial x) = (\partial/\partial x)\,(\delta\phi)$, and so on. Integrating the second term in Eq. (5.28) by parts and applying the Green–Gauss theorem (given in the Appendix B) gives

$$\int_V \frac{\partial F}{\partial \phi_x}\frac{\partial}{\partial x}(\delta\phi)\,\mathrm{d}V = \int_V \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\delta\phi\right)\mathrm{d}V - \int_V \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right)\delta\phi\,\mathrm{d}V \tag{5.29}$$

$$= \int_S l_x \frac{\partial F}{\partial \phi_x} \delta\phi \, \mathrm{d}S - \int_V \frac{\partial F}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right)\delta\phi \, \mathrm{d}V \tag{5.30}$$

where l_x is the direction cosine of the normal to the outer surface with respect to the x axis. Similarly, the third and fourth terms in Eq. (5.28) can be integrated and δI can be expressed as

$$\delta I = \int_V \left[\frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \phi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial \phi_z}\right)\right]\delta\phi \, \mathrm{d}V + \int_S \left[\frac{\partial F}{\partial \phi_x} l_x + \frac{\partial F}{\partial \phi_y}\cdot l_y + \frac{\partial F}{\partial \phi_z} l_z\right] \delta\phi \cdot \mathrm{d}S \tag{5.31}$$

The functional I assumes a stationary value only if the bracketed terms within the integrals vanish. This requirement gives the governing differential equation and the boundary conditions of the problem. Eq. (5.31) is the one that is applicable to the finite element formulation of most field problems according to the variational approach.

5.2.6 Advantages of Variational Formulation

From the previous discussion it is evident that any continuum problem can be solved using a differential equation formulation or a variational formulation. The equivalence of these formulations is apparent from the previous equations, which show that the functional I is extremized or made stationary only when the corresponding Euler–Lagrange equations and boundary conditions are satisfied. These equations are precisely the governing differential equations of the given problem.

The variational formulation of a continuum problem has the following advantages over differential equation formulation:

1. The functional I usually possesses a clear physical meaning in most practical problems.
2. The functional I contains lower order derivatives of the field variable compared to the governing differential equation and hence an approximate solution can be obtained using a larger class of functions.
3. Sometimes the problem may possess a dual variational formulation, in which case the solution can be sought either by minimizing (or maximizing) the functional I or by maximizing (or minimizing) its dual functional. In such cases, we can find an upper and a lower bound to the solution and estimate the order of error in either of the approximate solutions obtained.
4. Using variational formulation, it is possible to prove the existence of solution in some cases.
5. The variational formulation permits us to treat complicated boundary conditions as natural or free boundary conditions. Thus, we need to explicitly impose only the geometric or forced boundary conditions in the finite element method, and the variational statement implicitly imposes the natural boundary conditions.

As stated in Section 1.4, the finite element method is applicable to all types of continuum problems, namely, equilibrium, eigenvalue, and propagation problems. We first present the solution of all three categories of problems using the variational approach and then derive the finite element equations using the variational approach.

5.3 SOLUTION OF EQUILIBRIUM PROBLEMS USING VARIATIONAL (RAYLEIGH–RITZ) METHOD

The differential equation formulation of a general equilibrium problem leads to the following equations:

$$A\phi = b \text{ in } V \tag{5.32}$$

$$B_j\phi = g_j, \quad j = 1, 2, \ldots, p \text{ on } S \tag{5.33}$$

where ϕ is the unknown field variable (assumed to be a scalar for simplicity), A and B_j are differential operators, b and g_j are functions of the independent variables, p is the number of boundary conditions, V is the domain, and S is the boundary of the domain.

In variational formulation, a functional $I(\phi)$, for which the conditions of stationariness or extremization give the governing differential equation, Eq. (5.32), is identified and the problem is stated as follows:

$$\text{Minimize} \quad I(\phi) \text{ in } V \tag{5.34}$$

subject to the essential or forced boundary conditions

$$B_j\phi = g_j, \quad j = 1, 2, \ldots, p \tag{5.35}$$

The functional I is some integral of ϕ and its derivatives over the domain V and/or the boundary S. If the integrand of the functional is denoted by F so that

$$I = \int_V F(x, \phi, \phi_x)\cdot \mathrm{d}x \tag{5.36}$$

it can be seen from the previous discussion that F satisfies the Euler–Lagrange equation, Eq. (5.14). This Euler–Lagrange equation in ϕ is the same as the original field equation, Eq. (5.32).

In the most widely used variational method, the Rayleigh–Ritz method, an approximate solution of the following type is assumed for the field variable $\phi\,(x)$:

$$\underset{\sim}{\phi}\,(x) = \sum_{i=1}^{n} C_i f_i(x) \tag{5.37}$$

where $f_i(x)$ are known linearly independent functions (also called trial functions) defined over V and S, and C_i are unknown parameters to be determined. When $\underset{\sim}{\phi}\,(x)$ of Eq. (5.37) is substituted, Eq. (5.36) becomes a function of the unknowns C_i. The necessary conditions for the functional to be stationary are given by

$$\frac{\partial I(\underset{\sim}{\phi})}{\partial C_i} = 0, \quad i = 1, 2, \ldots, n \tag{5.38}$$

which yields n equations in the n unknowns C_i. If I is a quadratic function of ϕ and ϕ_x, Eq. (5.38) gives a set of n linear simultaneous equations.

It can be seen that the accuracy of the assumed solution $\underset{\sim}{\phi}\,(x)$ depends on the choice of the trial functions $f_i(x)$. The functions $f_j(x)$, $j = 1, 2, \ldots, n$ have to be continuous up to degree $r - 1$, where r denotes the highest degree of differentiation in the functional I [$r = 1$ in Eq. 5.36] and have to satisfy the essential boundary conditions, Eq. (5.35). In addition, the functions $f_j(x)$ must be part of a complete set for the solution to converge to the correct solution. To assess the convergence of the process, we need to take two or more trial functions, $f_j(x)$. When the method is applied for the stationariness of a given functional I, we can study the convergence by comparing the results obtained with the following sequence of assumed solutions:

$$\underset{\sim}{\phi}^{(1)}(x) = C_1^{(1)} f_1(x)$$

$$\underset{\sim}{\phi}^{(2)}(x) = C_1^{(2)} f_1(x) + C_2^{(2)} f_2(x)$$

$$\vdots$$

$$\underset{\sim}{\phi}^{(i)}(x) = C_1^{(i)} f_1(x) + C_2^{(i)} f_2(x) + \cdots + C_i^{(i)} f_i(x) \tag{5.39}$$

where the i-th assumed solution includes all the functions $f_j(x)$ included in the previous solutions. Usually, the functions $f_j(x)$ are taken as polynomials or trigonometric functions. If I is a quadratic functional, the sequence of solutions given by Eq. (5.39) leads to

$$I^{(1)} \geq I^{(2)} \geq \cdots \geq I^{(i)} \tag{5.40}$$

This behavior is called monotonic convergence to the minimum of I.

EXAMPLE 5.1

Find the approximate deflection of a simply supported beam under a uniformly distributed load p (Fig. 5.2) using the Rayleigh–Ritz method.

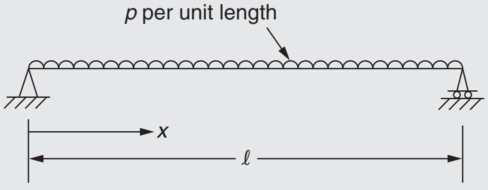

FIGURE 5.2 A simply supported beam under uniformly distributed load.

Continued

EXAMPLE 5.1 —cont'd

Approach: Find the functional I whose extremization yields the differential equation governing the deflection of the beam. Assume an approximate solution satisfying the boundary conditions in terms of two unknown constants and evaluate the constants using the conditions of extremization of I.

Solution

Let $w(x)$ denote the deflection of the beam (field variable). The differential equation formulation leads to the following statement of the problem.

Find $w(x)$ that satisfies the governing equation (with constant p and EI), given by:

$$EI\frac{d^4 w}{dx^4} - p = 0; \quad 0 \le x \le l \tag{E.1}$$

and the boundary conditions

$$\left.\begin{aligned} &w(x=0) = w(x=l) = 0 \text{ (deflection zero at ends)} \\ &EI\frac{d^2 w}{dx^2}(x=0) = EI\frac{d^2 w}{dx^2}(x=l) = 0 \text{ (bending moment zero at ends)} \end{aligned}\right\} \tag{E.2}$$

where E is Young's modulus, and I is the area moment of inertia of the beam. The variational formulation gives the following statement of the problem.

Find $w(x)$ that minimizes the integral [1]

$$A = \int_{x=0}^{l} F\cdot dx = \frac{1}{2}\int_0^l \left[EI\left(\frac{d^2 w}{dx^2}\right)^2 - 2p\cdot w\right] dx \tag{E.3}$$

and satisfies the boundary conditions stated in Eq. (E.2).

We shall approximate the solution $w(x)$ by orthogonal functions of the sinusoidal type. For example,

$$\underset{\sim}{w}(x) = C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{3\pi x}{l}\right) = C_1 f_1(x) + C_2 f_2(x) \tag{E.4}$$

where the functions $f_1(x) = \sin(\pi x/l)$ and $f_2(x) = \sin(3\pi x/l)$ satisfy the boundary conditions, Eq. (E.2). The substitution of Eq. (E.4) into Eq. (E.3) gives

$$\begin{aligned} A &= \int_0^l \left[\frac{EI}{2}\left\{C_1\left(\frac{\pi}{l}\right)^2 \sin\left(\frac{\pi x}{l}\right) + C_2\left(\frac{3\pi}{l}\right)^2 \sin\left(\frac{3\pi x}{l}\right)\right\}^2 - p\left\{C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{3\pi x}{l}\right)\right\}\right] dx \\ &= \int_0^l \left[\frac{EI}{2}\left\{C_1^2\left(\frac{\pi}{l}\right)^4 \sin^2\left(\frac{\pi x}{l}\right) + C_2^2\left(\frac{3\pi}{l}\right)^4 \sin^2\left(\frac{3\pi x}{l}\right) + 2C_1C_2\left(\frac{\pi}{l}\right)^2\left(\frac{3\pi}{l}\right)^2 \cdot \sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right\}\right. \\ &\qquad \left. -p\left\{C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{3\pi x}{l}\right)\right\}\right] dx \end{aligned} \tag{E.5}$$

By using the relations

$$\int_0^l \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0 & \text{if } m \ne n \\ l/2 & \text{if } m = n \end{cases} \tag{E.6}$$

and

$$\int_0^l \sin\left(\frac{m\pi x}{l}\right) dx = \frac{2l}{m\pi} \text{ if } m \text{ is an odd integer} \tag{E.7}$$

Eq. (E.5) can be written as

$$A = \frac{EI}{2}\left\{C_1^2\left(\frac{\pi}{l}\right)^4 \frac{l}{2} + C_2^2\left(\frac{3\pi}{l}\right)^4 \frac{l}{2}\right\} - p\left\{C_1\frac{2l}{\pi} + C_2\frac{2l}{3\pi}\right\} \tag{E.8}$$

EXAMPLE 5.1 —cont'd

where C_1 and C_2 are independent constants. For the minimum of A, we have

$$\left.\begin{aligned} \frac{\partial A}{\partial C_1} &= \frac{EI}{2}\left\{2C_1\left(\frac{\pi}{l}\right)^4\frac{l}{2}\right\} - p\frac{2l}{\pi} = 0 \\ \frac{\partial A}{\partial C_2} &= \frac{EI}{2}\left\{2C_2\left(\frac{3\pi}{l}\right)^4\frac{l}{2}\right\} - p\frac{2l}{3\pi} = 0 \end{aligned}\right\} \tag{E.9}$$

The solution of Eq. (E.9) gives

$$C_1 = \frac{4pl^4}{\pi^5 EI} \quad \text{and} \quad C_2 = \frac{4pl^4}{243\pi^5 EI} \tag{E.10}$$

Thus, the deflection of the beam is given by

$$\underset{\sim}{w}(x) = \frac{4pl^4}{\pi^5 EI}\left[\sin\left(\frac{\pi x}{l}\right) + \frac{1}{243}\sin\left(\frac{3\pi x}{l}\right)\right] \tag{E.11}$$

Thus, the deflection of the beam at the middle point is given by

$$\underset{\sim}{w}(x = l/2) = \frac{968}{243\pi^5}\frac{pl^4}{EI} = \frac{1}{76.5}\frac{pl^4}{EI} \tag{E.12}$$

which compares well with the exact solution

$$w(x = l/2) = \frac{5}{384}\frac{pl^4}{EI} = \frac{1}{76.8}\frac{pl^4}{EI} \tag{E.13}$$

We can find a more accurate solution by including more terms in Eq. (E.4). The n-term solution converges to the exact solution as $n \to \infty$.

1. It can be verified that the Euler–Lagrange equation corresponding to the functional A is the same as Eq. (E.1). The functional A represents the potential energy of the beam.

EXAMPLE 5.2

Find the solution of the differential equation

$$\frac{d^2\phi}{dx^2} + \phi + x = 0; \quad 0 \le x \le 1 \tag{E.1}$$

subject to the boundary conditions $\phi(0) = \phi(1) = 0$ using the Rayleigh–Ritz method.

Approach: Find the functional I whose extremization yields the differential Eq. (E.1). Assume an approximate solution satisfying the boundary conditions in terms of two unknown constants and evaluate the constants using the conditions of extremization of I.

Solution

The functional I corresponding to Eq. (E.1) is given by

$$I = \frac{1}{2}\int_0^1 \left\{-\left(\frac{d\phi}{dx}\right)^2 + \phi^2 + 2\ \phi\ x\right\}dx \tag{E.2}$$

with the boundary conditions $\phi(0) = \phi(1) = 0$. We assume a two-term approximate solution as

$$\overline{\phi}(x) = c_1 x(1 - x) + c_2 x^2(1 - x) \tag{E.3}$$

where c_1 and c_2 are unknown constants and each term is chosen to satisfy the specified boundary conditions, $\overline{\phi}(x = 0) = \overline{\phi}(x = 1) = 0$. Using

$$\begin{aligned} \left(\frac{d\overline{\phi}}{dx}\right)^2 &= \left\{c_1(1 - 2x) + c_2(2x - 3x^2)\right\}^2 \\ &= c_1^2(1 - 4x + 4x^2) + c_2^2(4x^2 + 9x^4 - 12x^3) + 2c_1c_2(2x - 7x^2 + 6x^3) \end{aligned} \tag{E.4}$$

Continued

EXAMPLE 5.2 —cont'd

Eq. (E.2) can be rewritten as

$$\begin{aligned} 2I &= \int_0^1 \{ -c_1^2(1-4x+4x^2) - c_2^2(4x^2-12x^3+9x^4) - 2c_1c_2(2x-7x^2+6x^3) \\ &\quad + c_1^2(x^2-2x^3+x^4) + c_2^2(x^4-2x^5+x^6) + 2c_1c_2(x^3-2x^4+x^5) \\ &\quad + c_1(2x^2-2x^3) + c_2(2x^3-2x^4)\}dx \\ &= -\frac{9}{30}c_1^2 - \frac{13}{105}c_2^2 - \frac{126}{420}c_1c_2 + \frac{1}{6}c_1 + \frac{1}{10}c_2 \end{aligned} \tag{E.5}$$

Using the conditions for the extremization of I, we obtain

$$\frac{\partial I}{\partial c_1} = -\frac{3}{5}c_1 - \frac{3}{10}c_2 + \frac{1}{6} = 0 \tag{E.6}$$

$$\frac{\partial I}{\partial c_2} = -\frac{3}{10}c_1 - \frac{26}{105}c_2 + \frac{1}{10} = 0 \tag{E.7}$$

The solution of Eqs. (E.6) and (E.7) gives $c_1 = \frac{1988}{10332} = 0.1924$ and $c_2 = \frac{7}{41} = 0.1707$. Thus, the solution becomes

$$\overline{\phi}(x) = 0.1924\, x(1-x) + 0.1707\, x^2(1-x) \tag{E.8}$$

Note

The exact solution of the problem is given by

$$\phi(x) = \frac{\sin x}{\sin 1} - x \tag{E.9}$$

5.4 SOLUTION OF EIGENVALUE PROBLEMS USING THE VARIATIONAL (RAYLEIGH–RITZ) METHOD

An eigenvalue problem, according to the differential equation formulation, can be stated as

$$A\phi = \lambda B\phi \ \text{ in } \ V \tag{5.41}$$

subject to the boundary conditions

$$E_j\phi = \lambda F_j\phi, \quad j = 1, 2, \ldots, p \ \text{ on } \ S \tag{5.42}$$

where A, B, E_j, and F_j are differential operators, and λ is the eigenvalue. Although the methods discussed for the solution of equilibrium problems can be extended to the case in which the boundary conditions are of the type shown in Eq. (5.42), only the special case in which $F_j = 0$ is considered here so that Eq. (5.42) becomes

$$E_j\phi = 0, \quad j = 1, 2, \ldots, p \ \text{ on } \ S \tag{5.43}$$

It is assumed that the solution of the problem defined by Eqs. (5.41) and (5.43) gives real eigenvalues λ.

In the variational formulation corresponding to Eq. (5.41), a functional $I(\phi, \lambda)$ to be made stationary is identified as

$$I(\phi, \lambda) = I_A(\phi) - I_B(\phi) \tag{5.44}$$

where the subscripts of the functionals I_A and I_B indicate that they are derived from A and B, respectively. It can be shown [5.4] that the stationary value of a function λ_R, called the Rayleigh quotient, defined by

$$\lambda_R = \frac{I_A(\underset{\sim}{\phi})}{I_B(\underset{\sim}{\phi})} \tag{5.45}$$

gives the eigenvalue, and the corresponding $\underset{\sim}{\phi}$ gives the eigenfunction. The trial solution $\underset{\sim}{\phi}(x)$ is chosen as

$$\underset{\sim}{\phi}(x) = \sum_{i=1}^{n} C_i f_i(x) \tag{5.46}$$

where $f_i(x)$ satisfy only the essential boundary conditions. In this case, the conditions for the stationariness of λ_R can be expressed as

$$\frac{\partial \lambda_R}{\partial C_i} = \frac{\partial}{\partial C_i}\left\{\frac{I_A\left(\sum_{i=1}^{n} C_i f_i\right)}{I_B\left(\sum_{i=1}^{n} C_i f_i\right)}\right\} = \frac{1}{I_B^2}\left(I_B\frac{\partial I_A}{\partial C_i} - I_A\frac{\partial I_B}{\partial C_i}\right) = 0, \quad i = 1, 2, \ldots, n \tag{5.47}$$

which shows that

$$\frac{\partial I_A}{\partial C_i} = \lambda \frac{\partial I_B}{\partial C_i}, \quad i = 1, 2, \ldots, n \tag{5.48}$$

Eq. (5.48) yields a set of n simultaneous linear equations. Moreover, if the functions $f_i(x)$ also satisfy the natural (free) boundary conditions, then the Rayleigh–Ritz method gives the same equations as the Galerkin method discussed in Section 5.8.

5.5 SOLUTION OF PROPAGATION PROBLEMS USING THE VARIATIONAL (RAYLEIGH–RITZ) METHOD

The differential equation formulation of a general propagation problem leads to the following equations:

Field equation:

$$A\phi = e \text{ in } V \quad \text{for } t > t_0 \tag{5.49}$$

Boundary conditions:

$$B_i\phi = g_i, i = 1, 2, \ldots, k \text{ on } S \text{ for } t \geq t_0 \tag{5.50}$$

Initial conditions:

$$E_j\phi = h_j, j = 1, 2, \ldots, l \text{ in } V \text{ for } t = t_0 \tag{5.51}$$

where A, B_i, and E_j are differential operators; e, g_i, and h_j are functions of the independent variable; and t_0 is the initial time. In variational methods, the functionals associated with specific types of propagation problems have been developed by several authors, such as Gurtin [5.5].

In the case of propagation problems, the trial solution $\underset{\sim}{\phi}(x, t)$ is taken as

$$\underset{\sim}{\phi}(x, t) = \sum_{i=1}^{n} C_i(t) f_i(x) \tag{5.52}$$

where C_i is now a function of time t. Alternatively, C_i can be taken as a constant and f_i as a function of both x and t as

$$\underset{\sim}{\phi}(x, t) = \sum_{i=1}^{n} C_i f_i(x, t) \tag{5.53}$$

As in the case of equilibrium and eigenvalue problems, the functions f_i have to satisfy the forced boundary conditions. The solution given by Eq. (5.52) or Eq. (5.53) is substituted into the functional, and the necessary conditions for the stationariness are applied to derive a set of equations in the unknowns $C_i(t)$ or C_i. These equations can then be solved to find the functions $C_i(t)$ or the constants C_i.

5.6 EQUIVALENCE OF FINITE ELEMENT AND VARIATIONAL (RAYLEIGH–RITZ) METHODS

If we compare the basic steps of the finite element method described in Section 1.5 with the Rayleigh–Ritz method discussed in this section, we find that both are essentially equivalent. In both methods, a set of trial functions are used for obtaining an approximate solution. Both methods seek a linear combination of trial functions that extremizes (or makes stationary) a given functional. The main difference between the methods is that in the finite element method, the assumed trial functions are not defined over the entire solution domain and they need not satisfy any boundary conditions. Since the trial functions have to be defined over the entire solution domain, the Rayleigh–Ritz method can be used only for domains of simple geometric shape. Similar geometric restrictions also exist in the finite element method, except for the elements. Since elements with simple geometric shape can be assembled to approximate even complex domains, the finite element method proves to be a more versatile technique than the Rayleigh–Ritz method. The only limitation of the finite element method is that the trial functions have to satisfy the convergence (continuity and completeness) conditions stated in Section 3.6.

5.7 DERIVATION OF FINITE ELEMENT EQUATIONS USING THE VARIATIONAL (RAYLEIGH–RITZ) APPROACH

Let the general problem (either physical or purely mathematical), when formulated according to variational approach, require the extremization of a functional I over a domain V. Let the functional I be defined as

$$I = \iiint_v F\left[\vec{\phi}, \frac{\partial}{\partial x}(\vec{\phi}), \ldots \mathrm{d}V\right] + \iint_s g\left[\vec{\phi}, \frac{\partial}{\partial x}(\vec{\phi}), \ldots \mathrm{d}S\right] \tag{5.54}$$

where, in general, the field variable or the unknown function $\vec{\phi}$ is a vector. The finite element procedure for solving this problem can be stated by the following steps:

Step 1: The solution domain V is divided into E smaller parts called subdomains that we call finite elements.
Step 2: The field variable $\vec{\phi}$, which we are trying to find, is assumed to vary in each element in a suitable manner as

$$\vec{\phi} = [N]\vec{\Phi}^{(e)} \tag{5.55}$$

where $[N]$ is a matrix of shape functions ($[N]$ will be a function of the coordinates), and $\vec{\Phi}^{(e)}$ is a vector representing the nodal values of the function $\vec{\phi}$ associated with the element.
Step 3: To derive the elemental equations, we use the conditions of extremization of the functional I with respect to the nodal unknowns $\vec{\underset{\sim}{\Phi}}$ associated with the entire domain.
These are

$$\frac{\partial I}{\partial \vec{\underset{\sim}{\Phi}}} = \begin{Bmatrix} \partial I/\partial \Phi_1 \\ \partial I/\partial \Phi_2 \\ \vdots \\ \partial I/\partial \Phi_M \end{Bmatrix} = \vec{0} \tag{5.56}$$

where M denotes the total number of nodal unknowns in the problem. If the functional I can be expressed as a summation of elemental contributions as

$$I = \sum_{e=1}^{E} I^{(e)} \tag{5.57}$$

where e indicates the element number, then Eq. (5.56) can be expressed as

$$\frac{\partial I}{\partial \Phi_i} = \sum_{e=1}^{E} \frac{\partial I^{(e)}}{\partial \Phi_i} = 0, \quad i = 1, 2, \ldots, M \tag{5.58}$$

In the special case in which I is a quadratic functional of $\vec{\phi}$ and its derivatives, we can obtain the element equations as

$$\frac{\partial I^{(e)}}{\partial \vec{\Phi}^{(e)}} = \left[K^{(e)}\right]\vec{\Phi}^{(e)} - \vec{P}^{(e)} \tag{5.59}$$

where $[K^{(e)}]$ and $\vec{P}^{(e)}$ are the element characteristic matrix and characteristic vector (or vector of nodal actions), respectively.
Step 4: To obtain the overall equations of the system, we rewrite Eq. (5.56) as

$$\frac{\partial I}{\partial \vec{\underset{\sim}{\Phi}}} = \left[\underset{\sim}{K}\right]\vec{\underset{\sim}{\Phi}} - \vec{\underset{\sim}{P}} = \vec{0} \tag{5.60}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E} [K^{(e)}] \tag{5.61}$$

$$\underset{\sim}{\vec{P}} = \sum_{e=1}^{E} \vec{P}^{(e)} \tag{5.62}$$

and the summation sign indicates assembly over all finite elements. The assembly procedure is described in Chapter 6.
Step 5: The linear simultaneous equations (5.60) can be solved, after applying the boundary conditions, to find the nodal unknowns $\underset{\sim}{\vec{\Phi}}$.
Step 6: The function (or field variable) $\vec{\phi}$ in each element is found by using Eq. (5.55).
If necessary, the derivatives of $\vec{\phi}$ are found by differentiating the function $\vec{\phi}$ in a suitable manner.

5.7.1 Convergence Requirements

As stated in Section 3.6, the following conditions have to be satisfied in order to achieve convergence of results as the subdivision is made finer:

1. As the element size decreases, the functions F and g of Eq. (5.54) must tend to be single valued and well behaved. Thus, the shape functions $[N]$ and nodal unknowns $\vec{\Phi}^{(e)}$ chosen must be able to represent any constant value of $\vec{\phi}$ or its derivatives present in the functional I in the limit as the element size decreases to zero.
2. In order to make the summation $I = \sum_{e=1}^{E} I^{(e)}$ valid, we must ensure that terms such as F and g remain finite at interelement boundaries. This can be achieved if the highest derivatives of the field variable $\vec{\phi}$ that occur in F and g are finite. Thus, the element shape functions $[N]$ are to be selected such that at element interface, $\vec{\phi}$ and its derivatives, of one order less than that occurring in F and g, are continuous.

The step-by-step procedure outlined previously can be used to solve any problem provided that the variational principle valid over the domain is known to us. This is illustrated by the following example.

EXAMPLE 5.3
Solve the differential equation

$$\frac{d^2\phi}{dx^2} + \phi + x = 0, \quad 0 \le x \le 1 \tag{E.1}$$

subject to the boundary conditions $\Phi(0) = \Phi(1) = 0$ using the variational finite element method.

Approach: Identify the functional I corresponding to the given differential equation. Divide the domain of the equation using line elements with linear interpolation model and apply the various steps of the finite element method to find the solution.

Solution
The functional I corresponding to Eq. (E.1) is given by[2]

$$I = \frac{1}{2}\int_0^1 \left[-\left(\frac{d\phi}{dx}\right)^2 + \phi^2 + 2\phi x \right] dx \tag{E.2}$$

Step 1: We discretize the domain ($x = 0$ to 1) using three nodes and two elements of equal length as shown in Fig. 5.3B.
Step 2: We assume a linear interpolation model within element e as

$$\phi(x) = [N(x)]\vec{\Phi}^{(e)} = N_i(x)\cdot\Phi_i^{(e)} + N_j(x)\cdot\Phi_j^{(e)} \tag{E.3}$$

where, from Eq. (3.26),

$$N_i(x) = (x_j - x)/l^{(e)} \tag{E.4}$$

Continued

2. The Euler–Lagrange equation corresponding to Eq. (E.2) can be verified to be the same as Eq. (E.1).

EXAMPLE 5.3 —cont'd

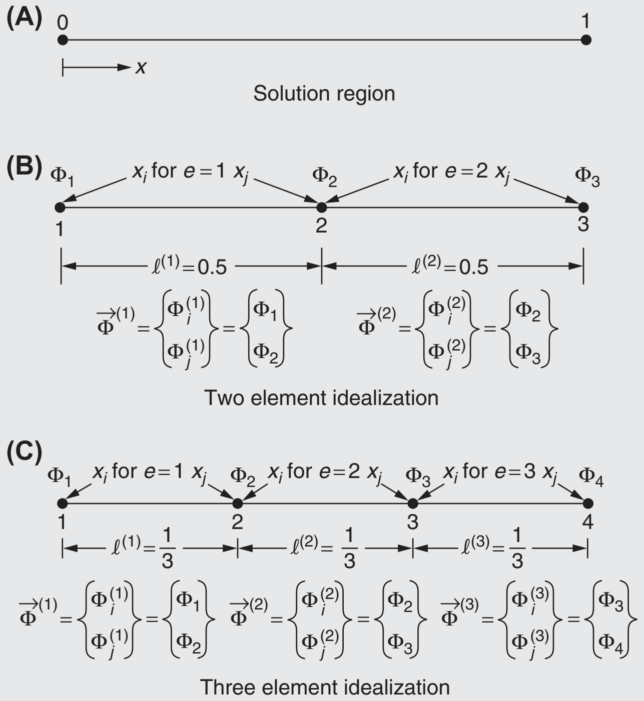

FIGURE 5.3 Discretization for Example 5.3.

$$N_j(x) = (x - x_i)/l^{(e)} \tag{E.5}$$

$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i^{(e)} \\ \Phi_j^{(e)} \end{Bmatrix}$ is the vector of nodal degrees of freedom; $l^{(e)}$ is the length; $\Phi_i^{(e)}$ and $\Phi_j^{(e)}$ are the values of $\phi(x)$ at nodes $i(x = x_i)$ and $j(x = x_j)$, respectively; and i and j are the first and the second (global) nodes of the element e.

Step 3: We express the functional I as a sum of E elemental quantities $I^{(e)}$ as

$$I = \sum_{e=1}^{E} I^{(e)} \tag{E.6}$$

where

$$I^{(e)} = \frac{1}{2}\int_{x_i}^{x_j}\left[-\left(\frac{d\phi}{dx}\right)^2 + \phi^2 + 2x\phi\right]\cdot dx \tag{E.7}$$

By substituting Eq. (E.3) into Eq. (E.7), we obtain

$$I^{(e)} = \frac{1}{2}\int_{x_i}^{x_j}\left\{-\vec{\Phi}^{(e)^T}\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right]\vec{\Phi}^{(e)} + \vec{\Phi}^{(e)^T}[N]^T[N]\vec{\Phi}^{(e)} + 2x[N]\vec{\Phi}^{(e)}\right\}\cdot dx \tag{E.8}$$

For the stationariness of I, we use the necessary conditions

$$\frac{\partial I}{\partial \Phi_i} = \sum_{e=1}^{E}\frac{\partial I^{(e)}}{\partial \Phi_i} = 0, \quad i = 1, 2, \ldots, M \tag{E.9}$$

where E is the number of elements, and M is the number of nodal degrees of freedom. Eq. (E.9) can also be expressed as

$$\sum_{e=1}^{E}\frac{\partial I^{(e)}}{\partial \vec{\Phi}^{(e)}} = \sum_{e=1}^{E}\int_{x_i}^{x_j}\left\{-\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right]\vec{\Phi}^{(e)} + [N]^T[N]\vec{\Phi}^{(e)} + x[N]^T\right\}dx = \vec{0}$$

or

EXAMPLE 5.3 —cont'd

$$\sum_{e=1}^{E}\left[K^{(e)}\right]\vec{\Phi}^{(e)} = \sum_{e=1}^{E}\vec{P}^{(e)} \tag{E.10}$$

where

$$\left[K^{(e)}\right] = \text{element characteristic matrix} = \int_{x_i}^{x_j}\left\{\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right] - [N]^T[N]\right\}dx \tag{E.11}$$

and

$$\vec{P}^{(e)} = \text{element characteristic vector} = \int_{x_i}^{x_j} x[N]^T dx \tag{E.12}$$

By substituting $[N(x)] = [N_i(x)\ N_j(x)] \equiv \left[\frac{x_j-x}{l^{(e)}}\ \frac{x-x_i}{l^{(e)}}\right]$ into Eqs. (E.11) and (E.12), we obtain

$$\left[K^{(e)}\right] = \int_{x_i}^{x_j}\left[\begin{Bmatrix} -\dfrac{1}{l^{(e)}} \\ \dfrac{1}{l^{(e)}} \end{Bmatrix}\left\{-\frac{1}{l^{(e)}}\ \frac{1}{l^{(e)}}\right\} - \begin{Bmatrix} \dfrac{x_j - x}{l^{(e)}} \\ \dfrac{x - x_i}{l^{(e)}} \end{Bmatrix}\left\{\frac{x_j - x}{l^{(e)}}\ \frac{x - x_i}{l^{(e)}}\right\}\right]dx$$

$$= \frac{1}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{l^{(e)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{E.13}$$

and

$$\vec{P}^{(e)} = \int_{x_i}^{x_j} x\begin{Bmatrix} \dfrac{x_j - x}{l^{(e)}} \\ \dfrac{x - x_i}{l^{(e)}} \end{Bmatrix}dx = \frac{1}{6}\begin{Bmatrix} \left(x_j^2 + x_i x_j - 2x_i^2\right) \\ \left(2x_j^2 - x_i x_j - x_i^2\right) \end{Bmatrix} \tag{E.14}$$

We shall compute the results for the cases of two and three elements.

For $E = 2$

Assuming the elements to be of equal length, we have $l^{(1)} = l^{(2)} = 0.5$, $x_i = 0.0$ and $x_j = 0.5$ for $e = 1$, and $x_i = 0.5$ and $x_j = 1.0$ for $e = 2$. Eqs. (E.13) and (E.14) yield

$$\left[K^{(1)}\right] = \left[K^{(2)}\right] = \frac{1}{0.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{0.5}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 22 & -25 \\ -25 & 22 \end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\vec{P}^{(2)} = \frac{1}{24}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$$

For $E = 3$

Assuming the elements to be of equal length, we have $l^{(1)} = l^{(2)} = l^{(3)} = 1/3$; $x_i = 0.0$ and $x_j = 1/3$ for $e = 1$; $x_i = 1/3$ and $x_j = 2/3$ for $e = 2$; and $x_i = 2/3$ and $x_j = 1$ for $e = 3$. Eqs. (E.13) and (E.14) give

Continued

EXAMPLE 5.3 —cont'd

$$[K^{(1)}] = [K^{(2)}] = [K^{(3)}] = 3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 52 & -55 \\ -55 & 52 \end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{1}{54}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\vec{P}^{(2)} = \frac{1}{54}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$$

$$\vec{P}^{(3)} = \frac{1}{54}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix}$$

Step 4: We assemble the element characteristic matrices and vectors and obtain the overall equations as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{P}} \tag{E.15}$$

where

$$[\underset{\sim}{K}] = \frac{1}{12}\begin{bmatrix} 22 & -25 & 0 \\ -25 & (22+22) & -25 \\ 0 & -25 & 22 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 22 & -25 & 0 \\ -25 & 44 & -25 \\ 0 & -25 & 22 \end{bmatrix}$$

$$\underset{\sim}{\vec{\Phi}} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix}$$

and

$$\underset{\sim}{\vec{P}} = \frac{1}{24}\begin{Bmatrix} 1 \\ 2+4 \\ 5 \end{Bmatrix} = \frac{1}{24}\begin{Bmatrix} 1 \\ 6 \\ 5 \end{Bmatrix} \text{for } E = 2$$

and

$$[\underset{\sim}{K}] = \frac{1}{18}\begin{bmatrix} 52 & -55 & 0 & 0 \\ -55 & (52+52) & -55 & 0 \\ 0 & -55 & (52+52) & -55 \\ 0 & 0 & -55 & 52 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 52 & -55 & 0 & 0 \\ -55 & 104 & -55 & 0 \\ 0 & -55 & 104 & -55 \\ 0 & 0 & -55 & 52 \end{bmatrix}$$

$$\underset{\sim}{\vec{\Phi}} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{Bmatrix}$$

and

$$\underset{\sim}{\vec{P}} = \frac{1}{54}\begin{Bmatrix} 1 \\ 2+4 \\ 5+7 \\ 8 \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 1 \\ 6 \\ 12 \\ 8 \end{Bmatrix} \text{for } E = 3$$

Step 5: We can solve the system equations (E.15) after incorporating the boundary conditions. For $E = 2$, the boundary conditions are $\Phi_1 = \Phi_3 = 0$.

EXAMPLE 5.3 —cont'd

The incorporation of these boundary conditions in Eq. (E.15) leads to (after deleting the rows and columns corresponding to Φ_1 and Φ_3 in $\left[\underset{\sim}{K}\right]$, $\underset{\sim}{\vec{\Phi}}$, and $\underset{\sim}{\vec{P}}$)

$$\frac{1}{12}(44)\,\Phi_2 = \frac{1}{24}(6)$$
$$\text{or}\quad \Phi_2 = \frac{3}{44} = 0.06817 \tag{E.16}$$

For $E = 3$, the boundary conditions are $\Phi_1 = \Phi_4 = 0$.
After incorporating these boundary conditions, Eq. (E.15) reduces to

$$\frac{1}{18}\begin{bmatrix} 104 & -55 \\ -55 & 104 \end{bmatrix}\begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 6 \\ 12 \end{Bmatrix}$$

the solution of which is given by

$$\begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.05493 \\ 0.06751 \end{Bmatrix} \tag{E.17}$$

There is no need for Step 6 in this example.
The exact solution of this problem is

$$\phi(x) = \left(\frac{\sin x}{\sin 1} - x\right)$$

which gives

$$\phi(0.5) = 0.0697 \tag{E.18}$$

and

$$\phi\left(\frac{1}{3}\right) = 0.0536, \quad \phi\left(\frac{2}{3}\right) = 0.0649 \tag{E.19}$$

Thus, the accuracy of the two- and three-element discretizations can be seen by comparing Eqs. (E.16) and (E.18), and Eqs. (E.17) and (E.19), respectively.

5.8 WEIGHTED RESIDUAL APPROACH

The weighted residual method is a technique that can be used to obtain approximate solutions to linear and nonlinear differential equations. If we use this method the finite element equations can be derived directly from the governing differential equations of the problem without any need of knowing the functional. We first consider the solution of equilibrium, eigenvalue, and propagation problems using the weighted residual method and then derive the finite element equations using the weighted residual approach.

5.8.1 Solution of Equilibrium Problems Using the Weighted Residual Method

A general equilibrium problem has been stated in Section 5.3 as

$$A\phi = b \text{ in } V \tag{5.63}$$

$$B_j\phi = g_j, \quad j = 1, 2, \ldots, p \text{ on } S \tag{5.64}$$

Eq. (5.63) can be expressed in a more general form as

$$F(\phi) = G(\phi) \text{ in } V \tag{5.65}$$

where F and G are functions of the field variable ϕ. In fact, Eqs. (5.41), (5.42), (5.49–5.51), (5.63), and (5.64) can be seen to be special cases of Eq. (5.65). In the weighted residual method, the field variable is approximated as

$$\underset{\sim}{\phi}(x) = \sum_{i=1}^{n} C_i f_i(x) \tag{5.66}$$

where C_i are constants and $f_i(x)$ are linearly independent functions chosen such that all boundary conditions are satisfied. A quantity R, known as the residual or error, is defined as

$$R = G\left(\underset{\sim}{\phi}\right) - F\left(\underset{\sim}{\phi}\right) \tag{5.67}$$

which is required to satisfy certain conditions that make this error R a minimum or maintain it small in some specified sense. More generally, a weighted function of the residual, $w\,f(R)$, where w is a weight or weighting function and $f(R)$ is a function of R, is taken to satisfy the smallness criterion. The function $f(R)$ is chosen so that $f(R) = 0$ when $R = 0$—that is, when $\underset{\sim}{\phi}(x)$ equals the exact solution $\phi(x)$. As stated, the trial function $\underset{\sim}{\phi}$ is chosen so as to satisfy the boundary conditions but not the governing equation in the domain V, and the smallness criterion is taken as

$$\int_V w\,f(R)\cdot \mathrm{d}V = 0 \tag{5.68}$$

where the integration is taken over the domain of the problem. In the following subsections, four different methods, based on a weighted residual criterion, are given.

5.8.2 Collocation (or Point Collocation) Method

In this method, the residual R is set equal to zero at n points in the domain V, thereby implying that the parameters C_i are to be selected such that the trial function $\underset{\sim}{\phi}(x)$ represents $\phi(x)$ at these n points exactly. This procedure yields n simultaneous algebraic equations in the unknowns C_i $(i = 1, 2, ..., n)$. The collocation points x_j at which $\underset{\sim}{\phi}(x_j) = \phi(x_j)$, $j = 1, 2, ..., n$ are usually chosen to cover the domain V more or less uniformly in some simple pattern. This approach is equivalent to taking, in Eq. (5.68),

$$f(R) = R \quad \text{and} \quad w = \delta(x_j - x) \tag{5.69}$$

where δ indicates the Dirac delta function, x_j denotes the position of the j-th point, and x gives the position of a general point in the domain V. Thus, $w = 1$ at point $x = x_j$ and zero elsewhere in the domain $V (j = 1, 2, ..., n)$.

EXAMPLE 5.4

Find the solution of the differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \phi + x = 0; \quad 0 \le x \le 1$$

subject to the boundary conditions $\phi(0) = \phi(1) = 0$ using the point collocation method. Use $x = \frac{1}{4}$ and $x = \frac{1}{2}$ as the collocation points.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Set the residue equal to zero at the collocation points to evaluate the constants.

Solution

The approximate solution satisfying the boundary conditions is taken as

$$\overline{\phi}(x) = c_1 x(1-x) + c_2 x^2(1-x) \tag{E.1}$$

where c_1 and c_2 are unknown constants. Using this solution, the residue can be expressed as

$$R = \frac{\mathrm{d}^2\overline{\phi}}{\mathrm{d}x^2} + \overline{\phi} + x = c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x \tag{E.2}$$

The residue is set equal to zero at each of the collocation points:

EXAMPLE 5.4 —cont'd

$$R\left(x=\frac{1}{4}\right)=c_1\left(-2+\frac{1}{4}-\frac{1}{16}\right)+c_2\left(2-\frac{3}{2}+\frac{1}{16}-\frac{1}{64}\right)+\frac{1}{4}$$
$$=-\frac{29}{16}c_1+\frac{35}{64}c_2+\frac{1}{4}=0 \tag{E.3}$$

$$R\left(x=\frac{1}{2}\right)=c_1\left(-2+\frac{1}{2}-\frac{1}{4}\right)+c_2\left(2-3+\frac{1}{4}-\frac{1}{8}\right)+\frac{1}{2}$$
$$=-\frac{7}{4}c_1+\frac{7}{8}c_2+\frac{1}{2}=0 \tag{E.4}$$

Eqs. (E.3) and (E.4) can be rewritten as

$$-116c_1+35c_2=-16 \tag{E.5}$$
$$-14c_1-7c_2=-4 \tag{E.6}$$

The solution of Eqs. (E.5) and (E.6) yields $c_1=\frac{18}{93}=0.1935$ and $c_2=\frac{40}{217}=0.1843$.
Thus, the solution given by the point collocation method is

$$\overline{\phi}(x)=0.1935x(1-x)+0.1843x^2(1-x) \tag{E.7}$$

Note
The exact solution of the problem is given by

$$\phi(x)=\frac{\sin x}{\sin 1}-x \tag{E.8}$$

5.8.3 Subdomain Collocation Method

Here, the domain V is first subdivided into n subdomains V_i, $i=1, 2, ..., n$, and the integral of the residual over each subdomain is then required to be zero:

$$\int_{V_i} R\,\mathrm{d}V_i=0,\quad i=1,2,\ldots,n \tag{5.70}$$

This yields n simultaneous algebraic equations for the n unknowns C_i, $i=1, 2, ..., n$. It can be seen that the method is equivalent to choosing

$$f(R)=R\quad\text{and}\quad w=\begin{cases}1 & \text{if } x \text{ is in } V_i\\ 0 & \text{if } x \text{ is not in } V_i,\quad i=1,2,\ldots,n\end{cases} \tag{5.71}$$

EXAMPLE 5.5
Find the solution of the differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}+\phi+x=0;\quad 0\le x\le 1$$

subject to the boundary conditions $\phi(0)=\phi(1)=0$ using the subdomain collocation method. Use the subdomains as $V_1=(0, ¼)$ and $V_2=(¼, ½)$.

Approach: Assume an approximate solution satisfying the boundary conditions with unknown constants.
Set the integral of the residue in each subdomain equal to zero to evaluate the constants.

Solution
The approximate solution satisfying the boundary conditions is taken as

Continued

EXAMPLE 5.5 —cont'd

$$\overline{\phi}(x) = c_1 x(1-x) + c_2 x^2(1-x) \quad \text{(E.1)}$$

where c_1 and c_2 are unknown constants. Using this solution, the residue can be expressed as

$$R = \frac{d^2\overline{\phi}}{dx^2} + \overline{\phi} + x = c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x \quad \text{(E.2)}$$

The integrals of the residue, $R(x)$, over the subdomains can be expressed as

$$\int_{x=0}^{\frac{1}{4}} R(x)\,dx = \left| c_1\left(-2x + \frac{x^2}{2} - \frac{x^3}{3}\right) + c_2\left(2x - 3x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right) + \frac{x^2}{2}\right|_0^{\frac{1}{4}}$$

$$= c_1\left(-\frac{1}{2} + \frac{1}{32} - \frac{1}{192}\right) + c_2\left(\frac{1}{2} - \frac{3}{16} + \frac{1}{192} - \frac{1}{1024}\right) + \frac{1}{32} = -\frac{91}{192}c_1 + \frac{973}{3072}c_2 + \frac{1}{32} = 0 \quad \text{(E.3)}$$

$$\int_{x=\frac{1}{4}}^{\frac{1}{2}} R(x)dx = \left| c_1\left(-2x + \frac{x^2}{2} - \frac{x^3}{3}\right) + c_2\left(2x - 3x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right) + \frac{x^2}{2}\right|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= c_1\left(-1 + \frac{1}{8} - \frac{1}{24}\right) + c_2\left(1 - \frac{3}{4} + \frac{1}{24} - \frac{1}{64}\right) + \frac{1}{8} - \left\{c_1\left(-\frac{1}{2} + \frac{1}{32} - \frac{1}{192}\right) + c_2\left(\frac{1}{2} - \frac{1}{16} + \frac{1}{192} - \frac{1}{1024}\right) + \frac{1}{32}\right\} \quad \text{(E.4)}$$

$$= -\frac{85}{192}c_1 - \frac{1661}{3072}c_2 + \frac{3}{32} = 0$$

Eqs. (E.3) and (E.4) can be rewritten as

$$-1456c_1 + 973c_2 = -96 \quad \text{(E.5)}$$

$$-1360c_1 - 1661c_2 = -288 \quad \text{(E.6)}$$

The solution of Eqs. (E.5) and (E.6) yields $c_1 = 0.1175$ and $c_2 = 0.0772$.
Thus, the solution given by of the subdomain collocation method is

$$\overline{\phi}(x) = 0.1175x(1-x) + 0.0772x^2(1-x) \quad \text{(E.7)}$$

Note

The exact solution of the problem is given by

$$\phi(x) = \frac{\sin x}{\sin 1} - x \quad \text{(E.8)}$$

5.8.4 Galerkin Method

Here the weights w_i are chosen to be the known functions $f_i\,(x)$ of the trial solution and the following n integrals of the weighted residual are set equal to zero:

$$\int_V f_i R dV = 0, \quad i = 1, 2, \ldots, n \quad (5.72)$$

Eq. (5.72) represent n simultaneous equations in the n unknowns, C_1, C_2, …, C_n. This method generally gives the best approximate solution.

EXAMPLE 5.6

Find the approximate deflection of a simply supported beam under a uniformly distributed load p (Fig. 5.2) using the Galerkin method.

Approach: Assume a two-term trial (approximate) solution with each term involving an unknown constant and a trial function satisfying the boundary conditions. Evaluate the constants by setting the integral of product of each of the trial functions and the residue equal to zero.

EXAMPLE 5.6 —cont'd

Solution

The differential equation governing the deflection of the beam (w) is given by (see Example 5.1)

$$EI\frac{d^4w}{dx^4} - p = 0; \quad 0 \le x \le l \tag{E.1}$$

The boundary conditions to be satisfied are

$$\left.\begin{aligned} &w(x=0) = w(x=l) = 0 && \text{(deflection zero at ends)} \\ &EI\frac{d^2w}{dx^2}(x=0) = EI\frac{d^2w}{dx^2}(x=l) = 0 && \text{(bending moment zero at ends)} \end{aligned}\right\} \tag{E.2}$$

where E is Young's modulus, and I is the area moment of inertia of the beam.

We shall assume the trial solution as

$$\underset{\sim}{w}(x) = C_1 \sin\left(\frac{\pi x}{l}\right) + C_2 \sin\left(\frac{3\pi x}{l}\right) \equiv C_1 f_1(x) + C_2 f_2(x) \tag{E.3}$$

where $f_1(x)$ and $f_2(x)$ satisfy the boundary conditions, Eq. (E.2), and C_1 and C_2 are the unknown constants. By substituting the trial solution of Eq. (E.3) into Eq. (E.1), we obtain the residual, R, as

$$R = EIC_1\left(\frac{\pi}{l}\right)^4 \sin\left(\frac{\pi x}{l}\right) + EIC_2\left(\frac{3\pi}{l}\right)^4 \sin\left(\frac{3\pi x}{l}\right) - p \tag{E.4}$$

By applying the Galerkin procedure, we obtain

$$\left.\begin{aligned} \int_0^l f_1(x)R\,dx &= EIC_1\left(\frac{\pi}{l}\right)^4 \frac{l}{2} - p\frac{2l}{\pi} = 0 \\ \int_0^l f_2(x)R\,dx &= EIC_2\left(\frac{3\pi}{l}\right)^4 \frac{l}{2} - p\frac{2l}{3\pi} = 0 \end{aligned}\right\} \tag{E.5}$$

The solution of Eq. (E.5) is

$$C_1 = \frac{4pl^4}{\pi^5 EI} \quad \text{and} \quad C_2 = \frac{4pl^4}{243\pi^5 EI} \tag{E.6}$$

which is the same as the one obtained in Example 5.1.

EXAMPLE 5.7

Find the solution of the differential equation

$$\frac{d^2\phi}{dx^2} + \phi + x = 0; \quad 0 \le x \le 1 \tag{E.1}$$

subject to the boundary conditions $\phi(0) = \phi(1) = 0$ using the Galerkin method.

Approach: Assume a two-term trial (approximate) solution with each term involving an unknown constant and a trial function satisfying the boundary conditions. Evaluate the constants by setting the integral of product of each of the trial functions and the residue equal to zero.

Solution

We assume a two-term trial or approximate solution as

$$\overline{\phi}(x) = c_1 f_1(x) + c_2 f_2(x) \equiv c_1 x(1-x) + c_2 x^2(1-x) \tag{E.2}$$

where c_1 and c_2 are unknown constants and each of the trial functions $f_1(x)$ and $f_2(x)$ is chosen to satisfy the specified boundary conditions, $f_i(x=0) = f_i(x=1) = 0$, $i = 1, 2$. By substituting the trial solution, Eq. (E.2) in Eq. (E.1), we obtain the residue $R(x)$ as

$$R = \frac{d^2\overline{\phi}}{dx^2} + \overline{\phi} + x = c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x \tag{E.3}$$

Continued

EXAMPLE 5.7 —cont'd

According to the Galerkin procedure,

$$\int_0^1 f_1(x)R(x)\,\mathrm{d}x = \int_0^1 (x-x^2)R(x)\,\mathrm{d}x = 0 \tag{E.4}$$

$$\int_0^1 f_2(x)R(x)\,\mathrm{d}x = \int_0^1 (x^2-x^3)R(x)\,\mathrm{d}x = 0 \tag{E.5}$$

Using the relations

$$\int_0^1 R(x)\mathrm{d}x = \frac{1}{2} - \frac{11}{6}c_1 - \frac{11}{12}c_2 \tag{E.6}$$

$$\int_0^1 xR(x)\,\mathrm{d}x = \frac{1}{3} - \frac{11}{12}c_1 - \frac{19}{20}c_2 \tag{E.7}$$

$$\int_0^1 x^2R(x)\mathrm{d}x = \frac{1}{4} - \frac{37}{60}c_1 - \frac{4}{5}c_2 \tag{E.8}$$

$$\int_0^1 x^3R(x)\,\mathrm{d}x = \frac{1}{5} - \frac{7}{15}c_1 - \frac{71}{105}c_2 \tag{E.9}$$

in Eqs. (E.4) and (E.5), we obtain

$$-0.3c_1 - 0.15c_2 = -0.0833 \tag{E.10}$$

$$-0.15c_1 - 0.1238c_2 = -0.05 \tag{E.11}$$

The solution of Eqs. (E.10) and (E.11) is $c_1 = 0.1924$ and $c_2 = 0.1708$. Thus, the approximate solution given by the Galerkin method is

$$\vec{\phi}(x) = 0.1924\, x(1-x) + 0.1708\, x^2(1-x) \tag{E.12}$$

Note

The exact solution of the problem is given by

$$\phi(x) = \frac{\sin x}{\sin 1} - x \tag{E.13}$$

5.8.5 Least Squares Method

In this method, the integral of the weighted square of the residual over the domain is required to be a minimum; that is,

$$\int_V wR^2\,\mathrm{d}V = \text{minimum} \tag{5.73}$$

By using Eqs. (5.66) and (5.63), Eq. (5.73) can be written as

$$\int_V w\left[b - A\left(\sum_{i=1}^{n} C_i f_i(x)\right)\right]^2 dV = \text{minimum} \tag{5.74}$$

where the unknowns in the integral are only C_i. The necessary conditions for minimizing the integral can be expressed as

$$\frac{\partial}{\partial C_i}\left[\int_V w\left\{b - A\left(\sum_{i=1}^{n} C_i f_i(x)\right)\right\}^2 \mathrm{d}V\right] = 0, \quad i = 1, 2, \ldots, n$$

or

$$\int_V wA(f_i(x))\left[b - A\left(\sum_{i=1}^{n} C_i f_i(x)\right)\right] \mathrm{d}V = 0, \quad i = 1, 2, \ldots, n \tag{5.75}$$

The weighting function w is usually taken as unity in this method. Eq. (5.75) leads to n simultaneous linear algebraic equations in terms of the unknowns $C_1, C_2, \ldots, C_n$.

EXAMPLE 5.8

Find the approximate deflection of a simply supported beam under a uniformly distributed load p (Fig. 5.2) using the least squares method.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Use the conditions to minimize the integral of the square of the residue over the length of the beam to evaluate the constants.

Solution

The governing differential equation and the boundary conditions are the same as those given in Eqs. (E.1) and (E.2), respectively, of Example 5.6. By assuming the trial solution as

$$\underset{\sim}{w}(x) = C_1 f_1(x) + C_2 f_2(x) \tag{E.1}$$

where

$$f_1(x) = \sin\left(\frac{\pi x}{l}\right) \quad \text{and} \quad f_2(x) = \sin\left(\frac{3\pi x}{l}\right) \tag{E.2}$$

the residual, R, becomes

$$R = EI\frac{d^4 w}{dx^4} - p = EIC_1\left(\frac{\pi}{l}\right)^4 \sin\left(\frac{\pi x}{l}\right) + EIC_2\left(\frac{3\pi}{l}\right)^4 \sin\left(\frac{3\pi x}{l}\right) - p \tag{E.3}$$

The application of the least squares method gives the following equations:

$$\begin{aligned}\frac{\partial}{\partial C_1}\left(\int_0^l R^2 dx\right) &= \frac{\partial}{\partial C_1}\left\{\int_0^l \left[(EI)^2 C_1^2\left(\frac{\pi}{l}\right)^8 \sin^2\left(\frac{\pi x}{l}\right) + (EI)^2 C_2^2\left(\frac{3\pi}{l}\right)^8 \sin^2\left(\frac{3\pi x}{l}\right) + p^2 + 2(EI)^2 C_1 C_2\left(\frac{\pi}{l}\right)^8 \sin\left(\frac{\pi x}{l}\right)\cdot\sin\left(\frac{3\pi x}{l}\right)\right.\right.\\ &\quad \left.\left. - 2EIpC_1\left(\frac{\pi}{l}\right)^4 \sin\left(\frac{\pi x}{l}\right) - 2EIpC_2\left(\frac{3\pi}{l}\right)^4 \sin\left(\frac{3\pi x}{l}\right)\right] dx\right\}\\ &= 0\end{aligned}$$

or

$$(EI)^2 C_1\left(\frac{\pi}{l}\right)^8 l \;-\; 4EIp\frac{l}{\pi}\left(\frac{\pi}{l}\right)^4 = 0 \tag{E.4}$$

$$\begin{aligned}\frac{\partial}{\partial C_2}\left(\int_0^l R^2\cdot dx\right) &= \frac{\partial}{\partial C_2}\left\{\int_0^l \left[(EI)^2 C_1^2\left(\frac{\pi}{l}\right)^8 \sin^2\left(\frac{\pi x}{l}\right) + (EI)^2 C_2\left(\frac{3\pi}{l}\right)^8 \cdot\sin^2\left(\frac{3\pi x}{l}\right) + p^2 + 2(EI)^2 C_1 C_2\left(\frac{\pi}{l}\right)^8 \sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right.\right.\\ &\quad \left.\left. - 2EIp\cdot C_1\left(\frac{\pi}{l}\right)^4 \sin\left(\frac{\pi x}{l}\right) - 2EIpC_2\left(\frac{3\pi}{l}\right)^4 \sin\left(\frac{3\pi x}{l}\right)\right] dx\right\}\\ &= 0\end{aligned}$$

or

$$(EI)^2 C_2\left(\frac{3\pi}{l}\right)^8 l - 4EIp\frac{l}{3\pi}\left(\frac{3\pi}{l}\right)^4 = 0 \tag{E.5}$$

The solution of Eqs. (E.4) to (E.5) leads to

$$C_1 = \frac{4pl^4}{\pi^5 EI} \quad \text{and} \quad C_2 = \frac{4pl^4}{243\pi^5 EI} \tag{E.6}$$

which can be seen to be identical to the solutions obtained in Examples 5.1 and 5.6 [3].

3. Although the solutions given by the Rayleigh–Ritz, Galerkin, and least squares methods happen to be the same for the example considered, in general they lead to different solutions.

EXAMPLE 5.9
Find the solution of the differential equation

$$\frac{d^2\phi}{dx^2} + \phi + x = 0; \quad 0 \le x \le 1$$

subject to the boundary conditions $\phi(0) = \phi(1) = 0$ using the least squares method.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Use the conditions to minimize the integral of the square of the residue over the domain of the problem to evaluate the constants.

Solution
The approximate solution satisfying the boundary conditions is taken as

$$\overline{\phi}(x) = c_1 x(1-x) + c_2 x^2(1-x) \tag{E.1}$$

where c_1 and c_2 are unknown constants. Using this solution, the residue can be expressed as

$$R = \frac{d^2\overline{\phi}}{dx^2} + \overline{\phi} + x = c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x \tag{E.2}$$

The integrals of the square of the residue, $R(x)$, over the domain of the problem is given by

$$I = \int_{x=0}^{1} w(x) R^2(x) dx \tag{E.3}$$

By assuming the weighting function to be unity, $w(x) = 1$, we obtain

$$I = \int_{x=0}^{1} R^2(x) dx \tag{E.4}$$

For the minimum value of the integral I, we get

$$\frac{\partial I}{\partial c_1} = \int_0^1 2R \frac{\partial R}{\partial c_1} dx = 0 \tag{E.5}$$

$$\frac{\partial I}{\partial c_2} = \int_0^1 2R \frac{\partial R}{\partial c_2} dx = 0 \tag{E.6}$$

Eqs. (E.5) and (E.6) can be evaluated as

$$\begin{aligned} \int_0^1 R\frac{\partial R}{\partial c_1} dx &= \int_0^1 \left[c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x\right]\left(-2 + x - x^2\right) dx \\ &= \frac{101}{30}c_1 + \frac{101}{60}c_2 - \frac{11}{12} = 0 \end{aligned} \tag{E.7}$$

$$\begin{aligned} \int_0^1 R\frac{\partial R}{\partial c_2} dx &= \int_0^1 \left[c_1\left(-2 + x - x^2\right) + c_2\left(2 - 6x + x^2 - x^3\right) + x\right]\left(2 - 6x + x^2 - x^3\right) dx \\ &= \frac{707}{420}c_1 + \frac{1572}{420}c_2 - \frac{399}{420} = 0 \end{aligned} \tag{E.8}$$

The solution of Eqs. (E.7) and (E.8) gives $c_1 = 0.1875$ and $c_2 = 0.1695$. Thus, the solution given by the least squares method is

$$\overline{\phi}(x) = 0.1875x(1-x) + 0.1695x^2(1-x) \tag{E.9}$$

Note
The exact solution of the problem is given by

$$\phi(x) = \frac{\sin x}{\sin 1} - x \tag{E.10}$$

5.9 SOLUTION OF EIGENVALUE PROBLEMS USING WEIGHTED RESIDUAL METHOD

An eigenvalue problem can be stated as

$$A\phi = \lambda B\phi \text{ in } V \tag{5.76}$$

$$E_j\phi = 0, \quad j = 1, 2, \ldots, p \text{ on } S \tag{5.77}$$

where A, B, and E_j are differential operators. By using Eqs. (5.66) and (5.76), the residual R can be expressed as[4]

$$R = \lambda B \underset{\sim}{\phi} - A \underset{\sim}{\phi} = \sum_{i=1}^{n} C_i(\lambda B f_i - A f_i) \tag{5.78}$$

If the trial solution of Eq. (5.66) contains any true eigenfunctions, then there exists sets of C_i and values of λ for which the residual R vanishes identically over the domain V. If $\underset{\sim}{\phi}(x)$ does not contain any eigenfunctions, then only approximate solutions will be obtained.

All four residual methods discussed in the case of equilibrium problems are also applicable to eigenvalue problems. For example, if we use the Galerkin method, we set the integral of the weighted residual equal to zero as

$$\int_V f_i(x) \cdot R \, \mathrm{d}V = 0, \quad i = 1, 2, \ldots, n \tag{5.79}$$

Eq. (5.79) gives the following algebraic (matrix) eigenvalue problem:

$$[A]\overrightarrow{C} = \lambda[B]\overrightarrow{C} \tag{5.80}$$

where $[A]$ and $[B]$ denote square symmetric matrices of size $n \times n$ given by

$$[A] = [A_{ij}] = \left[\int_V f_i A f_j \mathrm{d}V\right] \tag{5.81}$$

$$[B] = [B_{ij}] = \left[\int_V f_i B f_j \mathrm{d}V\right] \tag{5.82}$$

and $\overrightarrow{C}$ denotes the vector of unknowns C_i, $i = 1, 2, \ldots, n$. Now the solution of Eq. (5.80) can be obtained by any of the methods discussed in Section 7.3.

5.10 SOLUTION OF PROPAGATION PROBLEMS USING WEIGHTED RESIDUAL METHOD

A propagation problem has been stated earlier as

$$A\phi = e \text{ in } V \quad \text{for } t > t_0 \tag{5.83}$$

$$B_i\phi = g_i, \quad i = 1, 2, \ldots, k \text{ in } S \text{ for } t \geq t_0 \tag{5.84}$$

$$E_j\phi = h_j, \quad j = 1, 2, \ldots, l \text{ in } V \text{ for } t = t_0 \tag{5.85}$$

The trial solution of the problem is taken as

$$\underset{\sim}{\phi}(x, t) = \sum_{i=1}^{n} C_i(t) f_i(x) \tag{5.86}$$

where $f_i(x)$ are chosen to satisfy the boundary conditions, Eq. (5.84). Since Eqs. (5.83) and (5.85) are not satisfied by $\underset{\sim}{\phi}(x, t)$, there will be two residuals, one corresponding to each of these equations. For simplicity, we will assume that Eq. (5.85) gives the initial conditions explicitly as

$$\phi(x, t) = \phi_0 \text{ at } t = 0 \tag{5.87}$$

4. The trial functions $f_i(x)$ are assumed to satisfy the boundary conditions of Eq. (5.77).

Thus, the residual corresponding to the initial conditions (R_1) can be formulated as

$$R_1 = \phi_0 - \underset{\sim}{\phi}(x, t = 0) \ \text{ for all } x \text{ in } V \tag{5.88}$$

where

$$\underset{\sim}{\phi}(x, t = 0) = \sum_{i=1}^{n} C_i(0) f_i(x) \tag{5.89}$$

Similarly, the residual corresponding to the field equation (R_2) is defined as

$$R_2 = e - A\,\underset{\sim}{\phi}(x, t) \ \text{ for all } x \text{ in } V \tag{5.90}$$

Now any of the four residual methods discussed in the case of equilibrium problems can be applied to find the unknown functions $C_i(t)$. For example, if we apply the Galerkin procedure to each of the residuals R_1 and R_2, we obtain

$$\int_V f_i(x) R_1 \cdot \mathrm{d}V = 0, \quad i = 1, 2, \ldots, n \tag{5.91}$$

$$\int_V f_i(x) R_2 \cdot \mathrm{d}V = 0, \quad i = 1, 2, \ldots, n \tag{5.92}$$

Eqs. (5.91) and (5.92) lead to $2n$ equations in the $2n$ unknowns $C_i(0)$ and $C_i(t)$, $i = 1, 2, \ldots, n$, which can be solved either analytically or numerically.

5.11 DERIVATION OF FINITE ELEMENT EQUATIONS USING WEIGHTED RESIDUAL (GALERKIN) APPROACH

Let the governing differential equation of the (equilibrium) problem be given by

$$A(\phi) = b \ \text{ in } \ V \tag{5.93}$$

and the boundary conditions by

$$B_j(\phi) = g_j, \quad j = 1, 2, \ldots, p \ \text{ on } \ S \tag{5.94}$$

The Galerkin method requires that

$$\int_V \left[A\left(\underset{\sim}{\phi} \right) - b \right] f_i \mathrm{d}V = 0, \quad i = 1, 2, \ldots, n \tag{5.95}$$

where the trial functions f_i in the approximate solution

$$\underset{\sim}{\phi} = \sum_{i=1}^{n} C_i f_i \tag{5.96}$$

are assumed to satisfy the boundary conditions, Eq. (5.94). Note that f_i are defined over the entire domain of the problem.

Since the field equation (5.93) holds for every point in the domain V, it also holds for any set of points lying in an arbitrary subdomain or finite element in V. This permits us to consider any one element and define a local approximation similar to Eq. (5.96). Thus, we immediately notice that the familiar interpolation model for the field variable of the finite element will be applicable here also. If Eq. (5.96) is interpreted to be valid for a typical element e, the unknowns C_i can be recognized as the nodal unknowns $\Phi_i^{(e)}$ (nodal values of the field variable or its derivatives) and the functions f_i as the shape functions $N_i^{(e)}$. Eq. (5.95) can be made to be valid for element e as

$$\int_{V^{(e)}} \left[A\left(\phi^{(e)} \right) - b^{(e)} \right] N_i^{(e)} \cdot \mathrm{d}V^{(e)} = 0, \quad i = 1, 2, \ldots, n \tag{5.97}$$

where the interpolation model is taken in the standard form as

$$\phi^{(e)} = \left[N^{(e)}\right]\vec{\Phi}^{(e)} = \sum_i N_i^{(e)}\Phi_i^{(e)} \tag{5.98}$$

Eq. (5.97) gives the required finite element equations for a typical element. These element equations have to be assembled to obtain the system or overall equations as outlined in Section 6.2.

Notes

The shape functions of individual elements $N_i^{(e)}$ need not satisfy any boundary conditions, but they have to satisfy the interelement continuity conditions necessary for the assembly of the element equations. As stated earlier, to avoid any spurious contributions in the assembly process, we have to ensure that the (assumed) field variable ϕ and its derivatives up to one order less than the highest order derivative appearing under the integral in Eq. (5.97) are continuous along element boundaries. Since the differential operator A in the integrand usually contains higher order derivatives than the ones that appear in the integrand of the functional I in the variational formulation, we notice that the Galerkin method places more restrictions on the shape functions The boundary conditions of the problem have to be incorporated after assembling the element equations as outlined in Chapter 6.

EXAMPLE 5.10

Solve the differential equation

$$\frac{d^2\phi}{dx^2} + \phi + x = 0; \quad 0 \le x \le 1$$

subject to the boundary conditions $\phi(0) = \phi(1) = 0$ using the Galerkin finite element method.

Approach: Divide the domain of the equation using a suitable number of finite elements. Using a linear interpolation model, derive the system equations by setting the integral of product of each of the shape functions and the residue equal to zero over the domain of the equation.

Solution

In this case the residual is given by

$$R = \left(\frac{d^2\phi}{dx^2} + \phi + x\right) \tag{E.1}$$

Eq. (5.95) can be expressed as

$$\int_0^1 \left[\frac{d^2\phi}{dx^2} + \phi + x\right] N_k(x)dx = 0; \quad k = i, j$$

or

$$\sum_{e=1}^{E} \int_{x_i}^{x_j} \left[N^{(e)}\right]^T \left[\frac{d^2\phi^{(e)}}{dx^2} + \phi^{(e)} + x\right] dx = 0 \tag{E.2}$$

where E is the number of elements, and x_i and x_j are the values of x at the first and the second nodes of element e, respectively.

We shall assume a linear interpolation model for $\phi^{(e)}$ so that

$$\phi^{(e)}(x) = N_i(x)\Phi_i^{(e)} + N_j(x)\Phi_j^{(e)} \tag{E.3}$$

and hence

$$\left[N^{(e)}\right] = \left[N_i(x)\ N_j(x)\right] \tag{E.4}$$

where

$$N_i(x) = \frac{x_j - x}{l^{(e)}} \quad \text{and} \quad N_j(x) = \frac{x - x_i}{l^{(e)}} \tag{E.5}$$

The term $\int_{x_i}^{x_j} \left[N^{(e)}\right]^T (d^2\phi^{(e)}/dx^2)dx$ can be written, after substitution of Eqs. (E.3) and (E.4) and integration by parts, as

$$\int_{x_i}^{x_j} \left[N^{(e)}\right]^T \frac{d^2\phi^{(e)}}{dx^2} dx = \left[N^{(e)}\right]^T \frac{d\phi^{(e)}}{dx}\Bigg|_{x_i}^{x_j} - \int_{x_i}^{x_j} \frac{d\left[N^{(e)}\right]^T}{dx} \frac{d\phi^{(e)}}{dx} dx \tag{E.6}$$

Substitution of Eq. (E.6) into Eq. (E.2) yields, for element e,

Continued

EXAMPLE 5.10 —cont'd

$$\left[N^{(e)}\right]^T \frac{d\phi^{(e)}}{dx}\bigg|_{x_i}^{x_j} - \int_{x_i}^{x_j}\left\{\frac{d\left[N^{(e)}\right]^T}{dx}\frac{d\phi^{(e)}}{dx} - \left[N^{(e)}\right]^T\phi^{(e)} - \left[N^{(e)}\right]^T x\right\}dx = 0 \tag{E.7}$$

as the governing equation.

The first two terms in the integral of Eq. (E.7) yield the element characteristic matrix $[K^{(e)}]$ and the last term in the integral produces the element characteristic vector $\vec{P}^{(e)}$ in the equation

$$\left[K^{(e)}\right]\vec{\Phi}^{(e)} = \vec{P}^{(e)} \tag{E.8}$$

The left-most term in Eq. (E.7) contributes to the assembled vector $\vec{P}$ provided the derivative $(d\phi/dx)$ is specified at either end of the element e. This term is neglected if nothing is known about the value of $(d\phi/dx)$ at the nodal points. The evaluation of the integrals in Eq. (E.7) proceeds as follows:

$$\frac{d}{dx}\left[N^{(e)}\right]^T = \frac{d}{dx}\left\{\begin{matrix}(x_j - x)/l^{(e)} \\ (x - x_i)/l^{(e)}\end{matrix}\right\} = \frac{1}{l^{(e)}}\left\{\begin{matrix}-1 \\ 1\end{matrix}\right\} \tag{E.9}$$

$$\frac{d\phi^{(e)}}{dx} = \frac{d}{dx}\left[N^{(e)}\right]\vec{\Phi}^{(e)} = \frac{1}{l^{(e)}}\left[-1 \quad 1\right]\left\{\begin{matrix}\Phi_i^{(e)} \\ \Phi_j^{(e)}\end{matrix}\right\} \tag{E.10}$$

$$\int_{x_i}^{x_j}\frac{d}{dx}\left[N^{(e)}\right]^T\frac{d\Phi^{(e)}}{dx}dx = \frac{1}{l^{(e)}}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}\left\{\begin{matrix}\Phi_i^{(e)} \\ \phi_j^{(e)}\end{matrix}\right\} \tag{E.11}$$

$$\int_{x_i}^{x_j}\left[N^{(e)}\right]^T\Phi^{(e)}dx = \frac{l^{(e)}}{6}\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix}\left\{\begin{matrix}\Phi_i^{(e)} \\ \Phi_j^{(e)}\end{matrix}\right\} \tag{E.12}$$

$$\int_{x_i}^{x_j}\left[N^{(e)}\right]^T x\,dx = \frac{1}{6}\left\{\begin{matrix}\left(x_j^2 + x_i x_j + 2x_i^2\right) \\ \left(2x_j^2 - x_i x_j - x_i^2\right)\end{matrix}\right\} \tag{E.13}$$

Since the value of $(d\phi/dx)$ is not specified at any node, we neglect the left-most term in Eq. (E.7). Thus, we obtain from Eq. (E.7)

$$\sum_{e=1}^{E}\left[K^{(e)}\right]\vec{\Phi}^{(e)} = \sum_{e=1}^{E}\vec{P}^{(e)} \tag{E.14}$$

where

$$\left[K^{(e)}\right] = \frac{1}{l^{(e)}}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix} - \frac{l^{(e)}}{6}\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix} \tag{E.15}$$

$$\vec{P}^{(e)} = \frac{1}{6}\left\{\begin{matrix}\left(x_j^2 + x_i x_j - 2x_i^2\right) \\ \left(2x_j^2 - x_i x_j - x_i^2\right)\end{matrix}\right\} \tag{E.16}$$

It can be seen that Eqs. (E.14) to (E.16) are identical to those obtained in Example 5.3, and hence the solution will also be the same.

5.12 DERIVATION OF FINITE ELEMENT EQUATIONS USING WEIGHTED RESIDUAL (LEAST SQUARES) APPROACH

Let the differential equation to be solved be stated as

$$A(\phi) = f(x, y, z) \text{ in } V \tag{5.99}$$

subject to the boundary conditions

$$\phi = \phi_0 \text{ on } S_0 \tag{5.100}$$

$$B_j\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}, \ldots\right) = g_j(x, y, z) \text{ on } S_j \tag{5.101}$$

with

$$\sum_{j=0,\,1,\ldots} S_j = S \tag{5.102}$$

where $A(\phi)$ and $B\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}, \ldots\right)$ are linear differential operators involving the (unknown) field variable and its derivatives with respect to x, y, and z; f and g_j are known functions of x, y, and z; and V is the solution domain with boundary S.

Step 1: Divide the solution domain into E finite elements, each having n nodal points with m unknowns (degrees of freedom) per node. Thus, m denotes the number of parameters, such as ϕ, $(\partial\phi/\partial x)$, $(\partial\phi/\partial y)$, …, taken as unknowns at each node.

Step 2: Assume an interpolation model for the field variable inside an element e as

$$\phi^{(e)}(x, y, z) = \sum_i N_i(x, y, z)\Phi_i^{(e)} = [N(x, y, z)]\vec{\Phi}^{(e)} \tag{5.103}$$

where N_i is the shape function corresponding to the i-th degree of freedom of element of e, $\Phi_i^{(e)}$.

Step 3: Derive the element characteristic matrices and vectors. Substitution of the approximate solution of Eq. (5.103) into Eqs. (5.99) and (5.101) yields the residual errors as

$$R^{(e)}(x, y, z) = A\left([N]\vec{\Phi}^{(e)}\right) - f^{(e)} \equiv A^{(e)} - f^{(e)} \tag{5.104}$$

$$r_j^{(e)}(x, y, z) = B_j\left([N]\vec{\Phi}^{(e)}\right) - g_j^{(e)} \equiv B_j^{(e)} - g_j^{(e)} \tag{5.105}$$

where $R^{(e)}$ and $r_j^{(e)}$ represent the residual errors due to differential equation and j-th boundary condition, respectively, and $A^{(e)}$ and $B_j^{(e)}$ can be expressed in terms of the vector of nodal unknowns as

$$A^{(e)} \equiv \left[C^{(e)}(x, y, z)\right]\vec{\Phi}^{(e)} \tag{5.106}$$

$$B_j^{(e)} = \left[D_j^{(e)}(x, y, z)\right]\vec{\Phi}^{(e)} \tag{5.107}$$

In the least squares method we minimize the weighted square of the residual error over the domain; that is,

$$I = a\iiint_V R^2 dV + \sum_j b_j \iint_{S_j} r_j^2 \mathrm{d}S_j = \text{minimum} \tag{5.108}$$

where a and b_1, b_2, … are the weighting factors, all of which can be taken to be unity for simplicity; and the errors R and r_j can be expressed as the sum of element contributions as

$$R = \sum_{e=1}^{E} R^{(e)}, \quad r_j = \sum_{j=1}^{E} r_j^{(e)} \tag{5.109}$$

The conditions for the minimum of I are

$$\frac{\partial I}{\partial \underset{\sim}{\vec{\Phi}}} = \begin{Bmatrix} \partial I/\partial \Phi_1 \\ \partial I/\partial \Phi_2 \\ \vdots \\ \partial I/\partial \Phi_M \end{Bmatrix} = \sum_{e=1}^{E} \frac{\partial I^{(e)}}{\partial \vec{\Phi}^{(e)}} = \vec{0} \tag{5.110}$$

where M denotes the total number of nodal unknowns in the problem ($M = m \times$ total number of nodes), and $I^{(e)}$ represents the contribution of element e to the functional I:

$$I^{(e)} = \iiint_{V^{(e)}} R^{(e)^2} \mathrm{d}V + \sum_j \iint_{S_j^{(e)}} r_j^{(e)^2} \mathrm{d}S_j \tag{5.111}$$

The squares of the residues $R(e)$ and $r_j^{(e)}$ can be expressed as

$$\begin{aligned} R_j^{(e)^2} &= A^{(e)^2} - 2A^{(e)} f^{(e)} + f^{(e)^2} \\ &= \vec{\Phi}^{(e)^T} \left[C^{(e)}\right]^T \left[C^{(e)}\right] \vec{\Phi}^{(e)} - 2\left[C^{(e)}\right] \vec{\Phi}^{(e)} f^{(e)} + f^{(e)^2} \end{aligned} \tag{5.112}$$

$$\begin{aligned} r_j^{(e)^2} &= B_j^{(e)^2} - 2B_j^{(e)} g_j^{(e)} + g_j^{(e)^2} \\ &= \vec{\Phi}^{(e)^T} \left[D_j^{(e)}\right]^T \left[D_j^{(e)}\right] \vec{\Phi}^{(e)} - 2\left[D_j^{(e)}\right] \vec{\Phi}^{(e)} g_j^{(e)} + g_j^{(e)^2} \end{aligned} \tag{5.113}$$

Eqs. (5.110) and (5.111) lead to

$$\sum_{e=1}^{E} \left[\frac{\partial}{\partial \vec{\Phi}^{(e)}} \iiint_{V^{(e)}} R^{(e)} \mathrm{d}V + \frac{\partial}{\partial \vec{\Phi}^{(e)}} \left(\sum_j \iint_{S_j^{(e)}} r_j^{(e)^2} \mathrm{d}S_j \right) \right] = \vec{0} \tag{5.114}$$

with

$$\frac{\partial}{\partial \vec{\Phi}^{(e)}} \left(R^{(e)^2} \right) = 2\left[C^{(e)}\right]^T \left[C^{(e)}\right] \vec{\Phi}^{(e)} - 2\left[C^{(e)}\right]^T f^{(e)} \tag{5.115}$$

and

$$\frac{\partial}{\partial \vec{\Phi}^{(e)}} \left(r_j^{(e)^2} \right) = 2\left[D_j^{(e)}\right]^T \left[D_j^{(e)}\right] \vec{\Phi}^{(e)} - 2\left[D_j^{(e)}\right] g_j^{(e)} \tag{5.116}$$

By defining

$$\left[K_1^{(e)}\right] = \iiint_{V(e)} \left[C^{(e)}\right]^T \left[C^{(e)}\right] \mathrm{d}V \tag{5.117}$$

$$\left[K_2^{(e)}\right] = \sum_j \iint_{S_j^{(e)}} \left[D_j^{(e)}\right]^T \left[D_j^{(e)}\right] \mathrm{d}s_j \tag{5.118}$$

$$\vec{P}_1^{(e)} = \iiint_{V^{(e)}} f^{(e)} \left[C^{(e)}\right]^T \mathrm{d}V \tag{5.119}$$

$$\vec{P}_2^{(e)} = \sum_j \iint_{S_j^{(e)}} g_j^{(e)} \left[D_j^{(e)}\right]^T \mathrm{d}S_j \tag{5.120}$$

Eq. (5.114) becomes

$$\sum_{e=1}^{E} \left(\left[K^{(e)}\right] \vec{\Phi}^{(e)} - \vec{P}^{(e)} \right) = \vec{0} \tag{5.121}$$

where the summation sign indicates the familiar assembly over all the finite elements,

$$\left[K^{(e)}\right] = \left[K_1^{(e)}\right] + \left[K_2^{(e)}\right] = \text{element characteristic matrix} \tag{5.122}$$

and

$$\vec{P}^{(e)} = \vec{P}_1^{(e)} + \vec{P}_2^{(e)} = \text{element characteristic vector} \tag{5.123}$$

Step 4: Derive the overall system equations. The assembled set of equations (5.121) can be expressed in the standard form

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{P}} \tag{5.124}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E}[K^{(e)}] \quad \text{and} \quad \underset{\sim}{\vec{P}} = \sum_{e=1}^{E}\vec{P}^{(e)} \tag{5.125}$$

Step 5: Solve for the nodal unknowns. After incorporating the boundary conditions prescribed on S_0, Eq. (5.124) can be solved to find the vector $\underset{\sim}{\vec{\Phi}}$.
Step 6: Compute the element resultants. Once the nodal unknown vector $\underset{\sim}{\vec{\Phi}}$, and hence $\vec{\Phi}^{(e)}$, is determined the element resultants can be found, if required, by using Eq. (5.103).

EXAMPLE 5.11
Find the solution of the differential equation $(d^2\phi/dx^2) - \phi = x$ subject to the boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$ using the least squares finite element method.

Approach: Divide the domain of the equation into finite elements. Assume a suitable interpolation model for each finite element. Substitute the assumed solution into the differential equation to find the residue. Use the conditions to minimize the integral of the square of the residue over the domain of the problem to derive the system equations.

Solution
Here, the solution domain is given by $0 \le x \le 1$. Five one-dimensional elements, each having two nodes with two unknowns (ϕ and $(d\phi/dx)$) per node, are used for the idealization. Thus, the total number of degrees of freedom is $M = 12$. Since there are four nodal degrees of freedom per element, the first-order Hermite polynomials are used as interpolation functions (as in the case of a beam element). The exact solution of this problem is given by $\phi(x) = (2 \sinh x/\sinh 1) - x$. The finite element solution obtained by Akin and Sen Gupta [5.6] is compared with the exact solution at the six nodes in Table 5.1.

TABLE 5.1 Results of Example 5.11

	Value of ϕ		Value of $(d\phi/dx)$	
Value of x	Finite Element	Exact	Finite Element	Exact
0.0	0.000,000	0.000,000	0.701,837	0.701,837
0.2	0.142,641	0.142,641	0.735,988	0.735,987
0.4	0.299,034	0.299,034	0.839,809	0.839,808
0.6	0.483,481	0.483,480	1.017,47	1.017,47
0.8	0.711,411	0.711,412	1.276,09	1.276,10
1.0	1.000,000	1.000,000	1.626,07	1.626,07

5.13 STRONG AND WEAK FORM FORMULATIONS

The partial differential equations governing the equilibrium of a solid body are said to be of a strong form. The strong form of equations, as opposed to a weak form, requires strong continuity of the associated field variables, namely the displacement components u, v, and w in the case of a solid mechanics problem. Any function that defines any of these field variables needs to be differentiable up to the order of the governing partial differential equation. Usually, it is very difficult to find an exact solution of the strong form of partial differential equation.

The equations derived using an energy principle, such as the principle of minimum potential energy, or a weighted residual method, such as the Galerkin method, are usually of a weak form. The equations of weak form are usually in integral form and require a weaker continuity on the field variables. Because of the weaker requirement on the field variables and the integral form of the governing expression, a formulation based on a weak form is expected to lead to a set of equations for the discretized system that yield a more accurate result, particularly for systems involving complex geometry. Hence, a weak form type of formulation is preferred for obtaining an approximate solution. Thus, the finite element method, based on a weak form of formulation such as an energy principle or a weighted residual approach, has become quite popular. The weak form type of formulation yields a set of well-behaved algebraic system equations. The following example shows the advantages of a weak form formulation.

EXAMPLE 5.12

The equation governing the deflection of a beam, $w(x)$, is given by

$$EI\frac{d^4w}{dx^4} = p(x) \tag{E.1}$$

where $p(x)$ is the distributed force along the beam. For a cantilever beam subjected to an end load and an end moment as shown in Fig. 5.4, find the deflection of the beam using the Galerkin method with the assumed solution

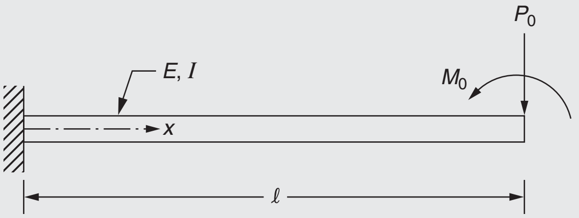

FIGURE 5.4 Cantilever beam subjected to end load and moment.

$$\tilde{w}(x) = C\,f(x) \equiv C\left(3x^2 l - x^3\right) \tag{E.2}$$

where $f(x)$ is a trial function and C is a constant. Also, indicate the advantage of the weak formulation.

Solution

Because the distributed load $p(x) = 0$ for the beam shown in Fig. 5.4, the governing equation becomes

$$EI\frac{d^4w}{dx^4} = 0 \tag{E.3}$$

In the Galerkin method, the constant C in the assumed solution is found using the relation

$$\int_0^l R(x)f(x)dx = 0 \tag{E.4}$$

where $R(x)$ is the residual and $f(x) = (3x^2\ l - x^3)$ is the weighting function given by Eq. (E.2). Eq. (E.4) can be rewritten as

$$\int_0^l \left(EI\frac{d^4\tilde{w}}{dx^4}\right)f(x)dx = 0 \tag{E.5}$$

Because the fourth derivative of $\tilde{w}(x)$ is zero, we reduce the order of the highest derivative of $\tilde{w}(x)$ by integrating Eq. (E.5) by parts:

$$EI\,f(x)\frac{d^3\tilde{w}}{dx^3}\bigg|_0^l - \int_0^l EI\frac{df}{dx}\frac{d^3\tilde{w}}{dx^3}dx = 0 \tag{E.6}$$

Integration of the second term on the left side of Eq. (E.6) by parts results in the equation

$$\int_0^l EI\frac{d^2f}{dx^2}\frac{d^2\tilde{w}}{dx^2}dx = \left[-EIf(x)\frac{d^3\tilde{w}}{dx^3}\bigg|_0^l + EI\frac{df}{dx}\frac{d^2\tilde{w}}{dx^2}\bigg|_0^l\right] \tag{E.7}$$

EXAMPLE 5.12 —cont'd

The boundary conditions yield

$$f(x=0)=0,\ \frac{\mathrm{d}f}{\mathrm{d}x}(x=0)=0,\ EI\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}x^2}(x=1)=M_0,\ EI\frac{\mathrm{d}^3\widetilde{w}}{\mathrm{d}x^3}(x=l)=P_0 \tag{E.8}$$

Using the first two conditions of Eq. (E.8), Eq. (E.7) can be expressed as

$$\int_0^l EI\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}x^2}\mathrm{d}x = EIf(x)\frac{\mathrm{d}^3\widetilde{w}}{\mathrm{d}x^3}\bigg|_l - EI\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}x^2}\bigg|_l \tag{E.9}$$

From Eq. (E.2) and Fig. 5.4, we have

$$f(l)=2l^3,\ \frac{\mathrm{d}f}{\mathrm{d}x}(l)=3l^2,\ EI\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}x^2}(l)=M_0,\ EI\frac{\mathrm{d}^3\widetilde{w}}{\mathrm{d}x^3}(l)=P_0 \tag{E.10}$$

The integral in Eq. (E.9) can be evaluated using the relations in Eq. (E.2) as

$$\int_0^l EI\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\frac{\mathrm{d}^2\widetilde{w}}{\mathrm{d}x^2}\mathrm{d}x = \int_0^l EI(6l-6x)C(6l-6x)\mathrm{d}x = \left(12EI\,l^3\right)C \tag{E.11}$$

Using Eqs. (E.10) and (E.11) in Eq. (E.9), the constant C can be found as follows:

$$C=\frac{P_0}{6EI}+\frac{M_0}{4EIl} \tag{E.12}$$

Thus, the approximate solution for the deflection of the beam becomes

$$\widetilde{w}(x)=\left(\frac{P_0}{6EI}+\frac{M_0}{4EIl}\right)\left(3x^2 l-x^3\right) \tag{E.13}$$

which yields the deflection at the free end ($x=l$) as

$$\widetilde{w}(x)=\frac{P_0 l^3}{3EI}+\frac{M_0 l^2}{2EI} \tag{E.14}$$

Advantages of weak formulation:

The original differential equation in $w(x)$ is of order four whereas the application of the weighted residual (Galerkin) method resulted in the integral form of equation (E.7) involving derivatives of $w(x)$ and $f(x)$ of order two only. Eq. (E.1) represents the strong form formulation while Eq. (E.7) denotes the weak form formulation of the beam deflection problem. Note that the solution of Eq. (E.1) requires the fourth derivative of any assumed (approximate) solution, while Eq. (E.7) requires only the second derivative of the assumed solution. Thus the function $w(x)$ in Eq. (E.1) is required to have C^4 continuity while the functions $w(x)$ and $f(x)$ in Eq. (E.7) are required to have C^2 continuity only. Thus, a relaxed continuity requirement can be used in the weak formulation.

REVIEW QUESTIONS

5.1 Give brief answers to the following questions.

1. What is a direct approach for deriving finite element equations?
2. What is variational calculus?
3. What is a functional?
4. What is a residual?
5. What is the difference between point collocation and subdomain collocation?

5.2 Fill in the blank with a suitable word.

1. In the direct method, the element matrices and element vectors are derived using ____________ principles.
2. The variational operator δ is ______________________________ with respect to both differentiation and integration.
3. The differential equation derived by setting the variation of the integral (δI) equal to zero is known as ______________________ equation.

4. When finite element equations are derived from the variational principles, the ______________ boundary conditions are automatically satisfied.
5. In the Rayleigh–Ritz method, an approximate solution, in the form of a ______________ of several known functions of the field variable, is assumed.
6. In the point collocation method, the residual is set equal to ______________ at the same selected points in the ______________.

5.3 Indicate whether each of the following statements is true or false.

1. The direct method can be used to derive the finite element equations of all types of problems.
2. The solutions obtained from the governing differential equations and those obtained from the variational principle of the problem represent exact and approximate solutions, respectively.
3. The Rayleigh–Ritz method can be considered as a variational method.
4. Variational formulation involves lower order derivatives compared to those in the governing differential equation.
5. The weighted residual methods can be used to find approximate solutions of even nonlinear differential equations.
6. The weighted residual method is applicable to eigenvalue problems.
7. The least squares method can be considered a residual method.
8. Usually it is easy to find an exact solution for the strong form of a partial differential equation.
9. Equations derived using an energy principle such as the principle of minimum potential energy are usually of weak form.
10. Equations derived from a strong form formulation are preferred to find an approximate solution.

5.4 Select the most appropriate answer from the multiple choices given.

1. Finite element equations cannot be derived using the following approach:
   (a) Least squares (b) Galerkin (c) Rayleigh

5.5 Match the indicated terms to the method to which they are associated.

1. Residual function	(a) Least squares method
2. Functional	(b) Galerkin method
3. Weighted residual	(c) Subdomain collocation
4. Weighted square of the residual	(d) Point collocation
5. Integral of the residual	(e) Rayleigh–Ritz method

PROBLEMS

5.1 Derive the Euler–Lagrange equation corresponding to the functional

$$I = \frac{1}{2}\iiint\limits_{V}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 - 2C\phi\right]\cdot \mathrm{d}V$$

What considerations would you take while selecting the interpolation polynomial for this problem?

5.2 Show that the equilibrium equations $[K]\overrightarrow{X} = \overrightarrow{P}$ where $[K]$ is a symmetric matrix, can be interpreted as the stationary requirement for the functional

$$I = \frac{1}{2}\overrightarrow{X}^T[K]\overrightarrow{X} - \overrightarrow{X}^T\overrightarrow{P}$$

5.3 The deflection of a beam on an elastic foundation is governed by the equation $(\mathrm{d}^4w/\mathrm{d}x^4) + w = 1$, where x and w are dimensionless quantities. The boundary conditions for a simply supported beam are given by transverse deflection $= w = 0$ and bending moment $= (\mathrm{d}^2w/\mathrm{d}x^2) = 0$. By taking a two-term trial solution as $w(x) = C_1f_1(x) + C_2f_2(x)$ with $f_1(x) = \sin \pi x$ and $f_2(x) = \sin 3\pi x$, find the solution of the problem using the Galerkin method.

5.4 Solve Problem 5.3 using the collocation method with collocation points at $x = 1/4$ and $x = 1/2$.

5.5 Solve Problem 5.3 using the least squares method.

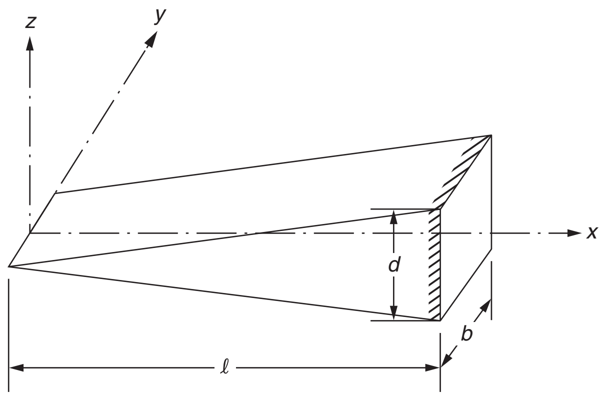

FIGURE 5.5 A tapered beam.

5.6 Derive the finite element equations for a simplex element in two dimensions using a variational approach for the biharmonic equation $\nabla^4\phi = C$. Discuss the continuity requirements of the interpolation model.

5.7 Derive the finite element equations for a simplex element in two dimensions using a residual method for the biharmonic equation.

5.8 The Rayleigh quotient (λ_R) for the vibrating tapered beam shown in Fig. 5.5 [5.7] is given by

$$\lambda_R = \frac{I_A(\phi)}{I_B(\phi)}$$

where

$$I_A(\phi) = \frac{1}{2}\int_0^l EI\left[\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2}\right]^2 \mathrm{d}x$$

$$I_B(\phi) = \frac{1}{2}\int_0^l \rho A[\phi(x)]^2 \mathrm{d}x$$

ϕ is the assumed solution for the deflection of the beam, E is Young's modulus, I is the area moment of inertia of the cross section $= (1/12)\, b[d\, x/l]^3$, ρ is the density, and A is the cross-sectional area $= b[d\, x/l]$. Find the eigenvalues (λ_R) of the beam using the Rayleigh–Ritz method with the assumed solution

$$\phi(x) = C_1\left[1-\frac{x}{l}\right]^2 + C_2\left[\frac{x}{l}\right]\left[1-\frac{x}{l}\right]^2$$

5.9 The differential equation governing the free transverse vibration of a string (Fig. 5.6) is given by

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0; \quad 0 \le x \le l$$

with the boundary conditions

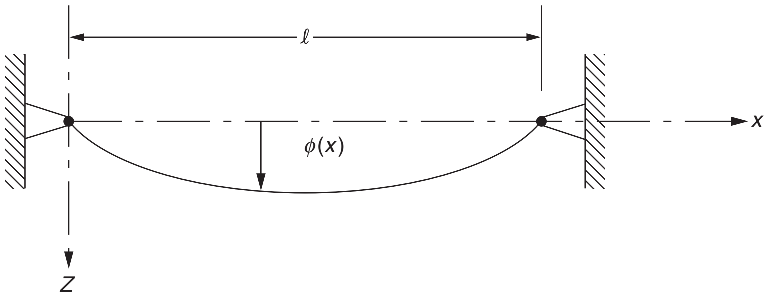

FIGURE 5.6 Transverse vibration of a string.

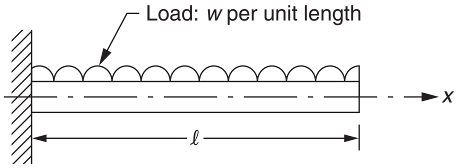

FIGURE 5.7 A cantilever beam subjected to a distributed load.

$$\phi(x) = 0 \quad \text{at} \quad x = 0, \ x = l$$

where $\lambda = (\rho\omega^2 l^2/T)$ is the eigenvalue, ρ is the mass per unit length, l is the length, T is the tension in string, ω is the natural frequency of vibration, and $x = (y/l)$. Using the trial solution

$$\phi(x) = C_1 x(l - x) + C_2 x^2(l - x)$$

where C_1 and C_2 are constants, determine the eigenvalues of the string using the Galerkin method.

5.10 Solve Problem 5.9 using the collocation method with $x = l/4$ and $x = 3\,l/4$ as the collocation points.

5.11 The cantilever beam shown in Fig. 5.7 is subjected to a uniform load of w per unit length. Assuming the deflection as

$$\phi(x) = c_1 \sin\frac{\pi x}{2l} + c_2 \sin\frac{3\pi x}{2l}$$

determine the constants c_1 and c_2 using the Rayleigh–Ritz method.

5.12 Solve Problem 5.11 using the Galerkin method.

5.13 Solve Problem 5.11 using the least squares method.

5.14 The transverse deflection (w) of a beam is governed by the equation

$$EI\frac{d^2w}{dx^2} = M(x) \qquad \text{(P.1)}$$

where E is Young's modulus, I is the area moment of inertia of the cross section, and $M(x)$ is the applied bending moment. If a simply supported beam, with the boundary conditions

$$w(x = 0) = 0, w(x = l) = 0 \qquad \text{(P.2)}$$

is subjected to a constant bending moment, $M(x) = M_0$, as shown in Fig. 5.8, find the approximate solution of the problem using the Rayleigh–Ritz method by assuming a one-term solution as

$$w(x) = C\sin\frac{\pi x}{l} \qquad \text{(P.3)}$$

where C is a constant (to be determined).

Hint: The functional I corresponding to Eq. (P.1) is given by

$$I = \int_0^l \left\{\frac{EI}{2}\left(\frac{dw}{dx}\right)^2 + M_0 w\right\} dx \qquad \text{(P.4)}$$

5.15 Find the approximate deflection of the simply supported beam described in Problem 5.14 by assuming the solution given by Eq. (P.3) using the collocation method. Take the collocation point as $x = l/2$.

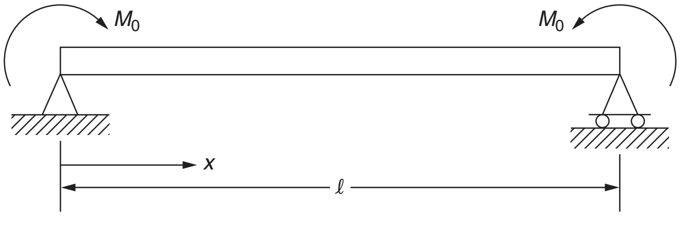

FIGURE 5.8 A beam subjected to constant bending moment.

5.16 Find the approximate deflection of the simply supported beam described in Problem 5.14 by assuming the solution given by Eq. (P.3) using the subdomain collocation method. Use the subdomain as (0, l).

5.17 Find the approximate deflection of the simply supported beam described in Problem 5.14 by assuming the solution given by Eq. (P.3) using the Galerkin method.

5.18 Find the approximate deflection of the simply supported beam described in Problem 5.14 by assuming the solution given by Eq. (P.3) using the least squares method.

5.19 Find the solution of the simply supported beam subjected to uniformly distributed load, described by Eqs. (E.1) to (E.3) of Example 5.1, by assuming a one-term solution

$$w(x) = C_1 \sin \frac{\pi x}{l}$$

using the Rayleigh—Ritz method.

5.20 Find the solution of the simply supported beam subjected to uniformly distributed load, described by Eqs. (E.1) and (E.2) of Example 5.1, by assuming a one-term solution

$$w(x) = C_1 \sin \frac{\pi x}{l}$$

using the point collocation method. Take the collocation point as $x = l/2$.

5.21 Find the solution of the simply supported beam subjected to uniformly distributed load, described by Eqs. (E.1) and (E.2) of Example 5.1, by assuming a one-term solution

$$w(x) = C_1 \sin \frac{\pi x}{l}$$

using the subdomain collocation method. Take the subdomain as (0, l).

5.22 Find the solution of the simply supported beam subjected to uniformly distributed load, described by Eqs. (E.1) and (E.2) of Example 5.1, by assuming a one-term solution

$$w(x) = C_1 \sin \frac{\pi x}{l}$$

using the Galerkin method.

5.23 Find the solution of the simply supported beam subjected to uniformly distributed load, described by Eqs. (E.1) and (E.2) of Example 5.1, by assuming a one-term solution

$$w(x) = C_1 \sin \frac{\pi x}{l}$$

using the least squares method.

5.24 Consider a simply supported beam subjected to a uniformly distributed load as shown in Fig. 5.2. The governing differential equation can also be expressed, in terms of the bending moment, $M(x)$, as

$$EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = M(x) = \frac{1}{2} px(l - x) \tag{P.5}$$

where E is Young's modulus, I is the area moment of inertia of the cross section, and $M(x) = \frac{1}{2} px(l - x)$ is the applied bending moment. The boundary conditions are

$$w(x = 0) = 0, \quad w(x = l) = 0 \tag{P.6}$$

Find the approximate solution of the problem using the Rayleigh—Ritz method by assuming a one-term solution as

$$w(x) = C \sin \frac{\pi x}{l} \tag{P.7}$$

where C is a constant (to be determined).
Hint: The functional I corresponding to Eq. (P.5) is given by

$$I = \int_0^l \left\{ \frac{EI}{2} \left(\frac{\mathrm{d}w}{\mathrm{d}x} \right)^2 + \frac{px(l - x)}{2} w \right\} \mathrm{d}x \tag{P.8}$$

5.25 Find the approximate deflection of the simply supported beam described in Problem 5.24 by assuming the solution given by Eq. (P.7) using the collocation method. Take the collocation point $x = l/2$.

5.26 Find the approximate deflection of the simply supported beam described in Problem 5.24 by assuming the solution given by Eq. (P.7) using the subdomain collocation method. Use the subdomain (0, l).

5.27 Find the approximate deflection of the simply supported beam described in Problem 5.24 by assuming the solution given by Eq. (P.7) using the Galerkin method.

5.28 Find the approximate deflection of the simply supported beam described in Problem 5.24 by assuming the solution given by Eq. (P.7) using the least squares method.

5.29 Consider a simply supported beam subjected to a concentrated load, P, acting at $x = l/2$ as shown in Fig. 5.9. The governing differential equation is

$$\begin{aligned} EI\frac{d^2w}{dx^2} &= M(x) = \frac{Px}{2}; \quad 0 \le x \le \frac{l}{2} \\ &= \frac{P(l-x)}{2}; \quad \frac{l}{2} \le x \le l \end{aligned} \tag{P.9}$$

where E is Young's modulus and I is the area moment of inertia of the cross section. The boundary conditions are

$$w(x=0) = 0, \quad w(x=l) = 0 \tag{P.10}$$

Find the approximate solution of the problem using the Rayleigh–Ritz method by assuming a one-term solution as

$$w(x) = C\sin\frac{\pi x}{l} \tag{P.11}$$

where C is a constant (to be determined).
Hint: The functional I corresponding to Eq. (P.9) is given by

$$I = \int_0^l \left\{ \frac{EI}{2}\left(\frac{dw}{dx}\right)^2 dx + \int_0^{\frac{l}{2}} \frac{Px}{2} w dx + \int_{\frac{l}{2}}^{l} \frac{P(l-x)}{2} w \right\} dx \tag{P.12}$$

5.30 Find the approximate deflection of the simply supported beam described in Problem 5.29 by assuming the solution given by Eq. (P.11) using the collocation method. Take the collocation point as $x = l/2$.

5.31 Find the approximate deflection of the simply supported beam described in Problem 5.29 by assuming the solution given by Eq. (P.11) using the subdomain collocation method. Use the subdomain as (0, l).

5.32 Find the approximate deflection of the simply supported beam described in Problem 5.29 by assuming the solution given by Eq. (P.11) using the Galerkin method.

5.33 Find the approximate deflection of the simply supported beam described in Problem 5.29 by assuming the solution given by Eq. (P.11) using the least squares method.

5.34 Solve Problem 5.29 using the Rayleigh–Ritz method by assuming a two-term solution as

$$w(x) = C_1\sin\frac{\pi x}{l} + C_2\sin\frac{3\pi x}{l}$$

5.35 Solve Problem 5.29 using the collocation method by assuming a two-term solution as

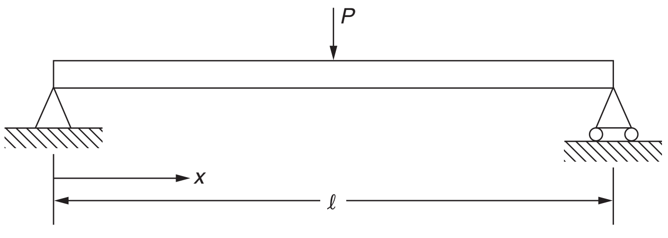

FIGURE 5.9 A simply supported beam subjected to a concentrated load.

$$w(x) = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$

5.36 Solve Problem 5.29 using the subdomain collocation method by assuming a two-term solution as

$$w(x) = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$

Take the subdomains as $x = (0, l/4)$ and $x = (l/4, l/2)$.

5.37 Solve Problem 5.29 using the Galerkin method by assuming a two-term solution as

$$w(x) = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$

5.38 Solve Problem 5.29 using the least squares method by assuming a two-term solution as

$$w(x) = C_1 \sin\frac{\pi x}{l} + C_2 \sin\frac{3\pi x}{l}$$

5.39 Consider the differential equation

$$\frac{d^2\phi}{dx^2} + 400x^2 = 0; \quad 0 \le x \le 1$$

with the boundary conditions

$$\phi(0) = 0, \quad \phi(1) = 0$$

The functional corresponding to this problem to be extremized is given by

$$I = \int_0^1 \left\{ -\frac{1}{2}\left[\frac{d\phi}{dx}\right]^2 + 400x^2\phi \right\} dx$$

Find the solution of the problem using the Rayleigh–Ritz method using a one-term solution $\phi(x) = c_1 x\,(1-x)$.

5.40 Find the solution of Problem 5.39 using the Rayleigh–Ritz method using a two-term solution
$\phi\,(x) = c_1 x\,(1-x) + c_2\,x^2(1-x)$

5.41 Find the solution of Problem 5.39 using the Galerkin method using the solution

$$\phi(x) = c_1 x(1-x) + c_2 x^2(1-x)$$

5.42 Find the solution of Problem 5.39 using the two-point collocation method with the trial solution

$$\phi(x) = c_1 x(1-x) + c_2 x^2(1-x)$$

Assume the collocation points as $x = 1/4$ and $x = 3/4$.

5.43 Find the solution of Problem 5.39 using the least squares approach with the trial solution

$$\phi(x) = c_1 x(1-x) + c_2 x^2(1-x)$$

5.44 Find the solution of Problem 5.39 using subdomain collocation with the trial solution

$$\phi(x) = c_1 x(1-x) + c_2 x^2(1-x)$$

Assume two subdomains as $x = 0$ to $1/4$ and $x = 3/4$ to 1.

5.45 Consider the coaxial cable shown in Fig. 5.10 with inside radius $r_i = 7$ mm, interface radius $r_m = 15$ mm, and outer radius $r_0 = 22$ mm. The permittivities of the inside and outside layers are $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$, respectively, and the charge densities of the inside and outside layers are $\sigma_i = 50$ and $\sigma_0 = 0$, respectively. If the electric potential is specified at the inner and outer surfaces as $\phi_i = 400$ and $\phi_0 = 0$, determine the variation of ϕ (r) using the finite element method based on the variational (Rayleigh–Ritz) approach. Use two linear finite elements in each layer.

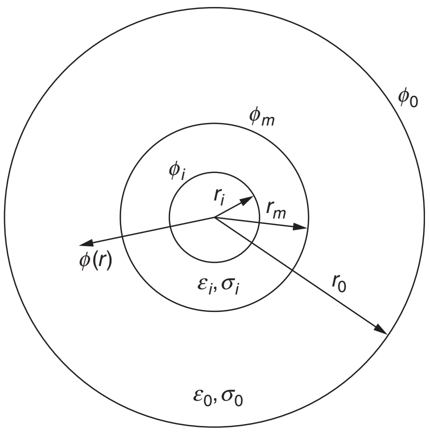

FIGURE 5.10 A coaxial cable.

Hint: The equation governing axisymmetric electrostatics is given by

$$c\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{c}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} + p = 0 \tag{P.13}$$

where c is the permittivity of the material, ϕ the electric potential, p is the charge density, and r is the radial distance. The variational function, I, corresponding to Eq. (P.13) is given by

$$I(\phi) = \int_{r_1}^{r_2} \left\{ \pi rc \left[\frac{\mathrm{d}\phi}{\mathrm{d}r}\right]^2 - 2\pi rp\phi \right\} \mathrm{d}r \tag{P.14}$$

where the relation $\mathrm{d}V = 2\pi \mathrm{r}\ \mathrm{d}r$ has been used.

5.46 Solve Problem 5.45 using the finite element method by adopting the Galerkin approach.

5.47 Solve Problem 5.45 using the finite element method by adopting the least squares approach.

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Chapter 6

Assembly of Element Matrices and Vectors, and Derivation of System Equations

Chapter Outline

6.1 COORDINATE TRANSFORMATION

The various methods of deriving element characteristic matrices and vectors have been discussed in Chapter 5. Before considering how these element matrices and vectors are assembled to obtain the characteristic equations of the entire system of elements, we need to discuss the aspect of coordinate transformation. The coordinate transformation is necessary when the field variable is a vector quantity such as displacement and velocity. Sometimes, the element matrices and vectors are computed in local coordinate systems suitably oriented for minimizing the computational effort. The local coordinate system may be different for different elements. When a local coordinate system is used, the directions of the nodal degrees of freedom (dof) will also be taken in a convenient manner. In such a case, before the element equations can be assembled, it is necessary to transform the element matrices and vectors derived in local coordinate systems so that all the elemental equations are referred to a common global coordinate system. The choice of the global coordinate system is arbitrary, and in practice it is generally taken to be the same as the coordinate system used in the engineering drawings, from which the coordinates of the different node points of the body can easily be found.

In general, for an equilibrium problem, the element equations in a local coordinate system can be expressed in the standard form

$$\left[k^{(e)}\right]\vec{\phi}^{(e)} = \vec{p}^{(e)} \tag{6.1}$$

where $[k^{(e)}]$ and $\vec{p}^{(e)}$ are the element characteristic matrix and vector, respectively, and $\vec{\phi}^{(e)}$ is the vector of nodal unknowns of element e. If the field variable is a directional quantity such as displacement, velocity, or force, a coordinate transformation exists between the local and global dof (unknowns) of the element e. We shall use lowercase and capital letters to denote the characteristics pertaining to the local and global coordinate systems, respectively. Let a transformation matrix $[\lambda^{(e)}]$ exist between the local and global coordinate systems such that [6.3]

$$\vec{\phi}^{(e)} = \left[\lambda^{(e)}\right]\vec{\Phi}^{(e)} \tag{6.2}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00006-7

and

$$\vec{p}^{(e)} = \left[\lambda^{(e)}\right]\vec{P}^{(e)} \tag{6.3}$$

The strain energy of the element, being a scalar quantity, will have the same value in both the local and global coordinate systems. Thus,

$$\frac{1}{2}\vec{\phi}^{(e)^T}\left[k^{(e)}\right]\vec{\phi}^{(e)} = \frac{1}{2}\vec{\Phi}^{(e)^T}\left[K^{(e)}\right]\vec{\Phi}^{(e)} \tag{6.4}$$

where $[K^{(e)}]$ denotes the stiffness matrix and $\vec{\Phi}^{(e)}$ the vector of nodal displacements of element e in the global coordinate system. By substituting Eq. (6.2) on the left-hand side of Eq. (6.4), we obtain

$$\frac{1}{2}\vec{\Phi}^{(e)^T}\left[\lambda^{(e)}\right]^T\left[k^{(e)}\right]\left[\lambda^{(e)}\right]\vec{\Phi}^{(e)} = \frac{1}{2}\vec{\Phi}^{(e)^T}\left[K^{(e)}\right]\vec{\Phi}^{(e)} \tag{6.5}$$

from which the stiffness matrix of the element in the global coordinate system can be expressed as

$$\left[K^{(e)}\right] = \left[\lambda^{(e)}\right]^T\left[k^{(e)}\right]\left[\lambda^{(e)}\right] \tag{6.6}$$

Note

If the vectors $\vec{\phi}^{(e)}$ and $\vec{\Phi}^{(e)}$ have the same size (i.e., the number of nodal displacement dof of the element is the same in both local and global systems), then the transformation matrix $[\lambda^{(e)}]$ will be a square matrix of direction cosines relating the two coordinate systems. In this case, the transformation matrix will be orthogonal and hence,

$$\left[\lambda^{(e)}\right]^{-1} = \left[\lambda^{(e)}\right]^T \tag{6.7}$$

so that the stiffness matrix in the global coordinate system can also be expressed as

$$\left[K^{(e)}\right] = \left[\lambda^{(e)}\right]^{-1}\left[k^{(e)}\right]\left[\lambda^{(e)}\right] \tag{6.8}$$

EXAMPLE 6.1

Show that the transformation matrix between two coordinate systems (x, y) and (X, Y) shown in Fig. 6.1 is orthogonal.

Approach: Show that the inverse of $[\lambda]$ is the same as the transpose of $[\lambda]$.

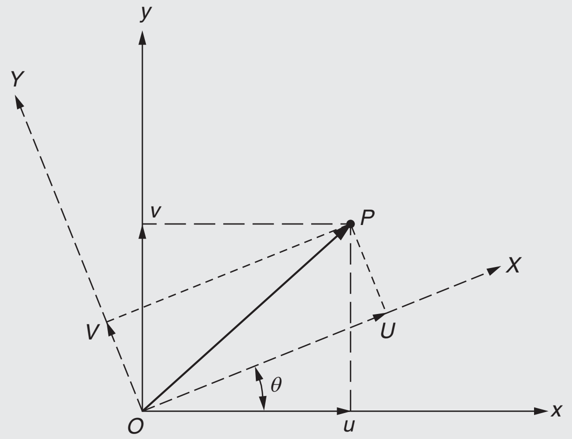

FIGURE 6.1 Two coordinate systems.

EXAMPLE 6.1 —cont'd

Solution

Consider a vector (such as the position of a point P or a displacement vector) in the two coordinate systems (x, y) and (X, Y) as shown in Fig. 6.1. The components of the vector are assumed to be (u, v) in the (x, y)-coordinate system and (U, V) in the (X, Y)-coordinate system. The relationships between the components (u, v) and (U, V) can be expressed in terms of a transformation matrix, $[\lambda]$, as

$$\begin{Bmatrix} U \\ V \end{Bmatrix} = [\lambda]\begin{Bmatrix} u \\ v \end{Bmatrix} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} \tag{E.1}$$

with

$$[\lambda] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{E.2}$$

The inverse of the transformation matrix, $[\lambda]^{-1}$, is given by

$$[\lambda]^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{E.3}$$

which can be seen to be same as the transpose, $[\lambda]^T$, of the matrix $[\lambda]$. Thus, the coordinate transformation matrix is orthogonal.

EXAMPLE 6.2

Consider the two-bar truss shown in Fig. 6.2A with the following properties:

Element 1: $A^{(1)} = 1$ in^2, $E^{(1)} = 30 \times 10^6$ psi; Element 2: $A^{(2)} = 0.5$ in^2, $E^{(2)} = 20 \times 10^6$ psi

a. Find the coordinate transformation matrices of the elements.

b. If the element stiffness matrices in the local coordinate system are given by

$$\left[k^{(e)}\right] = \frac{A^{(e)}E^{(e)}}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, e = 1, 2$$

where $A^{(e)}$, $E^{(e)}$, and $l^{(e)}$ denote, respectively, the area of cross section, Young's modulus, and length of element e, derive the stiffness matrices of the elements in the global (X, Y) coordinate system.

Approach: Use Eq. (6.8) after finding $[\lambda^{(e)}]$.

Solution

The assumed local coordinate systems (x-axes), the nodal displacement dof in the local system (q_1 and q_2), and the global displacement dof ($Q_i^{(e)}$, $i = 1, 2, 3, 4$) of the two elements ($e = 1, 2$) are shown in Fig. 6.2B.

a. To derive the coordinate transformation matrix for element 1, we note that local nodes 1 and 2 of the element are the same as global nodes 3 and 1, respectively. Thus, the local and global displacement dof at local node 1 are related as

$$q_1 = Q_5\cos\theta_1 + Q_6\sin\theta_1 \tag{E.1}$$

and at local node 2 as

$$q_2 = Q_1\cos\theta_1 + Q_2\sin\theta_1 \tag{E.2}$$

Noting that the length of element 1, $l^{(1)}$, is given by

$$l^{(1)} = l_{ij} = \sqrt{(0-(-20))^2 + (50-0)^2} = 53.8516 \text{ in}$$

the direction cosines of the x axis (or element 1) are given by

$$\cos\theta_1 = \frac{X_j - X_i}{l_{ij}} = \frac{X_1 - X_3}{l^{(1)}} = \frac{0-(-20)}{53.8516} = 0.3714$$

$$\sin\theta_1 = \frac{Y_j - Y_i}{l_{ij}} = \frac{Y_1 - Y_3}{l^{(1)}} = \frac{50-0}{53.8516} = 0.9285$$

Eqs. (E.1) and (E.2) can be written in matrix form as

$$\vec{q}^{(1)} = [\lambda^{(1)}]\vec{Q}^{(1)} \tag{E.3}$$

Continued

EXAMPLE 6.2 —cont'd

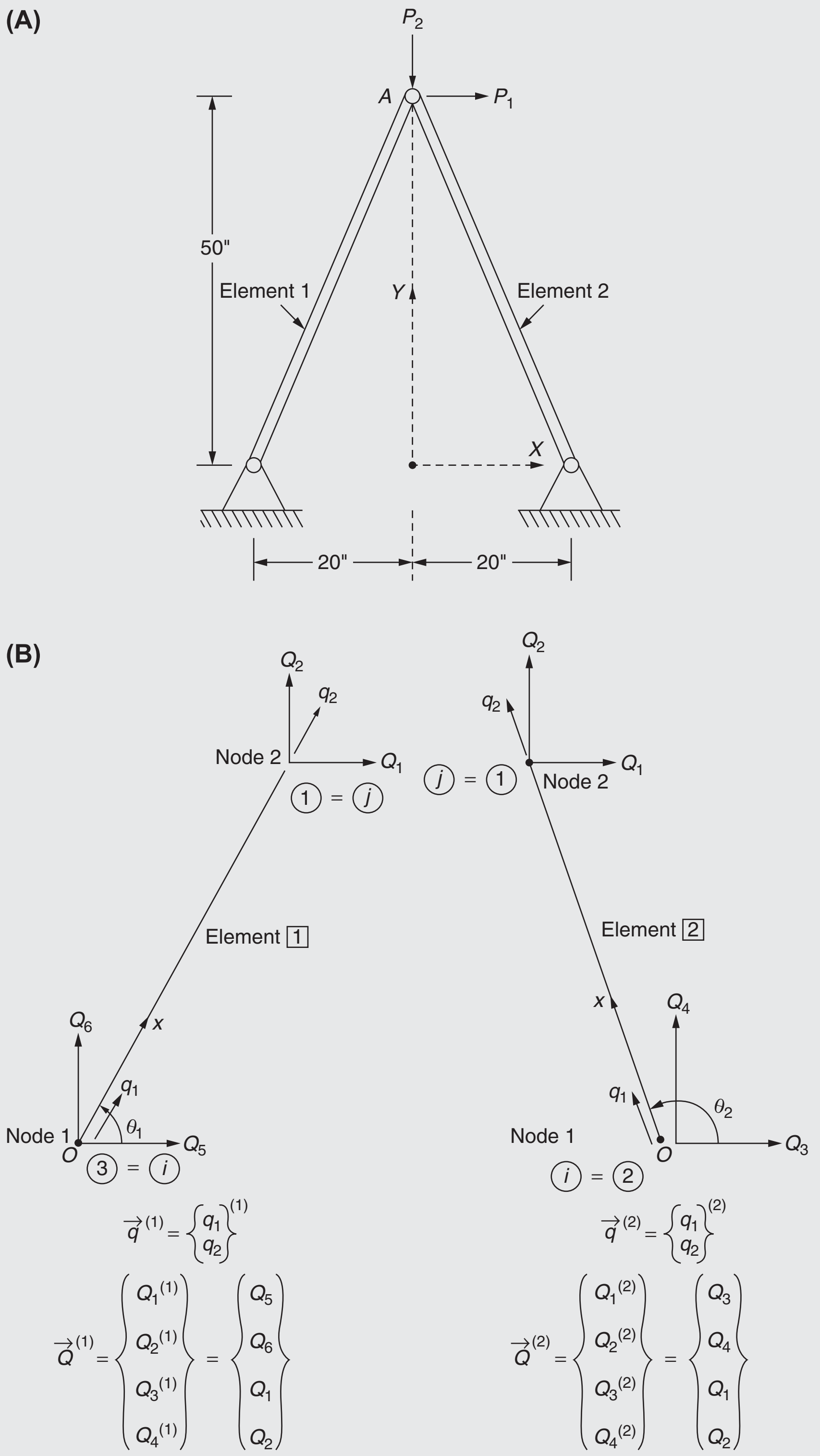

FIGURE 6.2 A two-bar truss.

EXAMPLE 6.2 —cont'd

where

$$\vec{q}^{(1)} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^{(1)}, \vec{Q}^{(1)} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_1 \\ Q_2 \end{Bmatrix}$$

and [λ (1)] is the coordinate transformation matrix of element 1 given by

$$\left[\lambda^{(1)}\right] = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \end{bmatrix}$$

By proceeding in a similar manner, the coordinate transformation matrix of element 2 can be derived as

$$\left[\lambda^{(2)}\right] = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & \sin\theta_2 \end{bmatrix}$$

where

$$\cos\theta_2 = \frac{X_j - X_i}{l_{ij}} = \frac{X_1 - X_2}{l^{(2)}} = \frac{0 - 20}{53.8516} = -0.3714$$

$$\sin\theta_2 = \frac{Y_j - Y_i}{l_{ij}} = \frac{Y_1 - Y_2}{l^{(2)}} = \frac{50 - 0}{53.8516} = 0.9285$$

with the length of element 2 equal to

$$l^{(2)} = l_{ij} = \sqrt{(0-20)^2 + (50-0)^2} = 53.8516 \text{ in}$$

b. The stiffness matrix of element 1 in the global coordinate system is given by Eq. (6.8):

$$\left[K^{(1)}\right] = \left[\lambda^{(1)}\right]^T \left[k^{(1)}\right]\left[\lambda^{(1)}\right]$$

$$= \frac{A^{(1)}E^{(1)}}{l^{(1)}} \begin{bmatrix} \cos^2\theta_1 & \sin\theta_1\cos\theta_1 & -\cos^2\theta_1 & -\sin\theta_1\cos\theta_1 \\ \sin\theta_1\cos\theta_1 & \sin^2\theta_1 & -\sin\theta_1\cos\theta_1 & -\sin^2\theta_1 \\ -\cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & \cos^2\theta_1 & \sin\theta_1\cos\theta_1 \\ -\sin\theta_1\cos\theta_1 & -\sin^2\theta_1 & \sin\theta_1\cos\theta_1 & \sin^2\theta_1 \end{bmatrix} \tag{E.4}$$

The known data yields

$$\frac{A^{(1)}E^{(1)}}{l^{(1)}} = \frac{(1)(30\times10^6)}{53.8516} = 5.5709\times10^5 \text{ lb/in},$$

$$\cos^2\theta_1 = (0.3714)^2 = 0.1379$$

$$\sin\theta_1\cos\theta_1 = (0.3714)(0.9285) = 0.3448,$$

$$\sin^2\theta_1 = (0.9285)^2 = 0.8621$$

and hence $[K^{(1)}]$ becomes

$$\left[K^{(1)}\right] = 5.5709\times10^5 \begin{bmatrix} 0.1379 & 0.3448 & -0.1379 & -0.3448 \\ 0.3448 & 0.8621 & -0.3448 & -0.8621 \\ -0.1379 & -0.3448 & 0.1379 & 0.3448 \\ -0.3448 & -0.8621 & 0.3448 & 0.8621 \end{bmatrix} \text{lb/in} \tag{E.5}$$

Continued

EXAMPLE 6.2 —cont'd

The stiffness matrix of element 2 in the global coordinate system is given by Eq. (6.8):

$$\left[K^{(2)}\right] = \left[\lambda^{(2)}\right]^T\left[k^{(2)}\right]\left[\lambda^{(2)}\right]$$

$$= \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix} \cos^2\theta_2 & \sin\theta_2\cos\theta_2 & -\cos^2\theta_2 & -\sin\theta_2\cos\theta_2 \\ \sin\theta_2\cos\theta_2 & \sin^2\theta_2 & -\sin\theta_2\cos\theta_2 & -\sin^2\theta_2 \\ -\cos^2\theta_2 & -\sin\theta_2\cos\theta_2 & \cos^2\theta_2 & \sin\theta_2\cos\theta_2 \\ -\sin\theta_2\cos\theta_2 & -\sin^2\theta_2 & \sin\theta_2\cos\theta_2 & \sin^2\theta_2 \end{bmatrix} \tag{E.6}$$

The known data yields

$$\frac{A^{(2)}E^{(2)}}{l^{(2)}} = \frac{(0.5)(20\times10^6)}{53.8516} = 1.8569\times10^5\,\text{lb/in},$$

$$\cos^2\theta_2 = (-0.3714)^2 = 0.1379$$

$$\sin\theta_2\cos\theta_2 = (-0.3714)(0.9285) = -0.3448,$$

$$\sin^2\theta_2 = (0.9285)^2 = 0.8621$$

and hence $[K^{(2)}]$ becomes

$$\left[K^{(2)}\right] = 1.8569\times10^5\begin{bmatrix} 0.1379 & -0.3448 & -0.1379 & 0.3448 \\ -0.3448 & 0.8621 & 0.3448 & -0.8621 \\ -0.1379 & 0.3448 & 0.1379 & -0.3448 \\ 0.3448 & -0.8621 & -0.3448 & 0.8621 \end{bmatrix}\text{lb/in} \tag{E.7}$$

6.2 ASSEMBLAGE OF ELEMENT EQUATIONS

Once the element characteristics, namely, the element matrices and element vectors, are found in a common global coordinate system, the next step is to construct the overall or system equations. The procedure for constructing the system equations from the element characteristics is the same regardless of the type of problem and the number and type of elements used.

The procedure of assembling the element matrices and vectors is based on the requirement of compatibility at the element nodes. This means that at the nodes where elements are connected, the value(s) of the unknown nodal dof or variable(s) is the same for all the elements joining at that node. In solid mechanics and structural problems, the nodal variables are usually generalized displacements, which can be translations, rotations, curvatures, or other spatial derivatives of translations: When the generalized displacements are matched at a common node, the nodal stiffnesses and nodal loads of each of the elements sharing the node are added to obtain the net stiffness and the net load at that node.

Let E and M denote the total number of elements and nodal dof (including the boundary and restrained dof), respectively. Let $\vec{\underset{\sim}{\Phi}}$ denote the vector of M nodal dof and $[\underset{\sim}{K}]$ the assembled system characteristic matrix of order $M\times M$. Since the element characteristic matrix $[K^{(e)}]$ and the element characteristic vector $\vec{P}^{(e)}$ are of the order $n\times n$ and $n\times 1$, respectively (with n indicating the number of element dof), they can be expanded to the order $M\times M$ and $M\times 1$, respectively, by including zeros in the remaining locations. Thus, the global characteristic matrix and the global characteristic vector can be obtained by algebraic addition as

$$\left[\underset{\sim}{K}\right] = \sum_{e=1}^{E}\left[\underline{K}^{(e)}\right] \tag{6.9}$$

and

$$\vec{\underset{\sim}{P}} = \sum_{e=1}^{E}\left[\underline{\vec{P}}^{(e)}\right] \tag{6.10}$$

where $\left[\underline{K}^{(e)}\right]$ is the expanded characteristic matrix of element e (of order $M \times M$), and $\underline{\vec{P}}^{(e)}$ is the expanded characteristic vector of element e (of order $M \times 1$). Even if the assemblage contains many different types of elements, Eqs. (6.9) and (6.10) will be valid, although the number of element dof, n, changes from element to element.

In actual computations, the expansion of the element matrix $\left[K^{(e)}\right]$ and the vector $\vec{P}^{(e)}$ to the sizes of the overall $[\underset{\sim}{K}]$ and $\underset{\sim}{\vec{P}}$ is not necessary. $[\underset{\sim}{K}]$ and $\underset{\sim}{\vec{P}}$ can be generated by identifying the locations of the elements of $\left[K^{(e)}\right]$ and $\vec{P}^{(e)}$ in $[\underset{\sim}{K}]$ and $\underset{\sim}{\vec{P}}$, respectively, and by adding them to the existing values as e changes from 1 to E [6.2]. This procedure is illustrated with reference to the assemblage of four one-dimensional elements for the planar truss structure shown in Fig. 6.3A. Since the elements lie in the XY plane, each element has four dof as shown in Fig. 6.3B. It is assumed that a proper coordinate transformation (Section 6.1) was used and $\left[K^{(e)}\right]$ of order 4×4 and $\vec{P}^{(e)}$ of order 4×1 of element e ($e = 1$ to 4) were obtained in the global coordinate system.

For assembling $\left[K^{(e)}\right]$ and $\vec{P}^{(e)}$, we consider the elements one after another. For $e = 1$, the element stiffness matrix $\left[K^{(1)}\right]$ and the element load vector $\vec{P}^{(1)}$ can be written as shown in Table 6.1. The location (row l and column m) of any component $K_{ij}^{(1)}$ in the global stiffness matrix $[\underset{\sim}{K}]$ is identified by the global dof Φ_l and Φ_m corresponding to the local dof $\Phi_i^{(1)}$ and $\Phi_j^{(1)}$, respectively, for $i = 1$ to 4 and $j = 1$ to 4. The correspondence between Φ_l and $\Phi_i^{(1)}$, and that between Φ_m and $\Phi_j^{(1)}$, is also shown in Table 6.1. Thus, the locations of the components $K_{ij}^{(1)}$ in $[\underset{\sim}{K}]$ will be as shown in Table 6.2(a). Similarly, the location of the components of the element load vector $\vec{P}^{(1)}$ in $\underset{\sim}{\vec{P}}$ will also be as shown in Table 6.2(b). For $e = 2$, the element stiffness matrix $\left[K^{(2)}\right]$ and the element load vector $\vec{P}^{(2)}$ can be written as shown in Table 6.3. As in the

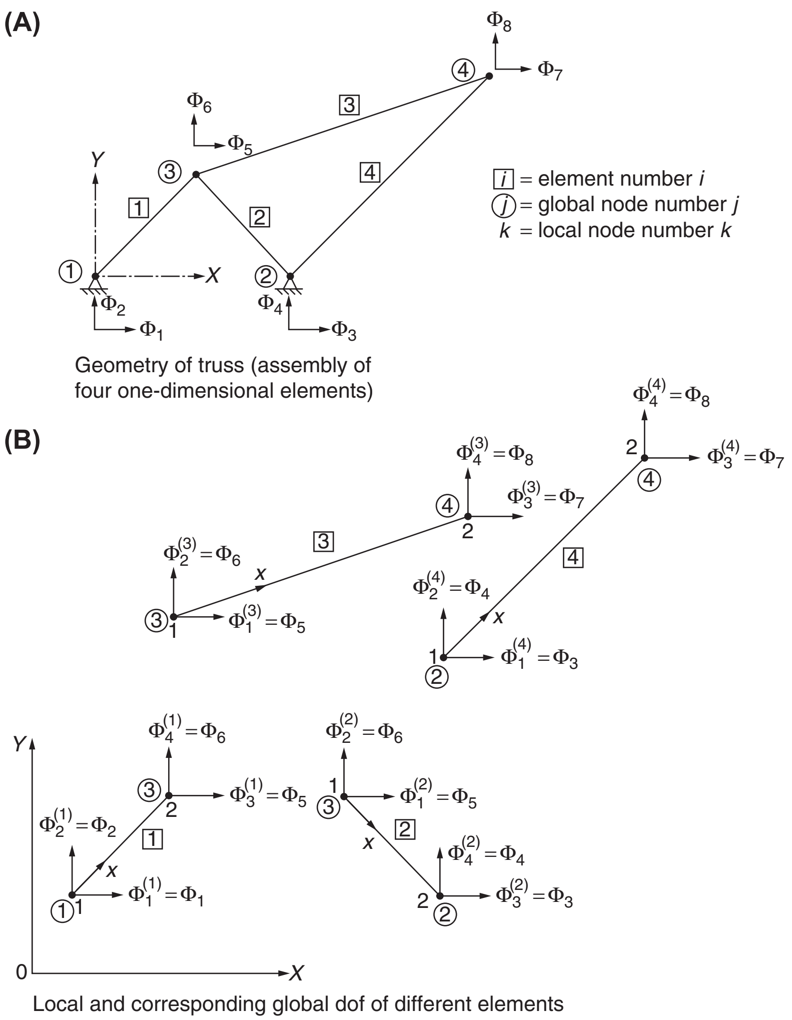

FIGURE 6.3 A planar truss as an assembly of one-dimensional elements.

TABLE 6.1 Stiffness Matrix and Load Vector of Element 1

$$
\begin{array}{cccc}
 & \text{Local dof } (\Phi_i^{(1)}) & \text{Corresponding global dof } (\Phi_l) & (\Phi_j^{(1)}) \rightarrow \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
 & & & (\Phi_m) \rightarrow \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} \\
 & \downarrow & \downarrow & \\
\underset{4\times 4}{[K^{(1)}]} = & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} \end{bmatrix}
\end{array}
$$

$$
\begin{array}{cccc}
 & & \text{Local dof } (\Phi_i^{(1)}) & \text{Corresponding global dof } (\Phi_l) \\
\underset{4\times 1}{\vec{P}^{(1)}} = & \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3^{(1)} \\ P_4^{(1)} \end{Bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}
\end{array}
$$

TABLE 6.2 Location of the Elements of $[K^{(1)}]$ and $\vec{P}^{(1)}$ in $[\underset{\sim}{K}]$ and $\underset{\sim}{\vec{P}}$

$$
\begin{array}{ccc}
 & \text{Global dof} \rightarrow & \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 \end{matrix} \\
 & \downarrow & \\
[K^{(1)}] = & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & & K_{13}^{(1)} & K_{14}^{(1)} & & \\ K_{21}^{(1)} & K_{22}^{(1)} & & & K_{23}^{(1)} & K_{24}^{(1)} & & \\ & & & & & & & \\ & & & & & & & \\ K_{31}^{(1)} & K_{32}^{(1)} & & & K_{33}^{(1)} & K_{34}^{(1)} & & \\ K_{41}^{(1)} & K_{42}^{(1)} & & & K_{43}^{(1)} & K_{44}^{(1)} & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}
\end{array}
$$

(a) Location of $[K^{(1)}]$ in $[\underset{\sim}{K}]$

$$
\begin{array}{ccc}
 & \text{Global dof} & \\
 & \downarrow & \\
\vec{P}^{(1)} = & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{Bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ \vdots \\ \\ P_3^{(1)} \\ P_4^{(1)} \\ \vdots \\ \\ \end{Bmatrix}
\end{array}
$$

(b) Location of $\vec{P}^{(1)}$ in $\underset{\sim}{\vec{P}}$

TABLE 6.3 Stiffness Matrix and Load Vector of Element 2

$$
\begin{array}{ccc}
 & & (\Phi_j^{(2)}) \longrightarrow 1 \quad 2 \quad 3 \quad 4 \\
\text{Local dof } (\Phi_i^{(2)}) & \text{Corresponding global dof } (\Phi_l) & (\Phi_m) \rightarrow 5 \quad 6 \quad 3 \quad 4 \\
\downarrow & \downarrow & \\
\end{array}
$$

$$
\underset{4\times 4}{[K^{(2)}]} = \begin{array}{cc} 1 & 5 \\ 2 & 6 \\ 3 & 3 \\ 4 & 4 \end{array}
\begin{bmatrix}
K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\
K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\
K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\
K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)}
\end{bmatrix}
$$

$$
\underset{4\times 1}{\vec{P}^{(2)}} = \begin{Bmatrix} P_1^{(2)} \\ P_2^{(2)} \\ P_3^{(2)} \\ P_4^{(2)} \end{Bmatrix}
\quad
\begin{array}{cc}
\text{Local dof } (\Phi_i^{(2)}) & \text{Corresponding global dof } (\Phi_l) \\
1 & 5 \\
2 & 6 \\
3 & 3 \\
4 & 4
\end{array}
$$

TABLE 6.4 Assembly of $[K^{(1)}]$, $\vec{P}^{(1)}$, $[K^{(2)}]$, and $\vec{P}^{(2)}$

$$
[K^{(1)}] + [K^{(2)}] =
\begin{array}{c|cccccccc}
\text{Global dof} \downarrow \;/\; \text{Global dof} \rightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & K_{11}^{(1)} & K_{12}^{(1)} & & & K_{13}^{(1)} & K_{14}^{(1)} & & \\
2 & K_{21}^{(1)} & K_{22}^{(1)} & & & K_{23}^{(1)} & K_{24}^{(1)} & & \\
3 & & & K_{33}^{(2)} & K_{34}^{(2)} & K_{31}^{(2)} & K_{32}^{(2)} & & \\
4 & & & K_{43}^{(2)} & K_{44}^{(2)} & K_{41}^{(2)} & K_{42}^{(2)} & & \\
5 & K_{31}^{(1)} & K_{32}^{(1)} & K_{13}^{(2)} & K_{14}^{(2)} & K_{33}^{(1)}+K_{11}^{(2)} & K_{34}^{(1)}+K_{12}^{(2)} & & \\
6 & K_{41}^{(1)} & K_{42}^{(1)} & K_{23}^{(2)} & K_{24}^{(2)} & K_{43}^{(1)}+K_{21}^{(2)} & K_{44}^{(1)}+K_{22}^{(2)} & & \\
7 & & & & & & & & \\
8 & & & & & & & &
\end{array}
$$

(a) Location of $[K^{(1)}]$ and $[K^{(2)}]$ in $[\underset{\sim}{K}]$

$$
\vec{P}^{(1)} + \vec{P}^{(2)} =
\begin{array}{c} \text{Global dof} \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{Bmatrix}
P_1^{(1)} \\ P_2^{(1)} \\ \vec{P}_3^{(2)} \\ \vec{P}_4^{(2)} \\ P_3^{(1)}+P_1^{(2)} \\ P_4^{(1)}+P_2^{(2)} \\ \cdot \\ \cdot
\end{Bmatrix}
$$

(b) Location of $\vec{P}^{(1)}$ and $\vec{P}^{(2)}$ in $\underset{\sim}{\vec{P}}$

case of $e = 1$, the locations of the elements $K_{ij}^{(2)}$ for $i = 1$ to 4 and $j = 1$ to 4 in the global stiffness matrix $[\underset{\sim}{K}]$ and $P_i^{(2)}$ for $i = 1$ to 4 in the global load vector $\underset{\sim}{\vec{P}}$ can be identified. Hence, these elements would be placed in $[\underset{\sim}{K}]$ and $\underset{\sim}{\vec{P}}$ at appropriate locations as shown in Table 6.4. It can be seen that if more than one element contributes to the stiffness K_{lm} of $[\underset{\sim}{K}]$, then the stiffnesses $K_{ij}^{(e)}$ for all the elements e contributing to K_{lm} are added together to obtain K_{lm}. A similar procedure is followed in obtaining P_l of $\underset{\sim}{\vec{P}}$.

For $e = 3$ and 4, the element stiffness matrices $\left[K^{(3)}\right]$ and $\left[K^{(4)}\right]$ and the element load vectors $\vec{P}^{(3)}$ and $\vec{P}^{(4)}$ are shown in Table 6.5. By proceeding with $e = 3$ and $e = 4$ as in the cases of $e = 1$ and $e = 2$, the final global stiffness matrix $[\underset{\sim}{K}]$ and load vector $\underset{\sim}{\vec{P}}$ can be obtained as shown in Table 6.6. If there is no contribution from any element to any K_{lm} in $[\underset{\sim}{K}]$, then the coefficient K_{lm} will be zero. Thus, each of the blank locations of the matrix $[\underset{\sim}{K}]$ in Table 6.6 is to be taken as zero. A similar argument applies to the blank locations, if any, of the vector $\underset{\sim}{\vec{P}}$. It is important to note that although a structure consisting of only four elements is considered in Fig. 6.3 for illustration, the same procedure is applicable for any structure having any number of finite elements. In fact, the procedure is applicable equally well to all types of problems. The general assembly procedure is shown as a flow diagram in Fig. 6.4.

TABLE 6.5 Element Stiffness Matrices and Load Vectors for e = 3 and 4

$$\begin{array}{cccc}
 & \text{Local dof } (\Phi_i^{(3)}) & (\Phi_j^{(3)}) \longrightarrow & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
 & & \text{Corresponding global dof } (\Phi_l) \quad (\Phi_m) \rightarrow & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\
 & \downarrow & \downarrow & \\
\underset{4\times4}{\left[K^{(3)}\right]} = & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}
\end{array}$$

$$\begin{array}{cccc}
 & \text{Local dof } (\Phi_i^{(4)}) & (\Phi_j^{(4)}) \longrightarrow & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
 & & \text{Corresponding global dof } (\Phi_l) \quad (\Phi_m) \rightarrow & \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\
 & \downarrow & \downarrow & \\
\underset{4\times4}{\left[K^{(4)}\right]} = & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array} & \begin{bmatrix} K_{11}^{(4)} & K_{12}^{(4)} & K_{13}^{(4)} & K_{14}^{(4)} \\ K_{21}^{(4)} & K_{22}^{(4)} & K_{23}^{(4)} & K_{24}^{(4)} \\ K_{31}^{(4)} & K_{32}^{(4)} & K_{33}^{(4)} & K_{34}^{(4)} \\ K_{41}^{(4)} & K_{42}^{(4)} & K_{43}^{(4)} & K_{44}^{(4)} \end{bmatrix}
\end{array}$$

(a) Element stiffness matrices

$$\begin{array}{ccc}
 & \text{Local dof } (\Phi_i^{(3)}) & \text{Corresponding global dof } (\Phi_l) \\
 & \downarrow & \downarrow \\
\underset{4\times1}{\vec{P}^{(3)}} = \begin{Bmatrix} P_1^{(3)} \\ P_2^{(3)} \\ P_3^{(3)} \\ P_4^{(3)} \end{Bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
\qquad
\begin{array}{ccc}
 & \text{Local dof } (\Phi_i^{(4)}) & \text{Corresponding global dof } (\Phi_l) \\
 & \downarrow & \downarrow \\
\underset{4\times1}{\vec{P}^{(4)}} = \begin{Bmatrix} P_1^{(4)} \\ P_2^{(4)} \\ P_3^{(4)} \\ P_4^{(4)} \end{Bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array}
\end{array}$$

(b) Load vectors

TABLE 6.6 Assembled Stiffness Matrix and Load Vector

$$\underset{8\times 8}{[\underset{\sim}{K}]} = \sum_{e=1}^{4}[K^{(e)}] = \begin{array}{c|cccccccc|}
\text{Global dof} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0 \\
2 & K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 \\
3 & 0 & 0 & K_{33}^{(2)}+K_{11}^{(4)} & K_{34}^{(2)}+K_{12}^{(4)} & K_{31}^{(2)} & K_{32}^{(2)} & K_{13}^{(4)} & K_{14}^{(4)} \\
4 & 0 & 0 & K_{43}^{(2)}+K_{21}^{(4)} & K_{44}^{(2)}+K_{22}^{(4)} & K_{41}^{(2)} & K_{42}^{(2)} & K_{23}^{(4)} & K_{24}^{(4)} \\
5 & K_{31}^{(1)} & K_{32}^{(1)} & K_{13}^{(2)} & K_{14}^{(2)} & K_{33}^{(1)}+K_{11}^{(2)}+K_{11}^{(3)} & K_{34}^{(1)}+K_{12}^{(2)}+K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\
6 & K_{41}^{(1)} & K_{42}^{(1)} & K_{23}^{(2)} & K_{24}^{(2)} & K_{43}^{(1)}+K_{21}^{(2)}+K_{21}^{(3)} & K_{44}^{(1)}+K_{22}^{(2)}+K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\
7 & 0 & 0 & K_{31}^{(4)} & K_{32}^{(4)} & K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)}+K_{33}^{(4)} & K_{34}^{(3)}+K_{34}^{(4)} \\
8 & 0 & 0 & K_{41}^{(4)} & K_{42}^{(4)} & K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)}+K_{43}^{(4)} & K_{44}^{(3)}+K_{44}^{(4)}
\end{array}$$

(a) Global stiffness matrix

$$\underset{8\times 1}{\vec{\underset{\sim}{P}}} = \sum_{e=1}^{4}\vec{P}^{(e)} = \left\{\begin{array}{l}
P_1^{(1)} \\
P_2^{(1)} \\
P_3^{(2)}+P_1^{(4)} \\
P_4^{(2)}+P_2^{(4)} \\
P_3^{(1)}+P_1^{(2)}+P_1^{(3)} \\
P_4^{(1)}+P_2^{(2)}+P_2^{(3)} \\
P_3^{(3)}+P_3^{(4)} \\
P_4^{(3)}+P_4^{(4)}
\end{array}\right\}
\begin{array}{c}
\text{Global dof} \downarrow \\
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}$$

(b) Global load vector

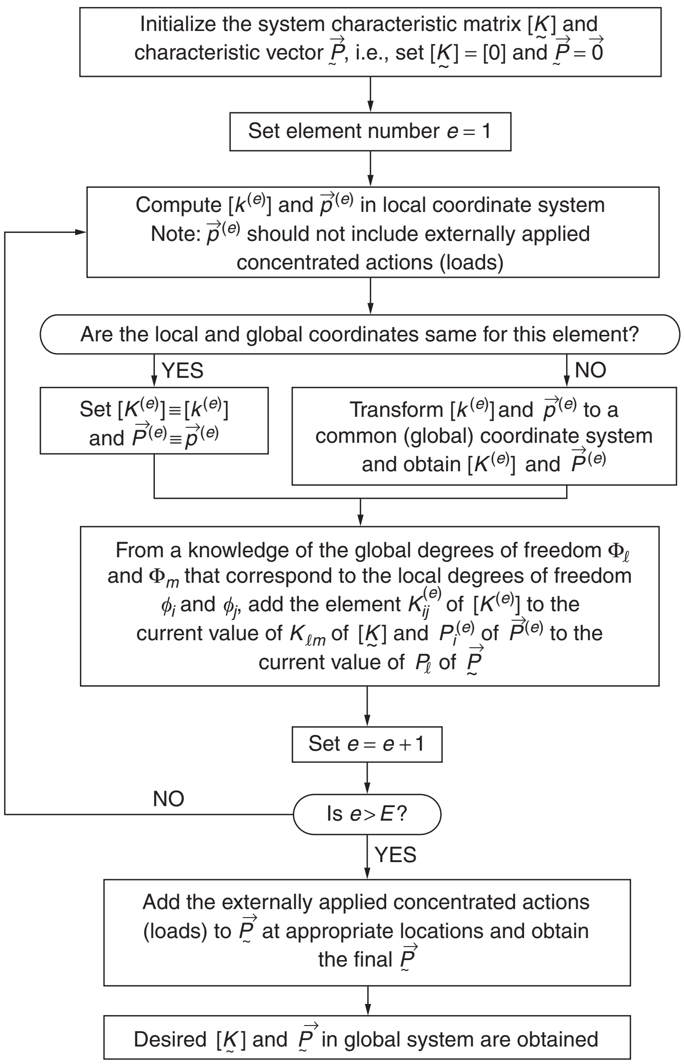

FIGURE 6.4 Assembly procedure.

EXAMPLE 6.3

Find the global stiffness matrix of the two-bar truss considered in Example 6.2.

Approach: Assemble the global element stiffness matrices of the two elements.

Solution

The global stiffness matrices of the two elements, given by Eqs. (E.5) and (E.7) of Example 6.2, can be rewritten as

$$[K^{(1)}] = 10^5 \begin{array}{c} \left[\begin{array}{cccc} 0.7682 & 1.9208 & -0.7682 & -1.9208 \\ 1.9208 & 4.8027 & -1.9208 & -4.8027 \\ -0.7682 & -1.9208 & 0.7682 & 1.9208 \\ -1.9208 & -4.8027 & 1.9208 & 4.8027 \end{array}\right] \begin{array}{c} Q_5 \\ Q_6 \\ Q_1 \\ Q_2 \end{array} \\ \begin{array}{cccc} Q_5 & \quad Q_6 & \quad Q_1 & \quad Q_2 \end{array} \end{array} \tag{E.1}$$

and

EXAMPLE 6.3 —cont'd

$$\left[K^{(2)}\right] = 10^5 \begin{array}{c} \left[\begin{array}{cccc} 0.2561 & -0.6402 & -0.2561 & 0.6402 \\ -0.6402 & 1.6008 & 0.6402 & -1.6008 \\ -0.2561 & 0.6402 & 0.2561 & -0.6402 \\ 0.6402 & -1.6008 & -0.6402 & 1.6008 \end{array}\right] \begin{array}{c} Q_3 \\ Q_4 \\ Q_1 \\ Q_2 \end{array} \\ \begin{array}{cccc} Q_3 & \quad Q_4 & \quad Q_1 & \quad Q_2 \end{array} \end{array} \tag{E.2}$$

where the displacement dof corresponding to the various rows and columns of $[K^{(1)}]$ and $[K^{(2)}]$ are also identified. To assemble these matrices to generate the global stiffness matrix, of order 6 × 6, of the system, we first open a 6 × 6 matrix with the rows and columns identified by the corresponding dof $Q_1, Q_2, \ldots, Q_6$ in sequence. Then each of the elements of the matrix $[K^{(1)}]$ is entered at the location of the global stiffness matrix whose row and column dof correspond to those of the element in the matrix $[K^{(1)}]$. A similar procedure is used to assemble the matrix $[K^{(2)}]$ in the global stiffness matrix $[\underset{\sim}{K}]$ of the two-bar truss. The resulting global stiffness matrix of the truss is given in Eq. (E.3).

$$[\underset{\sim}{K}] = 10^5 \begin{array}{c} \left[\begin{array}{cccccc} (0.7682+0.2561) & (1.9208-0.6402) & (-0.2561) & (0.6402) & (-0.7682) & (-1.9208) \\ (1.9208-0.6402) & (4.8027+1.6008) & (0.6402) & (-1.6008) & (-1.9208) & (-4.8027) \\ (-0.2561) & (0.6402) & (0.2561) & (-0.6402) & 0 & 0 \\ (0.6402) & (-1.6008) & (-0.6402) & (1.6008) & 0 & 0 \\ (-0.7682) & (-1.9208) & 0 & 0 & (0.7682) & (1.9208) \\ (-1.9208) & (-4.8027) & 0 & 0 & (1.9208) & (4.8027) \end{array}\right] \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array} \\ \begin{array}{cccccc} Q_1 & \quad Q_2 & \quad Q_3 & \quad Q_4 & \quad Q_5 & \quad Q_6 \end{array} \end{array} \tag{E.3}$$

6.3 INCORPORATION OF BOUNDARY CONDITIONS

After assembling the element characteristic matrices $[K^{(e)}]$ and the element characteristic vectors $\overrightarrow{P}^{(e)}$, the overall or system equations of the entire domain or body can be written (for an equilibrium problem) as

$$\underset{M\times M}{[\underset{\sim}{K}]} \; \underset{M\times 1}{[\overrightarrow{\underset{\sim}{\Phi}}]} = \underset{M\times 1}{[\overrightarrow{\underset{\sim}{P}}]} \tag{6.11}$$

These equations cannot be solved for $\overrightarrow{\underset{\sim}{\Phi}}$ since the matrix $[\underset{\sim}{K}]$ will be singular and hence its inverse does not exist. The physical significance of this, in the case of solid mechanics problems, is that the loaded body or structure is free to undergo unlimited rigid body motion unless some support constraints are imposed to keep the body or structure in equilibrium under the loads. Hence, some boundary or support conditions have to be applied to Eq. (6.11) before solving for $\overrightarrow{\underset{\sim}{\Phi}}$. In nonstructural problems, we have to specify the value of at least one and sometimes more than one nodal dof. The number of dof to be specified is dictated by the physics of the problem.

As seen in Eqs. (5.15) and (5.16), there are two types of boundary conditions: forced or geometric, or essential and free or natural. If we use a variational approach for deriving the system equations, we need to specify only the essential boundary conditions and the natural boundary conditions will be implicitly satisfied in the solution procedure. Thus, we need to apply only the geometric boundary conditions to Eq. (6.11). In the geometric boundary conditions, usually a displacement dof Φ_j, for example, is required to have a specified value Φ_j^*($\Phi_j^* = 0$ in many cases):

$$\Phi_j = \Phi_j^* \tag{6.11a}$$

The geometric boundary conditions can be incorporated into Eq. (6.11) by several methods as outlined in the following paragraphs. A boundary condition involving more than one nodal dof is known as a *multipoint constraint*. Several methods are available to incorporate linear multipoint constraints (see Section 6.5 and Problems 6.5 and 6.6). The processing of nonlinear multipoint constraints is described by Narayanaswamy [6.1].

6.3.1 Method 1

To understand this method, we partition Eq. (6.11) as

$$\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \begin{Bmatrix} \vec{\Phi}_1 \\ \vec{\Phi}_2 \end{Bmatrix} = \begin{Bmatrix} \vec{P}_1 \\ \vec{P}_2 \end{Bmatrix} \tag{6.12}$$

where $\vec{\Phi}_2$ is assumed to be the vector of specified nodal dof and $\vec{\Phi}_1$ as the vector of unrestricted (free) nodal dof. Then $\vec{P}_1$ will be the vector of known nodal actions, and $\vec{P}_2$ will be the vector of unknown nodal actions.[1] Eq. (6.12) can be written as

$$[K_{11}]\vec{\Phi}_1 + [K_{12}]\vec{\Phi}_2 = \vec{P}_1$$

or

$$[K_{11}]\vec{\Phi}_1 = \vec{P}_1 - [K_{12}]\vec{\Phi}_2 \tag{6.13}$$

and

$$[K_{12}]^T\vec{\Phi}_1 + [K_{22}]\vec{\Phi}_2 = \vec{P}_2 \tag{6.14}$$

Here, $[K_{11}]$ will not be singular and hence Eq. (6.13) can be solved to obtain

$$\vec{\Phi}_1 = [K_{11}]^{-1}\left(\vec{P}_1 - [K_{12}]\vec{\Phi}_2\right) \tag{6.15}$$

Once $\vec{\Phi}_1$ is known, the vector of unknown nodal actions $\vec{P}_2$ can be found from Eq. (6.14). In the special case in which all the prescribed nodal dof are equal to zero, we can delete the rows and columns corresponding to $\vec{\Phi}_2$ and state the equations simply as

$$[K_{11}]\vec{\Phi}_1 = \vec{P}_1 \tag{6.16}$$

6.3.2 Method 2

Since all the prescribed nodal dof usually do not come at the end of the vector $\underset{\sim}{\vec{\Phi}}$, the procedure of Method 1 involves an awkward renumbering scheme. Even when the prescribed nodal dof are not zero, it can be seen that the rearrangement of Eq. (6.11) and solution of Eqs. (6.13) and (6.14) are time-consuming and require tedious bookkeeping. Hence, the following equivalent method can be used for incorporating the prescribed boundary conditions $\vec{\Phi}_2$. Eqs. (6.13) and (6.14) can be written together as

$$\begin{bmatrix} [K_{11}] & [0] \\ [0] & [I] \end{bmatrix} \begin{Bmatrix} \vec{\Phi}_1 \\ \hline \vec{\Phi}_2 \end{Bmatrix} = \begin{Bmatrix} \vec{P}_1 - [K_{12}]\vec{\Phi}_2 \\ \hline \vec{\Phi}_2 \end{Bmatrix} \tag{6.17}$$

In actual computations, the process indicated in Eq. (6.17) can be performed without reordering the equations implied by the partitioning as follows.

Step 1: If Φ_j is prescribed as Φ_j^*, the characteristic vector $\underset{\sim}{\vec{P}}$ is modified as

$$P_i = P_i - K_{ij}\Phi_j^* \quad \text{for} \quad i = 1, 2, \ldots, M$$

Step 2: The row and column of $\left[\underset{\sim}{K}\right]$ corresponding to Φ_j are made zero except the diagonal element, which is made unity; that is,

$$K_{ji} = K_{ij} = 0 \quad \text{for} \quad i = 1, 2, \ldots, M$$

$$K_{jj} = 1$$

1. In the case of the solid mechanics problem, $\vec{\Phi}_2$ denotes the vector of nodal displacements that avoids the rigid body motion of the body, $\vec{P}_1$ the vector of known nodal loads, and $\vec{P}_2$ the unknown reactions at points at which the displacements $\vec{\Phi}_2$ are prescribed.

Step 3: The prescribed value of Φ_j is inserted in the characteristic vector as

$$P_j = \Phi_j^*$$

This procedure (Steps 1—3) is repeated for all prescribed nodal dof, Φ_j. This procedure retains the symmetry of the equations and the matrix $[\underset{\sim}{K}]$ can be stored in the band format with little extra programming effort.

6.3.3 Method 3

Another method of incorporating the prescribed condition $\Phi_j = \Phi_j^*$ is as follows.

Step 1: Multiply the diagonal term K_{jj} by a large number, such as 10^{10}, so that the new K_{jj} = old $K_{jj} \times 10^{10}$.
Step 2: Make the corresponding load P_j as

$$P_j = \text{new } K_{jj} \times \Phi_j^* = \text{old } K_{jj} \times 10^{10} \times \Phi_j^*$$

Step 3: Keep all other elements of the characteristic matrix and the characteristic vector unaltered so that

$$\text{new } K_{ik} = \text{old } K_{ik} \text{ for all } i \text{ and } k \text{ except } i = k = j$$

and

$$\text{new } P_i = \text{old } P_i \text{ for all } i \text{ except } i = j$$

This procedure (Steps 1—3) is repeated for all prescribed nodal dof, Φ_j. This procedure will yield a solution in which Φ_j is very nearly equal to Φ_j^*. This method can also be used when the characteristic matrix is stored in banded form. We represent the equations that result from the application of the boundary conditions to Eq. (6.11) as

$$[K]\vec{\Phi} = \vec{P} \tag{6.18}$$

where $[K]$, $\vec{\Phi}$, and $\vec{P}$ denote the final (modified) characteristic matrix, vector of nodal dof, and vector of nodal actions, respectively, of the complete body or system.

EXAMPLE 6.4

The cantilever beam shown in Fig. 6.5A is modeled using one beam element with four dof as shown in Fig. 6.5B. The resulting equilibrium equations of the system are given by

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{W}} = \underset{\sim}{\vec{P}} \tag{E.1}$$

where

$$[K] = \begin{array}{c} \begin{bmatrix} 12 & 600 & -12 & 600 \\ 600 & 40{,}000 & -600 & 20{,}000 \\ -12 & -600 & 12 & -600 \\ 600 & 20{,}000 & -600 & 40{,}000 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix} \\ \begin{matrix} W_1 \quad & W_2 \quad & W_3 \quad & W_4 \end{matrix} \end{array} \tag{E.2}$$

$$\underset{\sim}{\vec{W}} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} \tag{E.3}$$

$$\underset{\sim}{\vec{P}} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{E.4}$$

Continued

EXAMPLE 6.4 —cont'd

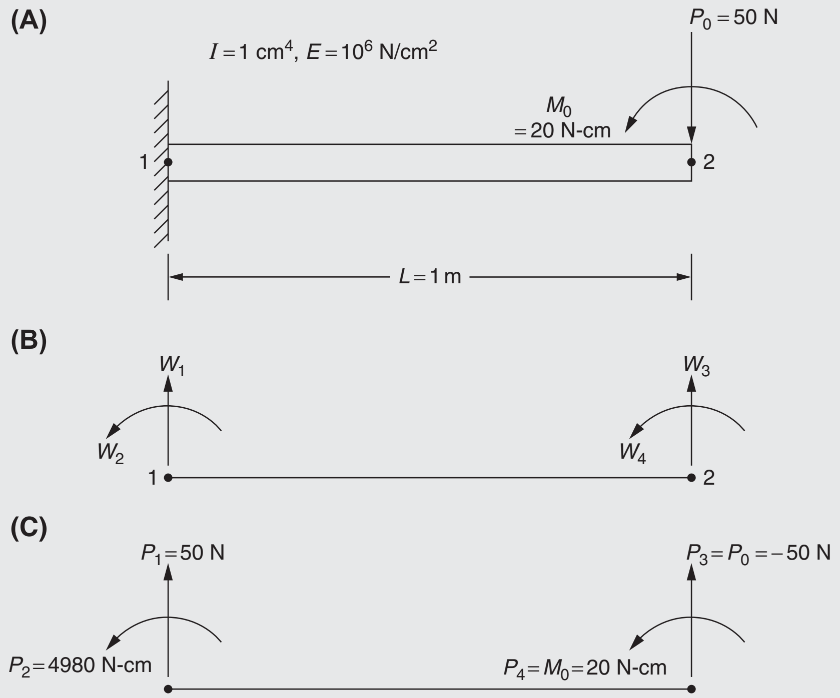

FIGURE 6.5 Cantilever beam.

Using the known boundary conditions, the displacements $W_1 = 0$, $W_2 = 0$ and the known external load values $P_3 = -50$ N, $P_4 = 20$ N-cm, determine the displacement components W_3 (cm) and W_4 (rad) of node 2 and the reactions P_1 (N) and P_2 (N-cm) at node 1.

Approach: Solve the system equations of (E.1) using Method 1.

Solution

Group the unknown and known displacement dof as components of the vectors $\overrightarrow{W}_1$ and $\overrightarrow{W}_2$, respectively, and the known and unknown loads as components of the vectors $\overrightarrow{P}_1$, and $\overrightarrow{P}_2$, respectively, as

$$\overrightarrow{W}_1 = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \text{vector of unknown displacements}$$

$$\overrightarrow{W}_2 = \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \text{vector of known displacements}$$

$$\overrightarrow{P}_1 = \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 20 \end{Bmatrix} = \text{vector of known loads}$$

$$\overrightarrow{P}_2 = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \text{vector of unknown loads (reactions)}$$

Rearrange and partition the system stiffness matrix $[\underset{\sim}{K}]$ by defining submatrices corresponding to the known and unknown displacement dof as

$$[\underset{\sim}{K}] = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} = \begin{array}{c} \left[\begin{array}{cc|cc} 12 & -600 & -12 & -600 \\ -600 & 40{,}000 & 600 & 20{,}000 \\ \hline -12 & 600 & 12 & 600 \\ -600 & 20{,}000 & 600 & 40{,}000 \end{array}\right] \begin{array}{c} W_3 \\ W_4 \\ W_1 \\ W_2 \end{array} \\ \begin{array}{cccc} W_3 & W_4 & W_1 & W_2 \end{array} \end{array} \tag{E.5}$$

EXAMPLE 6.4 —cont'd

so that

$$[K_{11}] = \begin{array}{cc} \begin{bmatrix} 12 & -600 \\ -600 & 40{,}000 \end{bmatrix} & \begin{array}{c} W_3 \\ W_4 \end{array} \\ \begin{array}{cc} W_3 & \quad W_4 \end{array} & \end{array} \tag{E.6}$$

$$[K_{12}] = \begin{array}{cc} \begin{bmatrix} -12 & -600 \\ 600 & 20{,}000 \end{bmatrix} & \begin{array}{c} W_3 \\ W_4 \end{array} \\ \begin{array}{cc} W_1 & \quad W_2 \end{array} & \end{array} \tag{E.7}$$

$$[K_{21}] = \begin{array}{cc} \begin{bmatrix} -12 & 600 \\ -600 & 20{,}000 \end{bmatrix} & \begin{array}{c} W_1 \\ W_2 \end{array} \\ \begin{array}{cc} W_3 & \quad W_4 \end{array} & \end{array} \tag{E.8}$$

$$[K_{22}] = \begin{array}{cc} \begin{bmatrix} 12 & 600 \\ 600 & 40{,}000 \end{bmatrix} & \begin{array}{c} W_1 \\ W_2 \end{array} \\ \begin{array}{cc} W_1 & \quad W_2 \end{array} & \end{array} \tag{E.9}$$

Noting that

$$[K_{11}]^{-1} = \begin{bmatrix} 0.3333 & 0.0050 \\ 0.0050 & 0.0001 \end{bmatrix} \tag{E.10}$$

the solution vector $\overrightarrow{W}_1$ can be found using Eq. (6.15):

$$\overrightarrow{W}_1 = [K_{11}]^{-1}\left(\overrightarrow{P}_1 - [K_{12}]\overrightarrow{W}_2\right) = [K_{11}]^{-1}\overrightarrow{P}_1 = \begin{bmatrix} 0.3333 & 0.0050 \\ 0.0050 & 0.0001 \end{bmatrix}\begin{Bmatrix} -50 \\ 20 \end{Bmatrix} = \begin{Bmatrix} -16.5667 \\ -0.2480 \end{Bmatrix} \tag{E.11}$$

The vector of reactions can be found using Eq. (6.14) as

$$\overrightarrow{P}_2 = [K_{12}]^T\overrightarrow{W}_1 + [K_{22}]\overrightarrow{W}_2 = [K_{12}]^T\overrightarrow{W}_1 = \begin{bmatrix} -12 & 600 \\ -600 & 20{,}000 \end{bmatrix}\begin{Bmatrix} -1.5667 \\ -0.2480 \end{Bmatrix} = \begin{Bmatrix} 50.0 \\ 4{,}980.0 \end{Bmatrix} \tag{E.12}$$

The actions (applied loads) and the reactions are shown in the free body diagram of the beam in Fig. 6.5C.

EXAMPLE 6.5

Find the displacement components of the cantilever beam problem stated in Example 6.4 using Method 2 of Section 6.3.

Approach: Method 2 of Section 6.3.

Solution

The equilibrium equations of the cantilever beam are given by Eq. (E.1) of Example 6.4:

$$\underset{\sim}{[K]}\ \underset{\sim}{\overrightarrow{W}} = \underset{\sim}{\overrightarrow{P}} \tag{E.1}$$

or

$$\begin{bmatrix} 12 & 600 & -12 & 600 \\ 600 & 40{,}000 & -600 & 20{,}000 \\ -12 & -600 & 12 & -600 \\ 600 & 20{,}000 & -600 & 40{,}000 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ -50 \\ 20 \end{Bmatrix} \tag{E.2}$$

where P_1 and P_2 are the reactions at node 1 along the directions of W_1 and W_2, respectively.

Continued

EXAMPLE 6.5 —cont'd

To incorporate the first known boundary condition $W_1 = W_1^* = 0$ according to Method 2 of Section 6.3, we modify the load components as $P_i = P_i - K_{1i}W_1^*$; $i = 1, 2, 3, 4$ and set $K_{1i} = K_{i1} = 0$; $i = 2, 3, 4$ and $K_{11} = 1$. Thus, Eq. (E.2) is modified as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 40,000 & -600 & 20,000 \\ 0 & -600 & 12 & -600 \\ 0 & 20,000 & -600 & 40,000 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} W_1^* = 0 \\ P_2 - 600(0) \\ -50 - (-12)(0) \\ 20 - 600(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_2 \\ -50 \\ 20 \end{Bmatrix} \tag{E.3}$$

Similarly, to incorporate the second known boundary condition $W_2 = W_2^* = 0$, we modify the load components as $P_i = P_i - K_{2i}\, W_2^*$; $i = 1, 2, 3, 4$ and set $K_{2i} = K_{i2} = 0$; $i = 1, 3, 4$ and $K_{22} = 1$. Thus Eq. (E.3) is modified as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 12 & -600 \\ 0 & 0 & -600 & 40,000 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ W_2^* = 0 \\ -50 - (-600)(0) \\ 20 - 20,000(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -50 \\ 20 \end{Bmatrix} \tag{E.4}$$

The solution of Eq. (E.4) gives the desired solution:

$$\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -16.5667 \\ -0.2480 \end{Bmatrix} \tag{E.5}$$

EXAMPLE 6.6

Find the displacement components of the cantilever beam problem stated in Example 6.4 using Method 3 of Section 6.3.

Approach: Method 3 of Section 6.3.

Solution

The equilibrium equations of the cantilever beam are given by Eq. (E.1) of Example 6.4:

$$[\underset{\sim}{K}]\, \overrightarrow{\underset{\sim}{W}} = \overrightarrow{\underset{\sim}{P}} \tag{E.1}$$

or

$$\begin{bmatrix} 12 & 600 & -12 & 600 \\ 600 & 40,000 & -600 & 20,000 \\ -12 & -600 & 12 & -600 \\ 600 & 20,000 & -600 & 40,000 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ -50 \\ 20 \end{Bmatrix} \tag{E.2}$$

where P_1 and P_2 are the reactions at node 1 along the directions of W_1 and W_2, respectively. To incorporate the first known boundary condition $W_1 = W_1^* = 0$ according to Method 3, we multiply the diagonal term K_{11} by a large number (10^{10}) and change the corresponding load component P_i to $K_{11} \times 10^{10} \times W_2^* = 0$. Thus, Eq. (E.2) is modified as

$$\begin{bmatrix} 12 \times 10^{10} & 600 & -12 & 600 \\ 600 & 40,000 & -600 & 20,000 \\ -12 & -600 & 12 & -600 \\ 600 & 20,000 & -600 & 40,000 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 12 \times 10^{10} \times 0 \\ P_2 \\ -50 \\ 20 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_2 \\ -50 \\ 20 \end{Bmatrix} \tag{E.3}$$

EXAMPLE 6.6 —cont'd

Similarly, to incorporate the second known boundary condition $W_2 = W_2^* = 0$ according to Method 3, we multiply the diagonal term K_{22} by a large number (10^{10}) and change the corresponding load component P_2 to $K_{22} \times 10^{10} \times W_2^* = 0$. Thus, Eq. (E.3) is modified as

$$\begin{bmatrix} 12\times 10^{10} & 600 & -12 & 600 \\ 600 & 40,000\times 10^{10} & -600 & 20,000 \\ -12 & -600 & 12 & -600 \\ 600 & 20,000 & -600 & 40,000 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 40,000\times 10^{10}\times 0 \\ -50 \\ 20 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -50 \\ 20 \end{Bmatrix} \tag{E.4}$$

The solution of Eq. (E.4) gives the desired solution:

$$\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -16.5667 \\ -0.2480 \end{Bmatrix} \tag{E.5}$$

EXAMPLE 6.7

The finite element analysis of heat transfer along a one-dimensional fin using two elements and three nodes leads to the equations

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{E.1}$$

or

$$\begin{bmatrix} 0.50893 & -0.49951 & 0 \\ -0.49951 & 1.01786 & -0.49951 \\ 0 & -0.49951 & 0.55919 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.23561 \\ 0.47122 \\ 0.73825 \end{Bmatrix} \tag{E.2}$$

If the temperature of node 1, T_1, is specified as 100 °C, find the temperatures of nodes 2 and 3 by solving Eq. (E.2) using Method 1 of Section 6.3.

Approach: Use Method 1 of Section 6.3.

Solution

Group the unknown and known temperatures as components of the vectors $\vec{T}_1$ and $\vec{T}_2$, respectively, and the corresponding components of $\vec{P}$ as the vectors $\vec{P}_1$ and $\vec{P}_2$, respectively, as

$$\vec{T}_1 = \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \text{vector of unknown temperatures}$$

$$\vec{T}_2 = \{T_1\} = \{100\} = \text{vector of known temperaturess}$$

$$\vec{P}_1 = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} 0.47122 \\ 0.73825 \end{Bmatrix} = \text{vector of right-hand side components corresponding to } \vec{T}_1$$

$$\vec{P}_2 = \{P_1\} = \text{vector of right-hand side components corresponding to } \vec{T}_2$$

Rearrange and partition the system characteristic matrix $[\underset{\sim}{K}]$ by defining submatrices corresponding to the known and unknown nodal temperatures as

$$[\underset{\sim}{K}] = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1.01786 & -0.49951 & -0.49951 \\ -0.49951 & 0.55919 & 0 \\ -0.49951 & 0 & 0.50893 \end{bmatrix} \begin{array}{l} T_2 \\ T_3 \\ T_1 \end{array} \\ \begin{array}{ccc} T_2 \qquad & \qquad T_3 \qquad & \qquad T_1 \end{array} \end{array} \tag{E.3}$$

so that

Continued

EXAMPLE 6.7 —cont'd

$$[K_{11}] = \begin{array}{cc} \left[\begin{array}{cc} 1.01786 & -0.49951 \\ -0.49951 & 0.55919 \end{array}\right] & \begin{array}{c} T_2 \\ T_3 \end{array} \\ \begin{array}{cc} T_2 & \quad\quad T_3 \end{array} & \end{array}$$

$$[K_{12}] = \begin{array}{cc} \left[\begin{array}{c} -0.49951 \\ 0 \end{array}\right] & \begin{array}{c} T_2 \\ T_3 \end{array} \\ T_1 & \end{array}$$

$$[K_{21}] = \begin{array}{cc} [\begin{array}{cc} -0.49951 & 0 \end{array}] & T_1 \\ \begin{array}{cc} T_2 & \quad T_3 \end{array} & \end{array}$$

$$[K_{21}] = \begin{array}{cc} [0.50893] & T_1 \\ T_1 & \end{array}$$

Noting that

$$[K_{11}]^{-1} = \begin{bmatrix} 1.7493 & 1.5626 \\ 1.5626 & 3.1841 \end{bmatrix}$$

the solution vector $\vec{T}_1$ can be found using Eq. (6.15):

$$\begin{aligned} \vec{T}_1 &= [K_{11}]^{-1}\left(\vec{P}_1 - [K_{12}]\vec{T}_2\right) = [K_{11}]^{-1}\left(\begin{Bmatrix} 0.47122 \\ 0.73825 \end{Bmatrix} - \begin{Bmatrix} -0.49951 \\ 0 \end{Bmatrix}\{100\}\right) \\ &= \begin{bmatrix} 1.7493 & 1.5626 \\ 1.5626 & 3.1841 \end{bmatrix}\begin{Bmatrix} 50.4222 \\ 0.7382 \end{Bmatrix} = \begin{Bmatrix} 89.3567 \\ 81.1402 \end{Bmatrix} {}^\circ\mathrm{C} \end{aligned} \tag{E.4}$$

EXAMPLE 6.8

The finite element analysis of heat transfer along a one-dimensional fin using two elements and three nodes leads to the equations

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{E.1}$$

or

$$\begin{bmatrix} 0.50893 & -0.49951 & 0 \\ -0.49951 & 1.01786 & -0.49951 \\ 0 & -0.49951 & 0.55919 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.23561 \\ 0.47122 \\ 0.73825 \end{Bmatrix} \tag{E.2}$$

If the temperature of node 1, T_1, is specified as 100 °C, find the temperatures of nodes 2 and 3 by solving Eq. (E.2) using Method 2 of Section 6.3.

Approach: Use Method 2 of Section 6.3.

EXAMPLE 6.8 —cont'd

Solution

To incorporate the known condition $T_1 = T_1^* = 100$ according to Method 2 of Section 6.3, we modify the right-hand side components as $P_i = P_i - K_{1i}\, T_1^*$; $i = 1, 2, 3$, set $K_{1i} = K_{i1} = 0$; $i = 2, 3$ and $K_{11} = 1$, and change the value of P_1 equal to $T_1^* = 100$ so that Eq. (E.2) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.01786 & -0.49951 \\ 0 & -0.49951 & 0.55919 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 0.47122 - (-0.49951)(100) \\ 0.73825 - (0)(100) \end{Bmatrix} = \begin{Bmatrix} 100.0 \\ 50.42222 \\ 0.73825 \end{Bmatrix} \tag{E.3}$$

The solution of Eq. (E.3) gives the desired solution

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 100.0 \\ 89.3567 \\ 81.1402 \end{Bmatrix} {}^\circ\mathrm{C} \tag{E.4}$$

EXAMPLE 6.9

Find the solution of Eq. (E.2) of Example 6.7 using Method 3 of Section 6.3 when the temperature of node 1, T_1, is specified as 100 °C.

Approach: Use Method 3 of Section 6.3.

Solution

The equations to be solved are given by

$$\begin{bmatrix} 0.50893 & -0.49951 & 0 \\ -0.49951 & 1.01786 & -0.49951 \\ 0 & -0.49951 & 0.55919 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.23561 \\ 0.47122 \\ 0.73825 \end{Bmatrix} \tag{E.1}$$

To incorporate the known condition $T_1 = T_1^* = 100$ according to Method 3, we multiply the diagonal term K_{11} by a large number (10^6) and change the corresponding right-hand side element P_1 to $K_{11} \times 10^6 \times\ T_2^* = 0.50893 \times\ 10^6 \times 100 = 50.893 \times 10^6$. Thus, Eq. (E.1) is modified as

$$\begin{bmatrix} 0.50893 \times 10^6 & -0.49951 & 0 \\ -0.49951 & 1.01786 & -0.49951 \\ 0 & -0.49951 & 0.55919 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 50.893\ \times\ 10^6 \\ 0.47122 \\ 0.73825 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives the desired solution

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 100.0 \\ 89.3567 \\ 81.1403 \end{bmatrix} {}^\circ\mathrm{C} \tag{E.3}$$

6.4 PENALTY METHOD

The penalty method can be used to implement the boundary conditions in a unified and simple manner. First, we consider the method for implementing a single point boundary condition:

$$\Phi_p = \Phi_p^* \tag{6.19}$$

where Φ_p^* is a constant denoting the specified value of the displacement dof Φ_p. Imagine a spring with large stiffness C oriented in the direction of the dof Φ_p with one end attached to the structure at point A, which undergoes a displacement Φ_p, and the other end fixed at point O as shown in Fig. 6.6.

When the point A of the structure undergoes a displacement Φ_p, assume that the other end of the spring, point O, moves by a distance Φ_p^* so that the net deformation of the spring is $\left(\Phi_p - \Phi_p^*\right)$. The strain energy of the spring due to this net deformation will be $\frac{1}{2}C\left(\Phi_p - \Phi_p^*\right)^2$. The total potential energy of the system, including the strain energy of the spring, will be

$$I_m = \frac{1}{2}\vec{\Phi}^T[K]\vec{\Phi} - \vec{\Phi}^T\vec{P} + \frac{1}{2}C\left(\Phi_p - \Phi_p^*\right)^2 \tag{6.20}$$

where $\vec{\Phi}$ and $\vec{P}$ represent the vectors of nodal displacements and the corresponding nodal loads of the system. The equilibrium equations of the system can be found by minimizing the total potential energy:

$$\frac{\partial I_m}{\partial \Phi_i} = 0;\quad i = 1,\ 2, \ldots, n$$

or

$$[K]\vec{\Phi} - \vec{P} + \left\{0 \quad 0 \quad \ldots \quad 0 \quad \left(C\Phi_p^*\right) \quad 0 \quad \ldots \quad 0\right\}^T \tag{6.21}$$

or

$$\begin{bmatrix} K_{1,1} & \ldots & K_{1,p-1} & K_{1,p} & K_{1,p+1} & \ldots & K_{i,M} \\ \vdots & & & & & & \\ K_{p-1,1} & \ldots & K_{p-1,p-1} & K_{p-1,p} & K_{p-1,p+1} & \ldots & K_{p-1,M} \\ K_{p,1} & \ldots & K_{p,p-1} & K_{p,p} & K_{p,p+1} & \ldots & K_{p,M} \\ K_{p+1,1} & \ldots & K_{p+1,p-1} & K_{p+1,p} & K_{p+1,p+1} & \ldots & K_{p+1,M} \\ \vdots & & & & & & \\ K_{M,1} & \ldots & K_{M,p-1} & K_{M,p} & K_{M,p+1} & \ldots & K_{M,M} \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \vdots \\ \Phi_{p-1} \\ \Phi_p \\ \Phi_{p+1} \\ \vdots \\ \Phi_M \end{Bmatrix} = \begin{Bmatrix} P_1 \\ \vdots \\ P_{p-1} \\ P_p + C\Phi_p^* \\ P_{p+1} \\ \vdots \\ P_M \end{Bmatrix} \tag{6.22}$$

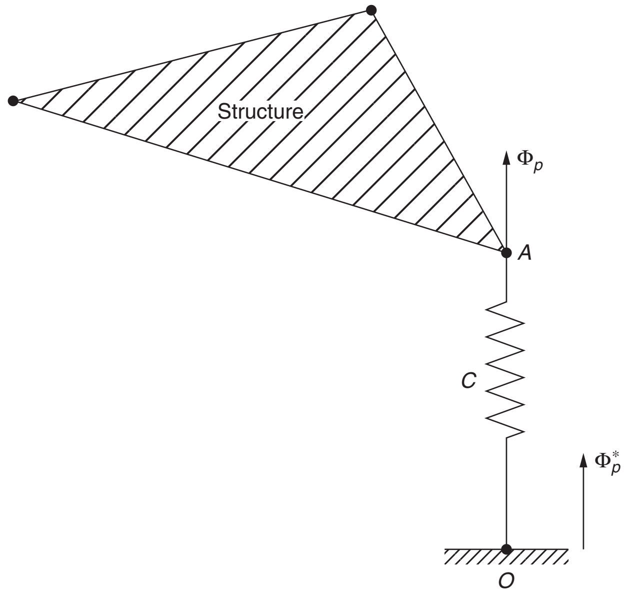

FIGURE 6.6 Spring connected to point A.

where P_p is the load applied along the dof Φ_p (if any). If the value of C is sufficiently large compared to the stiffness coefficients K_{ij}, the solution of Eq. (6.22) yields a value of Φ_p approximately equal to Φ_p^*. This can be seen by writing the pth row of Eq. (6.22) in scalar form:

$$K_{p,1}\Phi_1 + \ldots + K_{p,p-1}\Phi_{p-1} + K_{p,p}\Phi_p + K_{p,p+1}\Phi_{p+1} + \ldots + K_{p,M}\Phi_M = P_p + C\Phi_p^* \tag{6.23}$$

By dividing Eq. (6.23) throughout by C, we obtain

$$\frac{K_{p,1}}{C}\Phi_1 + \ldots + \frac{K_{p,p-1}}{C}\Phi_{p-1} + \frac{K_{p,p}}{C}\Phi_p + \frac{K_{p,p+1}}{C}\Phi_{p+1} + \ldots + \frac{K_{p,M}}{C}\Phi_M = \frac{P_p}{C} + \Phi_p^* \tag{6.24}$$

If C is chosen to be large compared the stiffness coefficients $K_{p,i}, i = 1, 2, \ldots, M$, Eq. (6.24) yields

$$\Phi_p \approx \Phi_p^* \tag{6.25}$$

which is the desired result. In practice, the value of C can be chosen as

$$C = \max_{i = 1,2,\ldots,M;\, j = 1,2,\ldots,M} |K_{i,j}| \times 10^6 \tag{6.26}$$

Notes

1. The spring with stiffness C is introduced only to facilitate the derivation of Eq. (6.22). In practice, the system equations are modified by adding C to the diagonal stiffness coefficient $K_{p,p}$ and $\left(C\,\Phi_p^*\right)$ to the corresponding element of the load, P_p.
2. If several boundary conditions of the type shown in Eq. (6.18) are to be incorporated, the preceding procedure is repeated for each boundary condition separately.
3. As can be seen in Eq. (6.25), the penalty method is an approximate method. The method can be seen to be similar to Method 3, presented in Section 6.3.

EXAMPLE 6.10

A stepped bar is subjected to an axial (vertical) force $P = 10^8$ N at node 2 as shown in Fig. 6.7. If the areas of the cross section of the steps are given by $A_1 = 0.1\ \text{m}^2$ and $A_2 = 0.05\ \text{m}^2$ and Young's moduli $E_1 = 200$ GPa and $E_2 = 70$ GPa, determine the following:

a. The displacement of node 3.
b. The displacements of nodes and the stresses in the two steps.

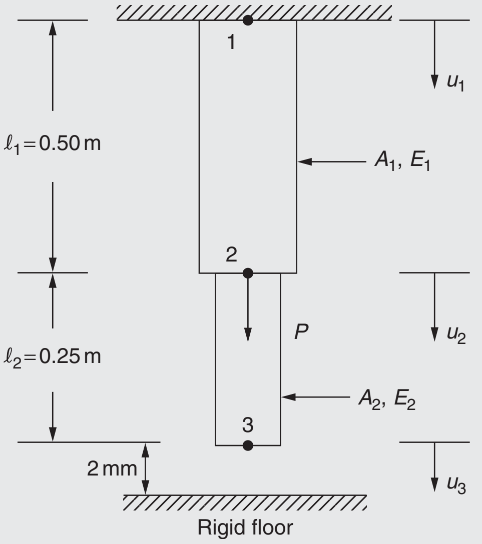

FIGURE 6.7 Stepped bar under axial force.

Continued

EXAMPLE 6.10 —cont'd

Solution

a. Using an axial bar element to model each step, the stiffness matrices of the elements can be found as

$$[k^{(1)}] = \frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(0.1)(200\times 10^9)}{0.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 40\times 10^9 \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array} \tag{E.1}$$

$$[k^{(2)}] = \frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(0.05)(70\times 10^9)}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 14\times 10^9 \begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ u_3 \end{array} \tag{E.2}$$

The system (assembled) stiffness matrix is

$$[k] = 10^9 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 40 & -40 & 0 \\ -40 & 54 & -14 \\ 0 & -14 & 14 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \tag{E.3}$$

The displacement and load vectors of the system are

$$\vec{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \vec{P} = \begin{Bmatrix} P_1 \\ 10^8 \\ 0 \end{Bmatrix} \tag{E.4}$$

where P_1 indicates the reaction at the fixed node 1. The solution of the system equations,

$$[k]\,\vec{u} = \vec{P} \tag{E.5}$$

using the boundary condition $u_1 = 0$, is given by

$$u_1 = 0,\quad u_2 = u_3 = 0.0025 \text{ m} = 2.5 \text{ mm} \tag{E.6}$$

Since the displacement u_3 is larger than the space (gap) available, the node contacts the floor and hence the maximum displacement of node 3 will be 2 mm.

b. Now we seek the solution of system Eq. (E.5) using the new boundary conditions

$$u_1 = 0 \text{ and } u_3 = u_3^* = 0.002 \text{ m} \tag{E.7}$$

In order to incorporate the boundary conditions of Eq. (E.7) using the penalty method, we find a large number C that is larger than the stiffness coefficients. The value of C is selected as

$$C = \max_{i,j} |k_{i,j}| \times 10^6 = (54\times 10^9)10^6 = 54\times 10^{15} \tag{E.8}$$

This value of C is added to the diagonal elements k [1,1] and k [3,3]. In addition, the value of $u_3^* = 0.002\ C$ is added to the third element of the load vector so that $p(3) = 0.002\ (54\times 10^{15}) = 0.108\times 10^{15}$. Thus, the modified system equations are given by

$$\begin{bmatrix} (40\times 10^9 + 54\times 10^{15}) & -40\times 10^9 & 0 \\ -40\times 10^9 & 54\times 10^9 & -14\times 10^9 \\ 0 & -14\times 10^9 & (14\times 10^9 + 54\times 10^{15}) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10^8 \\ 0.108\times 10^{15} \end{Bmatrix}$$

or

$$\begin{bmatrix} (40 + 54\times 10^6) & -40 & 0 \\ -40 & 54 & -14 \\ 0 & -14 & (14 + 54\times 10^6) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.1 \\ 0.108\times 10^6 \end{Bmatrix} \tag{E.9}$$

EXAMPLE 6.10 —cont'd

The solution of Eq. (E.9) gives the nodal displacements:

$$\vec{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.0000 \\ 0.0024 \\ 0.0020 \end{Bmatrix} \text{ mm} \tag{E.10}$$

The stresses in the elements can be computed as

$$\sigma_1 = E_1 \varepsilon_1 = E_1 \left(\frac{\Delta l_1}{l_1} \right) = E_1 \left(\frac{u_2 - u_1}{l_1} \right) = (200 \times 10^9) \left(\frac{0.0024 - 0.0000}{0.5} \right)$$

$$= 0.96 \times 10^9 \text{ Pa} = 0.96 \text{ GPa}$$

$$\sigma_2 = E_2 \varepsilon_2 = E_2 \left(\frac{\Delta l_2}{l_2} \right) = E_2 \left(\frac{u_3 - u_2}{l_2} \right) = (70 \times 10^9) \left(\frac{0.0020 - 0.0024}{0.25} \right)$$

$$= -0.112 \times 10^9 \text{ Pa} = -0.112 \text{ GPa}$$

6.5 MULTIPOINT CONSTRAINTS: PENALTY METHOD

The boundary conditions of the type indicated by Eq. (6.18) are also known as single point constraints. A multipoint constraint involves satisfying a relationship among multiple displacement dof. Thus, a multipoint constraint can be stated as

$$\alpha_1 \Phi_1 + \alpha_2 \Phi_2 + \ldots + \alpha_m \Phi_m = \alpha_0 \tag{6.27}$$

where α_i, $i = 0, 1, 2, \ldots, m$ are known constants and Φ_i, $i = 1, 2, \ldots, m$ are the displacement dof. To illustrate the procedure of incorporating a multipoint constraint in the solution of the problem, we consider a simple form of Eq. (6.27) as

$$\alpha_p \Phi_p + \alpha_q \Phi_q = \alpha_0 \tag{6.28}$$

Let the original potential energy of the system, without a consideration of Eq. (6.28), be

$$I_0 = \frac{1}{2} \vec{\Phi}^T [K] \vec{\Phi} - \vec{\Phi}^T \vec{P} \tag{6.29}$$

When Eq. (6.28) is to be considered, a penalty term is added to Eq. (6.29) and the modified potential energy of the system is defined as

$$I_m = \frac{1}{2} \vec{\Phi}^T [K] \vec{\Phi} + \frac{1}{2} C (\alpha_p \Phi_p + \alpha_q \Phi_q - \alpha_0)^2 - \vec{\Phi}^T \vec{P} \tag{6.30}$$

where C is a large number. If $C = 0$, the penalty term will be zero and the constraint is ignored. As the value of C increases, the value of the penalty term increases and the value of I_m also increases. Since the potential energy (I_m) assumes a minimum value only when the constraint, Eq. (6.28) is satisfied, we minimize the modified potential energy, I_m. By using the conditions for the minimum of I_m, we obtain

$$\frac{\partial I_m}{\partial \Phi_i} = 0; \quad i = 1, 2 \ldots m \tag{6.31}$$

The elements of the original stiffness matrix and the original load vector get modified because of the consideration of Eq. (6.28) as

$$\begin{bmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{bmatrix} \Rightarrow \begin{bmatrix} K_{pp} + C\alpha_p^2 & K_{pq} + C\alpha_p\alpha_q \\ K_{qp} + C\alpha_p\alpha_q & K_{qq} + C\alpha_q^2 \end{bmatrix} \tag{6.32}$$

$$\begin{Bmatrix} P_p \\ P_q \end{Bmatrix} \Rightarrow \begin{Bmatrix} P_p + C\alpha_0\alpha_p \\ P_q + C\alpha_0\alpha_q \end{Bmatrix} \tag{6.33}$$

The equations for the equilibrium of forces along the directions of Φ_p and Φ_q are given by

$$\frac{\partial I_m}{\partial \Phi_p} = 0$$

or

$$\begin{aligned} P_p &= -\frac{\partial}{\partial \Phi_p}\left[\sum_{i=1}^{m} K_{pi}\Phi_i^2 + \frac{1}{2}C(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0)^2\right] \\ &= -\sum_{i=1}^{m} K_{pi}\Phi_i - C\alpha_p(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0) \end{aligned} \tag{6.34}$$

$$\frac{\partial I_m}{\partial \Phi_q} = 0$$

or

$$\begin{aligned} P_q &= -\frac{\partial}{\partial \Phi_q}\left[\sum_{i=1}^{m} K_{qi}\Phi_i^2 + \frac{1}{2}C(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0)^2\right] \\ &= -\sum_{i=1}^{m} K_{qi}\Phi_i - C\alpha_q(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0) \end{aligned} \tag{6.35}$$

where P_p and P_q denote the reactions along the dof Φ_p and Φ_q, respectively. If the constant C is chosen to be substantially larger than the elements of the stiffness matrix, the reactions can be found as

$$P_p \approx -C\alpha_p(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0) \tag{6.36}$$

$$P_q \approx -C\alpha_q(\alpha_p\Phi_p + \alpha_q\Phi_q - \alpha_0) \tag{6.37}$$

EXAMPLE 6.11

Two springs with stiffnesses $k_1 = 1000$ lb/in and $k_2 = 2000$ lb/in are connected to a rigid bar OA of negligible mass, and a load $P = 500$ lb is applied at point B as shown in Fig. 6.8A.

a. By modeling the system using two spring elements, identify the boundary conditions of the system.
b. Find the displacements of the springs, v_2 and v_4.

Solution

a. When the springs are modeled as linear spring elements, the element stiffness matrices are given by

$$[k^{(1)}] = k_1 \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{matrix} v_1 \\ v_2 \end{matrix} \tag{E.1}$$

EXAMPLE 6.11 —cont'd

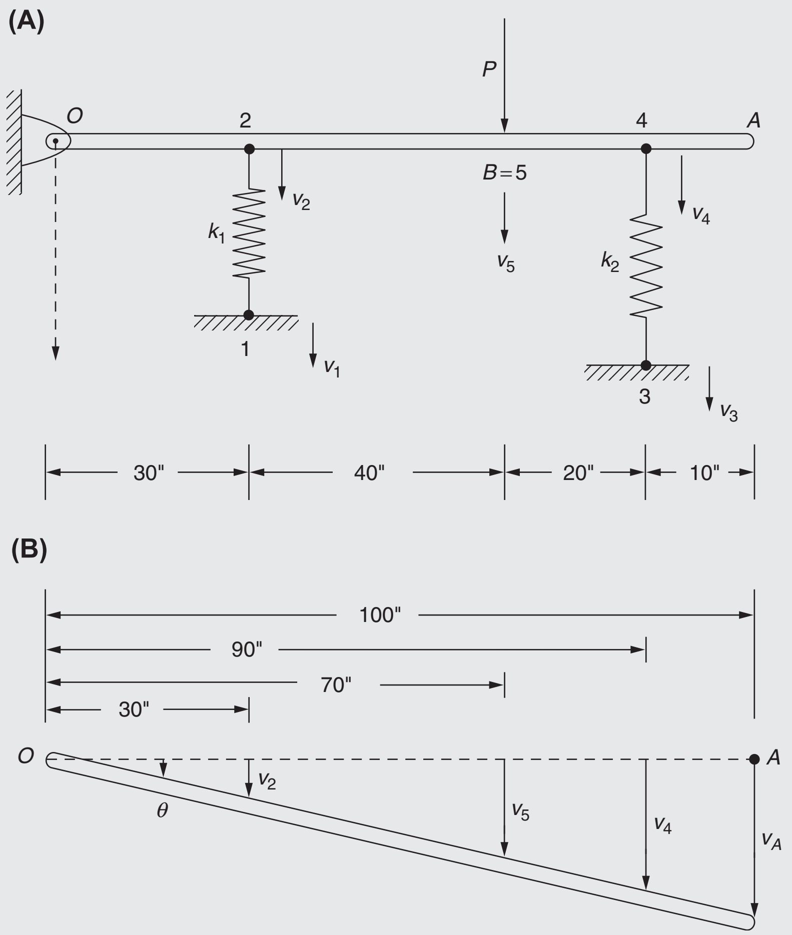

FIGURE 6.8 System involving a multipoint constraint.

$$[k^{(2)}] = k_2 \begin{array}{c} \begin{array}{cc} v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ v_3 \\ v_4 \end{array} \tag{E.2}$$

By introducing the displacement of the load-application point as an additional dof, v_5, the assembled stiffness matrix of the system can be expressed as

$$[K] = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 & 0 \\ 0 & 0 & -k_2 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} = \begin{bmatrix} 1000 & -1000 & 0 & 0 & 0 \\ -1000 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 2000 & -2000 & 0 \\ 0 & 0 & -2000 & 2000 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{E.3}$$

Continued

EXAMPLE 6.11 —cont'd

The system load vector is given by

$$\vec{P} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 500 \end{Bmatrix} \tag{E.4}$$

Boundary conditions of the system:
Since nodes 1 and 3 are fixed,

$$v_1 = 0, \; v_3 = 0 \tag{E.5}$$

Since the rigid bar remains straight (with rigid body rotation about the hinge point O), the nodal displacements are related as follows:

$$\tan\theta = \frac{v_2}{30} = \frac{v_5}{70} = \frac{v_4}{90}$$

or

$$\frac{v_2}{30} - \frac{v_5}{70} = 0, \quad \frac{v_4}{90} - \frac{v_5}{70} = 0$$

or

$$v_2 - \frac{3}{7}v_5 = 0, \quad v_4 - \frac{9}{7}v_5 = 0, \tag{E.6}$$

b. To incorporate the boundary conditions of Eq. (E.5), we multiply the stiffness elements $K(1, 1)$ and $K(3, 3)$ by a number $C = 10^6$, large in comparison to the values of the elements of the stiffness matrix, so that

$$\begin{bmatrix} K(1,1) & K(1,3) \\ K(3,1) & K(3,3) \end{bmatrix} \Rightarrow \begin{bmatrix} CK(1,1) & K(1,3) \\ K(3,1) & CK(3,3) \end{bmatrix} = \begin{array}{c} \begin{array}{cc} v_1 & \qquad\qquad v_3 \end{array} \\ \begin{bmatrix} 10^6(1000) & 0 \\ 0 & 10^6(2000) \end{bmatrix} \begin{array}{c} v_1 \\ v_3 \end{array} \end{array} \tag{E.7}$$

To apply the multipoint constraints of Eq. (E.6), we use Eq. (6.33). For the first constraint of (E.6), we note that $\alpha_2 = 1$, $\alpha_5 = -\frac{3}{7}$, and $\alpha_0 = 0$. This modifies the corresponding stiffness elements as

$$\begin{bmatrix} K(2,2) & K(2,5) \\ K(5,2) & K(5,5) \end{bmatrix} = \begin{bmatrix} 1000 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} K(2,2) + C\alpha_2^2 & K(2,5) + C\alpha_2\alpha_5 \\ K(5,2) + C\alpha_2\alpha_5 & K(5,5) + C\alpha_5^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1000 + 10^6(1^2) & 0 + 10^6(-\frac{3}{7})(1) \\ 0 + 10^6(-\frac{3}{7})(1) & 0 + 10^6(-\frac{3}{7})^2 \end{bmatrix} = 10^3 \begin{array}{c} \begin{array}{cc} v_2 & \qquad\qquad v_5 \end{array} \\ \begin{bmatrix} 1001.00 & -428.57 \\ -428.57 & 183.67 \end{bmatrix} \begin{array}{c} v_2 \\ v_5 \end{array} \end{array} \tag{E.8}$$

The corresponding elements of the load vector, $\begin{Bmatrix} P_2 \\ P_5 \end{Bmatrix}$, remain unaltered due to the first multipoint constraint of Eq. (E.6) because $\alpha_0 = 0$. Similarly, for the second constraint of Eq. (E.6), we note that $\alpha_4 = 1$, $\alpha_5 = -\frac{9}{7}$, and $\alpha_0 = 0$. This modifies the corresponding stiffness elements as

$$\begin{bmatrix} K(4,4) & K(4,5) \\ K(5,4) & K(5,5) \end{bmatrix} = \begin{bmatrix} 2000 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} K(4,4) + C\alpha_4^2 & K(4,5) + C\alpha_4\alpha_5 \\ K(5,4) + C\alpha_4\alpha_5 & K(5,5) + C\alpha_5^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 + 10^6(1^2) & 0 + 10^6(-\frac{9}{7})(1) \\ 0 + 10^6(-\frac{9}{7})(1) & 0 + 10^6(-\frac{9}{7})^2 \end{bmatrix} = 10^3 \begin{array}{c} \begin{array}{cc} v_4 & \qquad\qquad v_5 \end{array} \\ \begin{bmatrix} 1002.00 & -1285.71 \\ -1285.71 & 1653.06 \end{bmatrix} \begin{array}{c} v_4 \\ v_5 \end{array} \end{array} \tag{E.9}$$

EXAMPLE 6.11 —cont'd

The corresponding elements of the load vector, $\begin{Bmatrix} P_4 \\ P_5 \end{Bmatrix}$, remain unaltered due to the second multipoint constraint of Eq. (E.6) because $\alpha_0 = 0$. By incorporating the boundary conditions and the multipoint constraints as indicated in Eqs. (E.7) to (E.9), the final (modified) system equations can be expressed as

$$[K]\vec{V} = \vec{P} \tag{E.10}$$

where

$$[K] = \begin{bmatrix} (1000+10^6) & -1000 & 0 & 0 & 0 \\ -1000 & 1.001\times10^6 & 0 & 0 & -428.57\times10^3 \\ 0 & 0 & (2000+10^6) & -2000 & 0 \\ 0 & 0 & -2000 & 1.002\times10^6 & -1285.71\times10^3 \\ 0 & -428.57\times10^3 & 0 & -1285.71\times10^3 & (183.67\times10^3+1653.06\times10^3) \end{bmatrix} \tag{E.11}$$

$$\vec{V} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} \tag{E.12}$$

and

$$\vec{P} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 500.0 \end{Bmatrix} \tag{E.13}$$

The solution of Eq. (E.10) gives

$$\vec{V} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0.0001 \\ 0.0614 \\ 0.0004 \\ 0.1842 \\ 0.1435 \end{Bmatrix} \tag{E.14}$$

6.6 SYMMETRY CONDITIONS: PENALTY METHOD

To model a hexagonal pipe subjected to internal pressure shown in Fig. 6.9A, a 30° segment (one-twelfth of the cross section of the pipe) shown in Fig. 6.9B is used for finite element idealization. In order to maintain the symmetry of the pipe, the node points lying on the lines AB and CD should have zero displacement perpendicular to the lines AB and CD, respectively. For example, a typical point P lying on the line CD should have zero displacement along the line PN. If u_{2p-1} and u_{2p} denote the components of displacement of node P parallel to the x and y axes, respectively, the displacement of node P along the normal direction PN is constrained to be zero:

$$-u_{2p-1}\sin\theta + u_{2p}\cos\theta = 0 \tag{6.38}$$

Eq. (6.38) can be seen to be a multipoint constraint. To incorporate Eq. (6.38) using the penalty method, we formulate a modified potential energy function as

$$I_m = \frac{1}{2}\vec{\underset{\sim}{u}}^T\left[\underset{\sim}{k}\right]\vec{\underset{\sim}{u}} - \vec{\underset{\sim}{u}}^T\vec{\underset{\sim}{P}} + \frac{1}{2}C(-u_{2p-1}\sin\theta + u_{2p}\cos\theta)^2 \tag{6.39}$$

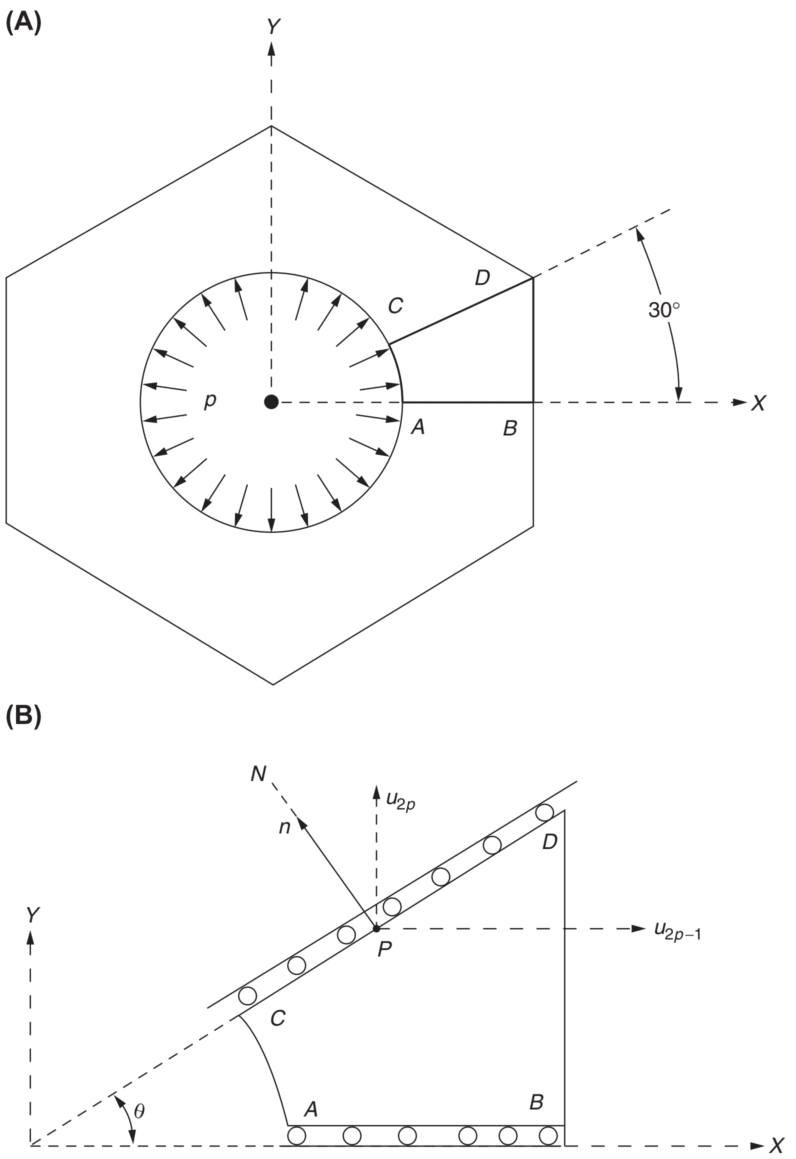

FIGURE 6.9 Symmetry conditions for hexagonal pipe.

6.6.1 Directional Constraint

Let a node p be constrained to move along a specified direction $\vec{d}$ in three-dimensional space as shown in Fig. 6.10. If the direction cosines of $\vec{d}$ are $l = \cos\left(\vec{d}, x\right), m = \cos\left(\vec{d}, y\right)$, and $n = \cos\left(\vec{d}, z\right)$, the plane perpendicular to the direction $\vec{d}$ is defined by the direction cosines (l, m, n). By using the penalty method, the elements of the stiffness matrix corresponding to the dof u_{3p-2}, u_{3p-1}, and u_{3p} will be modified by adding the following terms:

$$\begin{array}{cc} \begin{matrix} u_{3p-2} & u_{3p-1} & u_{3p} \end{matrix} & \\ \begin{bmatrix} Cl^2 & Clm & Cln \\ Clm & Cm^2 & Cmn \\ Cln & Cmn & Cn^2 \end{bmatrix} & \begin{matrix} u_{3p-2} \\ u_{3p-1} \\ u_{3p} \end{matrix} \end{array} \tag{6.40}$$

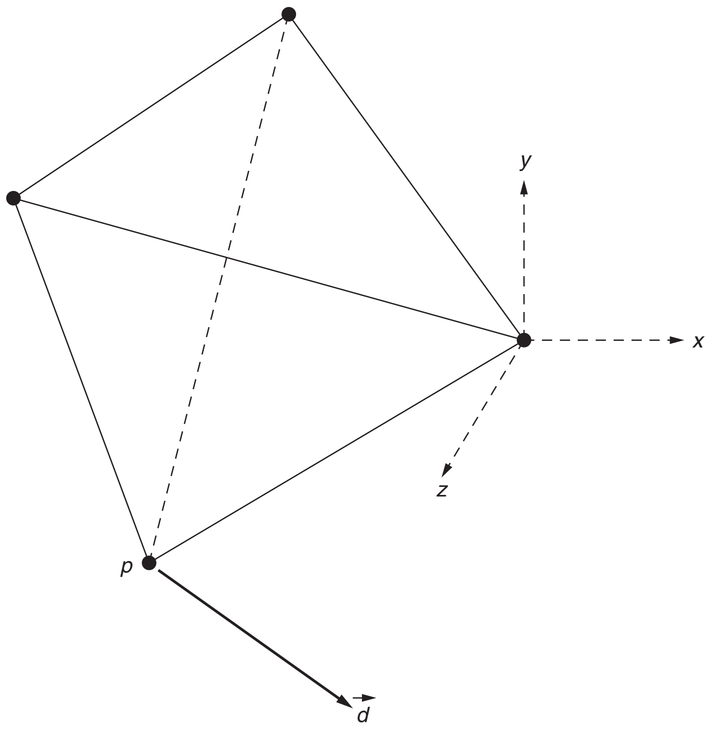

FIGURE 6.10 Directional constraint.

On the other hand, if a node p is constrained to move in a plane, defined by the direction cosines l, m, and n of its normal, the application of the penalty method leads to the addition of the following terms to the stiffness elements corresponding to the dof u_{3p-2}, u_{3p-1}, and u_{3p}:

$$\begin{array}{cc} \begin{array}{ccc} u_{3p-2} & u_{3p-1} & u_{3p} \end{array} & \\ \begin{bmatrix} C(1-l^2) & -Clm & -Cln \\ -Clm & C(1-m^2) & -Cmn \\ -Cln & -Cmn & C(1-n^2) \end{bmatrix} & \begin{array}{c} u_{3p-2} \\ u_{3p-1} \\ u_{3p} \end{array} \end{array} \tag{6.41}$$

6.7 RIGID ELEMENTS

Rigid elements are artificial or hypothetical elements that can be used to connect finite elements when (1) different elements have different types of dof and (2) the nodes of an element do not coincide with the nodes of another connecting element. The situation is shown in Fig. 6.11A in which a beam element, 4, is connected to a constant strain triangle (CST) element, 2, at node 6. The beam element 4 has the dof u_6, v_6, θ_6, u_7, v_7, and θ_7 as shown in Fig. 6.11B while the CST element 2 has the dof u_2, v_2, u_4, v_4, u_5, and v_5 as shown in Fig. 6.11C. Because node 6 does not coincide with any of the nodes 2, 4, and 5 of the CST element, the description of the complete set of dof associated with the nodes lying on element 2 should include u_4, v_4, u_5, v_5, u_6, v_6, and θ_6. It is possible to express the dof u_6, v_6, and θ_6 of node 6 in terms of the dof of the nodes 4 and 5 (u_4, v_4, u_5, v_5) lying on the edge 45 of the triangular element on which node 6 lies. The details of the procedure are given below.

Assuming that the u-component of displacement varies linearly between nodes 4 and 5 as

$$u(s) = c_1 + c_2 s, \quad c_1, c_2 = \text{constants} \tag{6.42}$$

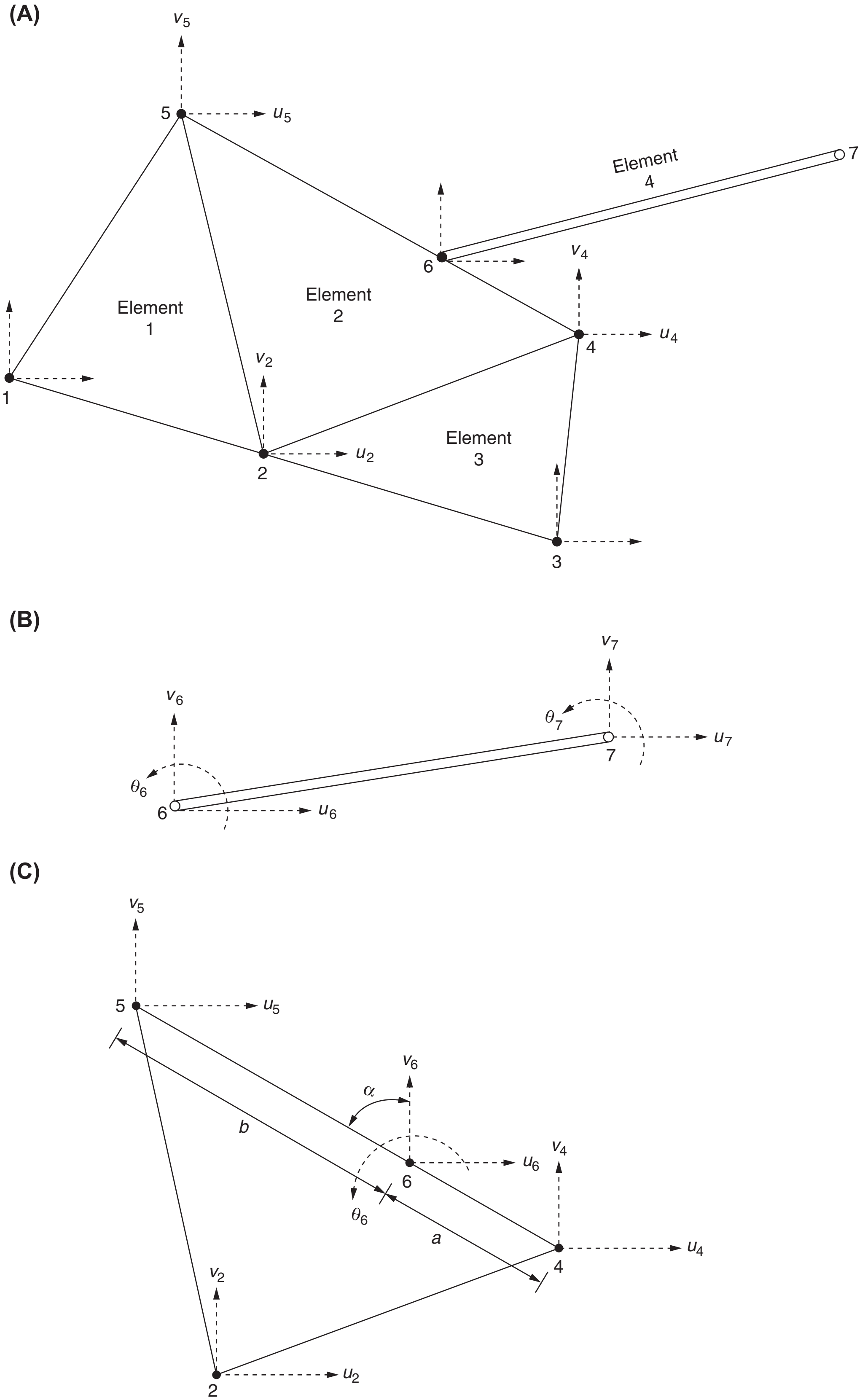

FIGURE 6.11 Connecting finite elements with different types of degrees of freedom.

where s is the distance from node 4 along the line 45, the u-component of displacement at node 6, u_6, can be expressed in terms of u_4 and u_5 as

$$u_6 = u(s = a) = \frac{b}{a+b}u_4 + \frac{a}{a+b}u_5 \tag{6.43}$$

where a and b are the distances of node 6 from nodes 4 and 5, respectively. Similarly, assuming a linear variation for the v-component of displacement between the nodes 4 and 5, the v-component of displacement at node 6, v_6, can be expressed in terms of v_4 and v_5 as

$$v_6 = v(s = a) = \frac{b}{a+b}v_4 + \frac{a}{a+b}v_5 \tag{6.44}$$

To relate the angular displacement θ_6 in terms of u_4, v_4, u_5, and v_5, we use the relation

$$\begin{aligned}\theta_6 &= (\text{moment due to } u_4 \text{ and } v_4 \text{ about } 6)/a + (\text{moment due to } u_5 \text{ and } v_5 \text{ about } 6)/b \\ &= \frac{(u_4 a \cos\alpha + v_4 a \sin\alpha)}{a} - \frac{(u_5 b \cos\alpha + v_5 b \sin\alpha)}{b} \\ &= u_4\cos\alpha + v_4 \sin\alpha - u_5 \cos\alpha - v_5 \sin\alpha\end{aligned} \tag{6.45}$$

Thus, the dof u_6, v_6, and θ_6 can be expressed in terms of u_4, v_4, u_5, and v_5 in matrix form as

$$\begin{Bmatrix} u_6 \\ v_6 \\ \theta_6 \end{Bmatrix} = \begin{bmatrix} \frac{b}{a+b} & 0 & \frac{a}{a+b} & 0 \\ 0 & \frac{b}{a+b} & 0 & \frac{a}{a+b} \\ \cos\alpha & \sin\alpha & -\cos\alpha & -\sin\alpha \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} \equiv [T_0] \begin{Bmatrix} u_4 \\ v_4 \\ u_5 \\ v_5 \end{Bmatrix} \tag{6.46}$$

One method of incorporating Eq. (6.47) is to treat the dof as multipoint constraints:

$$u_6 - \frac{b}{a+b}u_4 - \frac{a}{a+b}u_5 = 0 \tag{6.47}$$

$$v_6 - \frac{b}{a+b}v_4 - \frac{a}{a+b}v_5 = 0 \tag{6.48}$$

$$\theta_6 - u_4\cos\alpha - v_4\sin\alpha + u_5\cos\alpha + v_5\sin\alpha = 0 \tag{6.49}$$

Another method is to imagine the presence of a rigid link or rigid element that contains the dof to be related so that the transformation of the stiffness matrix $[k]$ of the rigid element can be expressed as

$$[k] = [T]^T [k_b][T] \tag{6.50}$$

where $[k_b]$ is the stiffness matrix of the beam element with dof $\{u_6, v_6, \theta_6, u_7, v_7, \theta_7\}^T$, $[k]$ is the stiffness matrix corresponding to the dof $\{ u_4, v_4, u_5, v_5, u_6, v_6, \theta_6\}^T$, and $[T]$ is the transformation matrix given by

$$[T] = \begin{bmatrix} \underset{3\times 4}{[T_0]} & \underset{3\times 3}{[0]} \\ \underset{3\times 4}{[0]} & \underset{3\times 3}{[I]} \end{bmatrix} \tag{6.51}$$

with

$$[T_0] = \begin{bmatrix} \dfrac{b}{a+b} & 0 & \dfrac{a}{a+b} & 0 \\ 0 & \dfrac{b}{a+b} & 0 & \dfrac{a}{a+b} \\ \cos\alpha & \sin\alpha & -\cos\alpha & -\sin\alpha \end{bmatrix} \tag{6.52}$$

and

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.53}$$

6.7.1 Stiffened Plates and Shells

Often, one-dimensional elements are used to stiffen plates and shells. Fig. 6.12 shows an axial load carrying (bar) element 2 attached to a quadrilateral element 1 undergoing translational deformation (with no bending). In this case, the dof of the bar element, in the (x, y) coordinate system, will be u_5, v_5, u_6, and v_6. The associated 4×4 stiffness matrix, $[k_b]$, can be transformed to the 8×8 stiffness matrix of the quadrilateral element $[k]$ with dof u_i, v_i, $i = 1, 2, 3, 4$ as

$$\underset{8\times8}{[k]} = \underset{8\times4}{[T]^T}\;\underset{4\times4}{[k_b]}\;\underset{4\times8}{[T]} \tag{6.54}$$

where the transformation matrix $[T]$ can be generated using equations similar to Eqs. (6.43) and (6.44). Eq. (6.54) can be considered to be associated with a *rigid element*.

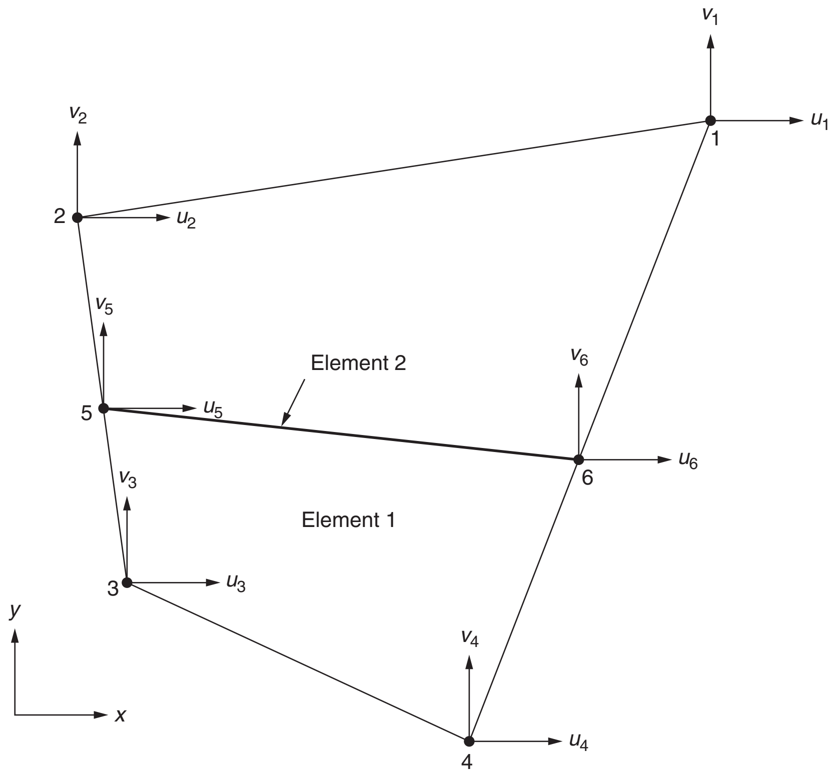

FIGURE 6.12 Stiffening element.

Notes

Rigid elements are available in many commercial finite element software packages. The user needs to create an element at the desired location and assign it as a *rigid element*. An alternate procedure is to treat the equations associated with the rigid element as multipoint constraint equations. When rigid elements are used, the analyst need not formulate the multipoint constraint equations.

Rigid elements can be used in other applications as in the case of spot welds used to connect two bodies (such as overlapping plates).

A limitation of the rigid elements is that they cannot expand or contract with temperature change.

REVIEW QUESTIONS

6.1 Give brief answers to the following questions.

1. What is the coordinate transformation matrix in two dimensions?
2. What is the relation between $\left[\lambda^{(e)}\right]$, $\left[\lambda^{(e)}\right]^T$, and $\left[\lambda^{(e)}\right]^{-1}$?
3. What is meant by compatibility at element nodes in the assembly of element matrices and element vectors?
4. What is a multipoint constraint?
5. What is the purpose of the penalty method?
6. What type of condition is to be satisfied to maintain symmetry on a surface in a two-dimensional problem?
7. Why is a local coordinate used to derive the element matrices?
8. Why is coordinate transformation required in finite element analysis?

6.2 Fill in the blank with a suitable word.

1. The number of dof to be specified at boundaries depends on the ________________ of the problem.
2. If the equations are derived using a variational method, the ______________ boundary conditions are implicitly satisfied.
3. To incorporate the dof Φ_i as zero, we can delete the ____________________ and ________________________in the global stiffness matrix of the system.
4. Rigid elements are used to connect two elements when the nodes of one element do not .... with the nodes of another connecting element.

6.3 Indicate whether each of the following statements is true or false.

1. Coordinate transformation of element matrices is not required when the field variable is temperature in a thermal problem.
2. Coordinate transformation of element matrices is not required when the field variable is displacement in a stress analysis problem.
3. In general, the local coordinates will have different directions for different elements.
4. The coordinate transformation matrix of element e, $\left[\lambda^{(e)}\right]$, is always a square matrix of direction cosines.
5. The coordinate transformation matrix is orthogonal.
6. Nodal unknowns are always displacements in a solid/structural mechanics problem.
7. The penalty method implements the boundary conditions exactly.
8. Rigid elements are hypothetical elements.
9. Rigid elements can expand or contract with temperature change.

6.4 Match the following.

1. Rigid element	(a) Multipoint constraint
2. Equation in terms of several dof	(b) Symmetry condition
3. Perpendicular displacement zero	(c) Stiffening of a plate
4. Implements all types of constraints	(d) Single-point constraint
5. Increase diagonal stiffness element and the right side elements	(e) Penalty method

PROBLEMS

6.1 Modify and solve the following system of equations using each of the methods described in Section 6.3 for the conditions $\Phi_1 = \Phi_2 = \Phi_3 = 2$, $\Phi_4 = 1$, $\Phi_8 = \Phi_9 = \Phi_{10} = 10$:

$$\begin{aligned}
&1.45\Phi_1 - 0.2\Phi_2 - 1.25\Phi_4 &&= 0\\
&-0.2\Phi_1 + 2.45\Phi_2 - 1.25\Phi_5 - \Phi_6 &&= 0\\
&\Phi_3 - 0.5\Phi_6 - 0.5\Phi_7 &&= 0\\
&-1.25\Phi_1 + 2.90\Phi_4 - 0.4\Phi_5 - 1.25\Phi_8 &&= 0\\
&-1.25\Phi_2 - 0.4\Phi_4 + 4.90\Phi_5 - \Phi_6 - 1.75\Phi_9 - 0.5\Phi_{10} &&= 0\\
&-\Phi_2 - 0.5\Phi_3 - \Phi_5 + 4\Phi_6 - \Phi_7 - 0.5\Phi_{10} &&= 0\\
&-0.5\Phi_3 - \Phi_6 + 2\Phi_7 - 0.5\Phi_{10} &&= 0\\
&-1.25\Phi_4 + 1.45\Phi_8 - 0.2\Phi_9 &&= 0\\
&-1.75\Phi_5 - 0.2\Phi_8 + 1.95\Phi_9 &&= 0\\
&-0.5\Phi_5 - 0.5\Phi_6 - 0.5\Phi_7 + 1.5\Phi_{10} &&= 0
\end{aligned}$$

6.2 Derive the coordinate transformation matrix for the one-dimensional element shown in Fig. 6.13, where q_i and Q_i denote, respectively, the local (x, y) and the global (X, Y) nodal displacements of the element.

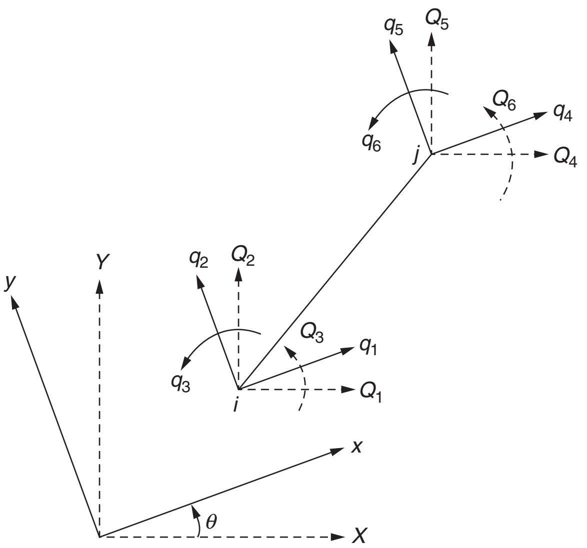

FIGURE 6.13 One-dimensional element in two coordinate systems.

6.3 If the element characteristic matrix of an element in the finite element grid shown in Fig. 6.14 is given by

$$\left[K^{(e)}\right] = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

find the overall or system characteristic matrix after applying the boundary conditions $\Phi_i = 0$, $i = 11$ to 15. Can the bandwidth be reduced by renumbering the nodes?

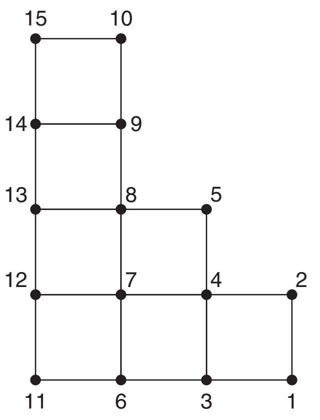

FIGURE 6.14 A finite element grid.

6.4 Incorporate the boundary conditions $\phi_1 = 3.0$ and $\phi_3 = -2.0$ using each of the methods described in Section 6.3 to the following system of equations:

$$\begin{bmatrix} 1.5 & -0.5 & 2.0 & 0.0 \\ -0.5 & 2.5 & -1.0 & -1.5 \\ 2.0 & -1.0 & 3.0 & 0.5 \\ 0.0 & -1.5 & 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} 3.0 \\ -1.0 \\ 1.5 \\ -0.5 \end{Bmatrix}$$

6.5 Consider a node that is supported by rollers as indicated in Fig. 6.15A. In this case, the displacement normal to the roller surface XY must be zero:

$$Q = Q_5 \cos\alpha + Q_6 \sin\alpha = 0 \tag{P.1}$$

where α denotes the angle between the normal direction to the roller surface and the horizontal (Fig. 6.15B). Constraints, in the form of linear equations, involving multiple variables are known as multipoint constraints. Indicate two methods of incorporating the boundary condition of Eq. (P.1) in the solution of equations.

6.6 As stated in Problem 6.5, the displacement normal to the roller support (surface) is zero. To incorporate the boundary condition, sometimes a stiff spring element is assumed perpendicular to the roller support surface as shown in Fig. 6.15C. In this case, the system will have four elements and eight dof. The boundary conditions $Q_1 = Q_2 = Q_7 = Q_8 = 0$ are incorporated in this method. Show the structure of the assembled equations for this case and discuss the advantages and disadvantages of the approach.

6.7 The stiffness matrix of a planar frame element in the local coordinate system is given as follows (see Fig. 6.16):

$$[k] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

where E is Young's modulus, A is the area of cross section, I is the moment of inertia, and L is the length. Using this, generate the stiffness matrices of the three elements shown in Fig. 6.17 in the local coordinate system and indicate the respective local dof.

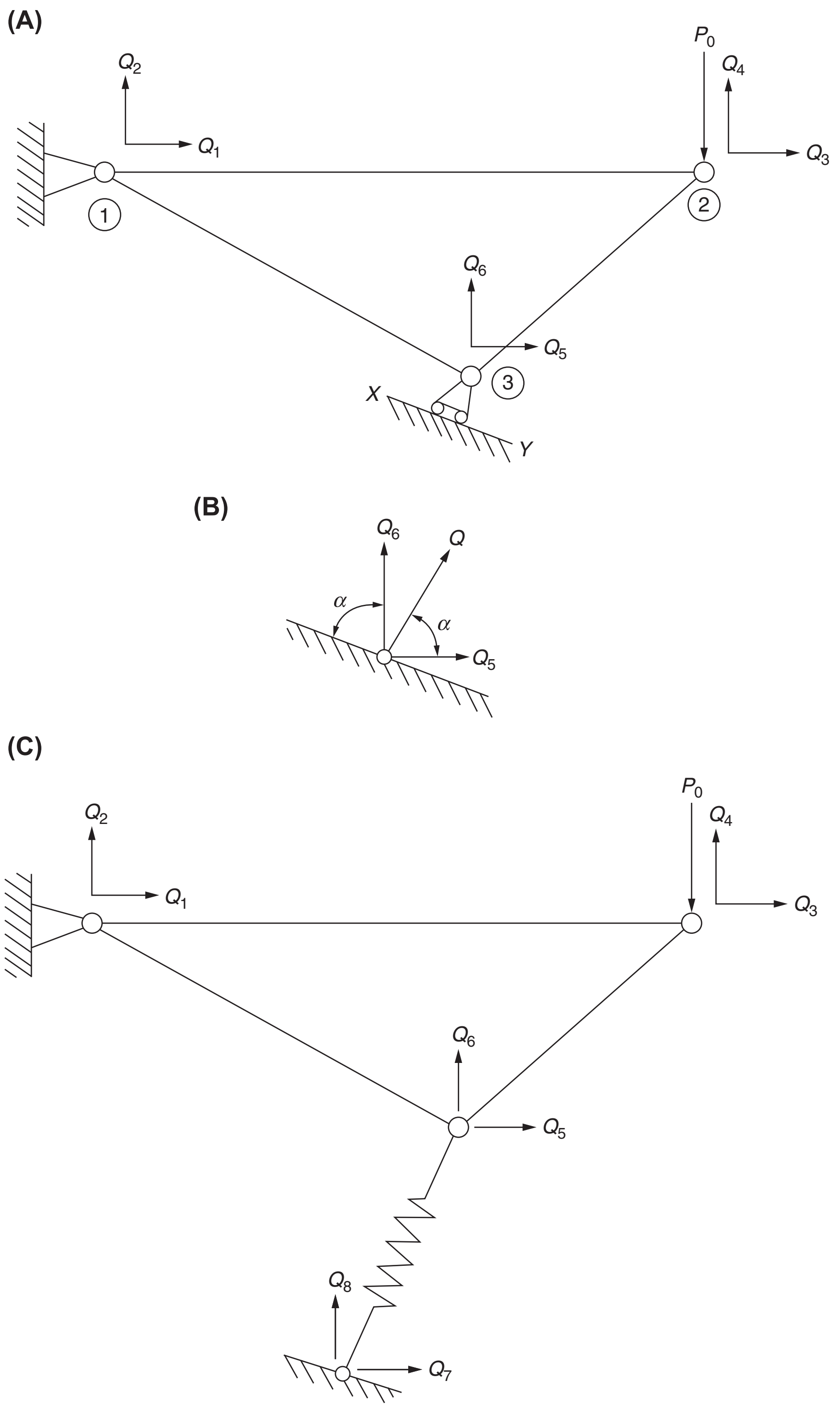

FIGURE 6.15 Node supported by a roller.

6.8 The transformation matrix between the local dof q_i and the global dof Q_i for the planar frame element shown in Fig. 6.16 is given by

$$[\lambda] = \begin{bmatrix} l_{ox} & m_{ox} & 0 & 0 & 0 & 0 \\ l_{oz} & m_{oz} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ox} & m_{ox} & 0 \\ 0 & 0 & 0 & l_{oz} & m_{oz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $l_{ox} = \cos\theta$, $m_{ox} = \sin\theta$, $l_{oz} = \cos(90 + \theta) = -\sin\theta$, and $m_{oz} = \sin(90 + \theta) = \cos\theta$. Using this, generate the transformation matrices for the three elements shown in Fig. 6.17.

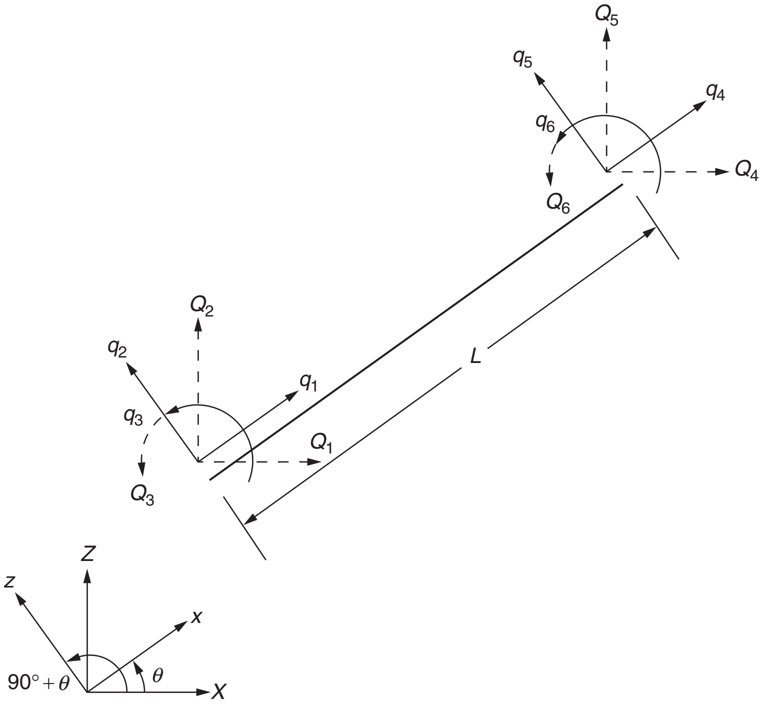

FIGURE 6.16 Planar frame element.

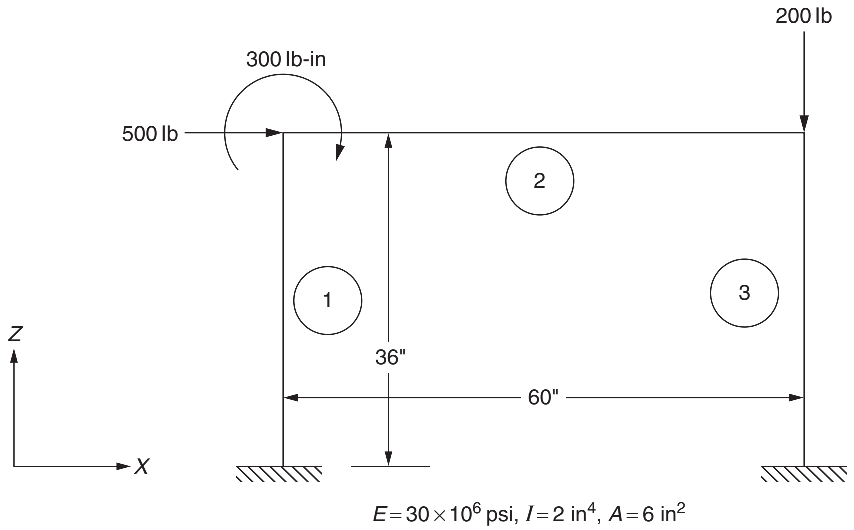

FIGURE 6.17 A planar frame.

6.9 Consider the coordinate transformation matrix, $[\lambda]$, of element 1 of Fig. 6.17 in Problem 6.7. Show that it is orthogonal—that is, show that $[\lambda]^{-1} = [\lambda]^T$.

6.10 Consider the coordinate transformation matrix, $[\lambda]$, of element 2 of Fig. 6.17 in Problem 6.7. Show that it is orthogonal—that is, show that $[\lambda]^{-1} = [\lambda]^T$.

6.11 Consider the coordinate transformation matrix, $[\lambda]$, of element 3 of Fig. 6.17 in Problem 6.7. Show that it is orthogonal—that is, show that $[\lambda]^{-1} = [\lambda]^T$.

6.12 Using the results of Problems 6.7 and 6.8, generate the stiffness matrices of the three elements shown in Fig. 6.17 in the global coordinate system. Derive the assembled stiffness matrix of the system.

6.13 For the assembled stiffness matrix derived in Problem 6.12, apply the boundary conditions, derive the final equilibrium equations, and solve the resulting equations.

6.14 The local stiffness matrix, $[k]$, and the corresponding coordinate transformation matrix, $[\lambda]$, of a planar truss element (see Fig. 6.18A) are given by

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad [\lambda] = \begin{bmatrix} l_{ox} & m_{ox} & 0 & 0 \\ 0 & 0 & l_{ox} & m_{ox} \end{bmatrix}$$

where A is the cross-sectional area, E is Young's modulus, L is the length, $l_{ox} = \cos\theta$, and $m_{ox} = \sin\theta$.

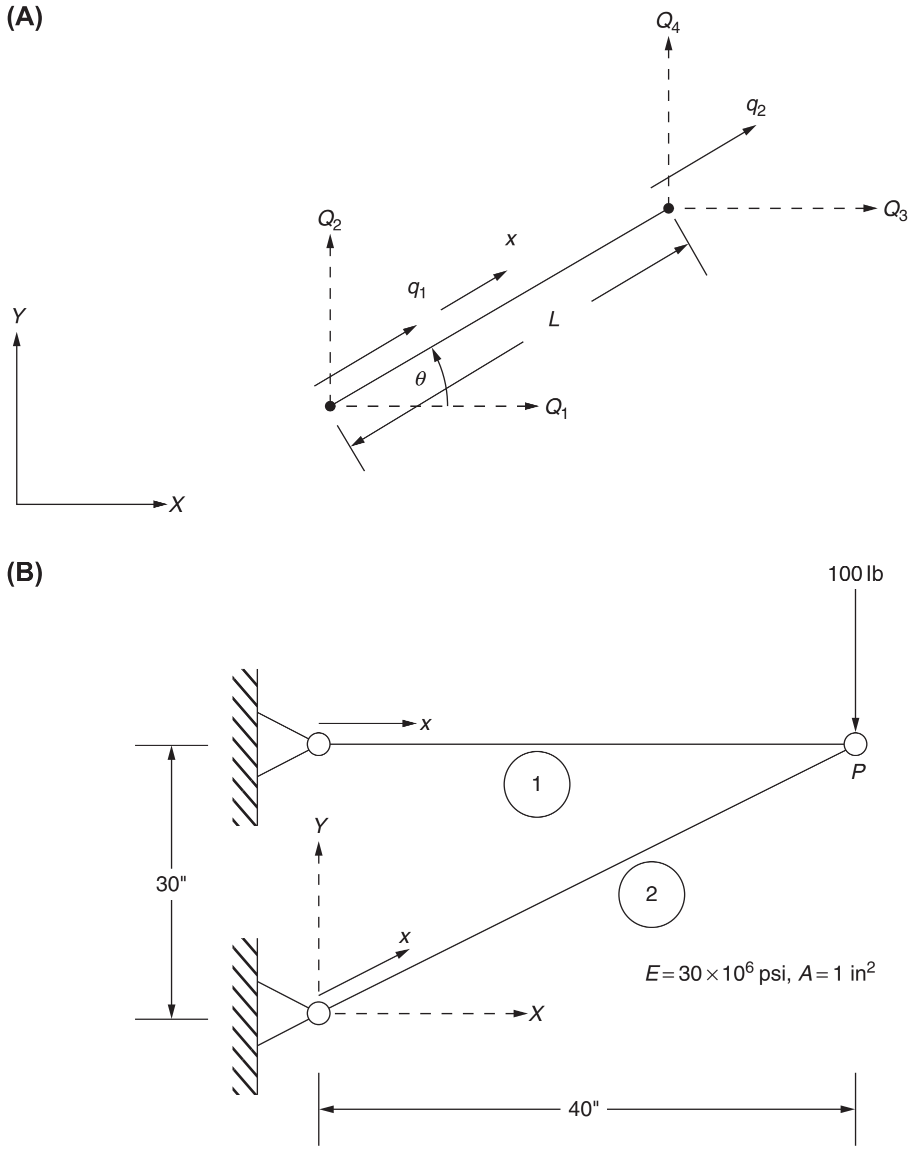

FIGURE 6.18 Planar truss.

a. Generate the global stiffness matrices of the two elements shown in Fig. 6.18B.
b. Find the assembled stiffness matrix, apply the boundary conditions, and find the displacement of node P of the two-bar truss shown in Fig. 6.18B.

6.15 To incorporate a specified displacement boundary condition, the *penalty method* can be used. To indicate the step-by-step approach used in the penalty approach, let the boundary condition to be enforced be of the form $W_i = a =$ specified value of the displacement W_i, and let the system equations be of the form $[K]\,\overrightarrow{W} = \overrightarrow{P}$.
a. Set the value of P_i in the load vector $\overrightarrow{P}$ as zero.
b. Modify the stiffness matrix $[K]$ by adding a large number C to the diagonal coefficient K_{ii}. Usually, the value of C is taken as 10^4 times the maximum stiffness coefficient in the matrix $[K]$; that is, $C =$ maximum $(K_{ij}) \times 10^4$.
c. Modify the load vector $\overrightarrow{P}$ by adding the number Ca to P_i.
d. Solve the resulting (modified) system of equations, $[K]\overrightarrow{W} = \overrightarrow{P}$ to find nodal displacement vector of the system, $\overrightarrow{W}$. This will yield a value of $W_i = a$ (approximately).
e. The reaction P_i corresponding to the specified displacement $W_i = a$ can be found as

$$P_i = -CW_i = -Ca$$

Using the penalty method, solve the equilibrium equations of Example 1.2:

$$10^6 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 = 0(\textit{set to zero}) \\ 0 \\ 1 \end{Bmatrix}$$

by enforcing the boundary condition, $\Phi_1 = 0$. Also find the reaction at the fixed end, P_1.

6.16 The finite element analysis of a fixed-fixed beam using two elements results in the following system equation:

$$[\underset{\sim}{K}]\,\overrightarrow{\underset{\sim}{W}} = \overrightarrow{\underset{\sim}{P}}$$

where

$$[\underset{\sim}{K}] = 10^4 \begin{array}{c} \begin{bmatrix} 1 & 10 & -1 & 10 & 0 & 0 \\ 10 & 133.3333 & -10 & 66.6667 & 0 & 0 \\ -1 & -10 & 1.125 & -7.5 & -0.125 & 2.5 \\ 10 & 66.6667 & -7.5 & 20.0 & -2.5 & 33.3333 \\ 0 & 0 & -0.125 & -2.5 & 0.125 & -2.5 \\ 0 & 0 & 2.5 & 33.3333 & -2.5 & 66.6667 \end{bmatrix} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix} \\ \begin{matrix} W_1 & \quad W_2 & \quad W_3 & \quad W_4 & \quad W_5 & \quad W_6 \end{matrix} \end{array}$$

and

$$\overrightarrow{\underset{\sim}{W}} = \begin{Bmatrix} W_1 = 0 \\ W_2 = 0 \\ W_3 \\ W_4 \\ W_5 = 0 \\ W_6 = 0 \end{Bmatrix}, \quad \overrightarrow{\underset{\sim}{P}} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 = -1000 \\ P_4 = 0 \\ P_5 \\ P_6 \end{Bmatrix}$$

The end nodes 1 and 3 of the finite element model are fixed (with dof $W_i = 0$, $i = 1, 2, 5, 6$) while the middle node (with dof W_3 and W_4) is subjected to loads of -1000 lb and 0 lb-in along the directions of W_3 and W_4, respectively. Find the values of the displacement components at node 2 and reactions at the fixed nodes 1 and 3 using Method 1 of Section 6.3.

6.17 Find the displacement solution of the fixed-fixed beam described in Problem 6.16 using Method 2 of Section 6.3.

6.18 Find the displacement solution of the fixed-fixed beam problem described in Problem 6.16 using Method 3 of Section 6.3.

6.19 The finite element analysis of a planar truss, with six elements or bars and four nodes, resulted in the following equilibrium equations:

$$\begin{bmatrix} 5 & 1 & 0 & 0 & -4 & 0 & -1 & -1 \\ 1 & 5 & 0 & -4 & 0 & 0 & -1 & -1 \\ 0 & 0 & 5 & -1 & -1 & 1 & -4 & 0 \\ 0 & -4 & -1 & 5 & 1 & -1 & 0 & 0 \\ -4 & 0 & -1 & 1 & 5 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 5 & 0 & -4 \\ -1 & -1 & -4 & 0 & 0 & 0 & 5 & 1 \\ -1 & -1 & 0 & 0 & 0 & -4 & 1 & 5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{Bmatrix}$$

The dof U_i, $i = 5, 6, 7, 8$ are zero (fixed) and loads of magnitude 0, 1000, 0, 0 lb are applied along the dof U_1, U_2, U_3, U_4, respectively. Determine the unknown nodal displacements and the reactions at the fixed nodes of the truss using Method 1 of Section 6.3.

6.20 Find the nodal displacements of the truss described in Problem 6.19 using Method 2 of Section 6.3.

6.21 Find the nodal displacements of the truss described in Problem 6.19 using Method 3 of Section 6.3.

6.22. The heat transfer analysis of a plate fin using four triangular simplex elements and six nodes lead to the following thermal equilibrium equations:

$$\begin{bmatrix} 29.606 & -8.5475 & -13.521 & 0 & 0 & 0 \\ -8.5475 & 27.0935 & 0 & -14.777 & 0 & 0 \\ -13.521 & 0 & 114.679 & -27.314 & -61.009 & 0 \\ 0 & -14.777 & -27.314 & 110.4005 & 0 & -61.892 \\ 0 & 0 & -61.009 & 0 & 134.778 & -68.471 \\ 0 & 0 & 0 & -61.892 & -68.471 & 133.012 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 680.9 \\ 1060.65 \\ 1233.05 \\ 656.0 \\ 127.15 \\ 95.35 \end{Bmatrix}$$

If each of the nodal temperatures T_5 and T_6 is specified as 50 °C, determine the other nodal temperatures using Method 1 of Section 6.3.

6.23 Find the nodal temperatures of the heat transfer problem stated in Problem 6.22 using Method 2 of Section 6.3.

6.24 Find the nodal temperatures of the heat transfer problem stated in Problem 6.22 using Method 3 of Section 6.3.

6.25 Consider the following equilibrium equations of a structure:

$$[\underset{\sim}{k}]\ \underset{\sim}{\vec{u}} = \underset{\sim}{\vec{p}} \tag{P.2}$$

where

$$\underset{\sim}{\vec{u}} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}, \quad \underset{\sim}{\vec{p}} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ -0.866 \\ 0 \\ 0 \\ P_5 \\ 0 \\ P_7 \\ P_8 \end{Bmatrix}$$

and

$$[k] = \begin{bmatrix} 57.5 & 0 & 0 & 0 & -57.5 & 0 & 0 & 0 \\ 0 & 491.0 & 0 & -491.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 225.0 & 71.5 & -175.5 & -88.0 & -49.0 & 16.5 \\ 0 & -491.0 & 71.5 & 540.0 & -88.0 & -44.0 & 165.0 & -5.5 \\ -57.5 & 0 & -175.5 & -88.0 & 233.0 & 86.0 & 0 & 0 \\ 0 & 0 & -88.0 & -44.0 & 88.0 & 338.5 & 0 & -294.5 \\ 0 & 0 & -49.0 & 16.5 & 0 & 0 & 49.0 & -16.5 \\ 0 & 0 & 16.5 & -5.5 & 0 & -294.5 & -16.5 & 300.0 \end{bmatrix}$$

If the value of each of the displacement variables u_5, u_7, and u_8 is known to be zero, determine the values of the remaining displacement components and the reactions along the fixed dof using Method 1 of Section 6.3.

6.26 Find the displacements of the structural problem stated in Problem 6.25 using Method 2 of Section 6.3.

6.27 Find the displacements of the structural problem stated in Problem 6.25 using Method 3 of Section 6.3.

6.28 Consider a bar element in three-dimensional space with nodal coordinates as indicated in Fig. 6.19. The displacement components of the ends in the local (x) and global (X, Y, Z) coordinate systems are also indicated in Fig. 6.19. Derive the coordinate transformation matrix of the element.

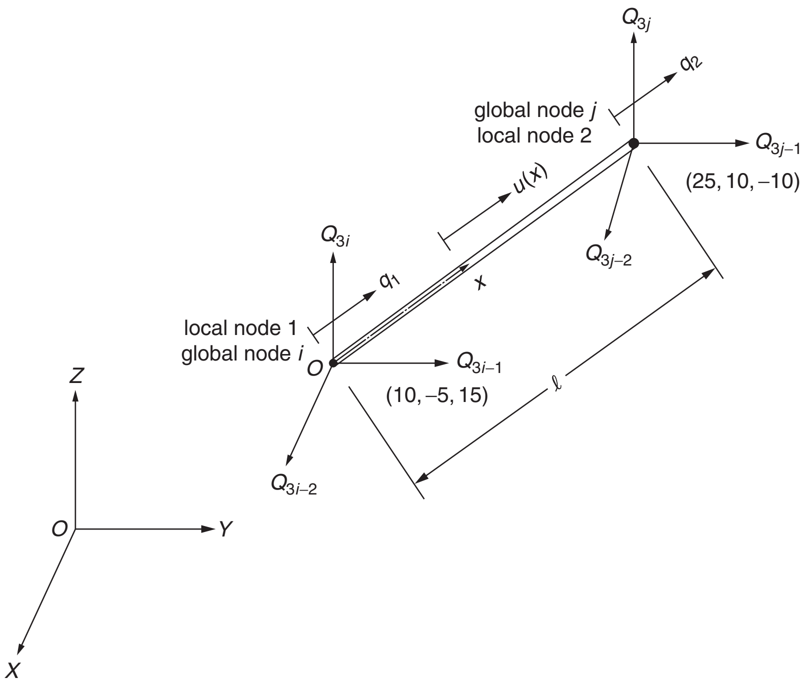

FIGURE 6.19 A space truss element.

6.29 For the space frame element shown in Fig. 6.20, indicate the procedure for deriving the coordinate transformation matrix between the local displacement components q_j, and the global displacement components Q_i.

6.30 For the planar four-bar truss shown in Fig. 6.3A, the global (X, Y)-coordinates of nodes 1, 2, 3, and 4 are given by (0, 0), (200, 0), (100, 100), and (400, 300) in, respectively. Determine the coordinate transformation matrices of the four elements.

6.31 In Problem 6.30, if the cross-sectional areas of elements 1, 2, 3, and 4 are given by 2, 2, 3, and 3 in^2, respectively, and Young's modulus of all the elements is 30×10^6 psi, find the global stiffness matrices of the four elements.

6.32 Find the global stiffness matrix of the truss using the global stiffness matrices of the four elements determined in Problem 6.31.

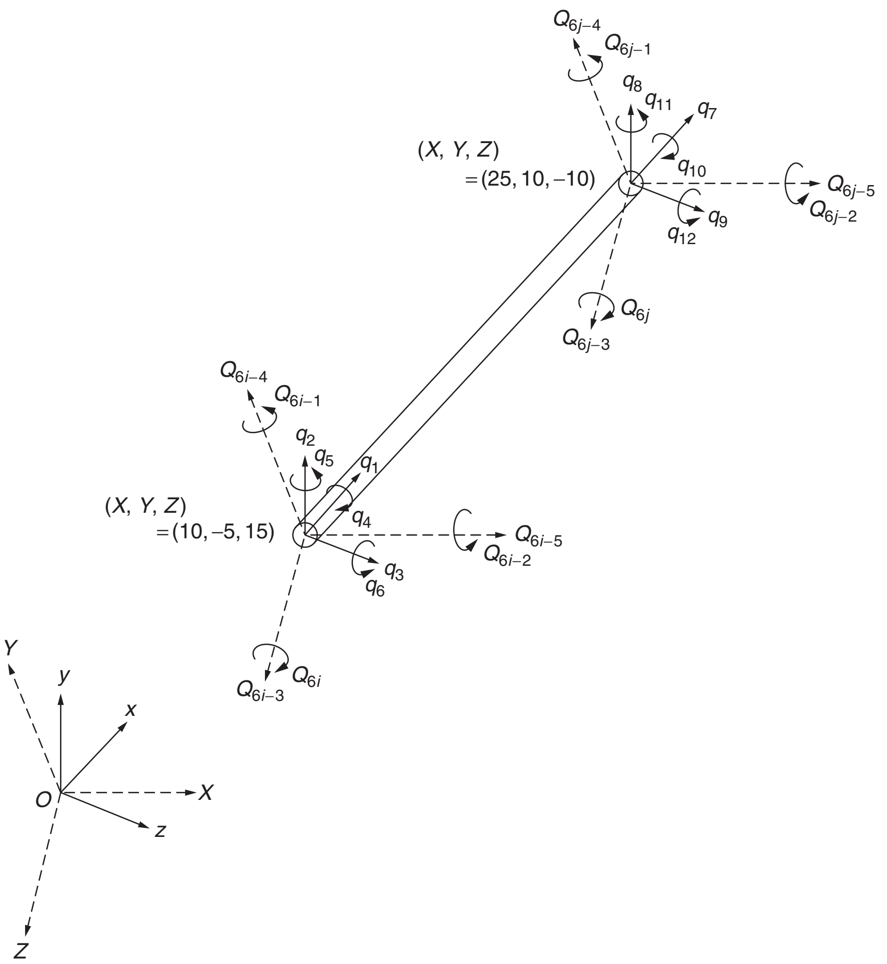

FIGURE 6.20 A space frame element.

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Chapter 7

Numerical Solution of Finite Element Equations

Chapter Outline

7.1 INTRODUCTION

Most problems in engineering mechanics can be stated as continuous or discrete problems. Continuous problems involve an infinite number of degrees of freedom, whereas discrete problems involve a finite number of degrees of freedom. All discrete and continuous problems can be classified as equilibrium (static), eigenvalue, and propagation (transient) problems. The finite element method is applicable for the solution of all three categories. As stated in Chapter 1, the finite element method is a numerical procedure that replaces a continuous problem by an equivalent discrete one. It will be quite convenient to use matrix notation in formulating and solving problems using the finite element procedure. When matrix notation is used in finite element analysis, the organizational properties of matrices allow for systematic compilation of the required data and the finite element analysis can then be defined as a sequence of matrix operations that can be programmed directly for a digital computer.

The governing finite element equations for various types of field problems can be expressed in matrix form as follows:

1. Equilibrium problems

$$[A]\overrightarrow{X} = \overrightarrow{b} \tag{7.1a}$$

subject to the boundary conditions

$$[B]\overrightarrow{X} = \overrightarrow{g} \tag{7.1b}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00007-9

2. Eigenvalue problems

$$[A]\vec{X} = \lambda[B]\vec{X} \tag{7.2a}$$

subject to the boundary conditions

$$[C]\vec{X} = \vec{g} \tag{7.2b}$$

3. Propagation problems

$$[A]\frac{d^2\vec{X}}{dt^2} + [B]\frac{d\vec{X}}{dt} + [C]\vec{X} = \vec{F}\left(\vec{X}, t\right), \quad t > 0 \tag{7.3a}$$

subject to the boundary conditions

$$[D]\vec{X} = \vec{g}, \quad t \geq 0 \tag{7.3b}$$

and the initial conditions

$$\vec{X} = \vec{X}_0, \quad t = 0 \tag{7.3c}$$

$$\frac{d\vec{X}}{dt} = \vec{Y}_0, \quad t = 0 \tag{7.3d}$$

where $[A]$, $[B]$, $[C]$, and $[D]$ are square matrices whose elements are known; $\vec{X}$ is the vector of unknowns (or field variables) in the problem $\vec{b}$, $\vec{g}$, $\vec{X}_0$, and $\vec{Y}_0$ are vectors of known constants; λ is the eigenvalue; t is the time parameter; and $\vec{F}$ is a vector whose elements are known functions of $\vec{X}$ and t.

This chapter provides an introduction to matrix techniques that are useful for the solution of Eqs. (7.1) to (7.3).

7.2 SOLUTION OF EQUILIBRIUM PROBLEMS

When the finite element method is used for the solution of equilibrium, steady-state, or static problems, we get a set of simultaneous linear equations that can be stated in the form of Eq. (7.1). We shall consider the solution of Eq. (7.1a) in this section by assuming that the boundary conditions of Eq. (7.1b) have been incorporated already.

Eq. (7.1a) can be expressed in scalar form as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{7.4}$$

where the coefficients a_{ij} and the constants b_i are either given or can be generated. The problem is to find the values of x_i ($i = 1, 2, \ldots, n$), if they exist, which satisfy Eq. (7.4). A comparison of Eqs. (7.1a) and (7.4) shows that

$$\underset{n\times n}{[A]} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}, \quad \underset{n\times 1}{\vec{X}} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \underset{n\times 1}{\vec{b}} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

In finite element analysis, the order of the matrix $[A]$ will be very large. The solution of some of the practical problems involves matrices of order 10,000 or more.

The methods available for solving systems of linear equations can be divided into two types: direct and iterative. Direct methods are those that, in the absence of round-off and other errors, will yield the exact solution in a finite number of elementary arithmetic operations. In practice, because a computer works with a finite word length, sometimes the direct methods do not give good solutions. Indeed, the errors arising from round-off and truncation may lead to extremely poor or even useless results. Hence, many researchers working in the field of finite element method are concerned with why and how the errors arise and with the search for methods that minimize the totality of such errors. The fundamental method used for direct solutions is Gaussian elimination, but even within this class there are a variety of choices of methods that vary in computational efficiency and accuracy.

Iterative methods are those that start with an initial approximation and that by applying a suitably chosen algorithm lead to successively better approximations. When the process converges, we can expect to get a good approximate solution to Eq. (7.1a). The accuracy and the rate of convergence of iterative methods vary with the algorithm chosen. The main advantages of iterative methods are the simplicity and uniformity of the operations to be performed, which make them well suited for use on digital computers, and their relative insensitivity to the growth of round-off errors.

Matrices associated with linear systems are also classified as dense or sparse. Dense matrices have very few zero elements, whereas sparse matrices have very few nonzero elements. Fortunately, in most finite element applications, the matrices involved are sparse (thinly populated) and symmetric. Hence, solution techniques that take advantage of the special character of such systems of equations have been developed.

7.2.1 Gaussian Elimination Method

The basic procedure available for the solution of Eq. (7.1) is the Gaussian elimination method, in which the given system of equations is transformed into an equivalent triangular system for which the solution can be easily found. The following example illustrates the procedure.

EXAMPLE 7.1

Solve the following system of equations using the Gaussian elimination method:

$$x_1 - x_2 + 3x_3 = 10 \tag{E.1}$$

$$2x_1 + 3x_2 + x_3 = 15 \tag{E.2}$$

$$4x_1 + 2x_2 - x_3 = 6 \tag{E.3}$$

Solution

To eliminate the x_1 terms from Eqs. (E.2) and (E.3), we multiply Eq. (E.1) by -2 and -4 and add, respectively, to Eqs. (E.2) and (E.3) leaving the first equation unchanged. We will then have

$$x_1 - x_2 + 3x_3 = 10 \tag{E.4}$$

$$5x_2 - 5x_3 = -5 \tag{E.5}$$

$$6x_2 - 13x_3 = -34 \tag{E.6}$$

To eliminate the x_2 term from Eq. (E.6), multiply Eq. (E.5) by $-6/5$ and add to Eq. (E.6). We will now have the triangular system

$$x_1 - x_2 + 3x_3 = 10 \tag{E.7}$$

$$5x_2 - 5x_3 = -5 \tag{E.8}$$

$$-7x_3 = -28 \tag{E.9}$$

This triangular system can now be solved by back substitution. From Eq. (E.9) we find $x_3 = 4$. Substituting this value for x_3 into Eq. (E.8) and solving for x_2, we obtain $x_2 = 3$. Finally, knowing x_3 and x_2, we can solve Eq. (E.1) for x_1, obtaining $x_1 = 1$. This solution can also be obtained by adopting the following equivalent procedure.

Eq. (E.1) can be solved for x_1 to obtain

$$x_1 = 10 + x_2 - 3x_3 \tag{E.10}$$

Substitution of this expression for x_1 into Eqs. (E.2) and (E.3) gives

$$5x_2 - 5x_3 = -5 \tag{E.11}$$

$$6x_2 - 13x_3 = -34 \tag{E.12}$$

The solution of Eq. (E.11) for x_2 leads to

$$x_2 = -1 + x_3 \tag{E.13}$$

By substituting Eq. (E.13) into Eq. (E.12) we obtain

$$-7x_3 = -28 \tag{E.14}$$

It can be seen that Eqs. (E.10), (E.11), and (E.14) are the same as Eqs. (E.7–E.9), respectively. Hence, we can obtain $x_3 = 4$ from Eq. (E.14), $x_2 = 3$ from Eq. (E.13), and $x_1 = 1$ from Eq. (E.10).

GENERALIZATION OF THE METHOD

Let the given system of equations be written as

$$\begin{aligned} a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 + \cdots + a_{1n}^{(0)}x_n &= b_1^{(0)} \\ a_{21}^{(0)}x_1 + a_{22}^{(0)}x_2 + \cdots + a_{2n}^{(0)}x_n &= b_2^{(0)} \\ &\vdots \\ a_{n1}^{(0)}x_1 + a_{n2}^{(0)}x_2 + \cdots + a_{nn}^{(0)}x_n &= b_n^{(0)} \end{aligned} \tag{7.5}$$

where the superscript (0) has been used to denote the original values. By solving the first equation of Eq. (7.5) for x_1, we obtain

$$x_1 = \frac{b_1^{(0)}}{a_{11}^{(0)}} - \frac{a_{12}^{(0)}}{a_{11}^{(0)}}x_2 - \frac{a_{13}^{(0)}}{a_{11}^{(0)}}x_3 - \cdots - \frac{a_{1n}^{(0)}}{a_{11}^{(0)}}x_n$$

Substitution of this x_1 into the remaining equations of Eq. (7.5) leads to

$$\begin{aligned} a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ &\vdots \\ a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \cdots + a_{nn}^{(1)}x_n &= b_n^{(1)} \end{aligned} \tag{7.6}$$

where

$$\left.\begin{aligned} a_{ij}^{(1)} &= a_{ij}^{(0)} - \left[a_{i1}^{(0)}a_{1j}^{(0)} \middle/ a_{11}^{(0)}\right] \\ b_i^{(1)} &= b_i^{(0)} - \left[a_{i1}^{(0)}b_1^{(0)} \middle/ a_{11}^{(0)}\right] \end{aligned}\right\} \quad i,j = 2,3,\ldots,n$$

Next, we eliminate x_2 from Eq. (7.6), and so on. In general, when x_k is eliminated we obtain

$$x_k = \frac{b_k^{(k-1)}}{a_{kk}^{(k-1)}} - \sum_{j=k+1}^{n} \frac{a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}}x_j \tag{7.7}$$

where

$$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - \left[a_{ik}^{(k-1)}a_{kj}^{(k-1)} \middle/ a_{kk}^{(k-1)}\right] \\ b_i^{(k)} &= b_i^{(k-1)} - \left[a_{ik}^{(k-1)}b_k^{(k-1)} \middle/ a_{kk}^{(k-1)}\right] \end{aligned}\right\} \quad i,j = k+1,\ldots,n$$

After applying the previous procedure $n-1$ times, the original system of equations reduces to the following single equation:

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

from which we can obtain

$$x_n = \left[b_n^{(n-1)} \middle/ a_{nn}^{(n-1)}\right]$$

The values of the remaining unknowns can be found in reverse order (x_{n-1}, x_{n-2}, …, x_1) by using Eq. (7.7).

Note

In the elimination process, if at any stage one of the pivot (diagonal) elements $a_{11}^{(0)}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots$ vanishes, we attempt to rearrange the remaining rows so as to obtain a nonvanishing pivot. If this is impossible, then the matrix [A] is singular and the system has no solution.

7.2.2 Choleski Method

The Choleski method is a direct method for solving a linear system that makes use of the fact that any square matrix $[A]$ can be expressed as the product of an upper and a lower triangular matrix. The method of expressing any square matrix as a product of two triangular matrices and the subsequent solution procedure are given next.

DECOMPOSITION OF $[A]$ INTO LOWER AND UPPER TRIANGULAR MATRICES

The given system of equations is

$$[A]\vec{X} = \vec{b} \tag{7.1a}$$

The matrix $[A]$ can be written as

$$[A] = [a_{ij}] = [L][U] \tag{7.8}$$

where $[L] = [l_{ij}]$ is a lower triangular matrix, and $[U] = [u_{ij}]$ is a unit upper triangular matrix, with

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [L][U] \tag{7.9}$$

$$[L] = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} = \text{a lower triangular matrix} \tag{7.10}$$

and

$$[U] = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \text{a unit upper triangular matrix} \tag{7.11}$$

The elements of $[L]$ and $[U]$ satisfying the unique factorization $[A] = [L][U]$ can be determined from the recurrence formulas

$$\begin{aligned} l_{ij} &= a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}, \quad i \geq j \\ u_{ij} &= \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}}, \quad i < j \\ u_{ii} &= 1 \end{aligned} \tag{7.12}$$

For the relevant indices i and j, these elements are computed in the order

$$l_{i1}, u_{1j}; l_{i2}, u_{2j}; \quad l_{i3}, u_{3j}; \quad \ldots; \quad l_{i,n-1}, u_{n-1,j}; \quad l_{nn}$$

SOLUTION OF EQUATIONS

Once the given system of equations $[A]\vec{X} = \vec{b}$ is expressed in the form $[L][U]\vec{X} = \vec{b}$, the solution can be obtained as follows:

By letting

$$[U]\vec{X} = \vec{Z} \tag{7.13}$$

the equations become $[L]\vec{Z} = \vec{b}$, which in expanded form can be written as

$$\begin{aligned} l_{11}z_1 &= b_1 \\ l_{21}z_1 + l_{22}z_2 &= b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 &= b_3 \\ &\cdots \\ &\cdots \\ &\cdots \\ l_{n1}z_1 + l_{n2}z_2 + l_{n3}z_3 + \cdots + l_{nn}z_n &= b_n \end{aligned} \tag{7.14}$$

The first of these equations can be solved for z_1, after which the second can be solved for z_2, the third for z_3, and so on. We can thus determine in succession $z_1, z_2, \ldots, z_n$, provided that none of the diagonal elements l_{ii} $(i = 1, 2, \ldots, n)$ vanishes. Once z_i is obtained the value of x_i can be found by writing Eq. (7.13) as

$$\begin{aligned} x_1 + u_{12}x_2 + u_{13}x_3 + \cdots + u_{1n}x_n &= z_1 \\ x_2 + u_{23}x_3 + \cdots + u_{2n}x_n &= z_2 \\ x_3 + \cdots + u_{3n}x_n &= z_3 \\ &\cdots \\ &\cdots \\ &\cdots \\ x_{n-1} + u_{n-1,n}x_n &= z_{n-1} \\ x_n &= z_n \end{aligned} \tag{7.15}$$

Just as in the Gaussian elimination process, this system can now be solved by back substitution for $x_n, x_{n-1}, \ldots, x_1$ in that order.

CHOLESKI DECOMPOSITION OF SYMMETRIC MATRICES

In most applications of finite element theory, the matrices involved will be symmetric, banded, and positive definite. In such cases, the symmetric positive definite matrix $[A]$ can be decomposed uniquely as[1]

$$[A] = [U]^T[U] \tag{7.16}$$

where

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & u_{nn} \end{bmatrix} \tag{7.17}$$

is an upper triangular matrix including the diagonal. The elements of $[U] = [u_{ij}]$ are given by

1. The matrix $[A]$ can also be decomposed as $[A] = [L][L]^T$, where $[L]$ represents a lower triangular matrix. The elements of $[L]$ can be found in Rao [7.1] and also in Problem 7.2.

$$\left.\begin{aligned}
u_{11} &= (a_{11})^{(1/2)} \\
u_{1j} &= a_{1j}/u_{11}, \quad j = 2,3,\ldots,n \\
u_{ii} &= \left(a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2\right)^{(1/2)}, \quad i = 2,3,\ldots,n \\
u_{ij} &= \frac{1}{u_{ii}}\left(a_{ij} - \sum_{k=1}^{i-1} u_{ki}u_{kj}\right), \quad \begin{aligned} &i = 2,3,\ldots,n, \text{ and} \\ &j = i+1, i+2, \ldots n \end{aligned} \\
u_{ij} &= 0, \quad i > j.
\end{aligned}\right\} \tag{7.18}$$

INVERSE OF A SYMMETRIC MATRIX

If the inverse of the symmetric matrix $[A]$ is needed, we first decompose it as $[A] = [U]^T[U]$ using Eq. (7.18), and then find $[A]^{-1}$ as

$$[A]^{-1} = \left[[U]^T[U]\right]^{-1} = [U]^{-1}\left([U]^T\right)^{-1} \tag{7.19}$$

The elements λ_{ij} of $[U]^{-1}$ can be determined from $[U][U]^{-1} = [I]$, which leads to

$$\left.\begin{aligned}
\lambda_{ii} &= \frac{1}{u_{ii}} \\
\lambda_{ij} &= \frac{-\left(\sum\limits_{k=i+1}^{j} u_{ik}\lambda_{kj}\right)}{u_{ii}}, \quad i < j \\
\lambda_{ij} &= 0, \quad i > j
\end{aligned}\right\} \tag{7.20}$$

Hence, the inverse of $[U]$ is also an upper triangular matrix. The inverse of $[U]^T$ can be obtained from the relation

$$\left([\mathrm{U}]^T\right)^{-1} = \left([\mathrm{U}]^{-1}\right)^T \tag{7.21}$$

Finally, the inverse of the symmetric matrix $[A]$ can be calculated as

$$[\mathrm{A}]^{-1} = [\mathrm{U}]^{-1}\left([\mathrm{U}]^{-1}\right)^T \tag{7.22}$$

EXAMPLE 7.2

Express the following matrix as a product of lower and upper triangular matrices using the Choleski method.

$$[A] = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \tag{E.1}$$

Solution

Since the matrix $[A]$ is symmetric, it can be expressed as

$$[A] = [U]^T[U] \tag{E.2}$$

where the elements of $[U]$ are denoted as

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{E.3}$$

Continued

EXAMPLE 7.2 —cont'd

with

$$u_{11} = \sqrt{a_{11}} = \sqrt{4} = 2,\ \ u_{12} = \frac{a_{12}}{u_{11}} = \frac{2}{2} = 1,\ \ u_{13} = \frac{a_{13}}{u_{11}} = \frac{14}{2} = 7$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{\frac{1}{2}} = (17 - 1)^{\frac{1}{2}} = 4,$$

$$u_{23} = \frac{1}{u_{22}}(a_{23} - u_{12}\ u_{13}) = \frac{1}{4}(-5 - 1(7)) = -3$$

$$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{\frac{1}{2}} = (83 - 49 - 9)^{\frac{1}{2}} = 5$$

This yields the desired result:

$$[A] = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \tag{E.4}$$

EXAMPLE 7.3

Find the inverse of the matrix

$$[A] = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \tag{E.1}$$

Solution

The inverse of the symmetric matrix $[A]$ is given by Eq. (7.22):

$$[A]^{-1} = [U]^{-1}\left([U]^{-1}\right)^T \tag{E.2}$$

where $[U]$ is the upper triangular matrix (given in Eq. (E.4) in Example 7.2):

$$[U] = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \tag{E.3}$$

The elements of $[U]^{-1} = [\lambda]$ are given by

$$\lambda_{11} = \frac{1}{u_{11}} = \frac{1}{2},\quad \lambda_{22} = \frac{1}{u_{22}} = \frac{1}{4},\quad \lambda_{33} = \frac{1}{u_{33}} = \frac{1}{5}$$

$$\lambda_{12} = \frac{-u_{12}\lambda_{22}}{u_{11}} = \frac{-(1)\left(\frac{1}{4}\right)}{2} = -\frac{1}{8},\quad \lambda_{23} = \frac{-u_{23}\ \lambda_{33}}{u_{22}} = \frac{-(-3)\left(\frac{1}{5}\right)}{4} = \frac{3}{20}$$

$$\lambda_{13} = \frac{-u_{12}\ \lambda_{23} - u_{13}\ \lambda_{33}}{u_{11}} = \frac{-(1)\left(\frac{3}{20}\right) - 7\left(\frac{1}{5}\right)}{2} = -\frac{31}{40}$$

Thus,

$$[U]^{-1} = [\lambda] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} & -\frac{31}{40} \\ 0 & \frac{1}{4} & \frac{3}{20} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \tag{E.4}$$

EXAMPLE 7.3 —cont'd

Eq. (E.2) can be used to find the inverse of the matrix $[A]$ as

$$[A]^{-1} = [U]^{-1}\left([U]^{-1}\right)^T$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} & -\frac{31}{40} \\ 0 & \frac{1}{4} & \frac{3}{20} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{1}{4} & 0 \\ -\frac{31}{40} & \frac{3}{20} & \frac{1}{5} \end{bmatrix} \tag{E.5}$$

$$= \frac{1}{800}\begin{bmatrix} 1386 & -118 & -124 \\ -118 & 68 & 24 \\ -124 & 24 & 32 \end{bmatrix}$$

EXAMPLE 7.4

Find the solution of the equations

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ -101 \\ 155 \end{Bmatrix} \tag{E.1}$$

using the Choleski decomposition of the matrix $[A]$.

Solution

Using the result of Example 7.2, Eq. (E.1) can be rewritten as

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ -101 \\ 155 \end{Bmatrix} \tag{E.2}$$

By denoting

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{E.3}$$

Eq. (E.2) can be expressed as

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 14 \\ -101 \\ 155 \end{Bmatrix} \tag{E.4}$$

or, equivalently,

$$2y_1 = 14, \quad y_1 + 4y_2 = -101, \quad 7y_1 - 3\ y_2 + 5y_3 = 155 \tag{E.5}$$

Eq. (E.5) yield, sequentially,

$$y_1 = 7, \quad 4y_2 = -101 - 1(7) \quad \text{or} \quad y_2 = -27$$

and

$$5y_3 = 155 - 7(7) + 3(-27) = 25 \quad \text{or} \quad y_3 = 5$$

Continued

EXAMPLE 7.4 —cont'd

Using the values of y_1, y_2, and y_3, Eq. (E.3) can be rewritten as

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ -27 \\ 5 \end{Bmatrix} \tag{E.6}$$

Eq. (E.6) can be written, from the bottom to the top, as

$$5x_3 = 5$$

$$4x_2 - 3x_3 = -27$$

$$2x_1 + x_2 + 7x_3 = 7$$

or

$$x_3 = 1$$

$$x_2 = \frac{1}{4}(-27 + 3(1)) = -6$$

$$x_1 = \frac{1}{2}(7 - 1(-6) - 7(1)) = 3$$

Thus, the desired solution is

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -6 \\ 1 \end{Bmatrix} \tag{E.7}$$

7.2.3 Other Methods

Most computer programs that implement the finite element method take advantage of the properties of symmetry and band form in storing the matrix [A]. In fact, the obvious advantage of small bandwidth has prompted engineers involved in finite element analysis to develop schemes to model systems so as to minimize the bandwidth of resulting matrices. Despite the relative compactness of band form storage, computer core space may be inadequate for the band form storage of matrices of extremely large systems. In such a case, the matrix is partitioned as shown in Fig. 7.1, and only a few of the triangular

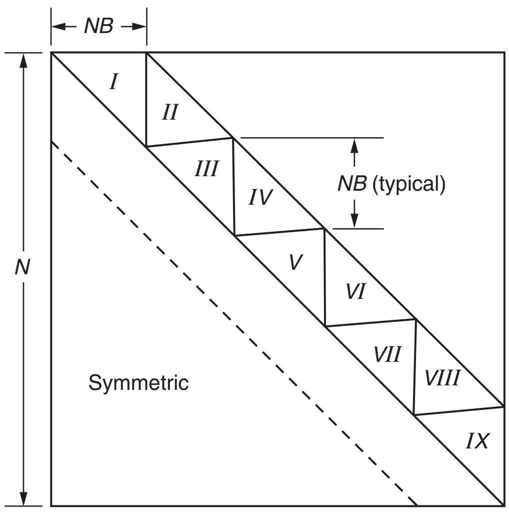

FIGURE 7.1 Partitioning of a large matrix.

submatrices are stored in the computer core at a given time; the remaining ones are kept in auxiliary storage, for example, on a disk. Several other schemes, such as the frontal or wavefront solution methods, have been developed for handling large matrices [7.2–7.5].

The Gauss elimination and Choleski decomposition schemes fall under the category of direct methods. In the class of iterative methods, the Gauss–Seidel method is well known [7.6]. The conjugate gradient and Newton's methods are other iterative methods based on the principle of unconstrained minimization of a function [7.7,7.8]. Note that the indirect methods are less popular than the direct methods in solving large systems of linear equations [7.9]. Special computer programs have been developed for the solution of finite element equations on small computers [7.10].

7.3 SOLUTION OF EIGENVALUE PROBLEMS

When the finite element method is applied for the solution of eigenvalue problems, we obtain an algebraic eigenvalue problem as stated in Eq. (7.2). We consider the solution of Eq. (7.2a) in this section, assuming that the boundary conditions, Eq. (7.2b), have been incorporated already. For most engineering problems, $[A]$ and $[B]$ will be symmetric matrices of order n. λ is a scalar (called the eigenvalue), and $\vec{X}$ is a column vector with n components (called the eigenvector). If the physical problem is the free vibration analysis of a structure, $[A]$ will be the stiffness matrix, $[B]$ will be the mass matrix, λ is the square of natural frequency, and $\vec{X}$ is the mode shape of the vibrating structure.

The eigenvalue problem given by Eq. (7.2a) can be rewritten as

$$([A] - \lambda[B])\vec{X} = \vec{0} \tag{7.23}$$

which can have solutions for

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

other than zero only if the determinant of the coefficient matrix vanishes; that is,

$$\begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} - \lambda b_{12} & \dots & a_{1n} - \lambda b_{1n} \\ a_{21} - \lambda b_{21} & a_{22} - \lambda b_{22} & \dots & a_{2n} - \lambda b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} - \lambda b_{n1} & a_{n2} - \lambda b_{n2} & \dots & a_{nn} - \lambda b_{nn} \end{vmatrix} = 0 \tag{7.24}$$

If the determinant in Eq. (7.24) is expanded, we obtain an algebraic equation of n-th degree for λ. This equation is called the characteristic equation of the system. The n roots of this equation are the n eigenvalues of Eq. (7.2a). The eigenvector corresponding to any λ_j, namely $\vec{X}_j$, can be found by inserting λ_j in Eq. (7.23) and solving for the ratios of the elements in $\vec{X}_j$. A practical way to do this is to set x_n, for example, equal to unity and solve the first $n-1$ equations for x_1, x_2, …, x_{n-1}. The last equation may be used as a check.

From Eq. (7.2a), it is evident that if $\vec{X}$ is a solution, then $k\vec{X}$ will also be a solution for any nonzero value of the scalar k. Thus, the eigenvector corresponding to any eigenvalue is arbitrary to the extent of a scalar multiplier. It is convenient to choose this multiplier so that the eigenvector has some desirable numerical property, and such vectors are called normalized vectors. One method of normalization is to make the component of the vector $\vec{X}_i$ having the largest magnitude equal to unity; that is,

$$\max_{j=1,2,\dots,n} (x_{ij}) = 1 \tag{7.25}$$

where x_{ij} is the j-th component of the vector $\vec{X}_i$. Another method of normalization commonly used in structural dynamics is as follows:

$$\vec{X}_i^T [B] \vec{X}_i = 1 \tag{7.26}$$

EXAMPLE 7.5
Find the eigenvalues of the matrix

$$[A] = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix} \tag{E.1}$$

Solution
The eigenvalues are given by the roots of the determinantal Eq. (7.24):

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 3 & 2-\lambda & -3 \\ 3 & 1 & -2-\lambda \end{vmatrix} = 0 \tag{E.2}$$

which can be expanded as

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} - 1\begin{vmatrix} 3 & -3 \\ 3 & -2-\lambda \end{vmatrix} - 1\begin{vmatrix} 3 & 2-\lambda \\ 3 & 1 \end{vmatrix} = 0$$

that is,

$$(2-\lambda)\left(-\left(4-\lambda^2\right)-(-3)\right)-1(-3\lambda+3)-1(3\lambda-3) = 0$$

that is,

$$\lambda^3-2\lambda^2-\lambda+2 = 0 \tag{E.3}$$

By inspection, the roots of Eq. (E.1) can be found as (can be verified by substitution):

$$\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2 \tag{E.4}$$

These roots denote the eigenvalues of the matrix $[A]$.

7.3.1 Standard Eigenvalue Problem

Although the procedure given previously for solving the eigenvalue problem appears to be simple, the roots of an n-th degree polynomial cannot be obtained easily for matrices of high order. Hence, in most of the computer-based methods used for the solution of Eq. (7.2a), the eigenvalue problem is first converted into the form of a standard eigenvalue problem, which can be stated as

$$[H]\vec{X} = \lambda\vec{X} \text{ or } \left([H]-\lambda[I]\right)\vec{X} = \vec{0} \tag{7.27}$$

By premultiplying Eq. (7.2a) by $[B]^{-1}$, we obtain Eq. (7.27), where

$$[H] = [B]^{-1}[A] \tag{7.28}$$

However, in this form the matrix $[H]$ is in general nonsymmetric, although $[B]$ and $[A]$ are both symmetric. Since a symmetric matrix is desirable from the standpoint of storage and computer time, we adopt the following procedure to derive a standard eigenvalue problem with symmetric $[H]$ matrix.

Assuming that $[B]$ is symmetric and positive definite, we use Choleski decomposition and express $[B]$ as $[B] = [U]^T[U]$. By substituting for $[B]$ in Eq. (7.2a), we obtain

$$[A]\vec{X} = \lambda[U]^T[U]\vec{X}$$

and hence

$$\left([U]^T\right)^{-1}[A][U]^{-1}[U]\vec{X} = \lambda[U]\vec{X} \tag{7.29}$$

By defining a new vector $\vec{Y}$ as $\vec{Y} = [U]\ \vec{X}$, Eq. (7.29) can be written as a standard eigenvalue problem as

$$([H]-\lambda[I])\vec{Y} = \vec{0} \tag{7.30}$$

where the matrix $[H]$ is now symmetric and is given by

$$[H] = \left([U]^T\right)^{-1}[A][U]^{-1} \tag{7.31}$$

To formulate $[H]$ according to Eq. (7.31), we decompose the symmetric matrix $[B]$ as $[B] = [U]^T[U]$, as indicated in Section 7.2.2 on Choleski decomposition. Find $[U]^{-1}$ and $\left([U]^T\right)^{-1}$ as shown in Section 7.2.2, "Inverse of a Symmetric Matrix," and then carry out the matrix multiplication as stated in Eq. (7.31). The solution of the eigenvalue problem stated in Eq. (7.30) yields λ_i and $\vec{Y}_i$. Then we apply the inverse transformation to obtain the desired eigenvectors as

$$\vec{X}_i = [U]^{-1}\vec{Y}_i \tag{7.32}$$

We now discuss some of the methods of solving the special eigenvalue problem stated in Eq. (7.27).

EXAMPLE 7.6

Convert the following eigenvalue problem into the form of a standard eigenvalue problem:

$$[A]\vec{X} = \lambda[B]\vec{X} \tag{E.1}$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad [B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{E.2}$$

Solution

Noting that the matrix $[B]$ is diagonal, it can be written as

$$[B] = [B]^{\frac{1}{2}}[B]^{\frac{1}{2}} \tag{E.3}$$

where the square root of the diagonal matrix $[B]$ and its inverse can be found as

$$[B]^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad [B]^{-\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{E.4}$$

Eq. (E.1) can be written as

$$[A][B]^{-\frac{1}{2}}[B]^{\frac{1}{2}}\vec{X} = \lambda[B]^{\frac{1}{2}}[B]^{\frac{1}{2}}\vec{X} \tag{E.5}$$

By premultiplying Eq. (E.5) by $[B]^{-\frac{1}{2}}$, we obtain

$$[B]^{-\frac{1}{2}}[A][B]^{-\frac{1}{2}}[B]^{\frac{1}{2}}\vec{X} = \lambda[B]^{\frac{1}{2}}\vec{X} \tag{E.6}$$

or

$$[H]\vec{Y} = \lambda\vec{Y} \tag{E.7}$$

where

$$[H] = [B]^{-\frac{1}{2}}[A][B]^{-\frac{1}{2}} \tag{E.8}$$

and

$$\vec{Y} = [B]^{\frac{1}{2}}\vec{X} \tag{E.9}$$

It can be seen that Eq. (E.7) is in the form of a standard eigenvalue problem. Once the solution of Eq. (E.7) is found, the original eigenvector can be generated, from Eq. (E.9), as

$$\vec{X} = [B]^{-\frac{1}{2}}\vec{Y} \tag{E.10}$$

Continued

EXAMPLE 7.6 —cont'd

In the present case,

$$[H] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -0.7071 \\ 0 & -0.7071 & 1.5 \end{bmatrix} \tag{E.11}$$

and hence the desired standard form of the eigenvalue problem is given by

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -0.7071 \\ 0 & -0.7071 & 1.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \lambda \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{E.12}$$

7.3.2 Methods of Solving Eigenvalue Problems

Two general types of methods, namely transformation methods and iterative methods, are available for solving eigenvalue problems. The transformation methods, such as Jacobi, Givens, and Householder schemes, are preferable when *all* the eigenvalues and eigenvectors are required. The iterative methods, such as the power method, are preferable when *few* eigenvalues and eigenvectors are required [7.11–7.13].

7.3.3 Jacobi Method

In this section, we present the Jacobi method for solving the standard eigenvalue problem:

$$[H]\overrightarrow{X} = \lambda \overrightarrow{X} \tag{7.33}$$

where $[H]$ is a symmetric matrix.

METHOD

The method is based on a theorem in linear algebra that states that a real symmetric matrix $[H]$ has only real eigenvalues and that there exists a real orthogonal matrix $[P]$ such that $[P]^T[H][P]$ is diagonal. The diagonal elements are the eigenvalues and the columns of the matrix $[P]$ are the eigenvectors.

In the Jacobi method, the matrix $[P]$ is obtained as a product of several rotation matrices of the form

$$\underset{n\times n}{[P_1]} = \begin{array}{c} \begin{array}{ccccccc} & & & i-\text{th} & & j-\text{th column} & \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta & & -\sin\theta & \\ & & & & \ddots & & \\ & & & \sin\theta & & \cos\theta & \\ & & & & & & \ddots \\ & & & & & & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ i-\text{th} \\ \\ j-\text{th row} \\ \\ \\ \end{array} \tag{7.34}$$

where all elements other than those appearing in columns and rows i and j are identical with those of the identity matrix $[I]$. If the sine and cosine entries appear in positions (i, i) (i, j), (j, i), and (j, j), then the corresponding elements of $[P_1]^T[H]\,[P_1]$ can be computed as

$$\begin{aligned} \underline{h}_{ii} &= h_{ii}\cos^2\theta + 2h_{ij}\sin\theta\cos\theta + h_{jj}\sin^2\theta \\ \underline{h}_{ij} &= \underline{h}_{ji} = (h_{jj} - h_{ii})\sin\theta\cos\theta + h_{ij}\left(\cos^2\theta - \sin^2\theta\right) \\ \underline{h}_{jj} &= h_{ii}\sin^2\theta - 2h_{ij}\sin\theta\cos\theta + h_{jj}\cos^2\theta \end{aligned} \tag{7.35}$$

If θ is chosen as

$$\tan 2\theta = 2h_{ij}/(h_{ii} - h_{jj}) \tag{7.36}$$

then it makes $\underline{h}_{ij} = \underline{h}_{ji} = 0$. Thus, each step of the Jacobi method reduces a pair of off-diagonal elements to zero. Unfortunately, in the next step, although the method reduces a new pair of zeros, it introduces nonzero contributions to formerly zero positions. However, successive matrices of the form

$$[P_2]^T[P_1]^T[H][P_1][P_2], \quad [P_3]^T[P_2]^T[P_1]^T[H][P_1][P_2][P_3], \ldots$$

converge to the required diagonal form and the desired matrix $[P]$ (whose columns give the eigenvectors) would then be given by

$$[P] = [P_1][P_2][P_3], \ldots \tag{7.37}$$

EXAMPLE 7.7

Find the solution of the standard eigenvalue problem

$$[H]\vec{Y} = \lambda\vec{Y} \tag{E.1}$$

where

$$[H] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -0.7071 \\ 0 & -0.7071 & 1.5 \end{bmatrix} \tag{E.2}$$

using the Jacobi method.

Solution

To decompose the matrix $[H]$, we start by eliminating the term h_{12} so that $i = 1$ and $j = 2$. The angle θ_1 for the first rotation matrix $[P_1]$ is given by

$$\tan 2\theta_1 = \frac{2h_{12}}{h_{12} - h_{22}} = \frac{2(-1)}{2-2} = \mp\infty \quad \text{or} \quad \theta_1 = 135°$$

so that the first rotation matrix becomes

$$[P_1] = \begin{bmatrix} -0.7071 & -0.7071 & 0 \\ 0.7071 & -0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives

$$[H_1] = [P_1]^T[H][P_1] = \begin{bmatrix} 3 & 0 & -0.5 \\ 0 & 1 & 0.5 \\ -0.5 & 0.5 & 1.5 \end{bmatrix}$$

Next, we select h_{13} in $[H_1]$ for elimination ($i = 1$ and $j = 3$) so that the angle θ_2 for the second rotation matrix $[P_2]$ is

$$\tan 2\theta_2 = \frac{2h_{13}}{h_{11} - h_{33}} = \frac{2(-0.5)}{3-1.5} = -0.6667 \quad \text{or} \quad \theta_2 = 163.15°$$

Thus, the second rotation matrix is given by

$$[P_2] = \begin{bmatrix} -0.9571 & 0 & -0.2899 \\ 0 & 1 & 0 \\ 0.2899 & 0 & -0.9571 \end{bmatrix}$$

which yields

$$[H_2] = [P_2]^T[H_1][P_2] = \begin{bmatrix} 3.1516 & 0.1450 & 0 \\ 0.1450 & 1 & -0.4786 \\ 0 & -0.4786 & 1.3485 \end{bmatrix}$$

Continued

EXAMPLE 7.7 —cont'd

By proceeding in a similar manner, the next sequence of rotation matrices can be generated as

$$[P_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5736 & -0.8192 \\ 0 & 0.8192 & -0.5736 \end{bmatrix}, \ldots [P_6] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.0039 \\ 0 & 0.0039 & 1 \end{bmatrix}$$

It is found that the sequence of matrices $[H_i]$ converged at this stage with

$$[H_6] = [P_6]^T[H_5][P_6] = \begin{bmatrix} 3.1619 & 0 & 0 \\ 0 & 1.6789 & 0 \\ 0 & 0 & 0.6594 \end{bmatrix} \approx \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and the product of the matrices $[P_i]$ converged to the matrix of eigenvectors:

$$[P_1][P_2]\ldots[P_6] = \begin{bmatrix} -0.6209 & 0.6074 & -0.4959 \\ 0.7215 & 0.1949 & -0.6646 \\ -0.3070 & -0.7702 & -0.5592 \end{bmatrix} \approx \begin{bmatrix} \vec{Y}_1 & \vec{Y}_2 & \vec{Y}_3 \end{bmatrix}$$

where $\vec{Y}_i$ denotes the eigenvector corresponding to the eigenvalue λ_i, $i = 1, 2, 3$.

7.3.4 Power Method

COMPUTING THE LARGEST EIGENVALUE BY THE POWER METHOD

The power method is the simplest iterative procedure for finding the largest or principal eigenvalue (λ_1) and the corresponding eigenvector of a matrix ($\vec{X}_1$). We assume that the $n \times n$ matrix $[H]$ is symmetric and real with n independent eigenvectors $\vec{X}_1, \ \vec{X}_2, \ \ldots, \ \vec{X}_n$. In this method, we choose an initial vector $\vec{Z}_0$ and generate a sequence of vectors $\vec{Z}_1$, $\vec{Z}_2, \ \ldots$, as

$$\vec{Z}_i = [H]\vec{Z}_{i-1} \tag{7.38}$$

so that, in general, the p-th vector is given by

$$\vec{Z}_P = [H]\vec{Z}_{P-1} = [H]^2\vec{Z}_{P-2} = \cdots = [H]^p\vec{Z}_0 \tag{7.39}$$

The iterative process of Eq. (7.38) is continued until the following relation is satisfied:

$$\frac{z_{p,1}}{z_{p-1,1}}, \frac{z_{p,2}}{z_{p-1,2}}, \cdots, \frac{z_{p,n}}{z_{p-1,n}} = \lambda_1 \tag{7.40}$$

where $z_{p,j}$ and $z_{p-1,j}$ are the j-th components of vectors $\vec{Z}_p$ and $\vec{Z}_{p-1}$, respectively. Here, λ_1 will be the desired eigenvalue.

The convergence of the method can be explained as follows. Since the initial (any arbitrary) vector $\vec{Z}_0$ can be expressed as a linear combination of the eigenvectors, we can write

$$\vec{Z}_0 = a_1\vec{X}_1 + a_2\vec{X}_2 + \cdots + a_n\vec{X}_n \tag{7.41}$$

where $a_1, a_2, \ldots, a_n$ are constants. If λ_i is the eigenvalue of $[H]$ corresponding to $\vec{X}_i$, then

$$\begin{aligned} [H]\vec{Z}_0 &= a_1[H]\vec{X}_1 + a_2[H]\vec{X}_2 + \cdots + a_n[H]\vec{X}_n \\ &= a_1\lambda_1\vec{X}_1 + a_2\lambda_2\vec{X}_2 + \cdots + a_n\lambda_n\vec{X}_n \end{aligned} \tag{7.42}$$

and

$$\begin{aligned} [H]^p\vec{Z}_0 &= a_1\lambda_1^p\vec{X}_1 + a_2\lambda_2^p\vec{X}_2 + \cdots + a_n\lambda_n^p\vec{X}_n \\ &= \lambda_1^p\left[a_1\vec{X}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^p a_2\vec{X}_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^p a_n\vec{X}_n\right] \end{aligned} \tag{7.43}$$

If λ_1 is the largest (dominant) eigenvalue,

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|, \quad \left|\frac{\lambda_i}{\lambda_1}\right| < 1 \tag{7.44}$$

and hence $\left(\frac{\lambda_i}{\lambda_1}\right)^p \to 0$ as $p \to \infty$.

Thus, Eq. (7.43) can be written, in the limit as $p \to \infty$, as

$$[H]^{p-1}\overrightarrow{Z}_0 = \lambda_1^{p-1} a_1 \overrightarrow{X}_1 \tag{7.45}$$

and

$$[H]^p \overrightarrow{Z}_0 = \lambda_1^p a_1 \overrightarrow{X}_1 \tag{7.46}$$

Therefore, if we take the ratio of any corresponding components of the vectors $\left([H]^p \overrightarrow{Z}_0\right)$ and $\left([H]^{p-1}\overrightarrow{Z}_0\right)$, it should have the same limiting value, λ_1. This property can be used to stop the iterative process. Moreover,

$$\left(\frac{[H]^p \overrightarrow{Z}_0}{\lambda_1^p}\right)$$

will converge to the eigenvector $a_1 \overrightarrow{X}_1$ as $p \to \infty$.

EXAMPLE 7.8

Find the dominant eigenvalue and the corresponding eigenvector of the matrix

$$[H] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution

By choosing the initial vector as

$$\overrightarrow{Z}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

we have

$$\overrightarrow{Z}_1 = [H]\overrightarrow{Z_0} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\overrightarrow{Z}_2 = [H]\overrightarrow{Z}_1 = [H]^2\overrightarrow{Z}_0 = \begin{Bmatrix} 2 \\ -2 \\ 2 \end{Bmatrix}$$

and

$$\overrightarrow{Z}_3 = [H]\overrightarrow{Z}_2 = [H]^3\overrightarrow{Z}_0 = \begin{Bmatrix} 6 \\ -8 \\ 6 \end{Bmatrix}$$

It is convenient here to divide the components of $\overrightarrow{Z}_3$ by 8 to obtain

$$\overrightarrow{Z_3} = k\begin{Bmatrix} 3/4 \\ -1 \\ 3/4 \end{Bmatrix}$$

Continued

EXAMPLE 7.8 —cont'd

where $k = 8$.

In the future, we continue to divide by some suitable factor to keep the magnitude of the numbers reasonable. Continuing the procedure, we find

$$\vec{Z}_7 = [H]^7\vec{Z}_0 = c\begin{Bmatrix} 99 \\ -140 \\ 99 \end{Bmatrix} \quad \text{and} \quad \vec{Z}_8 = [H]^8\vec{Z}_0 = c\begin{Bmatrix} 338 \\ -478 \\ 338 \end{Bmatrix}$$

where c is a constant factor. The ratios of the corresponding components of $\vec{Z}_8$ and $\vec{Z}_7$ are $338/99 = 3.41{,}414$ and $478/140 = 3.41{,}429$, which can be assumed to be the same for our purpose. The eigenvalue given by this method is thus $\lambda_1 = 3.41{,}414$ or $3.41{,}429$, whereas the exact solution is $\lambda_1 = 2 + \sqrt{2} = 3.41421$. By dividing the last vector $\vec{Z}_8$ by the magnitude of the largest component (478), we obtain the eigenvector as

$$\vec{X_1} = \begin{Bmatrix} 1/1.4142 \\ -1.0 \\ 1/1.4142 \end{Bmatrix}, \text{ which is very close to the correct solution}$$

$$\vec{X}_1 = \begin{Bmatrix} 1/\sqrt{2} \\ -1 \\ 1/\sqrt{2} \end{Bmatrix}$$

The eigenvalue λ_1 can also be obtained by using the Rayleigh quotient (R) defined as

$$R = \frac{\vec{X}^T[H]\vec{X}}{\vec{X}^T\vec{X}} \tag{7.47}$$

If $[H]\vec{X} = \lambda\vec{X}$, R will be equal to λ. Thus, we can compute the Rayleigh quotient at i-th iteration as

$$R_i = \frac{\vec{X}_i^T[H]\vec{X}_i}{\vec{X}_i^T\vec{X}_i}, \quad i = 1, 2, \ldots \tag{7.48}$$

Whenever R_i is observed to be essentially the same for two consecutive iterations $i - 1$ and i, we take $\lambda_1 = R_i$.

COMPUTING THE SMALLEST EIGENVALUE BY THE POWER METHOD

If we wish to solve

$$[H]\vec{X} = \lambda\vec{X} \tag{7.49}$$

to find the smallest eigenvalue and the associated eigenvector, we premultiply Eq. (7.49) by $[H]^{-1}$ and obtain

$$[H]^{-1}\vec{X} = \left(\frac{1}{\lambda}\right)\vec{X} \tag{7.50}$$

Eq. (7.50) can be written as

$$[\underset{\sim}{H}]\vec{X} = \underset{\sim}{\lambda}\vec{X} \tag{7.51}$$

where

$$[\underset{\sim}{H}] = [H]^{-1} \quad \text{and} \quad \underset{\sim}{\lambda} = \lambda^{-1} \tag{7.52}$$

This means that the absolutely smallest eigenvalue of $[H]$ can be found by solving the problem stated in Eq. (7.51) for the largest eigenvalue according to the procedure outlined in Section 7.3.4 on computing the largest eigenvalue. Note that $[\underset{\sim}{H}] = [H]^{-1}$ has to be found before finding $\lambda_{\text{smallest}}$. Although this involves additional computations (in finding $[H]^{-1}$), it may prove to be the best approach in some cases.

COMPUTING INTERMEDIATE EIGENVALUES

Let the dominant eigenvector $\overrightarrow{X_1}$ be normalized so that its first component is 1. Let

$$\overrightarrow{X_1} = \begin{Bmatrix} 1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}$$

Let $\vec{r}^T$ denote the first row of the matrix $[H]$—that is, $\vec{r}^T = \{h_{11}\ h_{12}\ \ldots\ h_{1n}\}$. Then form a matrix $[\underset{\sim}{H}]$ as

$$[\underset{\sim}{H}] = \vec{X}_1\vec{r}^T = \begin{Bmatrix} 1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix}\{h_{11}\quad h_{12}\quad \ldots\quad h_{1n}\} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ x_2h_{11} & x_2h_{12} & \cdots & x_2h_{1n} \\ \vdots & & & \\ x_nh_{11} & x_nh_{12} & \cdots & x_nh_{1n} \end{bmatrix} \tag{7.53}$$

Let the next dominant eigenvalue be λ_2 and normalize its eigenvector $(\vec{X}_2)$ so that its first component is 1.

If $\vec{X}_1$ or $\vec{X}_2$ has a zero first element, then a different element may be normalized and the corresponding row $\vec{r}^T$ of matrix $[H]$ is used. Since $[H]\vec{X}_1 = \lambda_1\vec{X}_1$ and $[H]\vec{X}_2 = \lambda_2\vec{X}_2$, we obtain, by considering only the row $\vec{r}^T$ of these products,

$$\vec{r}^T\vec{X}_1 = \lambda_1 \quad \text{and} \quad \vec{r}^T\vec{X}_2 = \lambda_2$$

This is a consequence of the normalizations. We can also obtain

$$[\underset{\sim}{H}]\vec{X}_1 = (\vec{X}_1\vec{r}^T)\vec{X}_1 = \vec{X}_1(\vec{r}^T\vec{X}_1) = \lambda_1\vec{X}_1 \tag{7.54}$$

and

$$[\underset{\sim}{H}]\vec{X}_2 = (\vec{X}_1\vec{r}^T)\vec{X}_2 = \vec{X}_1(\vec{r}^T\vec{X}_2) = \lambda_2\vec{X}_1 \tag{7.55}$$

so that

$$\left([H] - [\underset{\sim}{H}]\right)(\vec{X}_2 - \vec{X}_1) = \lambda_2\vec{X}_2 - \lambda_1\vec{X}_1 - \lambda_2\vec{X}_1 + \lambda_1\vec{X}_1 = \lambda_2(\vec{X}_2 - \vec{X}_1) \tag{7.56}$$

Eq. (7.56) shows that λ_2 is an eigenvalue and $\vec{X}_2 - \vec{X}_1$ is an eigenvector of the matrix $[H] - [\underset{\sim}{H}]$. Since $[H] - [\underset{\sim}{H}]$ has all zeros in its first row, whereas $\vec{X}_2 - \vec{X}_1$ has a zero as its first component, both the first row and first column of $[H] - [\underset{\sim}{H}]$ may be deleted to obtain the matrix $[H_2]$. We then determine the dominant eigenvalue and the corresponding eigenvector of $[H_2]$, and by attaching a zero first component, obtain a vector $\vec{Z}_1$. Finally, $\vec{X}_2 - \vec{X}_1$ must be a multiple of $\vec{Z}_1$ so that we can write

$$\vec{X}_2 = \vec{X}_1 + a\vec{Z}_1 \tag{7.57}$$

The multiplication factor a can be found by multiplying Eq. (7.57) by the row vector $\vec{r}^T$ so that

$$a = \frac{\lambda_2 - \lambda_1}{\vec{r}^T\vec{Z}_1} \tag{7.58}$$

A similar procedure can be adopted to obtain the other eigenvalues and eigenvectors. A procedure to accelerate the convergence of the power method has been suggested by Roberti [7.14].

EXAMPLE 7.9

Find the second and third eigenvalues of the matrix

$$[H] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

once

$$\vec{X}_1 = \begin{Bmatrix} 1.0 \\ -1.4142 \\ 1.0 \end{Bmatrix}$$

and

$$\lambda_1 = 3.4142$$

are known.

Solution

The first row of the matrix $[H]$ is given by $\vec{r}_1^{\,T} = \{2 \quad -1 \quad 0\}$ and hence

$$\left[\underset{\sim}{H}\right] = \vec{X}_1 \vec{r}_1^{\,T} = \begin{Bmatrix} 1.0 \\ -1.4142 \\ 1.0 \end{Bmatrix} \{2 \quad -1 \quad 0\} = \begin{bmatrix} 2.0 & -1.0 & 0.0 \\ -2.8284 & 1.4142 & 0.0 \\ 2.0 & -1.0 & 0.0 \end{bmatrix}$$

$$[H] - \left[\underset{\sim}{H}\right] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -2.8284 & 1.4142 & 0 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1.8284 & 0.5858 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

$$[H_2] = \begin{bmatrix} 0.5858 & -1 \\ 0 & 2 \end{bmatrix}$$

We apply the power method to obtain the dominant eigenvalue of $[H_2]$ by taking the starting vector as $\vec{X} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ and compute upto $[H_2]^{10}\vec{X} = c\begin{Bmatrix} -0.7071 \\ 1.0000 \end{Bmatrix}$, after which there is no significant change. As usual, c is some constant of no interest to us. Thus, the eigenvector of $[H_2]$ can be taken as $\begin{Bmatrix} -0.7071 \\ 1.0000 \end{Bmatrix}$.

The Rayleigh quotient corresponding to this vector gives $R_{10} = \lambda_2 = 2.0000$. By attaching a zero first element to the present vector $\begin{Bmatrix} -0.7071 \\ 1.0000 \end{Bmatrix}$, we obtain

$$\vec{Z}_2 = \begin{Bmatrix} 0.0 \\ -0.7071 \\ 1.0 \end{Bmatrix}$$

We then compute

$$a = \frac{\lambda_2 - \lambda_1}{\vec{r}_1^{\,T} \vec{Z}_2} = \frac{2.0000 - 3.4142}{(0.0 + 0.7071 + 0.0)} = -2.00002$$

Thus, we obtain the eigenvector $\vec{X}_2$ as

EXAMPLE 7.9 —cont'd

$$\vec{X}_2 = \vec{X}_1 + a\vec{Z}_2 = \begin{Bmatrix} 1.0 \\ -1.4142 \\ 1.0 \end{Bmatrix} - 2.00002 \begin{Bmatrix} 0.0 \\ -0.7071 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.00001 \\ -1.00002 \end{Bmatrix}$$

Next, to find $\vec{X}_3$, we take $\lambda_2 = 2$ and normalize the vector $\begin{Bmatrix} -0.7071 \\ 1.0 \end{Bmatrix}$ to obtain the vector $\vec{Y}_2 = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix}$.
The matrix $[H_2]$ is reduced as follows.
The first row of $[H_2]$ is given by $\vec{r}_2^{\,T} = \{0.5858 \;\; -1.0\}$, and

$$[H_2] - [\underset{\sim}{H}] = [H_2] - \vec{Y}_2 \vec{r}_2^{\,T} = \begin{bmatrix} 0.5858 & -1.0 \\ 0 & 2.0 \end{bmatrix} - \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} \{0.5858 \;\; -1.0\}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.08284 & 0.5858 \end{bmatrix}$$

By deleting the first row and first column, we obtain the new reduced matrix $[H_3]$ as $[H_3] = [0.5858]$. The eigenvalue of $[H_3]$ is obviously $\lambda_3 = 0.5858$, and we can choose its eigenvector as $\{1\}$. By attaching a leading zero, we obtain $\vec{U}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$. The value of a can be computed as

$$\frac{\lambda_3 - \lambda_2}{\vec{r}_2^{\,T} \vec{U}_2} = \frac{0.5858 - 2.000}{(0 - 1)} = 1.4142$$

and the corresponding eigenvector of $[H_2]$ can be obtained as

$$\vec{Y}_3 = \vec{Y}_2 + a\vec{U}_2 = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} + 1.4142 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 0.0 \end{Bmatrix}$$

The eigenvector of $[H]$ corresponding to λ_3 can be obtained by adding a leading zero to $\vec{Y}_3$ to obtain $\vec{Z}_3$ as

$$\vec{Z}_3 = \begin{Bmatrix} 0.0 \\ 1.0 \\ 0.0 \end{Bmatrix} \quad \text{and computing } a \text{ as} \quad \frac{\lambda_3 - \lambda_1}{\vec{r}_1^{\,T} \vec{Z}_3} = \frac{0.5858 - 3.4142}{(0 - 1 + 0)} = 2.8284$$

Finally, the eigenvector $\vec{X}_3$ corresponding to $[H]$ can be found as

$$\vec{X}_3 = \vec{X}_1 + a\vec{Z}_3 = \begin{Bmatrix} 1.0 \\ -1.4142 \\ 1.0 \end{Bmatrix} + 2.8284 \begin{Bmatrix} 0.0 \\ 1.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ 1.4142 \\ 1.0 \end{Bmatrix}$$

7.3.5 Rayleigh–Ritz Subspace Iteration Method

Another iterative method that can be used to find the lowest eigenvalues and the associated eigenvectors of the general eigenvalue problem, Eq. (7.2a), is the Rayleigh–Ritz subspace iteration method [7.15,7.16]. This method is very effective in finding the first few eigenvalues and the corresponding eigenvectors of large eigenvalue problems whose stiffness ($[A]$) and mass ($[B]$) matrices have large bandwidths. The various steps of this method are given here briefly. A detailed description of the method can be found in Bathe and Wilson [7.15].

ALGORITHM

Step 1: Start with q initial iteration vectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_q, q > p$, where p is the number of eigenvalues and eigenvectors to be calculated. Bathe and Wilson [7.15] suggested a value of $q = \min(2p, p + 8)$ for good convergence. Define the initial modal matrix $[X_0]$ as

$$[\mathrm{X}_0] = [\vec{\mathrm{X}}_1 \vec{\mathrm{X}}_2 \ldots \vec{\mathrm{X}}_\mathrm{q}] \tag{7.59}$$

and set the iteration number as $k = 0$. A computer algorithm for calculating efficient initial vectors for subspace iteration method was given in Cheu et al. [7.17].

Step 2: Use the following subspace iteration procedure to generate an improved modal matrix $[X_{k+1}]$:

a. Find $[\overline{X}_{k+1}]$ from the relation

$$[A][\overline{X}_{k+1}] = [B][X_k] \tag{7.60}$$

b. Compute

$$[A_{k+1}] = [\overline{X}_{k+1}]^T[A][\overline{X}_{k+1}] \tag{7.61}$$

$$[B_{k+1}] = [\overline{X}_{k+1}]^T[B][\overline{X}_{k+1}] \tag{7.62}$$

c. Solve for the eigenvalues and eigenvectors of the reduced system

$$[A_{k+1}][Q_{k+1}] = [B_{k+1}][Q_{k+1}][\Lambda_{k+1}] \tag{7.63}$$

and obtain $[\Lambda_{k+1}]$ and $[Q_{k+1}]$.

d. Find an improved approximation to the eigenvectors of the original system as

$$[X_{k+1}] = [\overline{X}_{k+1}][Q_{k+1}]. \tag{7.64}$$

Note

1. It is assumed that the iteration vectors converging to the exact eigenvectors $\vec{X}_1^{(\text{exact})}$, $\vec{X}_2^{(\text{exact})}$, ..., are stored as the first, second, ..., columns of the matrix $[X_{k+1}]$.
2. It is assumed that the vectors in $[X_0]$ are not orthogonal to one of the required eigenvectors.

Step 3: If $\lambda_i^{(k)}$ and $\lambda_i^{(k+1)}$ denote the approximations to the i-th eigenvalue in the iterations $k-1$ and k, respectively, we assume convergence of the process whenever the following criteria are satisfied:

$$\frac{\lambda_i^{(k+1)} - \lambda_i^{(k)}}{\lambda_i^{(k+1)}} \le \varepsilon, \quad i = 1, 2, \ldots, p \tag{7.65}$$

where $\varepsilon = 10^{-6}$. Note that although the iteration is performed with q vectors ($q > p$), the convergence is measured only on the approximations predicted for the p smallest eigenvalues.

7.4 SOLUTION OF PROPAGATION PROBLEMS

When the finite element method is applied for the solution of initial value problems (relating to an unsteady or transient state of phenomena), we obtain propagation problems involving a set of simultaneous linear differential equations.

Propagation problems involve time as one of the independent variables, and initial conditions on the dependent variables are given in addition to the boundary conditions. A general propagation problem can be expressed (after incorporating the boundary conditions) in standard form as

$$\left.\begin{aligned} \frac{d\vec{X}}{dt} &= \vec{F}(\vec{X}, t), \quad t > 0 \\ \vec{X} &= \vec{X}_0, \qquad t = 0 \end{aligned}\right\} \tag{7.66}$$

where the vectors of propagation variables, forcing functions, and initial conditions are given by

$$\vec{X} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} f_1(\vec{X}, t) \\ f_2(\vec{X}, t) \\ \vdots \\ f_n(\vec{X}, t) \end{Bmatrix}, \quad \vec{X}_0 = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{Bmatrix} \tag{7.67}$$

It can be seen that Eq. (7.66) represents a system of n simultaneous ordinary differential equations with n initial conditions.

In certain propagation problems, as in the case of damped mechanical and electrical systems, the governing equations are usually stated as

$$\left.\begin{aligned} &[A]\frac{\mathrm{d}^2\vec{X}}{\mathrm{d}t^2}+[B]\frac{\mathrm{d}\vec{X}}{\mathrm{d}t}+[C]\vec{X}=\vec{F}\left(\vec{X},t\right) && t>0 \\ &\vec{X}=\vec{X}_0 \quad \text{and} \quad \frac{\mathrm{d}\vec{X}}{\mathrm{d}t}=\vec{Y}_0 && t=0 \end{aligned}\right\} \tag{7.68}$$

where $[A]$, $[B]$, and $[C]$ denote known matrices of order $n \times n$. In the case of mechanical and structural systems, the matrices $[A]$, $[B]$, and $[C]$ denote mass, damping, and stiffness matrices, respectively, and the vector $\vec{F}$ represents the known spatial and time history of the external loads. It can be seen that Eq. (7.68) denotes a system of n coupled second-order (or equivalently, 2 n coupled first-order) ordinary differential equations with necessary initial conditions. Eq. (7.66) can be used to represent any n-th order differential equation.

7.4.1 Solution of a Set of First-Order Differential Equations

Eq. (7.66) can be written in scalar form as

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= f_1(t, x_1, x_2, \ldots, x_n) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \tag{7.69}$$

with the initial conditions

$$\begin{aligned} x_1(t=0) &= x_{01} \\ x_2(t=0) &= x_{02} \\ &\vdots \\ x_n(t=0) &= x_{0n} \end{aligned} \tag{7.70}$$

Eqs. (7.69) and (7.70) can be expressed in matrix form as

$$\begin{aligned} \frac{\mathrm{d}\vec{X}}{\mathrm{d}t} &= \vec{F}\left(t, \vec{X}\right) \\ \vec{X}(t=0) &= \vec{X}_0 \end{aligned}$$

where

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \quad \text{and} \quad \vec{F} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$$

These equations can be solved by any of the numerical integration methods, such as Runge–Kutta, Adams–Bashforth, Adams–Moulton, and Hamming [7.18]. In the fourth-order Runge–Kutta method, starting from the known initial vector $\vec{X}_0$ at $t = 0$, we compute the vector $\vec{X}$ after time Δt as

$$\vec{X}(t+\Delta t) = \vec{X}(t) + \frac{1}{6}\left[\vec{K}_1 + 2\vec{K}_2 + 2\vec{K}_3 + \vec{K}_4\right] \tag{7.71}$$

where

$$\begin{aligned}
\vec{K}_1 &= \Delta t \vec{F}\left(\vec{X}(t),\ t\right) \\
\vec{K}_2 &= \Delta t \vec{F}\left(\vec{X}(t) + \frac{\vec{K}_1}{2}, t + \frac{\Delta t}{2}\right) \\
\vec{K}_3 &= \Delta t \vec{F}\left(\vec{X}(t) + \frac{\vec{K}_2}{2}, t + \frac{\Delta t}{2}\right) \\
\vec{K}_4 &= \Delta t \vec{F}\left(\vec{X}(t) + \vec{K}_3,\ \ t + \Delta t\right)
\end{aligned} \tag{7.72}$$

EXAMPLE 7.10

Solve the following second-order differential equation using the fourth-order Runge method (also known as the classical fourth-order Runge–Kutta method):

$$\frac{d^2 y}{dt^2} = ty \tag{E.1}$$

with the initial conditions

$$y(t = t_0 = 0) = 0.3550$$

and

$$\frac{dy}{dt}(t = t_0 = 0) = -0.2588 \tag{E.2}$$

Solution

Eq. (E.1) can be expressed as a set of two first-order differential equations

$$\frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = ty_1 \tag{E.3}$$

which can be expressed in vector form as

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \vec{f} = \begin{Bmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{Bmatrix} = \begin{Bmatrix} y_2 \\ t \cdot y_1 \end{Bmatrix} \tag{E.4}$$

The initial conditions of Eq. (E.2) can be expressed in vector form as

$$\vec{y}(t = 0) = \vec{y}_0 = \begin{Bmatrix} y_{0,1} \\ y_{0,2} \end{Bmatrix} = \begin{Bmatrix} 0.3550 \\ -0.2588 \end{Bmatrix} \tag{E.5}$$

The iterative process of the Runge–Kutta method, given by Eqs. (7.71) and (7.72), can be used with $\vec{X}$ and $\vec{F}$ replaced by the two-component vectors $\vec{y}$ and $\vec{f}$, respectively. Starting from $\vec{y}_0$, using a step size of $h = 0.2$, the vector $\vec{y}_1 = \vec{y}(t = t_1 = h = 0.2)$ can be generated from Eq. (7.71):

$$\vec{y}_1 = \vec{y}_0 + \frac{h}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right) \tag{E.6}$$

where

EXAMPLE 7.10 —cont'd

$$\vec{k}_1 = \vec{f}(t_0, \vec{y}_0) = \begin{Bmatrix} y_{0,2} \\ t_0\, y_{0,1} \end{Bmatrix} = \begin{Bmatrix} -0.2588 \\ 0 \end{Bmatrix}$$

$$\vec{k}_2 = \vec{f}\left(t_0 + \frac{h}{2}, \vec{y}_0 + \frac{h}{2}\vec{k}_1\right) = \vec{f}\left(t = 0.1, \begin{Bmatrix} 0.3550 \\ -0.2588 \end{Bmatrix} + 0.1 \begin{Bmatrix} -0.2588 \\ 0 \end{Bmatrix}\right)$$

$$= \vec{f}\left(0.1, \begin{Bmatrix} 0.32912 \\ -0.2588 \end{Bmatrix}\right) = \begin{Bmatrix} -0.2588 \\ 0.1(0.32912) \end{Bmatrix} = \begin{Bmatrix} -0.2588 \\ 0.032912 \end{Bmatrix}$$

$$\vec{k}_3 = \vec{f}\left(t_0 + \frac{h}{2}, \vec{y}_0 + \frac{h}{2}\vec{k}_2\right) = \vec{f}\left(t = 0.1, \begin{Bmatrix} 0.3550 \\ -0.2588 \end{Bmatrix} + 0.1 \begin{Bmatrix} -0.2588 \\ 0.032912 \end{Bmatrix}\right)$$

$$= \vec{f}\left(0.1, \begin{Bmatrix} 0.32912 \\ -0.255509 \end{Bmatrix}\right) = \begin{Bmatrix} -0.255509 \\ 0.1(0.32912) \end{Bmatrix} = \begin{Bmatrix} -0.255509 \\ 0.032912 \end{Bmatrix}$$

$$\vec{k}_4 = \vec{f}\left(t_0 + h, \vec{y}_0 + h\,\vec{k}_3\right) = \vec{f}\left(t = 0.2, \begin{Bmatrix} 0.3550 \\ -0.2588 \end{Bmatrix} + 0.2 \begin{Bmatrix} -0.255509 \\ 0.032912 \end{Bmatrix}\right)$$

$$= \vec{f}\left(0.2, \begin{Bmatrix} 0.303898 \\ -0.252218 \end{Bmatrix}\right) = \begin{Bmatrix} -0.252218 \\ 0.2(0.303898) \end{Bmatrix} = \begin{Bmatrix} -0.252218 \\ 0.060780 \end{Bmatrix}$$

With these values, Eq. (E.6) can be used to generate $\vec{y}_1$ as follows:

$$\vec{y}_1 = \vec{y}_0 + \frac{h}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right)$$

$$= \begin{Bmatrix} 0.3550 \\ -0.2588 \end{Bmatrix} + \frac{0.2}{6}\left(\begin{Bmatrix} -0.2588 \\ 0 \end{Bmatrix} + 2\begin{Bmatrix} -0.2588 \\ 0.032912 \end{Bmatrix} + 2\begin{Bmatrix} -0.255509 \\ 0.032912 \end{Bmatrix} + \begin{Bmatrix} -0.252218 \\ 0.060780 \end{Bmatrix}\right) \tag{E.6}$$

$$= \begin{Bmatrix} 0.303679 \\ -0.252386 \end{Bmatrix}$$

With the value of $\vec{y}_1$ known, the procedure can be continued to generate $\vec{y}_2$, $\vec{y}_3$, ... at $t = 0.4, 0.6, \ldots$.

7.4.2 Numerical Solution of Eq. (7.68)

Several methods are available for the solution of Eq. (7.68). All of these methods can be divided into two classes: direct integration methods and the mode superposition method.

DIRECT INTEGRATION METHODS

In these methods, Eq. (7.68), or the special case, Eq. (7.66), is integrated numerically by using a step-by-step procedure [7.19]. The term direct denotes that no transformation of the equations into a different form is used prior to numerical integration. The direct integration methods are based on the following ideas:

a. Instead of trying to find a solution $\vec{X}(t)$ that satisfies Eq. (7.68) for any time t, we can try to satisfy Eq. (7.68) only at discrete time intervals Δt apart.

b. Within any time interval, the nature of variation of $\vec{X}$ (displacement), $\dot{\vec{X}}$(velocity), and $\ddot{\vec{X}}$ (acceleration) can be assumed in a suitable manner.

Here, the time interval Δt and the nature of variation of $\vec{X}$, $\dot{\vec{X}}$, and $\ddot{\vec{X}}$ within any Δt are chosen by considering factors such as accuracy, stability, and cost of solution. The finite difference, Houbolt, Wilson, and Newmark methods fall under the category of direct methods [7.20−7.22]. The finite difference method (a direct integration method) is outlined next.

FINITE DIFFERENCE METHOD

By using central difference formulas [7.23], the velocity and acceleration at any time t can be expressed as

$$\dot{\vec{X}}_t = \frac{1}{2\Delta t}\left(-\vec{X}_{t-\Delta t} + \vec{X}_{t+\Delta t}\right) \tag{7.73}$$

$$\ddot{\vec{X}}_t = \frac{1}{(\Delta t)^2}\left(\vec{X}_{t-\Delta t} - 2\vec{X}_t + \vec{X}_{t+\Delta t}\right) \tag{7.74}$$

If Eq. (7.68) is satisfied at time t, we have

$$[A]\ddot{\vec{X}}_t + [B]\dot{\vec{X}}_t + [C]\vec{X}_t = \vec{F}_t \tag{7.75}$$

By substituting Eqs. (7.73) and (7.74) in Eq. (7.75), we obtain

$$\left(\frac{1}{(\Delta t)^2}[A] + \frac{1}{2\Delta t}[B]\right)\vec{X}_{t+\Delta t} = \vec{F}_t - \left([C] - \frac{2}{(\Delta t)^2}[A]\right)\vec{X}_t - \left(\frac{1}{(\Delta t)^2}[A] - \frac{1}{2\Delta t}[B]\right)\vec{X}_{t-\Delta x} \tag{7.76}$$

Eq. (7.76) can now be solved for $\vec{X}_{t+\Delta t}$. Thus, the solution $\vec{X}_{t+\Delta t}$ is based on the equilibrium conditions at time t. Since the solution of $\vec{X}_{t+\Delta t}$ involves $\vec{X}_t$ and $\vec{X}_{t-\Delta t}$, we need to know $\vec{X}_{-\Delta t}$ for finding $\vec{X}_{\Delta t}$. For this we first use the initial conditions $\vec{X}_0$ and $\dot{\vec{X}}_0$ to find $\ddot{\vec{X}}_0$ using Eq. (7.75) for $t = 0$. Then we compute $\vec{X}_{-\Delta t}$ using Eqs. (7.73) to (7.75) as

$$\vec{X}_{-\Delta t} = \vec{X}_o - \Delta t\dot{\vec{X}}_0 + \frac{(\Delta t)^2}{2}\ddot{\vec{X}}_0 \tag{7.77}$$

A disadvantage of the finite difference method is that it is conditionally stable—that is, the time step Δt has to be smaller than a critical time step $(\Delta t)_{cri}$. If the time step Δt is larger than $(\Delta t)_{\mathrm{cri}}$, the integration is unstable in the sense that any errors resulting from the numerical integration or round-off in the computations grow and make the calculation of $\vec{X}$ meaningless in most cases.

NEWMARK METHOD

The basic equations of the Newmark method (or Newmark's β method) [7.20] follow:

$$\dot{\vec{X}}_{t+\Delta t} = \dot{\vec{X}}_t + (1-\gamma)\Delta t\ddot{\vec{X}}_t + \Delta t\gamma\ddot{\vec{X}}_{t+\Delta t} \tag{7.78}$$

$$\vec{X}_{t+\Delta t} = \vec{X}_t + \Delta t\dot{\vec{X}}_t + \left(\frac{1}{2} - \beta\right)(\Delta t)^2\ddot{\vec{X}}_t + \beta(\Delta t)^2\ddot{\vec{X}}_{t+\Delta t} \tag{7.79}$$

where γ and β are parameters that can be determined depending on the desired accuracy and stability. Newmark suggested a value of $\gamma = 1/2$ for avoiding artificial damping. The value of β depends on the way in which the acceleration, $\ddot{\vec{X}}$, is

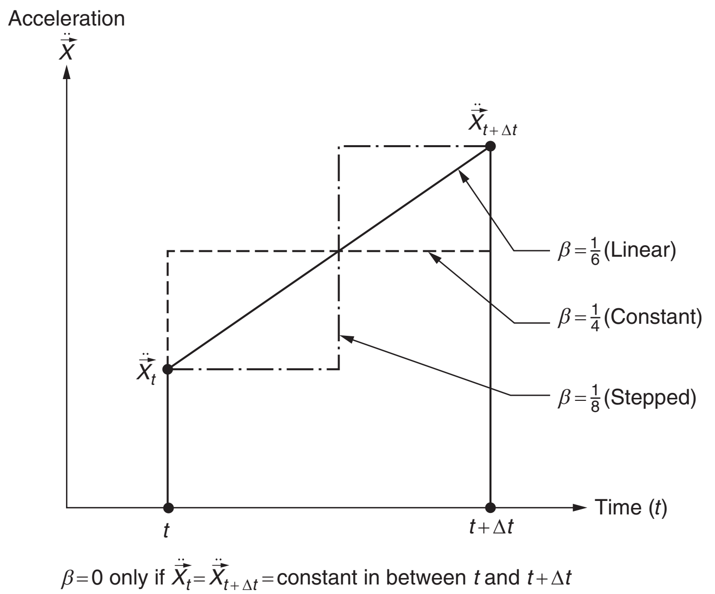

FIGURE 7.2 Values of β for different types of variation of $\ddot{\vec{X}}$.

assumed to vary during the time interval t and $t+\Delta t$. The values of β to be taken for different types of variation of $\ddot{\vec{X}}$ are shown in Fig. 7.2.

In addition to Eqs. (7.78) and (7.79), Eq. (7.68) is also assumed to be satisfied at time $t+\Delta t$ so that

$$[A]\ddot{\vec{X}}_{t+\Delta t}+[B]\dot{\vec{X}}_{t+\Delta t}+[C]\vec{X}_{t+\Delta t}=\vec{F}_{t+\Delta t} \tag{7.80}$$

To find the solution at the $t+\Delta t$, we solve Eq. (7.79) to obtain $\ddot{\vec{X}}_{t+\Delta t}$ in terms of $\vec{X}_{t+\Delta t}$, substitute this $\ddot{\vec{X}}_{t+\Delta t}$ into Eq. (7.78) to obtain $\dot{\vec{X}}_{t+\Delta t}$ in terms of $\vec{X}_{t+\Delta t}$, and then use Eq. (7.80) to find $\vec{X}_{t+\Delta t}$. Once $\vec{X}_{t+\Delta t}$ is known, $\dot{\vec{X}}_{t+\Delta t}$ and $\ddot{\vec{X}}_{t+\Delta t}$ can be calculated from Eqs. (7.78) and (7.79).

MODE SUPERPOSITION METHOD

It can be seen that the computational work involved in the direct integration methods is proportional to the number of time steps used in the analysis. Hence, in general, the use of direct integration methods is expected to be effective when the response over only a relatively short duration (involving few time steps) is required. On the other hand, if the integration has to be carried for many time steps, it may be more effective to transform Eq. (7.68) into a form in which the step-by-step solution is less costly. The mode superposition or normal mode method is a technique wherein Eq. (7.68) is first transformed into a convenient form before integration is carried. Thus, the vector X is transformed as

$$\underset{n\times 1}{\vec{X}(t)}=\underset{n\times r}{[T]}\;\underset{r\times 1}{\vec{Y}(t)} \tag{7.81}$$

where $[T]$ is a rectangular matrix of order $n\times r$, and $\vec{Y}(t)$ is a time-dependent vector of order $r(r\le n)$. The transformation matrix $[T]$ is still unknown and will have to be determined. Although the components of $\vec{X}$ have physical meaning (like displacements), the components of $\vec{Y}$ need not have any physical meaning and hence are called generalized displacements. By substituting Eq. (7.81) into Eq. (7.68), and premultiplying throughout by $[T]^T$, we obtain

$$[\underset{\sim}{A}]\ddot{\vec{Y}}+[\underset{\sim}{B}]\dot{\vec{Y}}+[\underset{\sim}{C}]\vec{Y}=\underset{\sim}{\vec{F}} \tag{7.82}$$

where

$$[\underset{\sim}{A}]=[T]^T[A][T] \tag{7.83}$$

$$[\underset{\sim}{B}] = [T]^T[B][T] \tag{7.84}$$

$$[\underset{\sim}{C}] = [T]^T[C][T] \tag{7.85}$$

and

$$\vec{\underset{\sim}{F}} = [T]^T\vec{F} \tag{7.86}$$

The basic idea behind using the transformation of Eq. (7.81) is to obtain the new system of Eq. (7.82) in which the matrices $[\underset{\sim}{A}]$, $[\underset{\sim}{B}]$, and $[\underset{\sim}{C}]$ will be of much smaller order than the original matrices [A], [B], and [C]. Furthermore, the matrix [T] can be chosen so as to obtain the matrices $[\underset{\sim}{A}]$, $[\underset{\sim}{B}]$, and $[\underset{\sim}{C}]$ in diagonal form, in which case Eq. (7.82) represents a system of r uncoupled second-order differential equations. The solution of these independent equations can be found by standard techniques, and the solution of the original problem can be found with the help of Eq. (7.81).

In the case of structural mechanics problems, the matrix [T] denotes the modal matrix and Eq. (7.82) can be expressed in scalar form as (see Section 12.5)

$$\ddot{Y}_i(t) + 2\zeta_i\omega_i\dot{Y}_i(t) + \omega_i^2 Y_i(t) = N_i(t), \quad i = 1, 2, \ldots, r \tag{7.87}$$

where the matrices $[\underset{\sim}{A}]$, $[\underset{\sim}{B}]$, and $[\underset{\sim}{C}]$ have been expressed in diagonal form (assuming proportional damping) as [7.20].

$$[\underset{\sim}{A}] = [I], \quad [\underset{\sim}{B}] = \begin{bmatrix} \ddots & & \\ & 2\zeta_i\omega_i & \\ & & \ddots \end{bmatrix}, \quad [\underset{\sim}{C}] = \begin{bmatrix} \ddots & & \\ & \omega_i^2 & \\ & & \ddots \end{bmatrix} \tag{7.88}$$

and the vector $\vec{\underset{\sim}{F}}$ as

$$\vec{\underset{\sim}{F}} = \begin{Bmatrix} N_1(t) \\ \vdots \\ N_r(t) \end{Bmatrix} \tag{7.89}$$

Here, ω_i is the rotational frequency (square root of the eigenvalue) corresponding to the i-th natural mode (eigenvector), and ζ_i is the modal damping ratio in the i-th natural mode.

7.5 PARALLEL PROCESSING IN FINITE ELEMENT ANALYSIS

Parallel processing is defined as the exploitation of parallel or concurrent events in the computing process [7.24]. Parallel processing techniques are being investigated because of the high degree of sophistication of the computational models required for future aerospace, transportation, nuclear, and microelectronic systems. Efforts have been devoted to the development of vectorized numerical algorithms for performing the matrix operations, solution of algebraic equations, and extraction of eigenvalues [7.25,7.26]. However, the progress has been slow, and no effective computational strategy exists that performs the entire finite element solution in the parallel processing mode.

The various phases of the finite element analysis can be identified as (1) input of problem characteristics, element and nodal data, and geometry of the system; (2) data preprocessing; (3) evaluation of element characteristics; (4) assembly of elemental contributions; (5) incorporation of boundary conditions; (6) solution of system equations; and (7) postprocessing of the solution and evaluation of secondary fields.

The input and preprocessing phases can be parallelized. Since the element characteristics require only information pertaining to the elements in question, they can be evaluated in parallel. The assembly cannot utilize the parallel operation efficiently since the element and global variables are related through a Boolean transformation. The incorporation of boundary conditions, although usually not time consuming, can be done in parallel.

The solution of system equations is the most critical phase. For static linear problems, the numerical algorithm should be selected to take advantage of the symmetric banded structure of the equations and the type of hardware used. A variety of efficient direct iterative and noniterative solution techniques have been developed for different computers by exploiting the parallelism, pipeline (or vector), and chaining capabilities [7.27]. For nonlinear steady-state problems, the data structure is essentially the same as for linear problems. The major difference lies in the algorithms for evaluating the nonlinear terms and solving the nonlinear algebraic equations. For transient problems, several parallel integration techniques have been proposed [7.28].

The parallel processing techniques are still evolving and are expected to be the dominant methodologies in the computing industry in the near future. Hence, it can be hoped that the full potentialities of parallel processing in finite element analysis will be realized in the next decade.

REVIEW QUESTIONS

7.1 Give brief answers to the following questions.

1. What is the difference between continuous and discrete problems?
2. State the general form of the governing finite element equations for a typical equilibrium problem with boundary conditions.
3. State the general form of the governing finite element equations for a typical eigenvalue problem with boundary conditions.
4. State the general form of the governing finite element equations for a typical propagation problem with boundary and initial conditions.
5. What is the important characteristic of direct methods of solving simultaneous linear algebraic equations?
6. What is the purpose of the Rayleigh—Ritz subspace iteration method?
7. What type of problem can be solved using the fourth-order Runge—Kutta method?
8. Indicate two methods of solving damped mechanical/structural dynamics problem using numerical integration methods.
9. Which method can be used to find an approximate solution of a propagation problem involving large size matrices by using only small size matrices?

7.2 Fill in the blank with a suitable word.

1. The basic procedure for solving a system of linear algebraic equations is called ———————— elimination method.
2. A sparse matrix is a thinly populated matrix having very few ———————— elements.
3. Gaussian elimination method transforms a given sent of linear equations into a ———————— form of equations.
4. If a pivot operation becomes impossible, it implies that the matrix $[A]$ is ———————— in the Gaussian elimination method.
5. The basic idea in the Choleski method is the ———————— of a square symmetric matrix into lower and upper triangular matrices.
6. A propagation problem can be expressed as asset of first order ———————— equations.

7.3 Indicate whether each of the following statements is true or false.

1. The static and steady-state problems are similar.
2. Finite element equations cannot be solved using parallel processing techniques.
3. Any square symmetric matrix can be expressed in terms of only upper triangular matrix or only lower triangular matrix.
4. If $[A] = [U]^T[U]$ where $[U]$ is an upper triangular matrix, then $[A]^{-1} = \left([U]^T\right)^{-1}[U]^{-1}$.
5. Eigenvectors of a matrix are orthogonal.
6. If $[A]$ and $[B]$ are symmetric matrices, then the matrix defined as $[B]^{-1}[A]$ is symmetric.
7. Jacobi method finds one eigenvalue at a time.
8. Any arbitrary vector can be expressed as a linear combination of the eigenvectors.
9. Unsteady state problems are same as propagation problems.
10. Any nth order differential equation can be expressed as a set of n first-order differential equations.

7.4 Select the most appropriate answer from the multiple choices given.

1. Jacobi method of solving eigenvalue problems is:
   (a) Indirect method (b) Transformation method (c) Iterative method

7.5 Match the following:

1. Static problem	(a) Propagation problem
2. Transient problem	(b) Eigenvalue problem
3. Characteristic value problem	(c) Equilibrium problem

7.6 Match the following:

1. Gaussian elimination method	(a) Solution of propagation problems
2. Cheloski method	(b) Lowest eigenvalues and eigenvectors
3. Gauss–Seidel method	(c) Largest eigenvalue
4. Jacobi method	(d) Direct method for solving linear equations
5. Power method	(e) Decomposing a matrix
6. Rayleigh–Ritz subspace iteration method	(f) Iterative solution of linear equations
7. Newmark method	(g) Finding all eigenvalues

PROBLEMS

7.1 Find the inverse of the following matrix using the decomposition $[A]=[U]^T[U]$:

$$[A]=\begin{bmatrix}5 & -1 & 1\\ -1 & 6 & -4\\ 1 & -4 & 3\end{bmatrix}$$

7.2 Find the inverse of the matrix $[A]$ given in Problem 7.1 using the decomposition $[A]=[L][L]^T$, where $[L]$ is a lower triangular matrix.

Hint: If a symmetric matrix $[A]$ of order n is decomposed as $[A]=[L][L]^T$, the elements of $[L]$ are given by

$$l_{ii}=\left[a_{ii}-\sum_{k=1}^{i-1} l_{ik}^2\right]^{(1/2)}, \quad i=1,\ 2,\ldots,n$$

$$l_{mi}=\frac{1}{l_{ii}}\left[a_{mi}-\sum_{k=1}^{i-1} l_{ik}l_{mk}\right], \quad m=i+1,\ldots,n \quad \text{and} \quad i=1,\ 2,\ldots,n$$

$$l_{\mathrm{ij}}=0, \quad i<j$$

The elements of $[L]^{-1}=[\lambda_{ij}]$ can be obtained from the relation $[L][L]^{-1}=[l_{ij}][\lambda_{ij}]=[I]$ as

$$\lambda_{ii}=\frac{1}{l_{ii}}, \quad i=1,2,\ldots,n$$

$$\lambda_{ij}=-\left(\sum_{k=j}^{i-1} l_{ik}\lambda_{kj}\right)\Big/l_{ii}, \quad i>j$$

$$\lambda_{ij}=0, \quad i<j$$

7.3 Express the following functions in matrix form as $f=(1/2)\ \vec{X}^T[A]\vec{X}$ and identify the matrix $[A]$:

$$f=6x_1^2+49x_2^2+51x_3^2-82x_2x_3+20x_1x_3-4x_1x_2$$

$$f=6x_1^2+3x_2^2+3x_3^2-4x_1x_2-2x_2x_3+4x_1x_3$$

7.4 Find the eigenvalues and eigenvectors of the following problem by solving the characteristic polynomial equation:

$$\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}$$

7.5 Find the eigenvalues and eigenvectors of the following matrix by solving the characteristic equation:

$$[A]=\begin{bmatrix}1 & 2 & 0\\ 2 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}$$

7.6 Find the eigenvalues and eigenvectors of the following matrix using the Jacobi method:

$$[A] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

7.7 Find the eigenvalues and eigenvectors of the matrix $[A]$ given in Problem 7.6 using the power method.

7.8 Solve the following system of equations using the finite difference method:

$$[A]\ddot{\vec{X}} + [C]\vec{X} = \vec{F}$$

where

$$[A] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}, \quad \text{and} \quad \vec{F} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

with the initial conditions

$$\vec{X}(t=0) = \dot{\vec{X}}(t=0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Take the time step Δt as 0.28 and find the solution at $t = 4.2$.

7.9 Find the solution of the following equations using the Gaussian elimination method:

$$\begin{bmatrix} 4 & 2 & 4 & 5 \\ 3 & 9 & 12 & 15 \\ 2 & 4 & 11 & 10 \\ 1 & 2 & 4 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

7.10 Find the solution of the following equations using the Gaussian elimination method:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

7.11 Find the inverse of the following matrix, for $n = 4$, using the Choleski decomposition method:

$$\begin{bmatrix} \frac{n+2}{2n+2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2n+2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2n+2} & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & \frac{n+2}{2n+2} \end{bmatrix}$$

Hint: The first, second, ..., nth columns of $[A]^{-1}$ are nothing but the solutions $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ corresponding to the right-hand-side vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n$, respectively, where

$$\vec{b}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \vec{b}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \ldots, \vec{b}_n = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix}$$

7.12 Find the inverse of the following matrix, for $n = 4$, using the Gaussian elimination method:

$$\underset{n \times n}{[A]} = \begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & n-1 & n-2 & & 2 & 1 \\ n-2 & n-2 & n-2 & & 2 & 1 \\ \vdots & & & & & \\ 2 & 2 & 2 & & 2 & 1 \\ 1 & 1 & 1 & & 1 & 1 \end{bmatrix}$$

7.13 a. Convert the following eigenvalue problem into a standard form:

$$\begin{bmatrix} 4 & -6 & 2 & 0 \\ -6 & 24 & 0 & 6 \\ 2 & 0 & 8 & 2 \\ 0 & 6 & 2 & 4 \end{bmatrix} \vec{X} = \frac{\lambda}{420} \begin{bmatrix} 4 & 13 & -3 & 0 \\ 13 & 312 & 0 & -13 \\ -3 & 0 & 8 & -3 \\ 0 & -13 & -3 & 4 \end{bmatrix} \vec{X}$$

b. Find the eigenvalues of the system by finding the roots of the determinantal equation.

7.14 Find the eigenvalues and eigenvectors of the matrix $[A]$ given in Problem 7.12 (with $n = 3$) using the Jacobi method.

7.15 Solve the following system of equations using the Choleski decomposition method using $[L][L]^T$ decomposition and $[U]^T[U]$ decomposition:

$$5x_1 + 3x_2 + x_3 = 14$$

$$3x_1 + 6x_2 + 2x_3 = 21$$

$$x_1 + 2x_2 + 3x_3 = 14$$

7.16 Express the following set of equations as a system of first-order equations:

$$\frac{d^2x}{dt^2} = x^2 - y + e^t$$

$$\frac{d^2y}{dt^2} = x + y^2 - e^t$$

$$\text{At} \quad t = 0: \quad x(0) = \frac{dx}{dt}(0) = 0; \quad y(0) = 1, \quad \frac{dy}{dt}(0) = -2$$

Obtain the solution of these equations using the fourth-order Runge–Kutta method.

7.17 Solve the following equations using the Gauss elimination method:

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

7.18 The finite element analysis of certain systems leads to a tridiagonal system of equations, $[A]\vec{x} = \vec{b}$, where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \dots & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \dots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}; \quad \vec{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Indicate a method of solving these equations.

7.19 Solve the following system of equations using a suitable procedure:

$$[A]\vec{x} = \vec{b}$$

with

$$[A] = \begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ -5 & 10 & -5 & 0 & 0 \\ 0 & -5 & 10 & -5 & 0 \\ 0 & 0 & -5 & 10 & -5 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{Bmatrix} \quad \text{and} \quad \vec{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{Bmatrix}$$

7.20 The elements of the Hillbert matrix, $[A] = [a_{ij}]$, are given by

$$a_{ij} = \frac{1}{i+j-1}; \quad i,j = 1,2,\dots,n$$

Find the inverse of the matrix, $[A]^{-1} = [b_{ij}]$, with $n = 4$ using the Gaussian elimination method, and compare the result with the exact solution given by

$$b_{ij} = \frac{(-1)^{i+j}(n+i-1)!(n+j-1)!}{(i+j-1)\{(i-1)!(j-1)!\}^2(n-i)!(n-j)!}, \quad i,j = 1,2,\dots,n$$

7.21 Express the n-th-order differential equation

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

as a set of n first-order differential equations.

7.22 Express the matrix

$$[A] = \begin{bmatrix} 15 & -5 & 0 & -5 \\ -5 & 12 & -2 & 0 \\ 0 & -2 & 6 & -2 \\ -5 & 0 & -2 & 9 \end{bmatrix} \tag{P.1}$$

as a product of lower and upper triangular matrices using the Choleski decomposition method.

7.23 Solve the following equations using the Choleski decomposition method:

$$[A]\vec{X} = \vec{b}$$

with the matrix $[A]$ given by Eq. (P.1) of Problem 7.22 and

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \quad \text{and} \quad \vec{b} = \begin{Bmatrix} -6.90 \\ 19.92 \\ 0 \\ 0 \end{Bmatrix}$$

7.24 Solve the following equations using the Gauss elimination method:

$$15x_1 - 5x_2 - 5x_4 = -6.90$$

$$-5x_1 + 12x_2 - 2x_3 = 19.92$$

$$-2x_2 + 6x_3 - 2x_4 = 0$$

$$-5x_1 - 2x_3 + 9x_4 = 0$$

7.25 Find the inverse of the matrix $[A]$ given in Eq. (P.1) of Problem 7.22 based on LU decomposition.

7.26 Solve the equations

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 4 \\ -6 \end{Bmatrix}$$

using the Gauss elimination method.

7.27 Solve the equations given in Problem 7.26 using the Choleski decomposition method.

7.28 Find the inverse of the matrix

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

using the Choleski decomposition.

7.29 Express the matrix $[A]$ in the form $[A] = [L][U]$ using the Choleski decomposition method when

$$[A] = \begin{bmatrix} 10 & -4 & 5 \\ -4 & 7 & 6 \\ 5 & 6 & -3 \end{bmatrix} \tag{P.2}$$

7.30 Find the inverse of the matrix $[A]$ given by Eq. (P.2) of Problem 7.29 using Choleski decomposition.

7.31 Solve the equations $[A]\,\vec{X} = \vec{b}$ when $[A]$ is given by Eq. (P.2) of Problem 7.29 and

$$\vec{b} = \begin{Bmatrix} 17 \\ 28 \\ 2 \end{Bmatrix}$$

using Choleski decomposition.

7.32 Find the eigenvalues of the matrix

$$[A] = \begin{bmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{bmatrix}$$

by finding the roots of the determinantal equation.

7.33 Find the eigenvectors of the matrix [A] given in Problem 7.32.

7.34 Find the largest eigenvalue and the corresponding eigenvector of the matrix [A] given in Problem 7.32 using the power method.

7.35 Find the second and third eigenvalues and the corresponding eigenvectors of the matrix [A] given in Problem 7.32 using the power method.

7.36 Find the eigenvalues and eigenvectors of the matrix [A] given in Problem 7.28 using the Jacobi method.

7.37 Find the eigenvalues of the matrix

$$[A] = \begin{bmatrix} 1 & 0.5 & 0.3333 \\ 0.5 & 0.3333 & 0.25 \\ 0.3333 & 0.25 & 0.2 \end{bmatrix}$$

using the Jacobi method.

7.38 Find the eigenvalues of the matrix [A] given in Problem 7.37 by finding the roots of the determinantal equation.

7.39 Find the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + 2x - 1; \quad 0 \le x \le 1$$

$$y(x = 0) = 1$$

using the fourth-order Runge–Kutta method.

7.40 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^3 + 12x^2 - 20x + 8.5; \quad 0 \le x \le 4$$

$$y(x = 0) = 1$$

using the fourth-order Runge–Kutta method.

7.41 The equation of motion of a simple pendulum, subjected to damping and external torque $M_t(t)$, is given by

$$ml^2\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + mgl \sin\theta = M_t(t)$$

where m is the mass of the bob, l is the length, c is the damping constant, g is the acceleration due to gravity, θ is the angular displacement, and t is the time. Express this equation as a system of two linear first-order differential equations.

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Part III

Application to Solid Mechanics Problems

Chapter 8

Basic Equations and Solution Procedure

Chapter Outline

8.1 INTRODUCTION

As stated in Chapter 1, the finite element method has been most extensively used in the field of solid and structural mechanics. The various types of problems solved by the finite element method in this field include elastic, elastoplastic, and viscoelastic analysis of trusses, frames, plates, shells, and solid bodies. Both static and dynamic analyses have been conducted using the finite element method. We consider the finite element elastic analysis of one-, two-, and three-dimensional problems as well as axisymmetric problems in this book.

In this chapter, the general equations of solid and structural mechanics are presented. The displacement method (or equivalently the principle of minimum potential energy) is used in deriving the finite element equations. The application of these equations to several specific cases is considered in subsequent chapters.

8.2 BASIC EQUATIONS OF SOLID MECHANICS

8.2.1 Introduction

The primary aim of any stress analysis or solid mechanics problem is to find the distribution of displacements and stresses under the stated loading and boundary conditions. If an analytical solution of the problem is to be found, we must satisfy the following basic or fundamental equations of solid mechanics.

	Number of Equations		
Type of Equations	In Three-Dimensional Problems	In Two-Dimensional Problems	In One-Dimensional Problems
Equilibrium equations	3	2	1
Stress—strain relations	6	3	1
Strain—displacement relations	6	3	1
Total number of equations	15	8	3

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00008-0

Unknowns	In Three-Dimensional Problems	In Two-Dimensional Problems	In One-Dimensional Problems
Displacements	u, v, w	u, v	u
Stresses	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz},$	$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	σ_{xx}
	$\sigma_{xy}, \sigma_{yz}, \sigma_{zx}$		
Strains	$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}$	$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$	ε_{xx}
	$\varepsilon_{yz}, \varepsilon_{zx}$		
Total number of unknowns	15	8	3

Thus, we have as many equations as there are unknowns to find the solution of any stress analysis problem. In practice, we will also have to satisfy some additional equations, such as external equilibrium equations (which pertain to the overall equilibrium of the body under external loads), compatibility equations (which pertain to the continuity of strains and displacements), and boundary conditions (which pertain to the prescribed conditions on displacements and/or forces at the boundary of the body).

Although any analytical (exact) solution has to satisfy all the equations stated previously, the numerical (approximate) solutions, like the ones obtained by using the finite element method, generally do not satisfy all the equations. However, a sound understanding of all the basic equations of solid mechanics is essential in deriving the finite element relations and also in estimating the order of error involved in the finite element solution by knowing the extent to which the approximate solution violates the basic equations, including the compatibility and boundary conditions. Hence, the basic equations of solid mechanics are summarized in the following section for ready reference in the formulation of finite element equations.

8.2.2 Equations

EXTERNAL EQUILIBRIUM EQUATIONS

If a body is in equilibrium under specified static loads, the reactive forces and moments developed at the support points must balance the externally applied forces and moments. In other words, the force and moment equilibrium equations for the overall body (overall or external equilibrium equations) have to be satisfied. If ϕ_x, ϕ_y, and ϕ_z are the body forces; Φ_x, Φ_y, and Φ_z are the surface (distributed) forces; P_x, P_y, and P_z are the external concentrated loads (including reactions at support points such as B, C, and D in Fig. 8.1); and Q_x, Q_y, and Q_z are the external concentrated

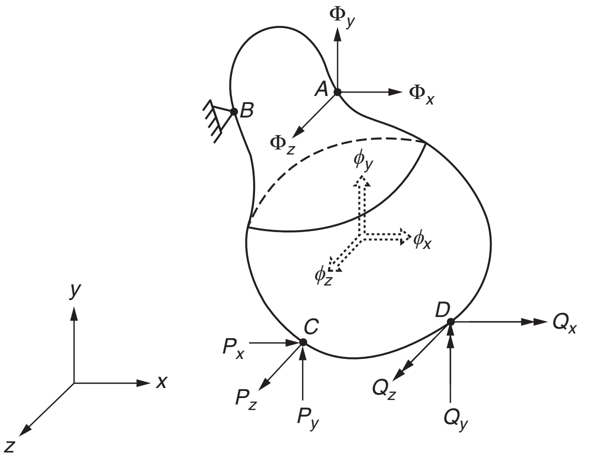

FIGURE 8.1 Force system for macroequilibrium of a body.

moments (including reactions at support points such as B, C, and D in Fig. 8.1); the external equilibrium equations can be stated as follows [8.1]:

$$\left.\begin{aligned}\int_S \Phi_x \mathrm{d}s + \int_V \phi_x \mathrm{dV} + \sum P_x &= 0\\ \int_S \Phi_y \mathrm{d}s + \int_V \phi_y \mathrm{dV} + \sum P_y &= 0\\ \int_S \Phi_z \mathrm{d}s + \int_V \phi_z \mathrm{dV} + \sum P_x &= 0\end{aligned}\right\} \tag{8.1}$$

For moment equilibrium:

$$\left.\begin{aligned}\int_S (\Phi_z y - \Phi_y z)\mathrm{d}s + \int_V (\phi_z y - \phi_y z)\mathrm{dV} + \sum Q_x &= 0\\ \int_S (\Phi_x z - \Phi_z x)\mathrm{d}s + \int_V (\phi_x z - \phi_z x)\mathrm{dV} + \sum Q_y &= 0\\ \int_S (\Phi_y x - \Phi_x y)\mathrm{d}s + \int_V (\phi_y x - \phi_x y)\mathrm{dV} + \sum Q_z &= 0\end{aligned}\right\} \tag{8.2}$$

where S is the surface and V is the volume of the solid body.

EQUATIONS OF INTERNAL EQUILIBRIUM

Due to the application of loads, stresses will be developed inside the body. If we consider an element of material inside the body, it must be in equilibrium due to the internal stresses developed. This leads to equations known as internal equilibrium equations.

Theoretically, the state of stress at any point in a loaded body is completely defined in terms of the nine components of stress σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yx}, σ_{yz}, σ_{zy}, σ_{zx}, and σ_{xz}, where the first three are the normal components and the latter six are the components of shear stress. The equations of internal equilibrium relating the nine components of stress can be derived by considering the equilibrium of moments and forces acting on the elemental volume shown in Fig. 8.2. The equilibrium of moments about the x, y, and z axes, assuming that there are no body moments, leads to the relations

$$\sigma_{yx} = \sigma_{xy}, \quad \sigma_{zy} = \sigma_{yz}, \quad \sigma_{xz} = \sigma_{zx} \tag{8.3}$$

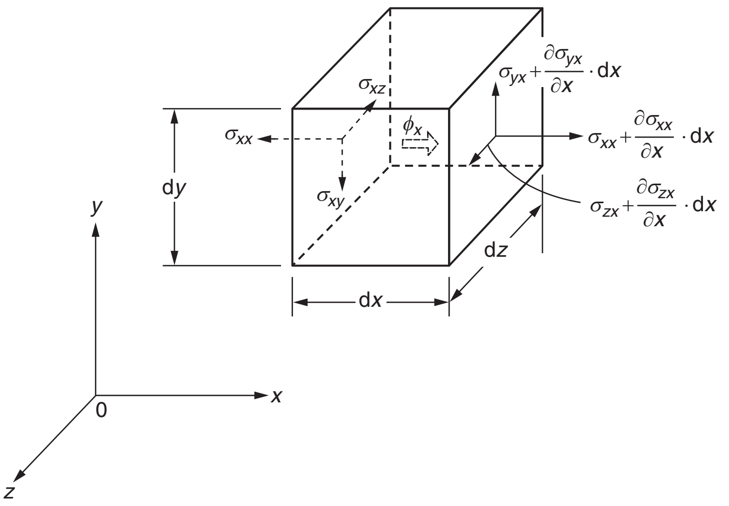

FIGURE 8.2 Elemental volume considered for internal equilibrium. (Only the components of stress acting on a typical pair of faces are shown for simplicity.)

These equations show that the state of stress at any point can be completely defined by the six components σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz}, and σ_{zx}. The equilibrium of forces in the x, y, and z directions gives the following differential equilibrium equations:

$$\left.\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}+\phi_x &= 0\\ \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{yz}}{\partial z}+\phi_y &= 0\\ \frac{\partial \sigma_{zx}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}+\phi_z &= 0\end{aligned}\right\} \tag{8.4}$$

where ϕ_x, ϕ_y, and ϕ_z are the body forces per unit volume acting along the directions x, y, and z, respectively.

For a two-dimensional problem, there will be only three independent stress components (σ_{xx}, σ_{yy}, σ_{xy}) and the equilibrium equations of Eq. (8.4) reduce to

$$\left.\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\phi_x &= 0\\ \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\phi_y &= 0\end{aligned}\right\} \tag{8.5}$$

In one-dimensional (axial) problems, only one component of stress, namely σ_{xx}, will be present and hence Eq. (8.4) reduces to

$$\frac{\partial \sigma_{xx}}{\partial x}+\phi_x = 0 \tag{8.6}$$

STRESS—STRAIN RELATIONS (CONSTITUTIVE RELATIONS) FOR ISOTROPIC MATERIALS

Three-dimensional case: In the case of a linear, elastic, isotropic three-dimensional solid, the stress—strain relations are given by Hooke's law as follows:

$$\vec{\varepsilon} = \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix} = [C]\vec{\sigma}+\vec{\varepsilon}_0 \equiv [C]\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}+\begin{Bmatrix}\varepsilon_{xx0}\\ \varepsilon_{yy0}\\ \varepsilon_{zz0}\\ \varepsilon_{xy0}\\ \varepsilon_{yz0}\\ \varepsilon_{zx0}\end{Bmatrix} \tag{8.7}$$

where $[C]$ is a matrix of elastic coefficients given by

$$[C] = \frac{1}{E}\begin{bmatrix}1 & -v & -v & 0 & 0 & 0\\ -v & 1 & -v & 0 & 0 & 0\\ -v & -v & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+v) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+v) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+v)\end{bmatrix} \tag{8.8}$$

$\vec{\varepsilon}_0$ is the vector of initial strains, E is Young's modules, and v is Poisson's ratio of the material. In the case of heating an isotropic material, the initial strain vector is given by

$$\vec{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{xx_0} \\ \varepsilon_{yy_0} \\ \varepsilon_{zz_0} \\ \varepsilon_{xy_0} \\ \varepsilon_{yz_0} \\ \varepsilon_{zx_0} \end{Bmatrix} = \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{8.9}$$

where α is the coefficient of thermal expansion and T is the temperature charge. Sometimes, the expressions for stresses in terms of strains will be needed. By including thermal strains, Eq. (8.7) can be inverted to obtain

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = [D](\vec{\varepsilon} - \vec{\varepsilon}_0) \equiv [D] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} - \frac{E\alpha T}{1-2v} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{8.10}$$

where the matrix $[D]$ is given by

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \tag{8.11}$$

In the case of two-dimensional problems, two types of stress distributions, namely plane stress and plane strain, are possible.

Two-dimensional case (plane stress): The assumption of plane stress is applicable for bodies whose dimension is very small in one of the coordinate directions. Thus, the analysis of thin plates loaded in the plane of the plate can be made using the assumption of plane stress. In plane stress distribution, it is assumed that

$$\sigma_{zz} = \sigma_{zx} = \sigma_{yz} = 0 \tag{8.12}$$

where z represents the direction perpendicular to the plane of the plate as shown in Fig. 8.3, and the stress components do not vary through the thickness of the plate (i.e., in the z direction). Although these assumptions violate some of the

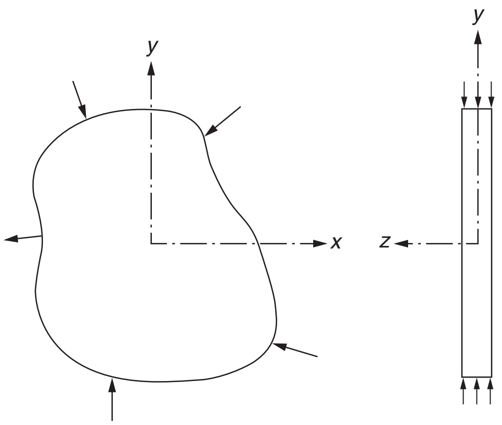

FIGURE 8.3 Example of a plane stress problem: thin plate under in-plane loading.

compatibility conditions, they are sufficiently accurate for all practical purposes provided the plate is thin. In this case, the stress–strain relations, Eqs. (8.7) and (8.10), reduce to

$$\vec{\varepsilon} = [C]\vec{\sigma} + \vec{\varepsilon}_0 \tag{8.13}$$

where

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}, \quad \vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$[C] = \frac{1}{E}\begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix} \tag{8.14}$$

$$\vec{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{xx0} \\ \varepsilon_{yy0} \\ \varepsilon_{xy0} \end{Bmatrix} = \alpha T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.15}$$

in the case of thermal strains, and

$$\vec{\sigma} = [D](\vec{\varepsilon} - \vec{\varepsilon}_0) = [D]\vec{\varepsilon} - \frac{E\alpha T}{1-v}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.16}$$

with

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \tag{8.17}$$

In the case of plane stress, the component of strain in the z direction, because of Poisson ratio effect, will be nonzero and is given by (from Eq. 8.7)

$$\varepsilon_{zz} = -\frac{v}{E}(\sigma_{xx} + \sigma_{yy}) + \alpha T = \frac{-v}{1-v}(\varepsilon_{xx} + \varepsilon_{yy}) + \frac{1+v}{1-v}\alpha T \tag{8.18}$$

while

$$\varepsilon_{yz} = \varepsilon_{zx} = 0 \tag{8.19}$$

EXAMPLE 8.1

The following stresses are developed in a plate under plane stress: $\sigma_{xx} = 10$ MPa, $\sigma_{yy} = -15$ MPa, and $\sigma_{xy} = 5$ MPa. Determine the strains induced in the plate, assuming that $E = 209$ GPa and $v = 0.3$.

Solution

Since the plate is in a state of plane stress, the strains ε_{xx}, ε_{yy}, and ε_{xy} are given by Eq. (8.13):

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$= \frac{1}{209(10^9)}\begin{bmatrix} 1 & -0.3 & 0 \\ -0.3 & 1 & 0 \\ 0 & 0 & 2.6 \end{bmatrix}\begin{Bmatrix} 10 \\ -15 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 70.0483 \\ -86.9565 \\ 62.8019 \end{Bmatrix} \times 10^{-6}$$

The shear strains ε_{yz} and ε_{zx} will be zero. The normal strain ε_{zz} can be computed using Eq. (8.18) as

$$\varepsilon_{zz} = -\frac{v}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{0.3}{209(10^9)}(10 - 15)(10^6) = 7.2464(10^{-6})$$

Two-dimensional case (plane strain): The assumption of plane strain is applicable for bodies that are long and whose geometry and loading do not vary significantly in the longitudinal direction. Thus, the analysis of dams, cylinders, and retaining walls shown in Fig. 8.4 can be made using the assumption of plane strain. In plane strain distribution, it is assumed that $w = 0$ and $(\partial w/\partial z) = 0$ at every cross section.

Here, the dependent variables are assumed to be functions of only the x and y coordinates provided we consider a cross section of the body away from the ends. In this case, the three-dimensional stress–strain relations given by Eqs. (8.7) and (8.10) reduce to

$$\vec{\varepsilon} = [C]\vec{\sigma} + \vec{\varepsilon}_0 \tag{8.20}$$

where

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}, \quad \vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$[C] = \frac{1+v}{E}\begin{bmatrix} 1-v & -v & 0 \\ -v & 1-v & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{8.21}$$

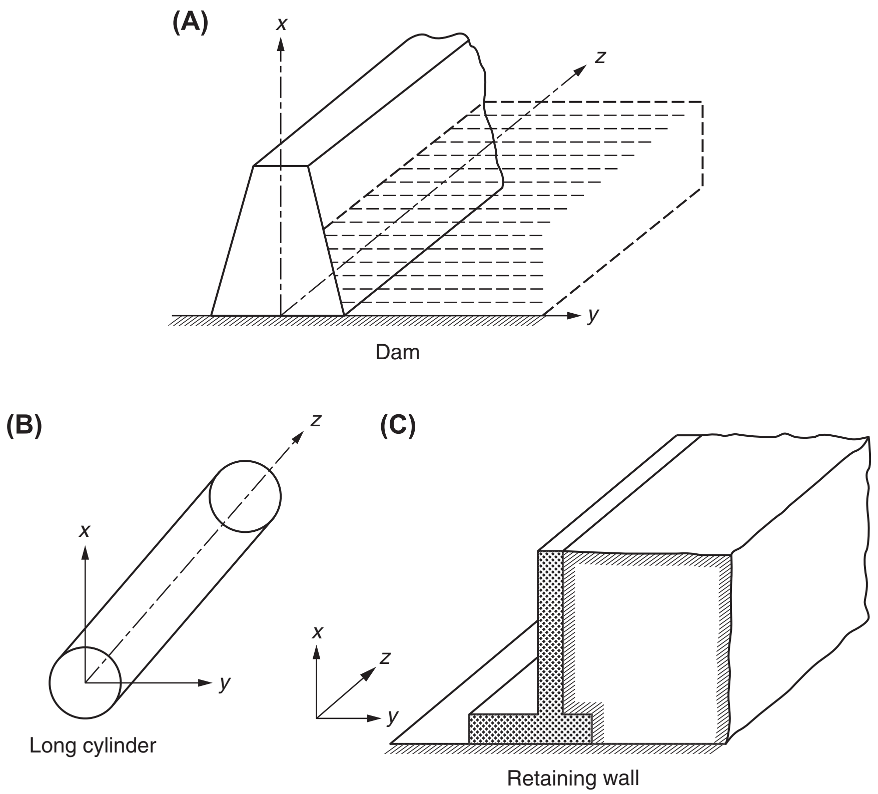

FIGURE 8.4 Examples of plane strain problems.

$$\vec{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{xx0} \\ \varepsilon_{yy0} \\ \varepsilon_{xy0} \end{Bmatrix} = (1+v)\alpha T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.22}$$

in the case of thermal strains, and

$$\vec{\sigma} = [D](\vec{\varepsilon} - \vec{\varepsilon}_0) = [D]\vec{\varepsilon} - \frac{E\alpha T}{1-2v} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.23}$$

with

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \tag{8.24}$$

The component of stress in the z direction will be nonzero because of the Poisson ratio effect and is given by

$$\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy}) - E\alpha T \tag{8.25}$$

and

$$\sigma_{yz} = \sigma_{zx} = 0 \tag{8.26}$$

EXAMPLE 8.2

The strains developed in a dam, considered to be in a state of plane strain, are given by $\varepsilon_{xx} = 0.00025$, $\varepsilon_{yy} = 0.0001$, and $\varepsilon_{xy} = -0.00015$. Determine the stresses developed in the dam. Assume that $E = 207$ GPa and $v = 0.3$.

Solution

Since the dam is in a state of plane strain, the stresses σ_{xx}, σ_{yy}, and σ_{xy} are given by Eq. (8.23):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \dfrac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$= \frac{207(10^9)}{1.3(0.4)} \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{Bmatrix} 0.00025 \\ 0.00010 \\ -0.00015 \end{Bmatrix} = \begin{Bmatrix} 81.6058 \\ 57.7811 \\ -11.9423 \end{Bmatrix} \text{MPa}$$

The shear stresses σ_{yz} and σ_{zx} will be zero. The normal stress σ_{zz} can be computed using Eq. (8.25) as

$$\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy}) = 0.3(81.6058 + 57.7811) = 41.7981 \text{ MPa}$$

One-dimensional case: In the case of one-dimensional axial stress problems, all stress components except for one normal stress are zero and the stress–strain relations degenerate to

$$\vec{\varepsilon} = [C]\vec{\sigma} + \vec{\varepsilon}_0 \tag{8.27}$$

where

$$\vec{\varepsilon} = \{\varepsilon_{xx}\}, \quad \vec{\sigma} = \{\sigma_{xx}\}$$
$$[C] = \left[\frac{1}{E}\right] \tag{8.28}$$

$$\vec{\varepsilon}_0 = \{\varepsilon_{xx_0}\} = \alpha T \tag{8.29}$$

in the case of thermal strains, and

$$\vec{\sigma} = [D](\vec{\varepsilon} - \vec{\varepsilon}_0) = [D]\vec{\varepsilon} - E\alpha T\{1\} \tag{8.30}$$

with

$$[D] = [E] \tag{8.31}$$

Axisymmetric case: In the case of solids of revolution (axisymmetric solids under axisymmetric loads), the stress–strain relations are given by

$$\vec{\varepsilon} = [C]\vec{\sigma} + \vec{\varepsilon}_0 \tag{8.32}$$

where

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{Bmatrix}, \quad \vec{\sigma} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{Bmatrix}$$

$$[C] = \frac{1}{E}\begin{bmatrix} 1 & -v & -v & 0 \\ -v & 1 & -v & 0 \\ -v & -v & 1 & 0 \\ 0 & 0 & 0 & 2(1+v) \end{bmatrix} \tag{8.33}$$

$$\vec{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_{rr_0} \\ \varepsilon_{\theta\theta_0} \\ \varepsilon_{zz_0} \\ \varepsilon_{rz_0} \end{Bmatrix} = \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.34}$$

in the case of thermal strains, and

$$\vec{\sigma} = [D](\vec{\varepsilon} - \vec{\varepsilon}_o) = [D]\vec{\varepsilon} - \frac{E\alpha T}{1-2v}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{8.35}$$

with

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \left(\frac{1-2v}{2}\right) \end{bmatrix} \tag{8.36}$$

In these equations, the subscripts r, θ, and z denote the radial, tangential, and axial directions, respectively.

STRESS–STRAIN RELATIONS FOR ANISOTROPIC MATERIALS

The stress–strain relations given earlier are valid for isotropic elastic materials. The term isotropic indicates that the material properties at a point in the body are not a function of orientation. In other words, the material properties are constant in any plane passing through a point in the material. There are certain materials (e.g., reinforced concrete, fiber-reinforced composites, brick, and wood) for which the material properties at any point depend on the orientation also. In general, such materials are called anisotropic materials. The generalized Hooke's law valid for anisotropic materials is given in this section. The special cases of the Hooke's law for orthotropic and isotropic materials will also be indicated.

For a linearly elastic anisotropic material, the strain–stress relations are given by the generalized Hooke's law as [8.2,8.3] follows:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ C_{12} & C_{22} & \cdots & C_{26} \\ \vdots & & & \\ C_{16} & C_{26} & \cdots & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{8.37}$$

where the matrix $[C]$ is symmetric and is called the compliance matrix. Thus, 21 independent elastic constants (equal to the number of independent components of $[C]$) are needed to describe an anisotropic material. Note that subscripts 1, 2, and 3 are used instead of x, y, and z in Eq. (8.37) for convenience.

Certain materials exhibit symmetry with respect to certain planes within the body. In such cases, the number of elastic constants will be reduced from 21. For an orthotropic material, which has three planes of material property symmetry, Eq. (8.37) reduces to

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{8.38}$$

where the elements C_{ij} are given by

$$\left.\begin{aligned} C_{11} &= \frac{1}{E_{11}}, \quad C_{12} = -\frac{\nu_{21}}{E_{22}}, \quad C_{13} = -\frac{\nu_{31}}{E_{33}}, \\ C_{22} &= \frac{1}{E_{22}}, \quad C_{23} = -\frac{\nu_{32}}{E_{33}}, \quad C_{33} = \frac{1}{E_{33}}, \\ C_{44} &= \frac{1}{G_{23}}, \quad C_{55} = \frac{1}{G_{13}}, \quad C_{66} = \frac{1}{G_{12}}, \end{aligned}\right\} \tag{8.39}$$

Here, E_{11}, E_{22}, and E_{33} denote the Young's modulus in the planes defined by axes 1, 2, and 3, respectively; G_{12}, G_{23}, and G_{13} represent the shear modulus in the planes 12, 23, and 13, respectively; and ν_{12}, ν_{13}, and ν_{23} indicate the major Poisson's ratios. Thus, nine independent elastic constants are needed to describe an orthotropic material under three-dimensional state of stress. For the specially orthotropic material that is in a state of plane stress, $\sigma_3 = \sigma_{23} = \sigma_{13} = 0$ and Eq. (8.38) reduces to

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} \tag{8.40}$$

which involves four independent elastic constants. The elements of the compliance matrix, in this case, can be expressed as

$$\begin{aligned} C_{11} &= \frac{1}{E_{11}} \\ C_{22} &= \frac{1}{E_{22}} \\ C_{12} &= -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \\ C_{66} &= \frac{1}{G_{12}} \end{aligned} \tag{8.41}$$

The stress–strain relations can be obtained by inverting the relations given by Eqs. (8.37), (8.38), and (8.40). Specifically, the stress–strain relations for a specially orthotropic material (under plane stress) can be expressed as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \equiv [Q]\vec{\varepsilon} \tag{8.42}$$

where the elements of the matrix $[Q]$ are given by

$$\left.\begin{aligned} Q_{11} &= \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} \\ Q_{12} &= \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} \\ Q_{66} &= 2G_{12} \end{aligned}\right\} \tag{8.43}$$

If the material is linearly elastic and isotropic, only two elastic constants are needed to describe the behavior and the stress–strain relations are given by Eq. (8.7) or Eq. (8.10).

STRAIN–DISPLACEMENT RELATIONS

The deformed shape of an elastic body under any given system of loads and temperature distribution conditions can be completely described by the three components of displacement u, v, and w parallel to the directions x, y, and z, respectively. In general, each of these components u, v, and w is a function of the coordinates x, y, and z. The strains induced in the body can be expressed in terms of the displacements u, v, and w. In this section, we assume the deformations to be small so that the strain–displacement relations remain linear.

To derive expressions for the normal strain components ε_{xx} and ε_{yy} and the shear strain component ε_{xy}, consider a small rectangular element $OACB$ whose sides (of lengths dx and dy) lie parallel to the coordinate axes before deformation. When the body undergoes deformation under the action of external load and temperature distributions, the element $OACB$ also deforms to the shape $O'A'C'B'$ as shown in Fig. 8.5. We can observe that the element $OACB$ has two basic types of deformation, one of change in size and the other of angular distortion.

Since the normal strain is defined as change in length divided by original length, the strain components ε_{xx} and ε_{yy} can be found as

$$\begin{aligned} \varepsilon_{xx} &= \frac{\substack{\text{change in length of the fiber } OA \text{ that lies} \\ \text{in the } x \text{ directon before deformation}}}{\text{original length of the fiber } OA} \\ &= \frac{\left[\mathrm{d}x + \left(u + \frac{\partial u}{\partial x}\cdot \mathrm{d}x\right) - u\right] - \mathrm{d}x}{\mathrm{d}x} = \frac{\partial u}{\partial x} \end{aligned} \tag{8.44}$$

and

$$\begin{aligned} \varepsilon_{yy} &= \frac{\substack{\text{change in length of the fiber } OB \text{ that lies} \\ \text{in the } y \text{ direction before deformation}}}{\text{original length of the fiber } OB} \\ &= \frac{\left[\mathrm{d}y + \left(v + \frac{\partial v}{\partial y}\cdot \mathrm{d}y\right) - v\right] - \mathrm{d}y}{\mathrm{d}y} = \frac{\partial v}{\partial y} \end{aligned} \tag{8.45}$$

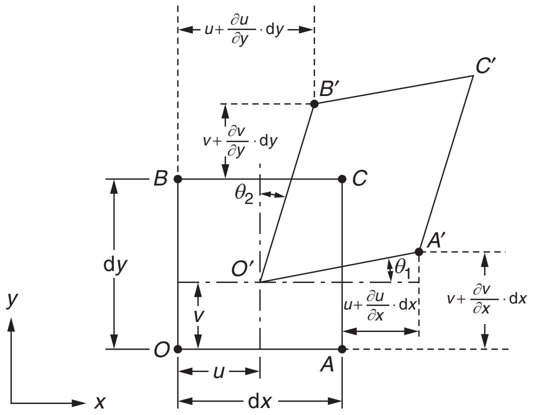

FIGURE 8.5 Deformation of a small element $OACB$.

The shear strain is defined as the decrease in the right angle between the fibers *OA* and *OB*, which were at right angles to each other before deformation. Thus, the expression for the shear strain ε_{xy} can be obtained as

$$\varepsilon_{xy} = \theta_1 + \theta_2 \simeq \tan\theta_1 + \tan\theta_2 \simeq \frac{\left(v + \frac{\partial v}{\partial x}\cdot dx\right) - v}{\left[dx + \left(u + \frac{\partial u}{\partial x}\cdot dx\right) - u\right]} + \frac{\left(u + \frac{\partial u}{\partial y}\cdot dy\right) - u}{\left[dy + \left(v + \frac{\partial v}{\partial y}\cdot dy\right) - v\right]}$$

If the displacements are assumed to be small, ε_{xy} can be expressed as

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{8.46}$$

The expressions for the remaining normal strain component ε_{zz} and shear strain components ε_{yz} and ε_{zx} can be derived in a similar manner as

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{8.47}$$

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{8.48}$$

and

$$\varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{8.49}$$

In the case of two-dimensional problems, Eqs. (8.44) to (8.46) are applicable, whereas Eq. (8.44) is applicable in the case of one-dimensional problems.

In the case of an axisymmetric solid, the strain−displacement relations can be derived as

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{u}{r} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ \varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \tag{8.50}$$

where u and w are the radial and the axial displacements, respectively.

EXAMPLE 8.3

For a planar deformation, find the most general displacement solution by integrating the following strain equations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \tag{E.1}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{E.2}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{E.3}$$

Give a physical interpretation of the resulting solution.

Solution

Integrate Eqs. (E.1) and (E.2) with respect to x and y, respectively, to obtain

$$u = f_1(y), \quad v = f_2(x) \tag{E.4}$$

Differentiate the functions $f_1(y)$ and $f_2(x)$ with respect to y and x, respectively, and substitute in Eq. (E.3):

$$\frac{df_1}{dy} + \frac{df_2}{dx} = 0 \tag{E.5}$$

Continued

EXAMPLE 8.3 —cont'd

Because, in Eq. (E.5), the first term is a function of y alone and the second term is a function of x alone, we can express

$$\frac{df_1}{dy} = -\frac{df_2}{dx} = C \tag{E.6}$$

where C is a constant (to be determined). By integrating the equations

$$\frac{df_1}{dy} = C, \quad -\frac{df_2}{dx} = C$$

with respect to y and x, respectively, we find

$$u = f_1(y) = Cy + D, \quad v = f_2(x) = -Cx + E \tag{E.8}$$

where D and E are constants (to be determined). Thus, the most general displacement solution corresponding to Eqs. (E.1) to (E.3) is indicated in Eq. (E.8). The constants D and E can be seen to represent rigid body translations along x and y directions, respectively. The constant C can be seen to denote an infinitesimal rigid body rotation. These rigid body motions are shown in Fig. 8.6.

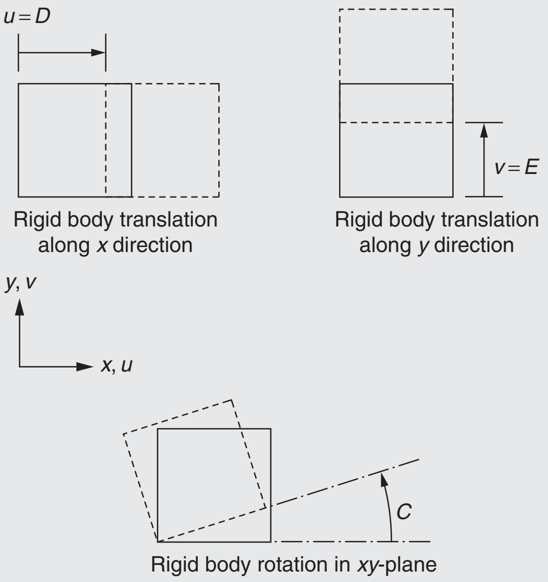

FIGURE 8.6 Rigid body motions in xy-plane.

EXAMPLE 8.4

The undeformed and deformed configurations of a rectangular plate are shown in Fig. 8.7. Determine the strains ε_{xx}, ε_{yy}, and ε_{xy} induced in the plate.

Solution

From the undeformed and deformed configurations of the plate shown in Fig. 8.7, the strains induced at the location $(x, y) = (0, 0)$ can be determined as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \cong \frac{\Delta u}{\Delta x} = \frac{4.0030 - 4}{4} = 0.00075$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \cong \frac{\Delta v}{\Delta y} = \frac{3.015 - 3}{3} = 0.005$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \cong \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x} = \frac{0.001}{3} + \frac{0.002}{4} = 0.0008333$$

EXAMPLE 8.4 —cont'd

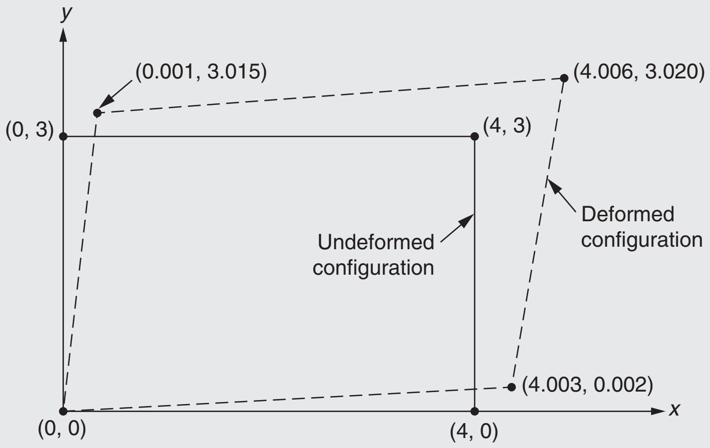

FIGURE 8.7 Deformation of a rectangular plate.

BOUNDARY CONDITIONS

Boundary conditions can be either on displacements or on surface forces (tractions). The boundary conditions on displacements require certain displacements to prevail at certain points on the boundary of the body, whereas the boundary conditions on stresses require that the stresses induced must be in equilibrium with the external forces applied at certain points on the boundary of the body. As an example, consider the flat plate under in-plane loading shown in Fig. 8.8.

In this case, the boundary conditions can be expressed as

$$u = v = 0 \text{ along the edge } AB$$

$$\text{(displacement boundary conditions)}$$

and

$$\sigma_{yy} = \sigma_{xy} = 0 \text{ along the edges } BC \text{ and } AD$$

$$\sigma_{xx} = -p, \sigma_{xy} = 0 \text{ along the edge } CD$$

$$\text{(surface forces or traction boundary conditions)}$$

It can be observed that the displacements are unknown and are free to assume any values dictated by the solution wherever stresses are prescribed and vice versa. This is true of all solid mechanics problems.

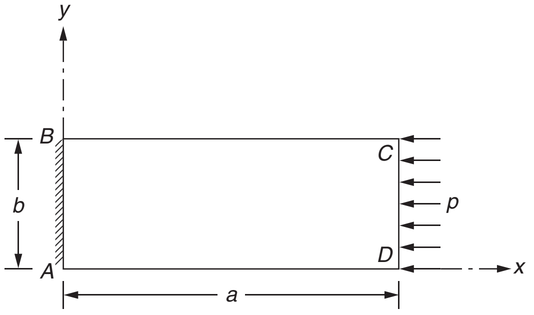

FIGURE 8.8 A flat plate under in-plane loading.

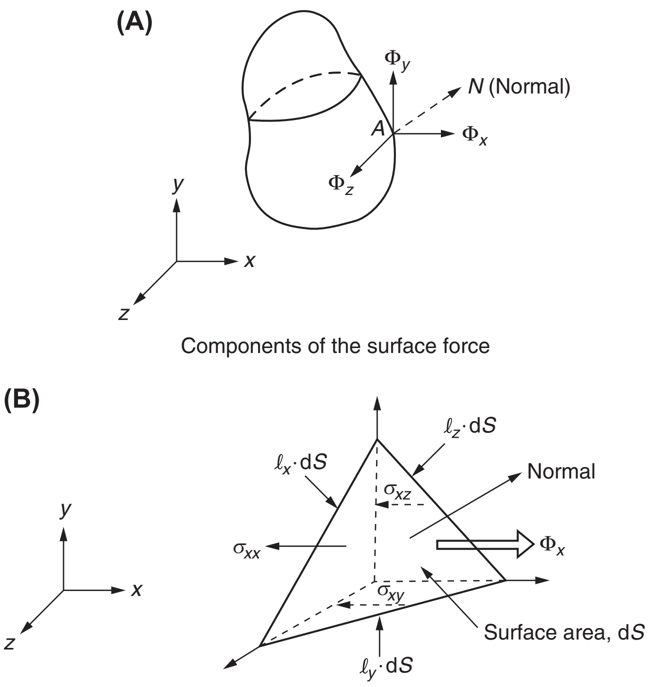

FIGURE 8.9 Forces acting at the surface of a body.

For the equilibrium of induced stresses and applied surface forces at point A of Fig. 8.9, the following equations must be satisfied:

$$\begin{aligned} \ell_x\sigma_{xx} + \ell_y\sigma_{xy} + \ell_z\sigma_{zx} &= \Phi_x \\ \ell_x\sigma_{xy} + \ell_y\sigma_{yy} + \ell_z\sigma_{yz} &= \Phi_y \\ \ell_x\sigma_{xz} + \ell_y\sigma_{yz} + \ell_z\sigma_{zz} &= \Phi_z \end{aligned} \tag{8.51}$$

where l_x, l_y, and l_z are the direction cosines of the outward drawn normal (AN) at point A; and Φ_x, Φ_y, and Φ_z are the components of surface forces (tractions) acting at point A in the directions x, y, and z, respectively. The surface (distributed) forces Φ_x, Φ_y, and Φ_z have dimensions of force per unit area. Eq. (8.51) can be specialized to two- and one-dimensional problems without much difficulty.

COMPATIBILITY EQUATIONS

When a body is continuous before deformation, it should remain continuous after deformation. In other words, no cracks or gaps should appear in the body and no part should overlap another due to deformation. Thus, the displacement field must be continuous as well as single-valued. This is known as the condition of compatibility. The condition of compatibility can also be seen from another point of view. For example, we can see from Eqs. (8.44) to (8.49) that the three strains ε_{xx}, ε_{yy}, and ε_{xy} can be derived from only two displacements, u and v. This implies that a definite relation must exist between ε_{xx}, ε_{yy}, and ε_{xy} if these strains correspond to a compatible deformation. This definite relation is called the compatibility equation. Thus, in three-dimensional elasticity problems, there are six compatibility equations [8.4]:

$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} \tag{8.52}$$

$$\frac{\partial^2\varepsilon_{yy}}{\partial z^2} + \frac{\partial^2\varepsilon_{zz}}{\partial y^2} = \frac{\partial^2\varepsilon_{yz}}{\partial y\partial z} \tag{8.53}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zx}}{\partial x \partial z} \tag{8.54}$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xy}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) = \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} \tag{8.55}$$

$$\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y}\right) = \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} \tag{8.56}$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) = \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} \tag{8.57}$$

In the case of two-dimensional plane strain problems, Eqs. (8.52) to (8.57) reduce to a single equation as

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{8.58}$$

For plane stress problems, Eqs. (8.52) to (8.57) reduce to the following equations:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = 0 \tag{8.59a -d}$$

In the case of one-dimensional problems, the conditions of compatibility will be automatically satisfied.

EXAMPLE 8.5

In the theory of elasticity, a stress function $\Phi(x, y)$ is defined in two-dimensional stress analysis problems with no body forces. The stress function can be used to determine the stresses as

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Determine the stress field corresponding to the following stress function:

$$\Phi(x, y) = c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3$$

where c_i, $i = 1, 2, 3, 4$ are constants. Indicate the conditions under which the stress field corresponds to that of a beam in pure bending.

Approach: The stress condition for a beam in pure bending is given by

$$\sigma_{xx} = \text{linear in } y, \quad \sigma_{yy} = 0, \quad \text{and } \tau_{xy} = 0.$$

Solution

Using the given stress function, the stresses can be determined as

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} = 2c_3 x + 6c_4 y$$

$$\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = 6c_1 x + 2c_2 y$$

$$\tau_{xy} = \frac{\partial^2 \Phi}{\partial x \partial y} = -2c_2 x - 2c_3 y$$

For a beam in pure bending, the normal stress in the y-direction (σ_{yy}) and the shear stress in the xy-plane (τ_{xy}) must be zero. This requires that c_1, c_2, and c_3 be zero. These conditions yield a linear variation of the stress σ_{xx} with respect to the vertical (y) direction, which is a valid variation for a beam in pure bending.

EXAMPLE 8.6

The displacement field in a cantilever beam of length l, fixed at $x = 0$, and subjected to a uniformly distributed load of magnitude $-p_0$ is given by

$$v(x,y) = -\frac{p_o}{24EI}x^2\left(x^2 - 4lx + 6l^2\right), \quad u(x,y) = -y\frac{\mathrm{d}v}{\mathrm{d}x}$$

Determine whether this displacement field satisfies the compatibility equations.
Approach: Find the strains in the beam and check the compatibility conditions.

Solution

From the given displacement field, the strains can be determined as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = y\frac{p_0}{24EI}\left(12x^2 - 24lx + 12l^2\right) \tag{E.1}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{E.2}$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{E.3}$$

The satisfaction of the compatibility conditions for a plane stress problem, given by Eq. (8.59a), can be verified as follows:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{E.4}$$

where

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = 0, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 0$$

and

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0.$$

This shows that Eq. (E.4) is satisfied.
The normal strain in the z-direction for a plane stress problem can be found using Eq. (8.18):

$$\varepsilon_{zz} = -\frac{v}{1-v}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) = -\frac{v}{1-v}\left[\frac{p_0 y}{24EI}\left(12x^2 - 24lx + 12l^2\right)\right] \tag{E.5}$$

The compatibility conditions related to ε_{zz} in Eq. (8.59) can be verified as follows:

$$\frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 0, \quad \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = -\frac{v}{1-v}\left[\frac{p_0 y}{24EI}(24)\right] \neq 0$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = -\frac{v}{1-v}\left[\frac{p_0}{24EI}(24x - 24l)\right] \neq 0$$

This shows that two of the compatibility conditions related to ε_{zz} are not satisfied.

PRINCIPAL STRESSES

Many finite element structural analyses involve the computation of principal stresses. In a three-dimensional solid body, the three principal stresses (σ_1, σ_2, σ_3) can be determined as the roots of the following cubic equation:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \sigma_{xy} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - \sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{yz} & \sigma_{zz} - \sigma \end{vmatrix} = \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{8.60}$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{8.61}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2 \tag{8.62}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \tag{8.63}$$

For a two-dimensional body (plane stress problem), the two in-plane principal stresses (σ_1, σ_2) can be found as

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \tag{8.64}$$

EXAMPLE 8.7

The stresses induced in a rectangular plate are given by $\sigma_{xx} = 6$ MPa, $\sigma_{yy} = 18$ MPa, and $\sigma_{xy} = 18$ MPa. Find the principal stresses.

Solution

Eq. (8.64) gives the principal stresses as

$$\sigma_{1,2} = \frac{6+18}{2} \pm \sqrt{\left(\frac{6-18}{2}\right)^2 + (-8)^2} = 22\text{ MPa},\ 2\text{ MPa}$$

EXAMPLE 8.8

By defining the principal strains ε_1, ε_2, and ε_3 as the strains in the directions of the principal stresses σ_1, σ_2, and σ_3, respectively, express the stress–strain relations and strain–stress relations in a biaxial state of stress.

Solution

Let the principal stresses in a biaxial state of stress be denoted as σ_1 and σ_2 with the corresponding principal strains denoted as ε_1, ε_2, and ε_3. Then the strain–stress relations can be expressed as

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{v\sigma_2}{E}, \quad \varepsilon_2 = \frac{\sigma_2}{E} - \frac{v\sigma_1}{E}, \quad \varepsilon_3 = -\frac{v\sigma_1}{E} - \frac{v\sigma_2}{E}$$

The corresponding stress–strain relations can be expressed as

$$\sigma_1 = \frac{E}{1-v^2}(\varepsilon_1 + v\varepsilon_2), \quad \sigma_2 = \frac{E}{1-v^2}(\varepsilon_2 + v\varepsilon_1), \quad \sigma_3 = 0$$

8.3 FORMULATIONS OF SOLID AND STRUCTURAL MECHANICS

As stated in Section 5.3, most continuum problems, including solid and structural mechanics problems, can be formulated according to one of two methods—differential equation and variational. Hence, the finite element equations can also be derived by using either a differential equation formulation method (e.g., Galerkin approach) or variational formulation method (e.g., Rayleigh–Ritz approach). In the case of solid and structural mechanics problems, each of the differential equation and variational formulation methods can be classified into three categories as shown in Table 8.1.

TABLE 8.1 Methods of Formulating Solid and Structural Mechanics Problems

Differential Equation Formulation Methods			Variational Formulation Methods		
Displacement method	Force method	Displacement-force method (mixed method)	Principle of minimum potential energy	Principle of minimum complementary energy	Principle of stationary Reissner energy

The displacement, force, and displacement−force methods of differential equation formulation are closely related to the principles of minimum potential energy, minimum complementary energy, and stationary Reissner energy formulations, respectively. We use the displacement method or the principle of minimum potential energy for presenting the various concepts of the finite element method because they have been extensively used in the literature.

8.3.1 Differential Equation Formulation Methods

DISPLACEMENT METHOD

As stated in Section 8.2.1, for a three-dimensional continuum or elasticity problem, there are six stress−strain relations (Eq. 8.10), six strain−displacement relations (Eqs. 8.44−8.49), and three equilibrium equations (Eq. 8.4), and the unknowns are six stresses (σ_{ij}), six strains (ε_{ij}), and three displacements (u, v, and w). By substituting Eqs. (8.44) to (8.49) into Eq. (8.10), we obtain the stresses in terms of the displacements. By substituting these stress−displacement relations into Eq. (8.4), we obtain three equilibrium equations in terms of the three unknown displacement components u, v, and w. Now these equilibrium equations can be solved for u, v, and w. Of course, the additional requirements such as boundary and compatibility conditions also have to be satisfied while finding the solution for u, v, and w. Since the displacements u, v, and w are made the final unknowns, the method is known as the displacement method.

FORCE METHOD

For a three-dimensional elasticity problem, there are three equilibrium equations, Eq. (8.4), in terms of six unknown stresses σ_{ij}. At the same time, there are six compatibility equations, Eqs. (8.52) to (8.57), in terms of the six strain components, ε_{ij}. Now we take any three strain components, for example, ε_{xy}, ε_{yz}, and ε_{zx}, as independent strains and write the compatibility equations in terms of ε_{xy}, ε_{yz}, and ε_{zx} only. By substituting the known stress−strain relations, Eq. (8.10), we express the three independent compatibility equations in terms of the stresses σ_{ij}. By using these three equations, three of the stresses out of σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz}, and σ_{zx} can be eliminated from the original equilibrium equations. Thus, we get three equilibrium equations in terms of three stress components only, and hence the problem can be solved. Since the final equations are in terms of stresses (or forces), the method is known as the force method.

DISPLACEMENT−FORCE METHOD

In this method, we use the strain−displacement relations to eliminate strains from the stress−strain relations. These six equations, in addition to the three equilibrium equations, will give us nine equations in the nine unknowns, σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz}, σ_{zx}, u, v, and w. Thus, the solution of the problem can be found by using the additional conditions such as compatibility and boundary conditions. Since both the displacements and the stresses (or forces) are taken as the final unknowns, the method is known as the displacement−force method.

8.3.2 Variational Formulation Methods

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

The potential energy of an elastic body π_p is defined as

$$\pi_p = \pi - W_p \tag{8.65}$$

where π is the strain energy, and W_p is the work done on the body by the external forces. The principle of minimum potential energy can be stated as follows: Of all possible displacement states (u, v, and w) a body can assume that satisfy compatibility and given kinematic or displacement boundary conditions, the state that satisfies the equilibrium equations makes the potential energy assume a minimum value. If the potential energy, π_p, is expressed in terms of the displacements u, v, and w, the principle of minimum potential energy gives, at the equilibrium state,

$$\delta\pi_p(u, v, w) = \delta\pi(u, v, w) - \delta W_p(u, v, w) = 0 \tag{8.66}$$

It is important to note that the variation is taken with respect to the displacements in Eq. (8.66), whereas the forces and stresses are assumed constant. The strain energy of a linear elastic body is defined as

$$\pi = \frac{1}{2}\iiint_V \vec{\varepsilon}^T \vec{\sigma}\ \mathrm{d}V \tag{8.67}$$

where V is the volume of the body. By using the stress–strain relations of Eq. (8.10), the strain energy, in the presence of initial strains $\vec{\varepsilon}_0$, can be expressed as

$$\pi = \frac{1}{2}\iiint_V \vec{\varepsilon}^T[D]\vec{\varepsilon}\,\mathrm{d}V - \iiint_V \vec{\varepsilon}^T[D]\vec{\varepsilon}_0\,\mathrm{d}V \tag{8.68}$$

The work done by the external forces can be expressed as

$$W_p = \iiint_V \vec{\overline{\phi}}^T\vec{U}\cdot\mathrm{d}V + \iint_{S_1}\vec{\overline{\Phi}}^T\vec{U}\cdot\mathrm{d}S_1 \tag{8.69}$$

where $\vec{\overline{\phi}} = \begin{Bmatrix}\overline{\phi}_x\\ \overline{\phi}_y\\ \overline{\phi}_z\end{Bmatrix}$ = known body force vector, $\vec{\overline{\Phi}} = \begin{Bmatrix}\overline{\Phi}_x\\ \overline{\Phi}_y\\ \overline{\Phi}_z\end{Bmatrix}$ = vector of prescribed surface forces (tractions), $\vec{U} = \begin{Bmatrix}u\\ v\\ w\end{Bmatrix}$ = vector of displacements, and S_1 is the surface of the body on which surface forces are prescribed. Using Eqs. (8.68) and (8.69), the potential energy of the body can be expressed as

$$\pi_p(u,v,w) = \frac{1}{2}\iiint_V \vec{\varepsilon}^T[D](\vec{\varepsilon} - 2\vec{\varepsilon}_0)\,\mathrm{d}V - \iiint_V \vec{\overline{\phi}}^T\vec{U}\cdot\mathrm{d}V - \iint_{S_1}\vec{\overline{\Phi}}^T\vec{U}\cdot\mathrm{d}S_1 \tag{8.70}$$

If we use the principle of minimum potential energy to derive the finite element equations, we assume a simple form of variation for the displacement field within each element and derive conditions that will minimize the functional I (same as π_p in this case). The resulting equations are the approximate equilibrium equations, whereas the compatibility conditions are identically satisfied. This approach is called the displacement or stiffness method of finite element analysis.

EXAMPLE 8.9

Find the axial deformation of a uniform bar fixed at one end and subjected to an axial load at the other end as shown in Fig. 8.10 using the principle of minimum potential energy.

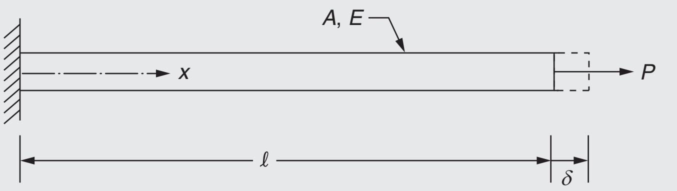

FIGURE 8.10 Bar under axial load.

Solution

Let δ be the axial deformation of the bar under the applied axial load. The potential energy of an elastic body is defined as

$$\text{Potential energy}(I) = \text{Strain energy}(\pi) - \text{Work done by the applied load} \tag{E.1}$$

The strain energy of the bar can be found as

$$\pi = \iiint_V (\text{Strain energy density})\,\mathrm{d}V = \iiint_V \frac{1}{2}\sigma\varepsilon\,\mathrm{d}V \tag{E.2}$$

For a bar, the stress (σ) and strain (ε) are constant and hence Eq. (E.2) can be written as

$$\pi = \frac{1}{2}\sigma\varepsilon(V) = \frac{1}{2}\sigma\varepsilon Al = \frac{1}{2}AEl\varepsilon^2 = \frac{1}{2}AEl\left(\frac{\delta}{l}\right)^2 = \frac{1}{2}\frac{AE}{l}\delta^2 \tag{E.3}$$

Continued

EXAMPLE 8.9 —cont'd

where A is the cross-sectional area, E is the Young's modulus, and l is the length of the bar. The work done by the applied load (P) due to deformation δ is given by

$$W = P\delta \tag{E.4}$$

Using Eqs. (E.3) and (E.4), the potential energy of the bar under axial load, Eq. (E.1), can be expressed as

$$I = \frac{1}{2}\frac{AE}{l}\delta^2 - P\delta \tag{E.5}$$

By minimizing the potential energy, we can find the axial deformation of the bar (δ) as

$$\frac{dI}{d\delta} = \frac{AE}{l}\delta - P = 0 \quad \text{or} \quad \delta = \frac{Pl}{AE} \tag{E.6}$$

PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

The complementary energy of an elastic body (π_c) is defined as

$$\begin{aligned}\pi_c = {} & \text{Complementary strain energy in terms of stresses } (\widetilde{\pi}) \\ & - \text{Work done by the applied loads during stress changes } \left(\widetilde{W}_p\right)\end{aligned}$$

The principle of minimum complementary energy can be stated as follows: Of all possible stress states that satisfy the equilibrium equations and the stress boundary conditions, the state that satisfies the compatibility conditions will make the complementary energy assume a minimum value.

If the complementary energy π_c is expressed in terms of the stresses σ_{ij}, the principle of minimum complementary energy gives, for compatibility,

$$\delta\pi_c(\sigma_{xx}, \sigma_{yy}, \ldots\sigma_{zx}) = \delta\widetilde{\pi}(\sigma_{xx}, \sigma_{yy}, \ldots\sigma_{zx}) - \delta\widetilde{W}_p(\sigma_{xx}, \sigma_{yy}, \ldots\sigma_{zx}) = 0 \tag{8.71}$$

It is important to note that the variation is taken with respect to the stress components in Eq. (8.71), whereas the displacements are assumed constant. The complementary strain energy of a linear elastic body is defined as

$$\widetilde{\pi} = \frac{1}{2}\iiint_V \vec{\sigma}^T \vec{\varepsilon}\, \mathrm{d}V \tag{8.72}$$

By using the strain–stress relations of Eq. (8.7), the complementary strain energy, in the presence of known initial strain $\vec{\varepsilon}_0$, can be expressed as[1]

$$\widetilde{\pi} = \frac{1}{2}\iiint_V \vec{\sigma}^T([C]\vec{\sigma} + 2\vec{\varepsilon}_0)\mathrm{d}V \tag{8.73}$$

The work done by applied loads during stress change (also known as complementary work) is given by

$$\widetilde{W}_p = \iint_{S_2}\left(\Phi_x\overline{u} + \Phi_y\overline{v} + \Phi_z\overline{w}\right)\mathrm{d}S_2 = \iint_{S_2}\vec{\Phi}^T\vec{\overline{U}}\,\mathrm{d}S_2 \tag{8.74}$$

where S_2 is the part of the surface of the body on which the values of the displacements are prescribed as $\vec{\overline{U}} = \begin{Bmatrix}\overline{u}\\ \overline{v}\\ \overline{w}\end{Bmatrix}$.

Eqs. (8.73) and (8.74) can be used to express the complementary energy of the body as

$$\pi_c(\sigma_{xx}, \sigma_{yy}, \ldots, \sigma_{zx}) = \frac{1}{2}\iiint_V \vec{\sigma}^T([C]\vec{\sigma} + 2\vec{\varepsilon}_0)\cdot\mathrm{d}V - \iint_{S_2}\vec{\Phi}^T\vec{\overline{U}}\cdot\mathrm{d}S_2 \tag{8.75}$$

1. The correctness of this expression can be verified from the fact that the partial derivative of $\widetilde{\pi}$ with respect to the stresses should yield the strain–stress relations of Eq. (8.7).

If we use the principle of minimum complementary energy in the finite element analysis, we assume a simple form of variation for the stress field within each element and derive conditions that will minimize the functional I (same as π_c in this case). The resulting equations are the approximate compatibility equations, whereas the equilibrium equations are identically satisfied. This approach is called the force or flexibility method of finite element analysis.

PRINCIPLE OF STATIONARY REISSNER ENERGY

In the case of the principle of minimum potential energy, we expressed π_p in terms of displacements and permitted variations of u, v, and w. Similarly, in the case of the principle of minimum complementary energy, we expressed π_c in terms of stresses and permitted variations of σ_{xx}, … σ_{zx}. In the present case, the Reissner energy (π_r) is expressed in terms of both displacements and stresses and variations are permitted in $\vec{U}$ and $\vec{\sigma}$. The Reissner energy for a linearly elastic material is defined as

$$\pi_R = \iiint_V [(\text{internal stresses}) \times (\text{strains expressed in terms of displacements})$$

$$-\text{complementary energy in terms of stresses}]\cdot \mathrm{d}V - \text{work done by applied forces}$$

$$= \iiint_V \left[\left\{\sigma_{xx}\bullet\frac{\partial u}{\partial x} + \sigma_{yy}\bullet\frac{\partial v}{\partial y} + \cdots + \sigma_{zx}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \widetilde{\pi}\right\}\right]\cdot \mathrm{d}V$$

$$-\iiint_V \left(\overline{\phi}_x\cdot u + \overline{\phi}_y\cdot v + \overline{\phi}_z\cdot w\right)\cdot \mathrm{d}V - \iint_{S_1} \left(\overline{\Phi}_x\cdot u + \overline{\Phi}_y\cdot v + \overline{\Phi}_z\cdot w\right)\cdot \mathrm{d}S_1$$

$$-\iint_{S_2} \left\{(u-\overline{u})\Phi_x + (v-\overline{v})\Phi_y + (w-\overline{w})\Phi_z\right\}\cdot \mathrm{d}S_2$$

$$= \iiint_V \left[\vec{\sigma}^T\vec{\varepsilon} - \frac{1}{2}\vec{\sigma}^T[C]\vec{\sigma} - \vec{\overline{\phi}}^T\vec{U}\right]\cdot \mathrm{d}V - \iint_{S_1} \vec{U}^T\vec{\overline{\Phi}}\mathrm{d}S_1 - \iint_{S_2} \left(\vec{U} - \vec{\overline{U}}\right)^T\vec{\Phi}\cdot \mathrm{d}S_2 \tag{8.76}$$

The variation of π_R is set equal to zero by considering variations in both displacements and stresses:

$$\delta\pi_R = \sum \underbrace{\frac{\partial\pi_R}{\partial\sigma_{ij}}}_{\substack{\text{Gives stress}-\\\text{displacement}\\\text{equations}}} \delta\sigma_{ij} + \underbrace{\left(\frac{\partial\pi_R}{\partial u}\delta u + \frac{\partial\pi_R}{\partial v}\delta v + \frac{\partial\pi_R}{\partial w}\delta w\right)}_{\substack{\text{Gives equilibrium equations}\\\text{and boundary conditions}}} = 0 \tag{8.77}$$

The principle of stationary Reissner energy can be stated as follows: Of all possible stress and displacement states the body can have, the particular set that makes the Reissner energy stationary gives the correct stress−displacement and equilibrium equations along with the boundary conditions. To derive the finite element equations using the principle of stationary Reissner energy, we must assume the form of variation for both displacement and stress fields within an element.

HAMILTON'S PRINCIPLE

The variational principle that can be used for dynamic problems is called the Hamilton's principle. In this principle, the variation of the functional is taken with respect to time. The functional (similar to π_p, π_c, and π_R) for this principle is the Lagrangian (L) defined as

$$L = T - \pi_p = \text{kinetic energy} - \text{potential energy} \tag{8.78}$$

The kinetic energy (T) of a body is given by

$$T = \frac{1}{2}\iiint_V \rho\dot{\vec{U}}^T\dot{\vec{U}}\mathrm{d}V \tag{8.79}$$

where ρ is the density of the material, and $\dot{\overrightarrow{U}} = \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix}$ is the vector of velocity components at any point inside the body.

Thus, the Lagrangian can be expressed as

$$L = \frac{1}{2}\iiint_V \left[\rho \dot{\overrightarrow{U}}^T \dot{\overrightarrow{U}} - \overrightarrow{\varepsilon}^T [D] \overrightarrow{\varepsilon} + 2\overrightarrow{U}^T \overrightarrow{\overrightarrow{\phi}}\right] \mathrm{d}V + \iint_{S_1} \overrightarrow{U}^T \overrightarrow{\overrightarrow{\Phi}}\, \mathrm{d}S_1 \tag{8.80}$$

Hamilton's principle can be stated as follows: Of all possible time histories of displacement states that satisfy the compatibility equations and the constraints or the kinematic boundary conditions and that also satisfy the conditions at initial and final times (t_1 and t_2), the history corresponding to the actual solution makes the Lagrangian functional a minimum. Thus, Hamilton's principle can be stated as

$$\delta \int_{t_1}^{t_2} L\, \mathrm{d}t = 0 \tag{8.81}$$

8.4 FORMULATION OF FINITE ELEMENT EQUATIONS (STATIC ANALYSIS)

We use the principle of minimum potential energy for deriving the equilibrium equations for a three-dimensional problem in this section. Since the nodal degrees of freedom are treated as unknowns in the present (displacement) formulation, the potential energy π_p has to be first expressed in terms of nodal degrees of freedom. Then the necessary equilibrium equations can be obtained by setting the first partial derivatives of π_p with respect to each of the nodal degrees of freedom equal to zero. The various steps involved in the derivation of equilibrium equations are:

Step 1: The solid body is divided into E finite elements.

Step 2: The displacement model within an element e is assumed as

$$\overrightarrow{U} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = [N]\overrightarrow{Q}^{(e)} \tag{8.82}$$

where $\overrightarrow{Q}^{(e)}$ is the vector of nodal displacement degrees of freedom of the element, and $[N]$ is the matrix of shape functions.

Step 3: The element characteristic (stiffness) matrices and characteristic (load) vectors are to be derived from the principle of minimum potential energy. For this, the potential energy functional of the body π_p is written as (by considering only the body and surface forces)

$$\pi_p = \sum_{e=1}^{E} \pi_p^{(e)}$$

where $\pi_p^{(e)}$ is the potential energy of element e given by (see Eq. 8.65)

$$\pi_p^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \overrightarrow{\varepsilon}^T [D](\overrightarrow{\varepsilon} - 2\overrightarrow{\varepsilon}_0)\mathrm{d}V - \iint_{S_1^{(e)}} \overrightarrow{U}^T \overrightarrow{\overrightarrow{\Phi}}\, \mathrm{d}S_1 - \iiint_{V^{(e)}} \overrightarrow{U}^T \overrightarrow{\overrightarrow{\phi}}\, \mathrm{d}V \tag{8.83}$$

where $V^{(e)}$ is the volume of the element, $S_1^{(e)}$ is the portion of the surface of the element over which distributed surface forces or tractions, $\overrightarrow{\overrightarrow{\Phi}}$, are prescribed, and $\overrightarrow{\overrightarrow{\phi}}$ is the vector of body forces per unit volume.

The strain vector $\vec{\varepsilon}$ appearing in Eq. (8.83) can be expressed in terms of the nodal displacement vector $\vec{Q}^{(e)}$ by differentiating Eq. (8.82) suitably as

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [B]\vec{Q}^{(e)} \tag{8.84}$$

where

$$[B] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \end{bmatrix} [N] \tag{8.85}$$

The stresses $\vec{\sigma}$ can be obtained from the strains $\vec{\varepsilon}$ using Eq. (8.10) as

$$\vec{\sigma} = [D](\vec{\varepsilon} - \vec{\varepsilon}_0) = [D][B]\vec{Q}^{(e)} - [D]\vec{\varepsilon}_0 \tag{8.86}$$

Substitution of Eqs. (8.82) and (8.84) into Eq. (8.83) yields the potential energy of the element as

$$\begin{aligned} \pi_p^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \vec{Q}^{(e)^T}[B]^T[D][B]\vec{Q}^{(e)}\,\mathrm{d}V - \iiint_{V^{(e)}} \vec{Q}^{(e)^T}[B]^T[D]\vec{\varepsilon}_0\,\mathrm{d}V - \iint_{S_1^{(e)}} \vec{Q}^{(e)^T}[N]^T\vec{\vec{\Phi}}\,\mathrm{d}S_1 \\ - \iiint_{V^{(e)}} \vec{Q}^{(e)^T}[N]^T\vec{\vec{\phi}}\,\mathrm{d}V \end{aligned} \tag{8.87}$$

In Eqs. (8.83) and (8.87), only the body and surface forces are considered. However, generally some external concentrated forces will also be acting at various nodes. If $\vec{\underset{\sim}{P}}_c$ denotes the vector of nodal forces (acting in the directions of the nodal displacement vector $\underset{\sim}{\vec{Q}}$ of the total structure or body), the total potential energy of the structure or body can be expressed as

$$\pi_p = \sum_{e=1}^{E} \pi_p^{(e)} - \underset{\sim}{\vec{Q}}^T \underset{\sim}{\vec{P}}_c \tag{8.88}$$

where $\underset{\sim}{\vec{Q}} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_M \end{Bmatrix}$ is the vector of nodal displacements of the entire structure or body, and M is the total number of nodal displacements or degrees of freedom.

Note that each component of the vector $\vec{Q}^{(e)}$, $e = 1, 2, \ldots, E$, appears in the global nodal displacement vector of the structure or body, $\underset{\sim}{\vec{Q}}$. Accordingly, $\vec{Q}^{(e)}$ for each element may be replaced by $\underset{\sim}{\vec{Q}}$ if the remaining element matrices and vectors (e.g., $[B]$, $[N]$, $\vec{\vec{\Phi}}$, and $\vec{\vec{\phi}}$) in the expression for $\pi_p^{(e)}$ are enlarged by adding the required number of zero elements and, where necessary, by rearranging their elements. In other words, the summation of Eq. (8.88) implies the expansion of element matrices to structure or body size followed by summation of overlapping terms. Thus, Eqs. (8.87) and (8.88) give

$$\pi_p = \frac{1}{2}\underset{\sim}{\vec{Q}}^T \left[\sum_{e=1}^{E} \iiint_{V^{(e)}} [B]^T[D][B]\,\mathrm{d}V\right] \underset{\sim}{\vec{Q}} - \underset{\sim}{\vec{Q}}^T \sum_{e=1}^{E} \left(\iiint_{V^{(e)}} [B]^T[D]\vec{\epsilon}_0 \mathrm{d}V \right.$$

$$\left. + \iint_{S_1^{(e)}} [N]^T \vec{\vec{\Phi}}\,\mathrm{d}S_1 + \iiint_{V^{(e)}} [N]^T \vec{\vec{\phi}}\,\mathrm{d}V \right) - \underset{\sim}{\vec{Q}}^T \underset{\sim}{\vec{P}}_c \tag{8.89}$$

Eq. (8.89) expresses the total potential energy of the structure or body in terms of the nodal degrees of freedom, $\underset{\sim}{\vec{Q}}$. The static equilibrium configuration of the structures can be found by solving the following necessary conditions (for the minimization of potential energy):

$$\frac{\partial \pi_p}{\partial \underset{\sim}{\vec{Q}}} = \vec{0} \quad \text{or} \quad \frac{\partial \pi_p}{\partial Q_1} = \frac{\partial \pi_p}{\partial Q_2} = \cdots = \frac{\partial \pi_p}{\partial Q_M} = 0 \tag{8.90}$$

With the help of Eq. (8.89), Eq. (8.90) can be expressed as

$$\underbrace{\left(\sum_{e=1}^{E} \underbrace{\iiint_{V^{(e)}} [B]^T[D]\,\mathrm{d}V}_{\text{Element stiffness matrix, } [K^{(e)}]} \right)}_{\text{Global or overall stiffness matrix of the structure or body, } [\underset{\sim}{K}]} \underbrace{\underset{\sim}{\vec{Q}}}_{\text{Global vector of nodal displacements}}$$

$$= \underbrace{\underbrace{\underset{\sim}{\vec{P}}_c}_{\text{Vector of concentrated loads}} + \underbrace{\sum_{e=1}^{E} \left(\underbrace{\iiint_{V^{(e)}} [B]^T[D]\vec{\varepsilon}_0\,\mathrm{d}V}_{\text{Vector of element nodal forces produced by initial strains, } \vec{P}_i^{(e)}} + \underbrace{\iint_{S_1^{(e)}} [N]^T \vec{\vec{\Phi}}\,\mathrm{d}S_1}_{\text{Vector of element nodal forces produced by surface forces, } \vec{P}_s^{(e)}} + \underbrace{\iint_{V^{(e)}} [N]^T \vec{\vec{\phi}}\,\mathrm{d}V}_{\text{Vector of element nodal forces produced by body forces, } \vec{P}_b^{(e)}} \right)}_{\text{Vector of element nodal forces, } \vec{P}^{(e)}}}_{\text{Total vector of nodal forces, } \underset{\sim}{\vec{P}}}$$

That is,

$$\left(\sum_{e=1}^{E}\left[K^{(e)}\right]\right)\vec{\underset{\sim}{Q}} = \vec{\underset{\sim}{P}}_c + \sum_{e=1}^{E}\left(\vec{P}_i^{(e)} + \vec{P}_s^{(e)} + \vec{P}_b^{(e)}\right) = \vec{\underset{\sim}{P}} \tag{8.91}$$

where

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\,\mathrm{d}V = \text{element stiffness matrix} \tag{8.92}$$

$$\vec{P}_i^{(e)} = \iiint_{V^{(e)}} [B]^T[D]\vec{\varepsilon_0}\,\mathrm{d}V = \text{element load vector due to intial strains} \tag{8.93}$$

$$\vec{P}_s^{(e)} = \iint_{S_1^{(e)}} [N]^T\vec{\vec{\Phi}}\,\mathrm{d}S_1 = \text{element load vector due to surface forces} \tag{8.94}$$

$$\vec{P}_b^{(e)} = \iiint_{V^{(e)}} [N]^T\vec{\vec{\phi}}\,\mathrm{d}V = \text{element load vector due to body forces} \tag{8.95}$$

Some of the contributions to the load vector $\vec{\underset{\sim}{P}}$ may be zero in a particular problem. In particular, the contribution of surface forces will be nonzero only for those element boundaries that are also part of the boundary of the structure or body that is subjected to externally applied distributed loading.

The load vectors $\vec{P}_i^{(e)}$, $\vec{P}_s^{(e)}$, and $\vec{P}_b^{(e)}$ given in Eqs. (8.93) to (8.95) are called kinematically consistent nodal load vectors [8.5]. Some of the components of $\vec{P}_i^{(e)}$, $\vec{P}_s^{(e)}$, and $\vec{P}_b^{(e)}$ may be moments or even higher-order quantities if the corresponding nodal displacements represent rotations or curvatures. These load vectors are called kinematically consistent because they satisfy the virtual work (or energy) equation. That is, the virtual work done by a particular generalized load P_j when the corresponding displacement δQ_j is permitted (while all other nodal displacements are prohibited) is equal to the work done by the distributed (nonnodal) loads in moving through the displacements dictated by δQ_j and the assumed displacement field.

Step 4: The desired equilibrium equations of the overall structure or body can now be expressed, using Eq. (8.91), as

$$\left[\underset{\sim}{K}\right]\vec{\underset{\sim}{Q}} = \vec{\underset{\sim}{P}} \tag{8.96}$$

where

$$\left[\underset{\sim}{K}\right] = \sum_{e=1}^{E}\left[K^{(e)}\right] = \text{assembled (global) stiffness matrix} \tag{8.97}$$

and

$$\vec{\underset{\sim}{P}} = \vec{\underset{\sim}{P}}_c + \sum_{e=1}^{E}\vec{P}_i^{(e)} + \sum_{e=1}^{E}\vec{P}_s^{(e)} + \sum_{e=1}^{E}\vec{P}_b^{(e)} = \text{assembled (global) nodal load vector} \tag{8.98}$$

Steps 5 and 6: The required solution for the nodal displacements and element stresses can be obtained after solving Eq. (8.96).

The following observations can be made from the previous derivation:

1. The formulation of element stiffness matrices, $[K^{(e)}]$, and element load vectors, $\vec{P}_i^{(e)}$, $\vec{P}_s^{(e)}$, and $\vec{P}_b^{(e)}$, which is basic to the development of finite element Eq. (8.96), requires integrations as indicated in Eqs. (8.92) to (8.95). For some elements, the evaluation of these integrals is simple. However, in certain cases, it is often convenient to perform the integrations numerically [8.6].
2. The formulas for the element stiffness and load vector in Eqs. (8.92) to (8.95) remain the same irrespective of the type of element. However, the orders of the stiffness matrix and load vector will change for different types of elements. For example, in the case of a triangular element under plane stress, the order of $[K^{(e)}]$ is 6×6 and of $\vec{Q}^{(e)}$ is 6×1. For a rectangular element under plane stress, the orders of $[K^{(e)}]$ and $\vec{Q}^{(e)}$ are 8×8 and 8×1, respectively. It is assumed that the displacement model is linear in both these cases.

3. The element stiffness matrix given by Eq. (8.92) and the assembled stiffness matrix given by Eq. (8.97) are always symmetric. In fact, the matrix $[D]$ and the product $[B]^T\,[D]\,[B]$ appearing in Eq. (8.92) are also symmetric.
4. In the analysis of certain problems, it is generally more convenient to compute the element stiffness matrices $[k^{(e)}]$ and element load vectors $\vec{p}_i^{(e)}$, $\vec{p}_s^{(e)}$, and $\vec{p}_b^{(e)}$ in local coordinate systems[2] suitably set up (differently for different elements) for minimizing the computational effort. In such cases, the matrices $[k^{(e)}]$ and vectors $\vec{p}_i^{(e)}$, $\vec{p}_s^{(e)}$, and $\vec{p}_b^{(e)}$ have to be transformed to a common global coordinate system before using them in Eqs. (8.97) and (8.98).
5. The equilibrium equations given by Eq. (8.96) cannot be solved since the stiffness matrices $[K^{(e)}]$ and $[\underset{\sim}{K}]$ are singular, and hence their inverses do not exist.
   The physical significance of this is that a loaded structure or body is free to undergo unlimited rigid body motion (translation and/or rotation) unless some support or boundary constraints are imposed on the structure or body to suppress the rigid body motion. These constraints are called boundary conditions. The methods of incorporating boundary conditions were considered in Chapter 6.
6. To obtain the (displacement) solution of the problem, we have to solve Eq. (8.96) after incorporating the prescribed boundary conditions. The methods of solving the resulting equations were discussed in Chapter 7.

8.5 NATURE OF FINITE ELEMENT SOLUTIONS

In an exact elasticity solution of a solid or structural mechanics problem, the differential equations of equilibrium are satisfied by every infinitesimally small element, the compatibility equations are satisfied throughout, and all the stress and displacement boundary conditions are met. The finite element solution of the same problem, on the other hand, will be approximate and will not satisfy all the relations of elasticity. The nature of displacement-based finite element solution for static problems from the point of view of an exact elastic solution can be stated as follows.

Equilibrium conditions

Internal equilibrium: For most cases, the internal (differential) equilibrium equations are not satisfied within the elements. They are only satisfied on the average within the elements. Thus, with mesh refinement using smaller size elements, the internal equilibrium equations will be satisfied to a larger extent. In some cases, such as in the case of the constant strain triangle (CST) element (Section 10.2), the internal equilibrium equations are satisfied.

Equilibrium across interelement boundaries: For most problems, the equilibrium of forces is not satisfied across element boundaries. This is true even in the case of the CST element. In addition, the equilibrium of induced stresses and the applied forces on the surface of the body, Eq. (8.51), are not satisfied exactly. These equations will be satisfied to a greater extent with the refinement of the finite element mesh.

Equilibrium at node points: All displacement-based finite element solutions satisfy the equilibrium of forces and moments at the node points. The nodal displacement vector (solution) $\vec{Q}$ satisfies the nodal equilibrium equations $[K]\,\vec{Q} = \vec{P}$ where each component of $\vec{P}$ denotes either the external force applied or an elastically generated internal reaction force.

Compatibility conditions

Internal compatibility: In all finite element solutions, the internal compatibility (within the finite elements) is satisfied because internal compatibility requires the variation of the displacement to be continuous, single-valued, and hence any polynomial model used in the finite element analysis will satisfy the required conditions.

Compatibility across interelement boundaries: Depending on the elements used, compatibility across interelement boundaries may or may not be satisfied. When the variation of the displacement along a side of the element depends only on the degrees of freedom of the nodes lying on that side and the adjacent elements share these nodes and the degrees of freedom, then the compatibility is satisfied across the common side of the adjacent elements. The CST element, for example, satisfies compatibility across interelement boundaries.

Compatibility at node points: In most finite element solutions, compatibility is satisfied at all the nodes because adjacent elements share the same nodal displacement components. In some situations, the adjacent elements may be having translational and rotational degrees of freedom at each node. But only the translational degrees of freedom may be connected at a particular node. In such a case, the node acts like a ball-and-socket joint and hence compatibility is not fully satisfied at the node.

2. When a local coordinate system is used, the resulting quantities are denoted by lowercase letters as $[k^{(e)}]$, $\vec{p}_i^{(e)}$, $\vec{p}_s^{(e)}$, and $\vec{p}_b^{(e)}$ instead of $[K^{(e)}]$, $\vec{P}_i^{(e)}$, $\vec{P}_s^{(e)}$, and $\vec{P}_b^{(e)}$.

REVIEW QUESTIONS

8.1 Give brief answers to the following questions.

1. What are the basic types of equations for a solid mechanics problem?
2. State the types of unknown quantities and their number for a one-dimensional solid mechanics problem.
3. State the additional types of equations to be satisfied in a solid mechanics problem.
4. How many independent internal equilibrium equations are there for a one-dimensional solid mechanics problem?
5. How are body forces defined?
6. Name the possible types of stress states for a two-dimensional body.
7. What are the nonzero stress components in a plane stress problem?
8. Give an example of a plane stress problem.
9. Give an example of a plane strain problem.
10. Under what circumstances does an axisymmetric state of stress exist?
11. State the boundary conditions of a three-dimensional solid body fixed at one point.
12. State the significance of compatibility conditions.
13. State the number of principal stresses for a three-dimensional body subjected to loads.
14. Define the potential energy of an elastic body.
15. Define the Lagrangian.

8.2 Fill in the blank with a suitable word.

1. External equilibrium equations ensure the force and ———————— equilibrium along or about the x, y, and z axes.
2. Internal ———————— equilibrium equations lead to symmetry of shear stresses in the absence of body moments.
3. The number of internal force equilibrium equations for a two-dimensional body is ————.
4. For a linear elastic isotropic body, the stress—strain relations are also known as ———————— law.
5. The nonzero strains in a plane strain problem are ε_{xx}, ε_{yy}, ε_{xy}, and ————————.
6. The nonzero stresses in a plane strain problem are σ_{xx}, σ_{yy}, τ_{xy}, and ————————.
7. The nonzero quantities (unknowns) in a one-dimensional solid body are ————, ————, and ε_x.
8. The linear strains along x, y, and z directions can be expressed in terms of u, v, and w as ————————.
9. Shear strain in xy-plane can be expressed in terms of u and v as ————————.
10. The application of the principle of minimum potential energy in the finite element method yields ———————— equations while ———————— equations are automatically satisfied.
11. The variational principle that can be used for dynamic problems is called the ———————— principle.
12. Displacement-based finite element solution satisfies compatibility at ————————.

8.3 Indicate whether each of the following statements is true or false.

1. The finite element method satisfies all the basic and additional equations for a solid mechanics problem.
2. The constitutive relations are same as the stress—strain relations.
3. Heating a solid body can induce both normal and shear strains.
4. The nonzero stresses in an axisymmetric problem are σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz},and σ_{rz}.
5. The displacement method is also known as the stiffness method.
6. The application of the principle of minimum complimentary energy method yields compatibility equations while the equilibrium equations are automatically satisfied.
7. The displacement-based finite element satisfies internal equilibrium equations within each element.
8. The displacement-based finite element solution does not satisfy the equilibrium across interelement boundaries.
9. The displacement-based finite element satisfies equilibrium equations at node points.
10. The displacement-based finite element satisfies internal compatibility within each element.

8.4 Select the most appropriate answer from the multiple choices given.

1. The number of basic equations for a three-dimensional problem is:
   (a) 3 (b) 15 (c) 12

2. The number of basic equations for a two-dimensional problem is:
   (a) 8 (b) 15 (c) 2
3. The number of basic equations for a one-dimensional problem is:
   (a) 1 (b) 2 (c) 1
4. The number of independent elastic constants involved in the compliance matrix of an anisotropic solid body is
   (a) 9 (b) 3 (c) 6
5. The number of strain displacement relations that exist for an axisymmetric problem is:
   (a) 3 (b) 4 (c) 6
6. The number of displacement components in an axisymmetric problem is:
   (a) 1 (b) 3 (c) 2
7. The number of compatibility conditions for a three-dimensional solid body is:
   (a) 3 (b) 6 (c) 4
8. The number of compatibility conditions for a plane stress problem is:
   (a) 4 (b) 3 (c) 1
9. The number of compatibility conditions for a plane strain problem is:
   (a) 4 (b) 3 (c) 1
10. The number of compatibility conditions for a one-dimensional solid body is:
   (a) 1 (b) 0 (c) 2

8.5 Match the following formulations of solid mechanics problems.

1. Displacement method of differential equation formulation	(a) Reissner stationary energy formulation
2. Force method of differential equation formulation	(b) Minimum potential energy formulation
3. Force-displacement method of differential equation formulation	(c) Complementary energy formulation

PROBLEMS

8.1 Consider an infinitesimal element of a solid body in the form of a rectangular parallelepiped as shown in Fig. 8.2. In this figure, the components of stress acting on one pair of faces only are shown for simplicity. Apply the moment equilibrium equations about the x, y, and z axes and show that the shear stresses are symmetric; that is, $\sigma_{yx} = \sigma_{xy}$, $\sigma_{zy} = \sigma_{yz}$, and $\sigma_{xz} = \sigma_{zx}$.

8.2 Determine whether the following state of strain is physically realizable:

$$\varepsilon_{xx} = c\left(x^2 + y^2\right), \quad \varepsilon_{yy} = cy^2, \quad \varepsilon_{xy} = 2cxy, \quad \varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{zx} = 0$$

where c is a constant.

8.3 When a body is heated nonuniformly and each element of the body is allowed to expand nonuniformly, the strains are given by

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \alpha T, \quad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \tag{P.1}$$

where α is the coefficient of thermal expansion (constant), and $T = T(x, y, z)$ is the temperature. Determine the nature of variation of $T(x, y, z)$ for which the equations of Eq. (P.1) are valid.

8.4 Consider the following state of stress and strain:

$$\sigma_{xx} = x^2, \quad \sigma_{yy} = y^2, \quad \varepsilon_{xy} = -2xy, \quad \sigma_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Determine whether the equilibrium equations are satisfied.

8.5 Consider the following condition:

$$\sigma_{xx} = x^2, \quad \sigma_{yy} = y^2, \quad \varepsilon_{xy} = -2xy, \quad \sigma_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Determine whether the compatibility equations are satisfied.

8.6 Consider the following state of strain:

$$\varepsilon_{xx} = c_1 x, \quad \varepsilon_{yy} = c_2, \quad \varepsilon_{zz} = c_3 x + c_4 y + c_5, \quad \varepsilon_{yz} = c_6 y, \quad \varepsilon_{xy} = \varepsilon_{zx} = 0$$

where c_i, $i = 1, 2, \ldots, 6$ are constants. Determine whether the compatibility equations are satisfied.

8.7 The internal equilibrium equations of a two-dimensional solid body, in polar coordinates, are given by

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \phi_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r} + \phi_\theta = 0$$

where ϕ_r and ϕ_θ are the body forces per unit volume in the radial (r) and circumferential (θ) directions, respectively.

If the state of stress in a loaded, thick-walled cylinder is given by

$$\sigma_{rr} = \frac{a^2 p}{b^2 - a^2}\left[1 - \frac{b^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{a^2 p}{b^2 - a^2}\left[1 + \frac{b^2}{r^2}\right]$$

$$\sigma_{r\theta} = 0$$

where a, b, and p denote the inner radius, outer radius, and internal pressure, respectively, determine whether this state of stress satisfies the equilibrium equations.

8.8 Determine whether the following displacement represents a feasible deformation state in a solid body:

$$u = ax, \quad v = ay, \quad w = az$$

where a is a constant.

8.9 Consider a plate with a hole of radius a subjected to an axial stress p. The state of stress around the hole [8.7] follows:

$$\sigma_{rr} = \frac{1}{2}p\left[1 - \frac{a^2}{r^2}\right] + \frac{1}{2}p\left[1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right]\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{1}{2}p\left[1 + \frac{a^2}{r^2}\right] - \frac{1}{2}p\left[1 + 3\frac{a^4}{r^4}\right]\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{1}{2}p\left[1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right]\sin 2\theta$$

Determine whether these stresses satisfy the equilibrium equations stated in Problem 8.7.

8.10 Consider a uniform bar of length l and cross-sectional area A rotating about a pivot point O as shown in Fig. 8.11. Using the centrifugal force as the body force, determine the stiffness matrix and load vector of the element using a linear displacement model:

$$u(x) = N_1(x)q_1 + N_2(x)q_2$$

where $N_1(x) = 1 - (x/l)$, $N_2(x) = (x/l)$ and the stress–strain relation, $\sigma_{xx} = E\varepsilon_{xx}$, where E is the Young's modulus.

8.11 A stress function $\Phi(r)$ can be defined for an axisymmetric (planar) problem so that the stresses can be determined as

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = 0$$

where r and θ denote the radial and tangential directions, respectively. If the stress function of a thick-walled cylindrical pressure vessel of inner radius r_i and outer radius r_o, subjected to both internal and external pressures, is given by

$$\Phi(r) = c_1 + c_2 \ln r + c_3 r^2 + c_4 r^2 \ln r$$

determine the radial and tangential stresses in the pressure vessels. Also indicate the conditions to be used to determine the constants c_i, $i = 1, 2, 3, 4$.

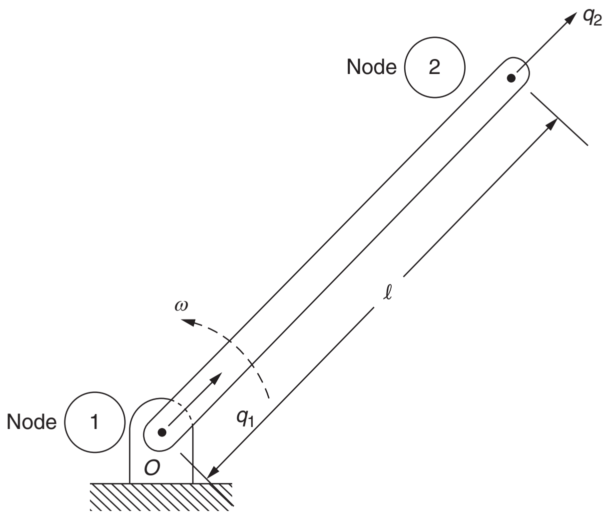

FIGURE 8.11 Rotating uniform bar.

8.12 Consider the stress function for a two-dimensional state of stress, with no body forces:

$$\Phi(x, y) = c_1 x^4 + c_2 x^3 y + c_3 x^2 y^2 + c_4 x y^3 + c_5 y^4$$

Determine the relationship between the constants c_1, c_2, …, c_5 for satisfying the compatibility conditions.

8.13 A simple stress function for a planar problem corresponding to a concentrated force acting on a flat boundary (shown in Fig. 8.12) is given by

$$\Phi(r, \theta) = c\, r\, \theta \sin\theta, \quad c = \text{constant}$$

with the stress components given by

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}, \quad \sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}, \quad \tau_{xy} = \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}$$

Determine the resulting stresses at point P and find the value of the constant c.

8.14 Express the differential equations of equilibrium of a planar elasticity problem in terms of the displacement components.

8.15 Consider the following displacement field in a plane elasticity problem (with negligible value of the Poisson's ratio):

$$u(x, y) = cxy^2, \quad v(x, y) = -cx^2 y; \quad c = \text{constant}$$

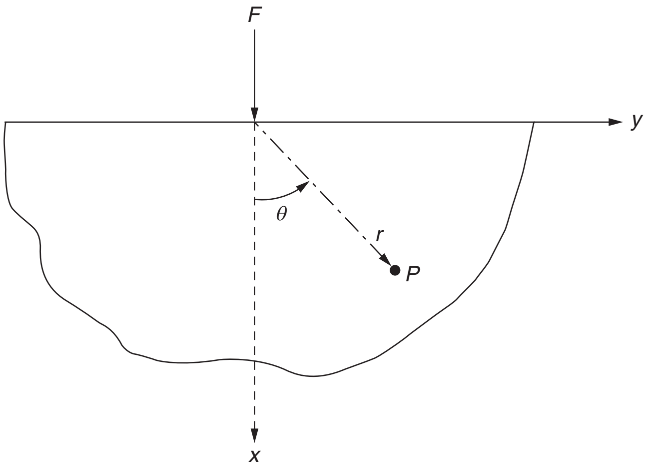

FIGURE 8.12 Concentrated force on flat boundary.

Determine whether this displacement field is a valid solution. If not, state the condition that is not satisfied.

8.16 Consider the displacement variations

$$u(x,y) = c_1\left(x^2+y^2\right) - c_2 y + c_3, v(x,y) = 2c_1 xy + c_4; \quad c_i = \text{constants}, \quad i = 1,2,3,4$$

Determine whether this displacement field corresponds to a valid solution of a plane stress problem.

8.17 Consider the displacement variations in a planar problem:

$$u(x,y) = -c\left(vx^2y^2\right), \quad v(x,y) = 2cxy; \quad c = \text{constant}$$

Find the stresses corresponding to this displacement field.

8.18 Derive Eq. (8.50).

8.19 Consider the following displacement field in a plane elasticity problem (with negligible value of the Poisson's ratio):

$$u(x,y) = 4c\left(x^2y+y^3\right), \quad v(x,y) = 2cy^3; \quad c = \text{constant}$$

Determine whether this displacement field is a valid solution. If not, state the condition that is not satisfied.

8.20 The stress field in a plane elasticity problem is given by

$$\sigma_{xx} = c_1 x + c_2 y, \sigma_{yy} = c_3 x + c_4 y, \tau_{xy} = -c_1 y - c_4 x; c_i = \text{constants}, \quad i = 1,2,3,4$$

Determine whether this stress field is a valid solution. If not, state the condition that is not satisfied.

8.21 Determine the type of problem for which the following stress function is valid:

$$\Phi(r) = c\ln r + \frac{pr_i^2 r^2}{2(r_0^2 - r_i^2)}; \quad r_i \le r \le r_0$$

where c is a constant and r_i and r_0 are inner and outer radii of an annular region. Evaluate the constant c by using the boundary condition $\sigma_r = 0$ at $r = r_0$.

8.22 Prove that the equilibrium of moments about the x, y, and z axes for the element shown in Fig. 8.2 gives $\sigma_{yx} = \sigma_{xy}$, $\sigma_{zy} = \sigma_{yz}$, and $\sigma_{xz} = \sigma_{zx}$. State the assumptions involved.

8.23 **a.** Show the various stresses acting on a rectangular element of size $dx \times dy$ in a two-dimensional state of stress.
b. Show that the equilibrium of moments about the z direction leads to $\sigma_{yx} = \sigma_{xy}$.
c. Prove that the equilibrium of forces along the x and y directions leads to Eq. (8.5).

8.24 Derive the strain–displacement relations for an axisymmetric problem (Eq. 8.50). Assume the displacements to be small.

8.25 Consider an element of size $r\Delta\theta\,\Delta r\,\Delta z$ of a solid body in cylindrical coordinates (Δz is the thickness of the element in the z direction) and derive the differential equations for the equilibrium of forces in the radial and tangential directions as

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \phi_r = 0$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + 2\frac{\sigma_{r\theta}}{r} + \phi_\theta = 0$$

where ϕ_r and ϕ_θ are, respectively, the body forces in the radial and tangential directions.

8.26 Consider a solid circular of length $l = 5$ m and radius $r_0 = 15$ mm. Use r,θ,z - coordinate system with origin at the centroid of the shaft at one end with the z axis along the centroidal axis towards the other end of the shaft. The shaft is subjected to both tensile and torsional forces that resulted in an elongation of $\Delta l = 15$ mm and a rotation of 45° at one end relative to the other end of the shaft. Assume that the deformations occur while the volume of the shaft remains unchanged. If the end of the shaft at $z = 0$ is constrained so that it undergoes only radical displacement (Δr), determine the following: (a) Displacement components at a general point in the shaft, and (b) Components of strain at the outer radius of the shaft.

8.27 A state of plane stress exists at a point where $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = -100\ \mu$, and $\varepsilon_{xy} = 300\ \mu$. If $E = 150$ MPa and $v = 0.3$, determine the complete strain and stress vectors.

8.28 The strains in a plane stress problem are given by $\varepsilon_{xx} = -120\ \mu$, $\varepsilon_{yy} = -90\ \mu$, and $\varepsilon_{xy} = 150\ \mu$. If $E = 207$ GPa and $\nu = 0.3$, determine the complete strain and stress vectors.

8.29 Find the stress vector if the modulus of elasticity is 200 MPa, the Poisson's ratio is 0.3, and the strain vector is $\vec{\varepsilon}^T = \{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{zx} \} = \{ 10 \quad -6 \quad 4 \quad -4 \quad 2 \quad 6 \} \mu$.

8.30 If the stress vector is given by

$$\vec{\sigma}^T = \{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx} \} = \{ -24 \quad 10 \quad -16 \quad 9 \quad 3 \quad -12 \} \text{ MPa},$$

and the elastic constants by $E = 207$ GPa and $\nu = 0.3$, determine the strain vector.

8.31 The stress components in a plane stress problem are given by $\sigma_{xx} = 15$ MPa, $\sigma_{yy} = 10$ MPa, and $\sigma_{xy} = -20$ MPa. Determine all the strain components.

8.32 The strain components in a plane strain problem are given by $\varepsilon_{xx} = 0.002$, $\varepsilon_{yy} = 0.015$, and $\varepsilon_{xy} = 0.005$. Assuming that $E = 70$ GPa and $\nu = 0.33$, determine all the stress components.

8.33 The undeformed and deformed configurations of a rectangular plate are shown in Fig. 8.13. Determine the strains ε_{xx}, ε_{yy}, and ε_{xy} induced in the plate.

8.34 Find the most general displacement solution by integrating the following strain equations:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} = 0$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = 0$$

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0$$

Give a physical interpretation of the resulting solution.

8.35 A solid body, in the form of a rectangular parallelepiped (length l, width w, and height h), is subjected to a uniform normal stress so that the stress state is given by $\sigma_{xx} = \sigma_0 =$ constant, $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$. Express the change in the volume of the solid body in terms of σ_0, E, and ν.

8.36 A solid body in the form of a rectangular parallelepiped (length l along the x direction, width w along the y direction, and height h along the z direction), is heated uniformly by a temperature ΔT. If the motion of the body is completely constrained in the y and z directions and is free to move in the x direction, determine the following: (a) change in the length l and (b) stresses developed in the y and z directions.

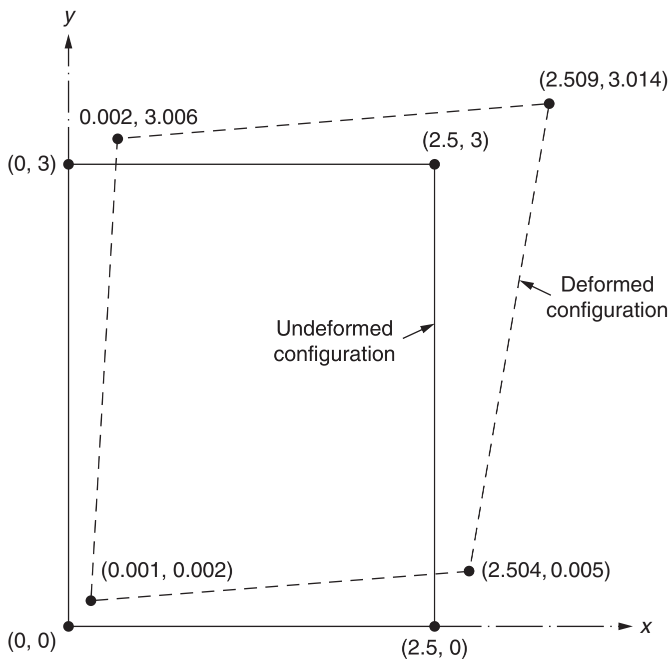

FIGURE 8.13 Deformation of a rectangular plate.

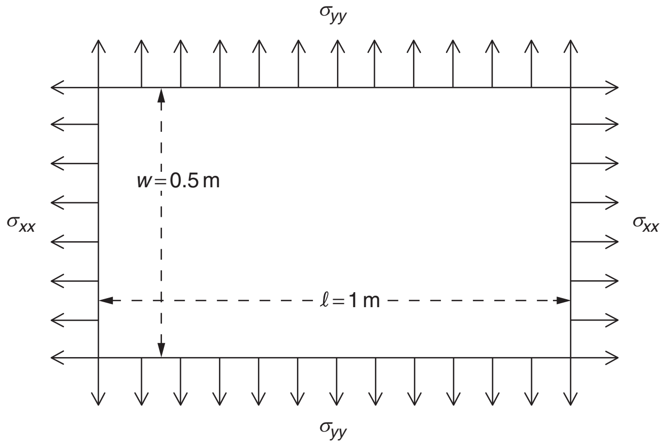

FIGURE 8.14 Plate subjected to in-plane normal stresses.

8.37 A solid body, in the form of a rectangular parallelepiped (length l along the x direction, width w along the y direction, and height h along the z direction), is heated uniformly by a temperature ΔT. If the body is completely free to deform, find the change in the volume of the solid body.

8.38 A solid body, in the form of a rectangular parallelepiped (length l along the x direction, width w along the y direction, and height h along the z direction), is heated uniformly by a temperature ΔT. If the motion of the body is completely constrained in the x direction and is free to move in the y and z directions, determine the following: (a) change in the volume of the body and (b) stress developed in the body in the x direction.

8.39 A 0.02-m thick rectangular plate of length $l = 1$ m and width $w = 0.5$ m is subjected to the stresses $\sigma_{xx} = 200$ MPa and $\sigma_{yy} = -100$ MPa as shown in Fig. 8.14. If $E = 209$ GPa and $v = 0.3$, determine the following: (a) deformations Δl and Δw and (b) change in the volume of the plate.

8.40 If a two-dimensional solid body is subjected to the distributed forces Φ_x and Φ_y along its curved edge, express the boundary conditions for the equilibrium of the induced stresses σ_{xx}, σ_{yy}, and σ_{xy} and the applied forces.
Hint: Use the two-dimensional version of Eq. (8.51).

8.41 The stresses applied along the curved boundary (edge) of a plate are given by $\vec{\vec{\sigma}} = \sigma_{xx}\vec{i} + \sigma_{yy}\vec{j}$ (see Fig. 8.15). Express the boundary conditions (for the equilibrium of the applied stresses and the induced stresses) in terms of the displacement components u and v.

8.42 Derive Eq. (8.59a–d) from Eqs. (8.55) through (8.58).

8.43 The stresses developed in a solid body under applied loads are given by $\vec{\vec{\sigma}}^T = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx}\} = \{6 \quad 0 \quad 0 \quad 2 \quad 4 \quad 2\}$ MPa. Determine the principal stresses.

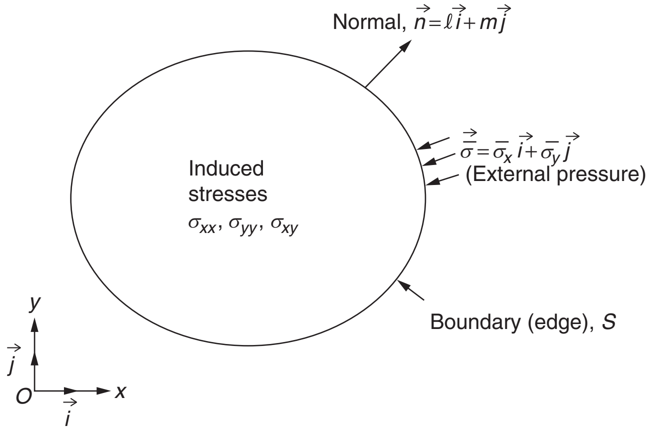

FIGURE 8.15 A plate in xy-plane.

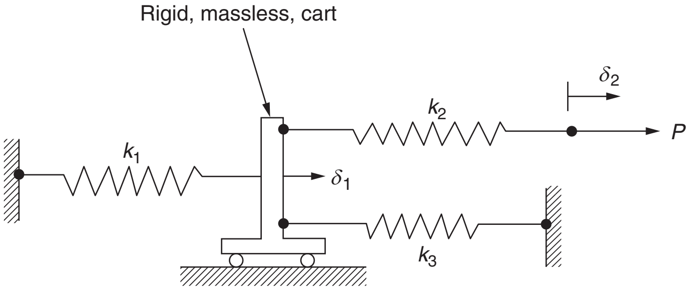

FIGURE 8.16 Rigid cart connected by springs.

8.44 The state of stress at a point in a solid body is given by the stress vector $\vec{\sigma}^T = \{-4 \quad 12 \quad -8 \;\; 64 - 10\}$ MPa. Determine the principal stresses.

8.45 The stresses in a plane stress problem are given by $\sigma_{xx} = 10$ MPa, $\sigma_{yy} = -10$ MPa, and $\sigma_{xy} = 10$ MPa. Determine the principal stresses.

8.46 The stresses developed in a thin plate are known to be $\sigma_{xx} = 0$, $\sigma_{yy} = 0$, and $\sigma_{xy} = 20$ MPa. Determine the principal stresses in the plate.

8.47 Express the stress–strain relations and the strain–stress relations in terms of the principal stresses (σ_1, σ_2, σ_3) and the principal strains (ε_1, ε_2, ε_3) for a uniaxial state of stress.
Hint: The principal strains are defined as the strains in the directions of the principal stresses.

8.48 Express the stress–strain relations and the strain–stress relations in terms of the principal stresses (σ_1, σ_2, σ_3) and the principal strains (ε_1, ε_2, ε_3) for a triaxial state of stress.
Hint: The principal strains are defined as the strains in the directions of the principal stresses.

8.49 Express the stress–strain relations and the strain–stress relations in terms of the principal stresses (σ_1, σ_2, σ_3) and the principal strains (ε_1, ε_2, ε_3) for a biaxial state of stress.
Hint: The principal strains are defined as the strains in the directions of the principal stresses.

8.50 Three springs, with stiffnesses k_1, k_2, and k_3, are connected to a rigid massless cart as shown in Fig. 8.16. The free end of the spring of stiffness k_2 is subjected to a force P. If the cart is constrained to move horizontally, determine the displacements δ_1 and δ_2 by applying the principle of minimum potential energy.

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Chapter 9

Analysis of Trusses, Beams, and Frames

Chapter Outline

9.1 INTRODUCTION

The derivation of element equations for one-dimensional structural elements is considered in this chapter. These elements can be used for the analysis of skeletal-type systems such as planar trusses, space trusses, beams, continuous beams, planar frames, grid systems, and space frames. Pin-jointed bar elements are used in the analysis of trusses. A truss element is a bar that can resist only axial forces (compressive or tensile) and can deform only in the axial direction. It will not be able to carry transverse loads or bending moments. In planar truss analysis, each of the two nodes can have components of displacement parallel to X and Y axes. In three-dimensional truss analysis, each node can have displacement components in X, Y, and Z directions. Rigidly jointed bar (beam) elements are used in the analysis of frames. Thus, a frame or a beam element is a bar that can resist not only axial forces but also transverse loads and bending moments. In the analysis of planar frames, each of the two nodes of an element will have two translational displacement components (parallel to X and Y axes) and a rotational displacement (in the plane XY). For a space frame element, each of the two ends is assumed to have three translational displacement components (parallel to X, Y, and Z axes) and three rotational displacement components (one in each of the three planes XY, YZ, and ZX). In the present development, we assume the members to be uniform and linearly elastic.

9.2 SPACE TRUSS ELEMENT

Consider the pin-jointed bar element shown in Fig. 9.1, in which the local x axis is taken in the axial direction of the element with origin at corner (or local node) 1. A linear displacement model is assumed as

$$u(x) = q_1 + (q_2 - q_1)\frac{x}{l}$$

or

$$\underset{1\times 1}{\{u(x)\}} = \underset{1\times 2}{[N]}\ \underset{2\times 1}{\vec{q}^{(e)}} \tag{9.1}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00009-2

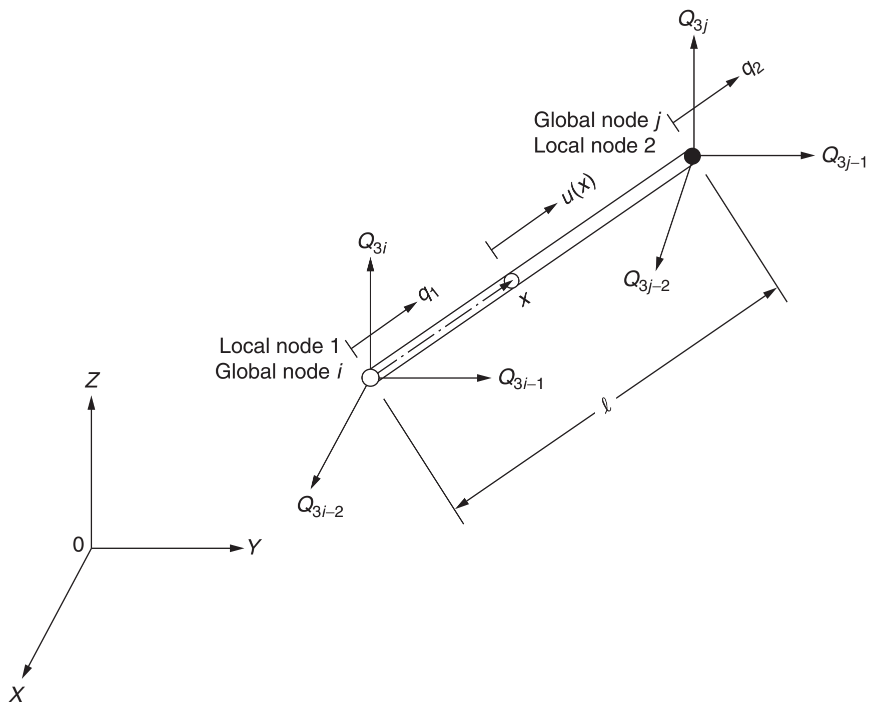


FIGURE 9.1 A space truss element.

where

$$[N] = \left[\left(1 - \frac{x}{l}\right) \frac{x}{l}\right] \tag{9.2}$$

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{9.3}$$

where q_1 and q_2 represent the nodal degrees of freedom in the local coordinate system (unknowns), l denotes the length of the element, and the superscript e denotes the element number. The axial strain can be expressed as

$$\varepsilon_{xx} = \frac{\partial u(x)}{\partial x} = \frac{q_2 - q_1}{l}$$

or

$$\underset{1\times1}{\{\varepsilon_{xx}\}} = \underset{1\times2}{[B]} \quad \underset{2\times1}{\vec{q}^{(e)}} \tag{9.4}$$

where

$$[B] = \left[-\frac{1}{l} \quad \frac{1}{l}\right] \tag{9.5}$$

The stress−strain relation is given by

$$\sigma_{xx} = E(\varepsilon_{xx} - \varepsilon_{xx_0})$$

which can be expressed in matrix form as

$$\underset{1\times1}{\{\sigma_{xx}\}} = \underset{1\times1}{[D]}\left(\underset{1\times1}{\{\varepsilon_{xx}\}} - \underset{1\times1}{\{\varepsilon_{xx_0}\}}\right) \tag{9.6}$$

where $[D] = [E]$, $\varepsilon_{xx_0} = \alpha T$, $E =$ Young's modulus, $\alpha =$ coefficient of thermal expansion, and $T =$ temperature change. The stiffness matrix of the element (in the local coordinate system) can be obtained, from Eq. (8.87), as

$$\underset{2\times 2}{\left[k^{(e)}\right]} = \iiint_{V^{(e)}} [B]^T [D][B]\, \mathrm{d}V = A \int_{x=0}^{l} \begin{Bmatrix} -\dfrac{1}{l} \\ \\ \dfrac{1}{l} \end{Bmatrix} E \left\{ -\frac{1}{l} \quad \frac{1}{l} \right\} \mathrm{d}x \tag{9.7}$$

$$= \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ \\ -1 & 1 \end{bmatrix}$$

where A is the area of cross section of the bar. To find the stiffness matrix of the bar in the global coordinate system, we need to find the transformation matrix. In general, the element under consideration will be one of the elements of a space truss. Let the (local) nodes 1 and 2 of the element correspond to nodes i and j, respectively, of the global system as shown in Fig. 9.1. The local displacements q_1 and q_2 can be resolved into components Q_{3i-2}, Q_{3i-1}, Q_{3i}; and Q_{3j-2}, Q_{3j-1}; and Q_{3j} parallel to the global X, Y, Z axes, respectively. Then the two sets of displacements are related as

$$\vec{q}^{(e)} = [\lambda]\vec{Q}^{(e)} \tag{9.8}$$

where the transformation matrix $[\lambda]$ and the vector of nodal displacements of element e in the global coordinate system, $\vec{Q}^{(e)}$, are given by

$$[\lambda] = \begin{bmatrix} l_{ij} & m_{ij} & n_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ij} & m_{ij} & n_{ij} \end{bmatrix} \tag{9.9}$$

$$\vec{Q}^{(e)} = \begin{Bmatrix} Q_{3i-2} \\ Q_{3i-1} \\ Q_{3i} \\ Q_{3j-2} \\ Q_{3j-1} \\ Q_{3j} \end{Bmatrix} \tag{9.10}$$

and l_{ij}, m_{ij}, and n_{ij} denote the direction cosines of angles between the line ij and the directions OX, OY, and OZ, respectively. The direction cosines can be computed in terms of the global coordinates of nodes i and j as

$$l_{ij} = \frac{X_j - X_i}{l}, \quad m_{ij} = \frac{Y_j - Y_i}{l}, \quad n_{ij} = \frac{Z_j - Z_i}{l} \tag{9.11}$$

where (X_i, Y_i, Z_i) and (X_j, Y_j, Z_j) are the global coordinates of nodes i and j, respectively, and l is the length of the element ij given by

$$l = \left\{ (X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2 \right\}^{1/2} \tag{9.12}$$

Thus, the stiffness matrix of the element in the global coordinate system can be obtained, using Eq. (6.6), as

$$\underset{6\times6}{\left[K^{(e)}\right]} = \underset{6\times2}{[\lambda]^T}\ \underset{2\times2}{\left[k^{(e)}\right]}\ \underset{2\times6}{[\lambda]}$$

$$= \frac{AE}{l}\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} \\ l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} & -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} \\ l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 & -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 \\ -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} & l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} \\ -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} & l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} \\ -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 & l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 \end{bmatrix} \tag{9.13}$$

9.2.1 Consistent Load Vector

The consistent load vectors can be computed using Eqs. (8.88) to (8.90):

$$\vec{p}_i^{(e)} = \text{load vector due to initial (thermal) strains} = \iiint_{V^{(e)}} [B]^T[D]\vec{\varepsilon}_0\, dV \tag{9.14}$$

$$= A\int_0^l \begin{Bmatrix} -1/l \\ 1/l \end{Bmatrix}[E]\{\alpha T\}\cdot dx = AE\alpha T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{9.15}$$

$$\vec{p}_b^{(e)} = \text{load vector due to constant body force } (\phi_0) = \iiint_{V^{(e)}} [N]^T\vec{\vec{\phi}}\, dV \tag{9.16}$$

$$= A\int_0^l \begin{Bmatrix} \left(1-\frac{x}{l}\right) \\ (x/l) \end{Bmatrix}\{\phi_0\}\, dx = \frac{\phi_0 A l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{9.17}$$

The total consistent load vector in the local coordinate system is given by

$$\vec{p}^{(e)} = \vec{p}_i^{(e)} + \vec{p}_b^{(e)} \tag{9.18}$$

This load vector, when referred to the global coordinate system, will be

$$\overline{P}^{(e)} = [\lambda]^T\vec{p}^{(e)} \tag{9.19}$$

where $[\lambda]$ is given by Eq. (9.9).

9.2.2 Computation of Stresses

After finding the displacement solution of the system, the nodal displacement vector $\vec{Q}^{(e)}$ of element e can be identified. The stress induced in element e can be determined using Eqs. (9.6), (9.4), and (9.8) as

$$\sigma_{xx} = E\left([B][\lambda]\vec{Q}^{(e)} - \alpha T\right) \tag{9.20}$$

where $[B]$ and $[\lambda]$ are given by Eqs. (9.5) and (9.9), respectively.

EXAMPLE 9.1

A bar element, with cross-sectional area 2 in^2 and Young's modulus 30×10^6 psi, is shown in Fig. 9.2. The coordinates of the two nodes of the element in the global coordinate system (X, Y, Z) are also indicated in Fig. 9.2. Find (a) the stiffness matrix of the bar in the local coordinate system (x, y, z), (b) the coordinate transformation matrix $[\lambda]$ of the element, and (c) the global stiffness matrix of the element.

Approach: Use Eqs. (9.7), (9.9), and (9.13).

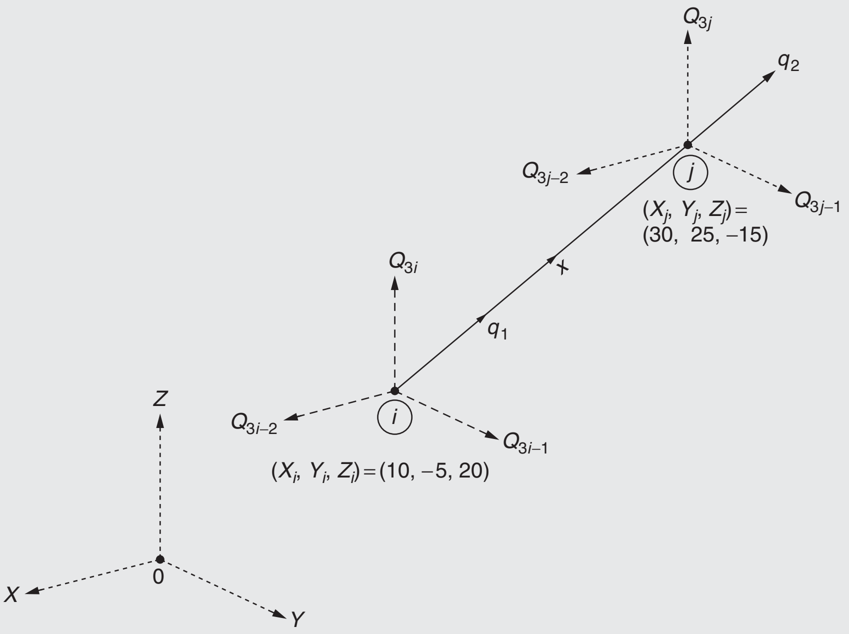


FIGURE 9.2 Bar element with global degrees of freedom.

Solution

The length of the bar element can be computed as

$$l^{(e)} = \left\{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2\right\}^{\frac{1}{2}} = \left\{(30-10)^2 + (25-(-5))^2 + (-15-20)^2\right\}^{\frac{1}{2}}$$

$$= 50.2494 \text{ in.}$$

The direction cosines of the x-axis with respect to the X, Y, and Z axes are given by

$$l_{ij} = \frac{X_j - X_i}{l^{(e)}} = \frac{30-10}{50.2494} = 0.3980, \qquad m_{ij} = \frac{Y_j - Y_i}{l^{(e)}} = \frac{25-(-5)}{50.2494} = 0.5970$$

$$n_{ij} = \frac{Z_j - Z_i}{l^{(e)}} = \frac{-15-20}{50.2494} = -0.6965$$

a. The local stiffness matrix of the element is given by Eq. (9.7):

$$\left[k^{(e)}\right] = \frac{A^{(e)}E^{(e)}}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(2)(30 \times 10^6)}{50.2494}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 1.1940 & -1.1940 \\ -1.1940 & 1.1940 \end{bmatrix} \text{lb/in.}$$

Continued

EXAMPLE 9.1 —cont'd

b. The coordinate transformation matrix of the element is given by Eq. (9.9):

$$\left[\lambda^{(e)}\right] = \begin{bmatrix} l_{ij} & m_{ij} & n_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ij} & m_{ij} & n_{ij} \end{bmatrix}$$

$$= \begin{bmatrix} 0.3980 & 0.5970 & -0.6965 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3980 & 0.5970 & -0.6965 \end{bmatrix}$$

c. The global stiffness matrix of the element is given by Eq. (9.13): Using

$$\frac{A^{(e)}E^{(e)}}{l^{(e)}} = 1.1940 \times 10^6$$

and

$$\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} \\ m_{ij}l_{ij} & m_{ij}^2 & m_{ij}n_{ij} \\ n_{ij}l_{ij} & n_{ij}m_{ij} & n_{ij}^2 \end{bmatrix} = \begin{bmatrix} (0.3980)^2 & (0.3980)(0.5970) & (0.3980)(-0.6965) \\ (0.5970)(0.3980) & (0.5970)^2 & (0.5970)(-0.6965) \\ (-0.6965)(0.3980) & (-0.6965)(0.5970) & (-0.6965)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1584 & 0.2376 & -0.2772 \\ 0.2376 & 0.3564 & -0.4158 \\ -0.2772 & -0.4158 & 0.4851 \end{bmatrix}$$

the global stiffness matrix of the element can be written as

$$\left[K^{(e)}\right] = 1.1940 \times 10^6 \begin{bmatrix} 0.1584 & 0.2376 & -0.2772 & -0.1584 & -0.2376 & 0.2772 \\ 0.2376 & 0.3564 & -0.4158 & -0.2376 & -0.3564 & 0.4158 \\ -0.2772 & -0.4158 & 0.4851 & 0.2772 & 0.4158 & -0.4851 \\ -0.1584 & -0.2376 & 0.2772 & 0.1584 & 0.2376 & -0.2772 \\ -0.2376 & -0.3564 & 0.4158 & 0.2376 & 0.3564 & -0.4158 \\ 0.2772 & 0.4158 & -0.4851 & -0.2772 & -0.4158 & 0.4851 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 0.1891 & 0.2837 & -0.3310 & -0.1891 & -0.2837 & 0.3310 \\ 0.2837 & 0.4255 & -0.4965 & -0.2837 & -0.4255 & 0.4965 \\ -0.3310 & -0.4965 & 0.5792 & 0.3310 & 0.4965 & -0.5792 \\ -0.1891 & -0.2837 & 0.3310 & 0.1891 & 0.2837 & -0.3310 \\ -0.2837 & -0.4255 & 0.4965 & 0.2837 & 0.4255 & -0.4965 \\ 0.3310 & 0.4965 & -0.5792 & -0.3310 & -0.4965 & 0.5792 \end{bmatrix} \text{lb/in.}$$

EXAMPLE 9.2

The bar element considered in Example 9.1 has a coefficient of thermal expansion of $\alpha = 5.5 \times 10^{-6}$ in/in °F. If the temperature of the bar increases by $T = 100$°F, determine the load vectors of the element in the local and global coordinate systems.

Approach: Use Eq. (9.14).

Solution

a. The load vector in the local coordinate system due to a temperature rise is given by Eq. (9.14):

$$\vec{p}_i^{(e)} = A^{(e)}E^{(e)}\alpha^{(e)}T\begin{Bmatrix}-1\\1\end{Bmatrix} = (2)(30\times10^6)(5.5\times10^{-6})(100)\begin{Bmatrix}-1\\1\end{Bmatrix} = 33{,}000\begin{Bmatrix}-1\\1\end{Bmatrix}\text{lb}$$

where the coefficient of thermal expansion of element e, $\alpha^{(e)}$, is the same as α.

b. Using the coordinate transformation matrix $[\lambda]$ determined in Example 9.1, the load vector of the element in the global coordinate system can be found from Eq. (9.19):

$$\vec{P}_i^{(e)} = [\lambda]^T\vec{p}_i^{(e)} = \begin{bmatrix} l_{ij} & m_{ij} & n_{ij} & 0 & 0 & 0\\ 0 & 0 & 0 & l_{ij} & m_{ij} & n_{ij}\end{bmatrix}^T \vec{p}_i^{(e)}$$

$$= \begin{bmatrix} 0.3980 & 0.5970 & -0.6965 & 0 & 0 & 0\\ 0 & 0 & 0 & 0.3980 & 0.5970 & -0.6965\end{bmatrix}^T (33{,}000)\begin{Bmatrix}-1\\1\end{Bmatrix}$$

$$= \begin{Bmatrix} -13{,}134.0\\ -19{,}701.0\\ 22{,}984.5\\ 13{,}134.0\\ 19{,}701.0\\ -22{,}984.5\end{Bmatrix}\text{lb}$$

EXAMPLE 9.3

Two bars made of different materials are connected as shown in Fig. 9.3. The properties of the two bars are given by:

$A^{(1)} = 500$ mm^2, $E^{(1)} = 75$ GPa, $\alpha^{(1)} = 20 \times 10^{-6}$ per °C

$A^{(2)} = 1000$ mm^2, $E^{(2)} = 200$ GPa, $\alpha^{(2)} = 15 \times 10^{-6}$ per °C

If the temperature of the two-bar system is raised by 50°C, determine the displacement of the middle node and the stresses developed in the two bars.

Approach: Solve the system equilibrium equations using load vector due to a temperature rise.

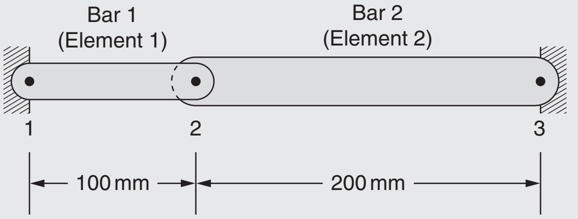


FIGURE 9.3 Two bars made of different materials.

Solution

Since both the bars lie along the same axis, there is no need to modify the local element stiffness matrices and load vectors. The stiffness matrices of the elements are given by

$$[k^{(1)}] = \frac{A^{(1)}E^{(1)}}{l^{(1)}}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{(500)(75\times10^3)}{100}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 375\times10^3 \begin{array}{c} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{matrix}u_1\\u_2\end{matrix} \\ \begin{matrix}u_1 & \quad u_2\end{matrix}\end{array} \text{N/mm}$$

Continued

EXAMPLE 9.3 —cont'd

$$[k^{(2)}] = \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(1000)(200\times 10^3)}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1000\times 10^3 \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{matrix} u_2 & u_3 \end{matrix} \end{array} \begin{matrix} u_2 \\ u_3 \end{matrix} \text{ N/mm}$$

The assembled stiffness matrix can be found as

$$[K] = 10^3 \begin{array}{c} \begin{bmatrix} 375 & -375 & 0 \\ -375 & 1375 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix} \\ \begin{matrix} u_1 & \quad u_2 & \quad u_3 \end{matrix} \end{array} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \text{ N/mm}$$

The load vectors of the elements can be determined as

$$\vec{p}^{(1)} = A^{(1)}E^{(1)}\alpha^{(1)}T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = (500)(75\times 10^3)(20\times 10^{-6})(50)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 37,500\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} u_1 \\ u_2 \end{matrix} N$$

$$\vec{p}^{(2)} = A^{(2)}E^{(2)}\alpha^{(2)}T\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = (1000)(200\times 10^3)(15\times 10^{-6})(50)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$= 150,000\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix} N$$

The assembled load vector can be found as

$$\vec{P} = \begin{Bmatrix} -37,500 \\ 37,500 - 150,000 \\ 150,000 \end{Bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} N$$

The equilibrium equations of the system can be expressed as

$$[K]\vec{U} = \vec{P}$$

or

$$10^3\begin{bmatrix} 375 & -375 & 0 \\ -375 & 1,375 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -37,500 \\ -112,500 \\ 150,000 \end{Bmatrix} \tag{E.1}$$

Because the end nodes 1 and 3 are fixed, $u_1 = u_3 = 0$ and hence the rows and columns corresponding to the degrees of freedom u_1 and u_3 are deleted in Eq. (E.1) to obtain the equation

$$10^3(1375)u_2 = -112,500 \text{ or } u_2 = -0.0818181 \text{ mm}$$

Stress in Element 1

The strain in element 1 can be found as

$$\varepsilon^{(1)} = \frac{u_2 - u_1}{l^{(1)}} = \frac{-0.0818181 - 0}{100} = -81.8181\times 10^{-5}$$

and the stress is related to strain as

$$\sigma^{(1)} = E^{(1)}\left(\varepsilon^{(1)} - \varepsilon_0^{(1)}\right) = E^{(1)}\left(\varepsilon^{(1)} - \alpha^{(1)}\ T\right)$$

$$= 75\times 10^3\{-81.8181\times 10^{-5} - 20\times 10^{-6}(50)\}$$

$$= -136.3636 \text{ N/mm}^2 = -136.3636 \text{ MPa}$$

EXAMPLE 9.3 —cont'd

Stress in Element 2

The strain in element 2 can be found as

$$\varepsilon^{(2)} = \frac{u_3 - u_2}{l^{(2)}} = \frac{0 - (-0.0818181)}{200} = 40.9090 \times 10^{-5}$$

and the stress is related to strain as

$$\sigma^{(2)} = E^{(2)}\left(\varepsilon^{(2)} - \varepsilon_0^{(2)}\right) = E^{(2)}\left(\varepsilon^{(2)} - \alpha^{(2)} T\right)$$

$$= 200 \times 10^3 \left\{40.9090 \times 10^{-5} - 15 \times 10^{-6}(50)\right\}$$

$$= 70.0 \text{ N/mm}^2 = 70.0 \text{ MPa}$$

EXAMPLE 9.4

Find the nodal displacements developed in the planar truss shown in Fig. 9.4 when a vertically downward load of 1000 N is applied at node 4. The pertinent data are given in Table 9.1.

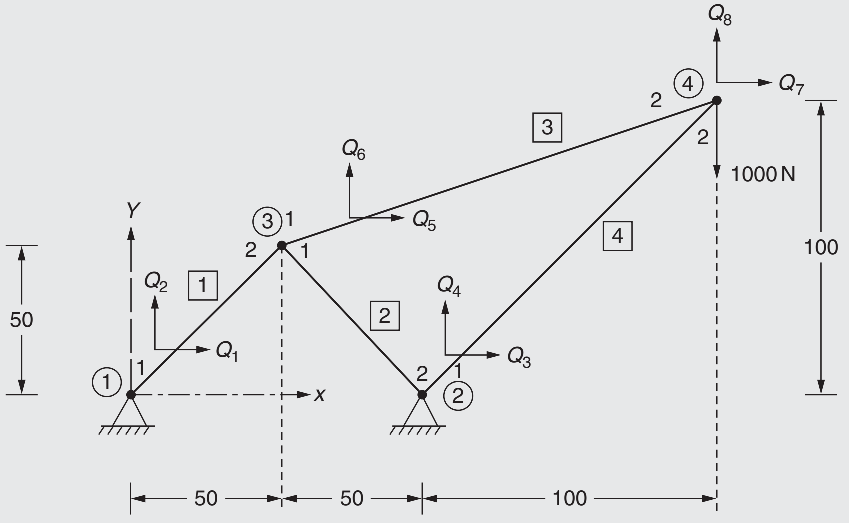


FIGURE 9.4 Geometry of the planar truss of Example 9.4 (dimensions in cm).

TABLE 9.1 Data of Members of Truss

Member Number "e"	Cross-Sectional Area $A^{(e)}$ cm^2	Length $l^{(e)}$ cm	Young's Modulus $E^{(e)}$ N/cm^2
1	2.0	$\sqrt{2}\,50$	2×10^6
2	2.0	$\sqrt{2}\,50$	2×10^6
3	1.0	$\sqrt{2.5}\,100$	2×10^6
4	1.0	$\sqrt{2}\,100$	2×10^6

Continued

EXAMPLE 9.4 —cont'd

Solution

The numbering system for the nodes, members, and global displacements is indicated in Fig. 9.4. The nodes 1 and 2 in the local system and the local x direction assumed for each of the four elements are shown in Fig. 9.5. For convenience, the global node numbers i and j corresponding to the local nodes 1 and 2 for each element and the direction cosines of the line ij (x axis) with respect to the global X and Y axes are given in Table 9.2.

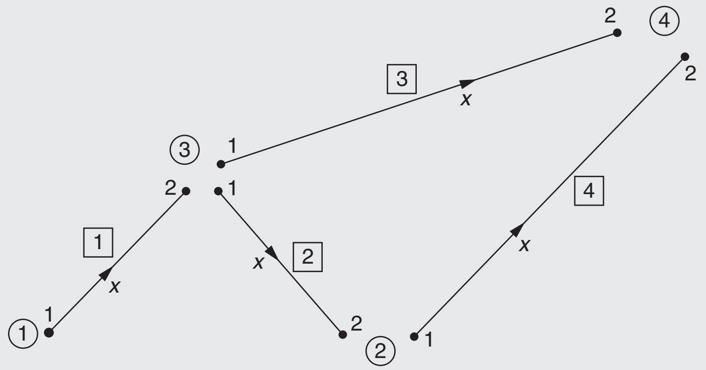


FIGURE 9.5 Finite element idealization.

TABLE 9.2 Direction Cosines of Members

Member Number	Global Node Corresponding to		Coordinates of Local Nodes 1 (i) and 2 (j) in Global System				Direction Cosines of Line ij	
	Local Node 1	Local Node 2						
"e"	(i)	(j)	X_i	Y_i	X_j	Y_j	l_{ij}	m_{ij}
1	1	3	0.0	0.0	50.0	50.0	$1/\sqrt{2}$	$1/\sqrt{2}$
2	3	2	50.0	50.0	100.0	0.0	$1/\sqrt{2}$	$-1/\sqrt{2}$
3	3	4	50.0	50.0	200.0	100.0	$1.5/\sqrt{2.5}$	$0.5/\sqrt{2.5}$
4	2	4	100.0	0.0	200.0	100.0	$1/\sqrt{2}$	$1/\sqrt{2}$

The stiffness matrix of element e in the global coordinate system can be computed from the following (obtained by deleting the rows and columns corresponding to the Z degrees of freedom from Eq. (9.13)):

$$[K^{(e)}] = \frac{A^{(e)}E^{(e)}}{l^{(e)}} \begin{array}{c} \begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} \\ l_{ij}m_{ij} & m_{ij}^2 & -l_{ij}m_{ij} & -m_{ij}^2 \\ -l_{ij}^2 & -l_{ij}m_{ij} & l_{ij}^2 & l_{ij}m_{ij} \\ -l_{ij}m_{ij} & -m_{ij}^2 & l_{ij}m_{ij} & m_{ij}^2 \end{bmatrix} \\ \underbrace{\begin{matrix} Q_{2i-1} & \quad Q_{2i} & \quad Q_{2j-1} & \quad Q_{2j} \end{matrix}}_{\text{Global degrees of freedom corresponding to different columns}} \end{array} \begin{array}{l} \left.\begin{matrix} Q_{2i-1} \\ Q_{2i} \\ Q_{2j-1} \\ Q_{2j} \end{matrix}\right\} \text{Global degrees of freedom corresponding to different rows} \\ \\ \end{array}$$

EXAMPLE 9.4 —cont'd

Hence,

$$\left[K^{(1)}\right]=\frac{(2.0)(2\times10^6)}{\sqrt{2}\,50}\begin{array}{c c}
\begin{array}{cccc} Q_1 & Q_2 & Q_5 & Q_6 \end{array} & \\
\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} & \begin{array}{c} Q_1 \\ Q_2 \\ Q_5 \\ Q_6 \end{array}
\end{array}$$

$$=\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}(2\sqrt{2})\times10^4 N/\text{cm}$$

$$\left[K^{(2)}\right]=\frac{(2.0)(2\times10^6)}{\sqrt{2}\,50}\begin{array}{c c}
\begin{array}{cccc} Q_5 & Q_6 & Q_3 & Q_4 \end{array} & \\
\begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} & \begin{array}{c} Q_5 \\ Q_6 \\ Q_3 \\ Q_4 \end{array}
\end{array}$$

$$=\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}(2\sqrt{2})\times10^4 N/\text{cm}$$

$$\left[K^{(3)}\right]=\frac{(1.0)(2\times10^6)}{\sqrt{2.5}\,100}\begin{array}{c c}
\begin{array}{cccc} Q_5 & Q_6 & Q_7 & Q_8 \end{array} & \\
\begin{bmatrix} \frac{2.25}{2.50} & \frac{0.75}{2.50} & \frac{-2.25}{2.50} & \frac{-0.75}{2.50} \\ \frac{0.75}{2.50} & \frac{0.25}{2.50} & \frac{-0.75}{2.50} & \frac{-0.25}{2.50} \\ \frac{-2.25}{2.50} & \frac{-0.75}{2.50} & \frac{2.25}{2.50} & \frac{0.75}{2.50} \\ \frac{-0.75}{2.50} & \frac{-0.25}{2.50} & \frac{0.75}{2.50} & \frac{0.25}{2.50} \end{bmatrix} & \begin{array}{c} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{array}
\end{array}$$

$$=\begin{bmatrix} 9 & 3 & -9 & -3 \\ 3 & 1 & -3 & -1 \\ -9 & -3 & 9 & 3 \\ -3 & -1 & 3 & 1 \end{bmatrix}(8\sqrt{2.5})\times10^2 N/\text{cm}$$

$$\left[K^{(4)}\right]=\frac{(1.0)(2\times10^6)}{\sqrt{2}\,100}\begin{array}{c c}
\begin{array}{cccc} Q_3 & Q_4 & Q_7 & Q_8 \end{array} & \\
\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} & \begin{array}{c} Q_3 \\ Q_4 \\ Q_7 \\ Q_8 \end{array}
\end{array}$$

$$=\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}(5\sqrt{2})\times10^3\, N/\text{cm}$$

Continued

EXAMPLE 9.4 —cont'd

These element matrices can be assembled to obtain the global stiffness matrix, $[\underset{\sim}{K}]$, as

$$[\underset{\sim}{K}] = 10^3 \begin{array}{cc} \begin{array}{cccccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & Q_8 \end{array} & \\ \begin{bmatrix} 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & -20\sqrt{2} & -20\sqrt{2} & 0 & 0 \\ 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & -20\sqrt{2} & -20\sqrt{2} & 0 & 0 \\ 0 & 0 & \begin{matrix} 20\sqrt{2} \\ +5\sqrt{2} \end{matrix} & \begin{matrix} -20\sqrt{2} \\ +5\sqrt{2} \end{matrix} & -20\sqrt{2} & 20\sqrt{2} & -5\sqrt{2} & -5\sqrt{2} \\ 0 & 0 & \begin{matrix} -20\sqrt{2} \\ +5\sqrt{2} \end{matrix} & \begin{matrix} 20\sqrt{2} \\ +5\sqrt{2} \end{matrix} & 20\sqrt{2} & -20\sqrt{2} & -5\sqrt{2} & -5\sqrt{2} \\ -20\sqrt{2} & -20\sqrt{2} & -20\sqrt{2} & 20\sqrt{2} & \begin{matrix} 20\sqrt{2} \\ +20\sqrt{2} \\ +7.2\sqrt{2.5} \end{matrix} & \begin{matrix} 20\sqrt{2} \\ -20\sqrt{2} \\ +2.4\sqrt{2.5} \end{matrix} & -7.2\sqrt{2.5} & -2.4\sqrt{2.5} \\ -20\sqrt{2} & -20\sqrt{2} & 20\sqrt{2} & -20\sqrt{2} & \begin{matrix} 20\sqrt{2} \\ -20\sqrt{2} \\ +2.4\sqrt{2.5} \end{matrix} & \begin{matrix} 20\sqrt{2} \\ +20\sqrt{2} \\ +0.8\sqrt{2.5} \end{matrix} & -2.4\sqrt{2.5} & -0.8\sqrt{2.5} \\ 0 & 0 & -5\sqrt{2} & -5\sqrt{2} & -7.2\sqrt{2.5} & -2.4\sqrt{2.5} & \begin{matrix} 7.2\sqrt{2.5} \\ +5\sqrt{2} \end{matrix} & \begin{matrix} 2.2\sqrt{2.5} \\ +5\sqrt{2} \end{matrix} \\ 0 & 0 & -5\sqrt{2} & -5\sqrt{2} & -2.4\sqrt{2.5} & -0.8\sqrt{2.5} & \begin{matrix} 2.4\sqrt{2.5} \\ +5\sqrt{2} \end{matrix} & \begin{matrix} 0.8\sqrt{2.5} \\ +5\sqrt{2} \end{matrix} \end{bmatrix} & \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{array} \end{array} N/\text{cm}$$

Thus, the global equations of equilibrium can be expressed as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{Q}} = \underset{\sim}{\vec{P}} \tag{E.1}$$

where

$$\underset{\sim}{\vec{Q}} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_8 \end{Bmatrix} \quad \text{and} \quad \underset{\sim}{\vec{P}} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_8 \end{Bmatrix}$$

By deleting the rows and columns corresponding to the restrained degrees of freedom ($Q_1 = Q_2 = Q_3 = Q_4 = 0$), Eq. (E.1) can be written as

$$[K]\vec{Q} = \vec{P} \tag{E.2}$$

where

$$[K] = 10^3 \begin{bmatrix} 40\sqrt{2} + 7.2\sqrt{2.5} & 2.4\sqrt{2.5} & -7.2\sqrt{2.5} & -2.4\sqrt{2.5} \\ 2.4\sqrt{2.5} & 40\sqrt{2} + 0.8\sqrt{2.5} & -2.4\sqrt{2.5} & -0.8\sqrt{2.5} \\ -7.2\sqrt{2.5} & -2.4\sqrt{2.5} & 5\sqrt{2} + 7.2\sqrt{2.5} & 5\sqrt{2} + 2.4\sqrt{2.5} \\ -2.4\sqrt{2.5} & -0.8\sqrt{2.5} & 5\sqrt{2} + 2.4\sqrt{2.5} & 5\sqrt{2} + 0.8\sqrt{2.5} \end{bmatrix} \text{N/cm}$$

$$\vec{Q} = \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} \quad \text{and} \quad \vec{P} = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \end{Bmatrix} \equiv \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix} \text{N}$$

The solution of Eq. (E.2) gives the displacements as

$$Q_5 = 0.026517 \text{ cm}$$

$$Q_6 = 0.008839 \text{ cm}$$

$$Q_7 = 0.347903 \text{ cm}$$

$$Q_8 = -0.560035 \text{ cm}$$

EXAMPLE 9.5

Find the stresses developed in the various elements of the truss considered in Example 9.4.

Solution

The nodal displacements of the truss in the global coordinate system (including the fixed degrees of freedom) are given by

$$\vec{\underset{\sim}{Q}} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.026517 \\ 0.008839 \\ 0.347903 \\ -0.560035 \end{Bmatrix} \text{ cm} \tag{E.1}$$

The nodal degrees of freedom of various elements, in the global coordinate system, can be identified from Figs. 9.4 and 9.5 and Eq. (E.1) as

$$\vec{Q}^{(1)} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \\ Q_4^{(1)} \end{Bmatrix} \equiv \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0.026517 \\ 0.008839 \end{Bmatrix} \text{ cm} \tag{E.2}$$

$$\vec{Q}^{(2)} = \begin{Bmatrix} Q_1^{(2)} \\ Q_2^{(2)} \\ Q_3^{(2)} \\ Q_4^{(2)} \end{Bmatrix} \equiv \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0.026517 \\ 0.008839 \\ 0 \\ 0 \end{Bmatrix} \text{ cm} \tag{E.3}$$

$$\vec{Q}^{(3)} = \begin{Bmatrix} Q_1^{(3)} \\ Q_2^{(3)} \\ Q_3^{(3)} \\ Q_4^{(3)} \end{Bmatrix} \equiv \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = \begin{Bmatrix} 0.026517 \\ 0.008839 \\ 0.347903 \\ -0.560035 \end{Bmatrix} \text{ cm} \tag{E.4}$$

$$\vec{Q}^{(4)} = \begin{Bmatrix} Q_1^{(4)} \\ Q_2^{(4)} \\ Q_3^{(4)} \\ Q_4^{(4)} \end{Bmatrix} \equiv \begin{Bmatrix} Q_3 \\ Q_4 \\ Q_7 \\ Q_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0.347903 \\ -0.560035 \end{Bmatrix} \text{ cm} \tag{E.5}$$

As indicated in Eq. (9.20), the axial stress developed in element e is given by

$$\sigma_{xx}^{(e)} = E^{(e)} \left[B^{(e)}\right] \left[\lambda^{(e)}\right] \vec{Q}^{(e)}$$

$$= E^{(e)} \left[-\frac{1}{l^{(e)}} \quad \frac{1}{l^{(e)}} \right] \begin{bmatrix} l_{ij}^{(e)} & m_{ij}^{(e)} & 0 & 0 \\ 0 & 0 & l_{ij}^{(e)} & m_{ij}^{(e)} \end{bmatrix} \begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ Q_3^{(e)} \\ Q_4^{(e)} \end{Bmatrix} \tag{E.6}$$

Eq. (E.6) can be simplified as

$$\sigma_{xx}^{(e)} = E^{(e)} \left\{ -\frac{1}{l^{(e)}} \left(l_{ij}^{(e)} Q_1^{(e)} + m_{ij}^{(e)} Q_2^{(e)} \right) + \frac{1}{l^{(e)}} \left(l_{ij}^{(e)} Q_3^{(e)} + m_{ij}^{(e)} Q_4^{(e)} \right) \right\} \tag{E.7}$$

Continued

EXAMPLE 9.5 —cont'd

which yields the following results:

Element 1: $E^{(1)} = 2 \times 10^6$ N/cm^2, $l^{(1)} = 70.7107$ cm, $l_{ij}^{(1)} = m_{ij}^{(1)} = 0.707,107$, $Q_i^{(1)}$, $i = 1, 2, 3, 4$, are given by Eq. (E.2) so that $\sigma_{xx}^{(1)} = 707.1200$ N/cm^2.

Element 2: $E^{(2)} = 2 \times 10^6$ N/cm^2, $l^{(2)} = 70.7107$ cm, $l_{ij}^{(2)} = 0.707,107$, $m_{ij}^{(1)} = -0.707,107$, $Q_i^{(2)}$, $i = 1, 2, 3, 4$, are given by Eq. (E.3) so that $\sigma_{xx}^{(2)} = -353.560$ N/cm^2.

Element 3: $E^{(3)} = 2 \times 10^6$ N/cm^2, $l^{(3)} = 158.114$ cm, $l_{ij}^{(3)} = 0.948,683$, $m_{ij}^{(2)} = 0.316,228$, $Q_i^{(3)}$, $i = 1, 2, 3, 4$, are given by Eq. (E.4) so that $\sigma_{xx}^{(3)} = 1581.132$ N/cm^2.

Element 4: $E^{(4)} = 2 \times 10^6$ N/cm^2, $l^{(4)} = 141.421$ cm, $l_{ij}^{(4)} = m_{ij}^{(4)} = 0.707,107$, $Q_i^{(4)}$, $i = 1, 2, 3, 4$, are given by Eq. (E.5) so that $\sigma_{xx}^{(4)} = -2121.326$ N/cm^2.

9.3 BEAM ELEMENT

A beam is a straight bar element that is primarily subjected to transverse loads. Assuming that the beam is symmetric about y and z axes (with $I_{yz} = 0$), the deformed shape of a beam is described by the transverse displacement and slope (rotation) of the beam. Hence, the transverse displacement and rotation at each end of the beam element are treated as the unknown degrees of freedom. For a beam element of length l lying in the xy plane, as shown in Fig. 9.6, the 4 degrees of freedom in the local (xy) coordinate system are indicated as q_1, q_2, q_3, and q_4. Note that the counterclockwise rotational degree of freedom is considered positive (i.e., rotation as vector points along the z axis). Because there are four nodal displacements, we assume a cubic displacement model for $v(x)$ as (Fig. 9.6)

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{9.21}$$

where the constants α_1 to α_4 can be found by using the conditions

$$v(x) = q_1 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x}(x) = q_2 \quad \text{at} \quad x = 0$$

and

$$v(x) = q_3 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x}(x) = q_4 \quad \text{at} \ x = l$$

Eq. (9.21) can thus be expressed as

$$\underset{1\times1}{v(x)} = \underset{1\times4}{[N]} \ \underset{4\times1}{\vec{q}} \tag{9.22}$$

where $[N]$ is given by

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

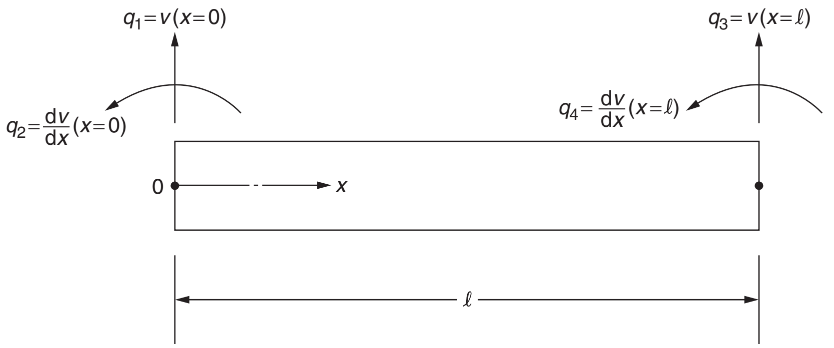


FIGURE 9.6 Beam element with degrees of freedom.

with

$$N_1(x) = \left(2x^3 - 3lx^2 + l^3\right)/l^3$$
$$N_2(x) = \left(x^3 - 2lx^2 + l^2x\right)/l^2$$
$$N_3(x) = -\left(2x^3 - 3lx^2\right)/l^3 \tag{9.23}$$
$$N_4(x) = \left(x^3 - lx^2\right)/l^2$$

and

$$\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{9.24}$$

According to simple beam theory, plane sections of the beam remain plane after deformation and hence the axial displacement u due to the transverse displacement v can be expressed as (from Fig. 9.7)

$$u = -y\frac{\partial v}{\partial x}$$

where y is the distance from the neutral axis. The axial strain, in the absence of thermal effects, is given by[1]

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = [B]\vec{q} \tag{9.25}$$

where

$$[B] = -\frac{y}{l^3}\{(12x - 6l) \quad l(6x - 4l) - (12x - 6l) \quad l(6x - 2l)\} \tag{9.26}$$

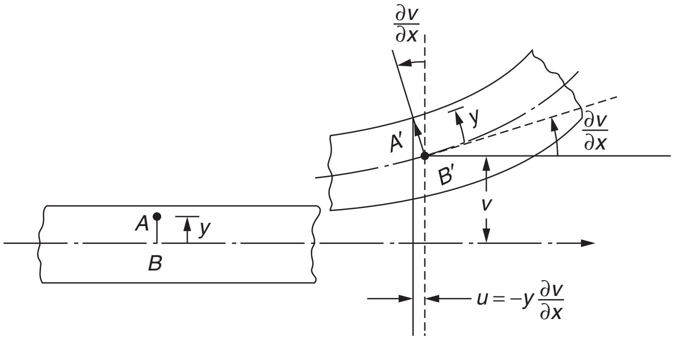


FIGURE 9.7 Deformation of an element of a beam in an xy plane.

1. If the nodal displacements of the elements q_1, q_2, q_3, and q_4 are known, the stress distribution in the element, σ_{xx} can be found as

$$\sigma_{xx} = \sigma_{xx}(x, y) = E\varepsilon_{xx} = [E][B]\vec{q}$$

where $\sigma_{xx}(x, y)$ denotes the stress in the element at a point located at distance x from the origin (in the horizontal direction from the left node) and y from the neutral axis (in the vertical direction).

Using Eqs. (9.26) and (8.87) with $[D] = [E]$, the stiffness matrix can be found as

$$\left[k^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V = E \int_o^l \mathrm{d}x \iint_A [B]^T [B] \mathrm{d}A$$

$$= \frac{EI_{zz}}{l^3} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \tag{9.27}$$

where $I_{zz} = \iint_A y^2 \cdot \mathrm{d}A$ is the area moment of inertia of the cross section about the z axis. Note that the nodal interpolation functions $N_i(x)$ of Eq. (9.23) are the same as the first-order Hermite polynomials defined in Section 4.4.5.

The consistent load vector of a beam element can be found using different approaches as indicated in the following examples.

EXAMPLE 9.6

A beam element is subjected to a uniformly distributed transverse load along its length as shown in Fig. 9.8A. Determine the consistent load vector using the basic concepts of solid mechanics.

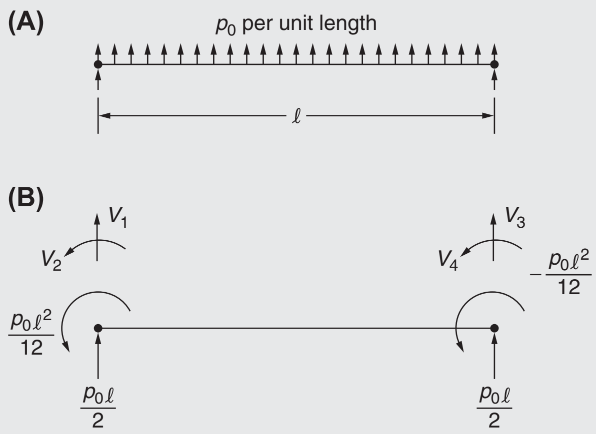


FIGURE 9.8 Consistent load vector.

Solution

When a uniform beam is subjected to a uniformly distributed load of intensity p_0 along its length l, the reactions at the ends given by the simple beam theory, when both the ends are fixed, are shown in Fig. 9.8A. The negative values of the reactions can be taken as the nodal loads corresponding to a uniform beam element as shown in Fig. 9.8B so that the element load vector due to a uniform distributed load can be taken as

$$\vec{P}^{(e)} = \begin{Bmatrix} \frac{p_0 l}{2} \\ \frac{p_0 l^2}{12} \\ \frac{p_0 l}{2} \\ -\frac{p_0 l^2}{12} \end{Bmatrix} \tag{E.1}$$

The element load vectors of a fixed-fixed beam for other types of loads such as a linearly varying distributed load or a concentrated load or moment acting at any point along the length of the beam element can be generated in a similar manner using results from the simple beam theory [9.4].

The consistent load vector of a beam element corresponding to a specified load distribution can be generated, in a simpler way, using Eq. (8.89) as shown in the following examples.

EXAMPLE 9.7

A uniformly distributed load of magnitude p_0 per unit length is applied along the length of a beam element. Derive the corresponding consistent load vector in the local coordinate system.

Solution

The consistent load vector, $\vec{P}_s^{(e)}$, for a uniformly distributed load, $\Phi(x) = p_0$, can be found using Eq. (8.89):

$$\vec{P}_s^{(e)} = \iint_{S_1^{(e)}} [N]^T \vec{\Phi}\, dS_1 = \int_{x=0}^{l^{(e)}} p_0 [N]^T dx = \int_{x=0}^{l^{(e)}} p_0 \begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{Bmatrix} dx$$

$$= \int_{x=0}^{l^{(e)}} p_0 \begin{Bmatrix} (2x^3 - 3lx^2 + l^3)/l^3 \\ (x^3 - 2lx^2 + l^2 x)/l^2 \\ -(2x^3 - 3lx^2)/l^3 \\ (x^3 - lx^2)/l^2 \end{Bmatrix} dx = \begin{Bmatrix} \dfrac{p_0 l}{2} \\ \dfrac{p_0 l^2}{12} \\ \dfrac{p_0 l}{2} \\ -\dfrac{p_0 l^2}{12} \end{Bmatrix} \tag{E.1}$$

which can be seen to be identical to Eq. (E.1) of Example 9.6.

EXAMPLE 9.8

A concentrated transverse load P_0 is applied at the point $x = a$ of a beam element of length l. Derive the corresponding consistent load vector.

Solution

The consistent load vector $\vec{P}_c^{(e)}$, can be determined from the expression of the work done by the load (W) as follows:

$$\begin{aligned} W &= \int_{x=0}^{l^{(e)}} p(x) v(x)\, dx = \int_{x=0}^{l^{(e)}} P_0 \delta(a-x) [N(x)]^T \vec{q}^{(e)} dx = P_0 [N(x=a)]^T \vec{q}^{(e)} \\ &= \vec{P}_c^{(e)T} \vec{q}^{(e)} \end{aligned} \tag{E.1}$$

where $p(x)$ denotes the distributed load applied along the beam and $v(x) = [N(x)]^T \vec{q}^{(e)}$ is the deflection of the beam and $\delta(x-a) = \begin{cases} 1 & \text{if} \quad x = a \\ 0 & \text{if} \quad x \neq a \end{cases}$ is the Dirac delta function. From Eq. (E.1), the consistent load vector can be identified as

$$\vec{P}_c^{(e)} = \begin{Bmatrix} N_1(x=a) \\ N_2(x=a) \\ N_3(x=a) \\ N_4(x=a) \end{Bmatrix} P_0 = \begin{Bmatrix} 2a^3 - 3la^2 + l^3 \\ a^3 l - 2l^2 a^2 + l^3 a \\ -2a^3 + 3la^2 \\ a^3 l - l^2 a^2 \end{Bmatrix} \frac{P_0}{l^3} \tag{E.2}$$

EXAMPLE 9.9

Find the consistent load vector of a beam element when a concentrated transverse load of magnitude P_0 acts at $x = l^{(e)}/4$.

Solution

The consistent load vector of the beam element corresponding to the given concentrated load can be determined from Eq. (E.2) of Example 9.8:

$$\vec{P}_c^{(e)} = \begin{Bmatrix} N_1(x = l^{(e)}/4) \\ N_2(x = l^{(e)}/4) \\ N_3(x = l^{(e)}/4) \\ N_4(x = l^{(e)}/4) \end{Bmatrix} P_0 = \frac{P_0}{64} \begin{Bmatrix} 54 \\ 9l \\ 10 \\ -3l \end{Bmatrix} \tag{E.1}$$

EXAMPLE 9.10

A pinned-pinned uniform beam of length 1 m is subjected to a uniformly distributed load of $p_0 = 100$ N/cm along its length as shown in Fig. 9.8A. If the material of the beam is steel with $E = 207$ GPa and the cross section of the beam is rectangular with width 1 cm (in z direction) and depth 2 cm (in y direction), determine the following using a one-beam-element model:

a. The maximum deflection (at the middle) of the beam

b. The variation of the stress induced in the beam

Solution

a. The equilibrium equations of the beam, using a one-element model, can be expressed as (Fig. 9.8B):

$$[\underset{\sim}{K}]\underset{\sim}{\vec{V}} = \underset{\sim}{\vec{P}} \tag{E.1}$$

where

$$[\underset{\sim}{K}] = [K^{(e)}] = \frac{EI_{zz}}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\underset{\sim}{\vec{V}} = \vec{V}^{(e)} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{Bmatrix}, \quad \underset{\sim}{\vec{P}} = \vec{P}^{(e)} = \begin{Bmatrix} \frac{p_0 l}{2} \\ \frac{p_0 l^2}{12} \\ \frac{p_0 l}{2} \\ -\frac{p_0 l^2}{12} \end{Bmatrix}$$

After incorporating the boundary conditions of the pinned-pinned beam ($V_1 = V_3 = 0$), the equilibrium equations of the beam take the form

$$[K]\vec{V} = \vec{P} \tag{E.2}$$

where

$$[K] = \frac{EI_{zz}}{l^3}\begin{bmatrix} 4l^2 & 2l^2 \\ 2l^2 & 4l^2 \end{bmatrix}, \quad \vec{V} = \begin{Bmatrix} V_2 \\ V_4 \end{Bmatrix}, \quad \vec{P} = \begin{Bmatrix} P_2 \\ P_4 \end{Bmatrix} = \frac{p_0 l^2}{12}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

From the known data, $\frac{EI_{zz}}{l^3} = \frac{(207\times10^9)\left(\frac{2}{3}\times 10^{-8}\right)}{(1)^3} = 1380$ N/m, $\frac{p_0 l^2}{12} = \frac{(10{,}000)(1^2)}{12} = 833.3333$ N-m, Eq. (E.2) can be solved to find $V_2 = 0.3019$ rad and $V_4 = -0.3019$ rad. The deflection along the beam is given by

$$v(x) = [N(x)]\vec{V}^{(e)} = N_1(x)V_1 + N_2(x)V_2 + N_3(x)V_3 + N_4(x)V_4 \tag{E.3}$$

EXAMPLE 9.10 —cont'd

Since $V_1 = V_3 = 0$, $V_2 = 0.3019$ and $V_4 = -0.3019$, Eq. (E.3) reduces to

$$v(x) = N_2(x)(0.3019) + N_4(x)(-0.3019) \text{ m} \tag{E.4}$$

where $N_2(x)$ and $N_4(x)$ are given by Eq. (9.23). The maximum deflection of the beam at the middle ($x = l/2 = 0.5$ m) can be found as $v_{\max} = 0.075475 \text{ m} = 7.5475 \text{ cm}$.

This value can be compared with the value of 9.4354 cm given by the simple beam theory.

b. The variation of the normal stress, σ_{xx}, along the beam can be determined as

$$\sigma_{xx}(x) = -yE\frac{d^2v}{dx^2} = -yE\left\{\frac{d^2N_2(x)}{dx^2}V_2 + \frac{d^2N_4(x)}{dx^2}V_4\right\}$$

$$= -y\,E(-0.6038) = y\left(207 \times 10^9\right)(0.6038) = 124.9866 \times 10^9 y$$

The maximum induced stress (at the top and bottom of the beam cross section) occurs when $y = \pm 1 \text{ cm} = \pm 0.01$ m:

$$\sigma_{xx}|_{\max} = 1.25 \times 109 \text{ N/m}^2 = 1.25 \text{ GPa}$$

9.4 SPACE FRAME ELEMENT

A space frame element is a straight bar of uniform cross section that is capable of resisting axial forces, bending moments about the two principal axes in the plane of its cross section, and twisting moment about its centroidal axis. The corresponding displacement degrees of freedom are shown in Fig. 9.9A. It can be seen that the stiffness matrix of a frame element will be of order 12×12. If the local axes (xyz system) are chosen to coincide with the principal axes of the cross-section, it is possible to construct the 12×12 stiffness matrix from 2×2 and 4×4 submatrices. According to the engineering theory of bending and torsion of beams, the axial displacements q_1 and q_7 depend only on the axial forces, and the torsional displacements q_4 and q_{10} depend only on the torsional moments. However, for arbitrary choice of xyz coordinate system, the bending displacements in xy plane, namely q_2, q_6, q_8, and q_{12}, depend not only on the bending forces acting in that plane (i.e., shear forces acting in the y direction and the bending moments acting in the xy plane) but also on the bending forces acting in the plane xz. On the other hand, if the xy and xz planes coincide with the principal axes of the cross section, the bending displacements and forces in the two planes can be considered to be independent of each other.

In this section, we choose the local xyz coordinate system to coincide with the principal axes of the cross section with the x axis representing the centroidal axis of the frame element. Thus, the displacements can be separated into four groups, each of which can be considered independently of the others. We first consider the stiffness matrices corresponding to different independent sets of displacements and then obtain the total stiffness matrix of the element by superposition.

9.4.1 Axial Displacements

The nodal displacements are q_1 and q_7 (Fig. 9.9B) and a linear displacement model leads to the stiffness matrix (corresponding to the axial displacement) as

$$[k_a^{(e)}] = \iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V = \frac{AE}{l}\begin{array}{c} \begin{array}{cc} q_1 & q_7 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ q_1 \\ q_7 \end{array} \tag{9.28}$$

where A, E, and l are the area of cross section, Young's modulus, and length of the element, respectively. Note that the elements of the matrix $\left[k_a^{(e)}\right]$ are identified by the degrees of freedom indicated at the top and right-hand side of the matrix in Eq. (9.28).

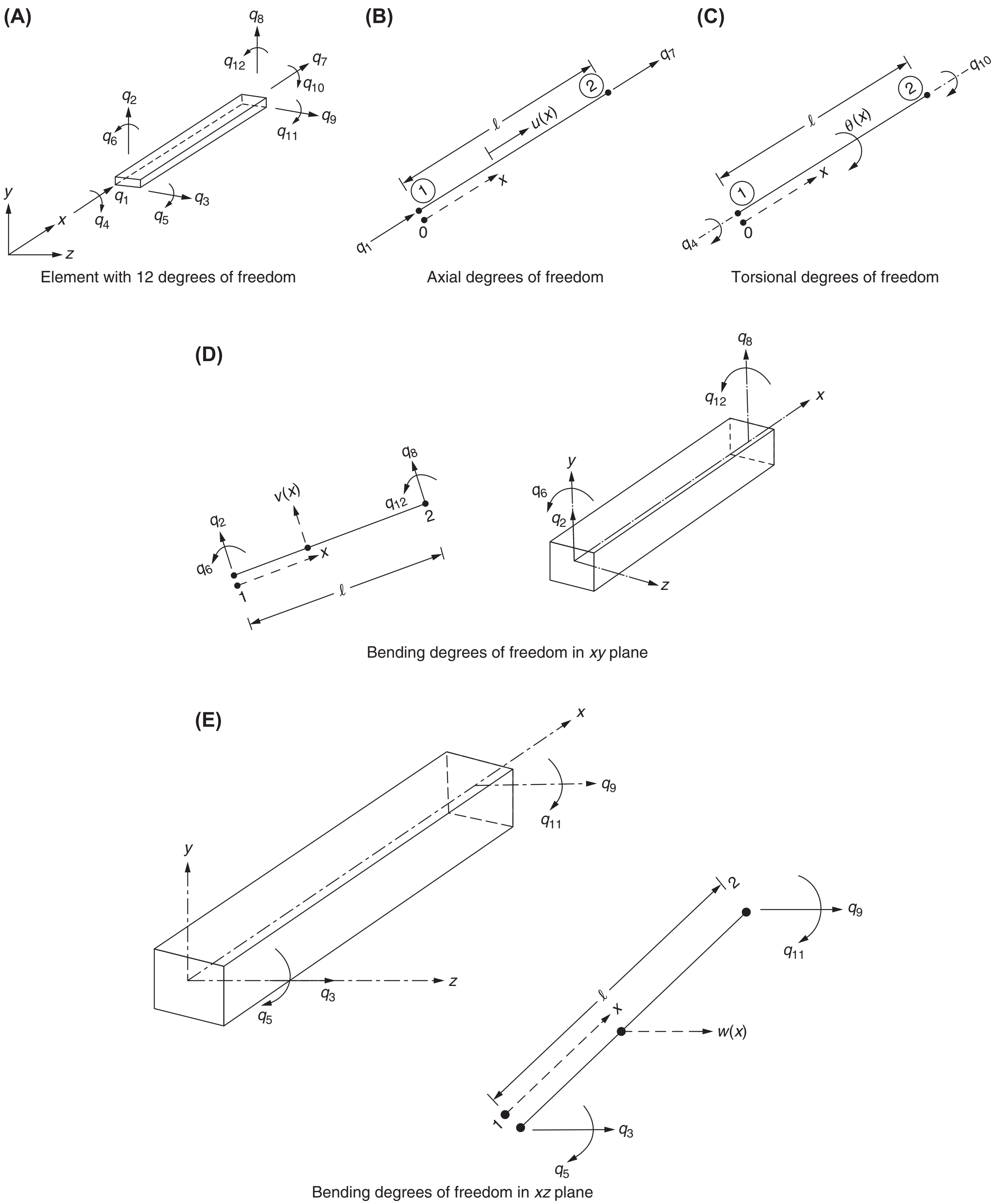


FIGURE 9.9 A space frame element.

9.4.2 Torsional Displacements

Here the degrees of freedom (torsional displacements) are given by q_4 and q_{10} as indicated in Fig. 9.9C. By assuming a linear variation of the torsional displacement or rotation angle (about the x axis), the displacement model can be expressed as

$$\theta(x) = [N]\vec{q}_t \tag{9.29}$$

where

$$[N] = \left[\left(1-\frac{x}{l}\right)\left(\frac{x}{l}\right)\right] \tag{9.30}$$

and

$$\vec{q}_t = \begin{Bmatrix} q_4 \\ q_{10} \end{Bmatrix} \tag{9.31}$$

Assuming the cross-section of the frame element is circular, the shear strain induced in the element can be expressed as [9.1]

$$\varepsilon_{\theta x} = r\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{9.32}$$

where r is the distance of the fiber from the centroidal axis of the element. Thus, the strain−displacement relation can be expressed, in the absence of thermal effects, as

$$\vec{\varepsilon} = [B]\vec{q}_t \tag{9.33}$$

where

$$\vec{\varepsilon} = \{\varepsilon_{\theta x}\} \quad \text{and} \quad [B] = \left[-\frac{r}{l} \quad \frac{r}{l}\right]$$

From Hooke's law, the stress−strain relation can be expressed as

$$\vec{\sigma} = [D]\vec{\varepsilon} \tag{9.34}$$

where

$$\vec{\sigma} = \{\sigma_{\theta x}\}, \quad [D] = [G]$$

and G is the shear modulus of the material. The stiffness matrix of the element corresponding to torsional displacement degrees of freedom can be derived as

$$\left[k_t^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\,\mathrm{d}V$$

$$= G\int_{x=0}^{l}\mathrm{d}x\iint_A r^2\mathrm{d}A\begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix}\left\{-\frac{1}{l} \quad \frac{1}{l}\right\} \tag{9.35}$$

Since $\iint_A r^2\,\mathrm{d}A = J =$ polar moment of inertia of the cross section, Eq. (9.35) can be rewritten, assuming the cross section of the element to be uniform, as

$$[k_t^{(e)}] = \frac{GJ}{l}\begin{array}{c} \begin{array}{cc} q_4 & q_{10} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ q_4 \\ q_{10} \end{array} \tag{9.36}$$

Note that the quantity GJ/l is called the torsional stiffness of the frame element [9.1]. If the cross section of the frame element is rectangular as shown in Fig. 9.10, the torsional stiffness is given by $(GJ/L) = cG(ab^3/l)$, where the value of the constant c is given as shown here [9.3]:

Value of a/b	1.0	1.5	2.0	3.0	5.0	10.0
Value of c	0.141	0.196	0.229	0.263	0.291	0.312

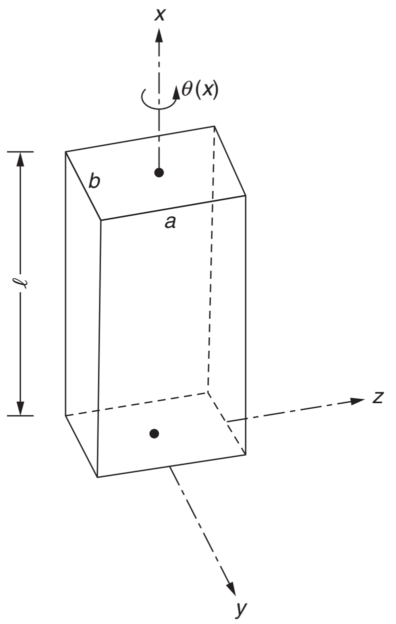


FIGURE 9.10 Rectangular section of a frame element.

9.4.3 Bending Displacements in the Plane *xy*

The four bending degrees of freedom are q_2, q_6, q_8, and q_{12} (Fig. 9.9D) and the corresponding stiffness matrix can be derived as (see Section 9.3)

$$[k_{xy}^{(e)}] = \frac{EI_{zz}}{l^3} \begin{array}{c} \begin{array}{cccc} q_2 & q_6 & q_8 & q_{12} \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} \\ q_2 \\ q_6 \\ q_8 \\ q_{12} \end{array} \tag{9.37}$$

where $I_{zz} = \iint_A r^2 \, dA$ is the area moment of inertia of the cross section about the z axis.

9.4.4 Bending Displacements in the Plane *xz*

Here bending of the element takes place in the xz plane instead of the xy plane. Thus, we have the degrees of freedom q_3, q_5, q_9, and q_{11} (Fig. 9.9E) in place of q_2, q_6, q_8, and q_{12} (Fig. 9.9D), respectively. By proceeding as in the case of bending in the plane xy, we can derive the stiffness matrix as

$$[k_{xz}^{(e)}] = \frac{EI_{yy}}{l^3} \begin{array}{c} \begin{array}{cccc} q_3 & q_5 & q_9 & q_{11} \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} \\ q_3 \\ q_5 \\ q_9 \\ q_{11} \end{array} \tag{9.38}$$

where I_{yy} denotes the area moment of inertia of the cross-section of the element about the y axis.

9.4.5 Total Element Stiffness Matrix

The stiffness matrices derived for different sets of independent displacements can now be compiled (superposed) to obtain the overall stiffness matrix of the frame element as

$$
\underset{12\times 12}{[k^{(e)}]} = \begin{array}{c} \begin{array}{cccccccccccc} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 & q_8 & q_9 & q_{10} & q_{11} & q_{12} \end{array} \\ \left\{ \begin{array}{cccccccccccc}
\frac{EA}{l} & & & & & & & & & & & \\
0 & \frac{12EI_{zz}}{l^3} & & & & & \text{Symmetric} & & & & & \\
0 & 0 & \frac{12EI_{yy}}{l^3} & & & & & & & & & \\
0 & 0 & 0 & \frac{GJ}{l} & & & & & & & & \\
0 & 0 & \frac{6EI_{yy}}{l^2} & 0 & \frac{4EI_{yy}}{l} & & & & & & & \\
0 & \frac{6EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{l} & & & & & & \\
-\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & & & & & \\
0 & -\frac{12EI_{zz}}{l^3} & 0 & 0 & 0 & \frac{-6EI_{zz}}{l^2} & 0 & \frac{12EI_{zz}}{l^3} & & & & \\
0 & 0 & \frac{-12EI_{yy}}{l^3} & 0 & \frac{-6EI_{yy}}{l^2} & 0 & 0 & 0 & \frac{12EI_{yy}}{l^3} & & & \\
0 & 0 & 0 & \frac{-GJ}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{l} & & \\
0 & 0 & \frac{6EI_{yy}}{l^2} & 0 & \frac{2EI_{yy}}{l} & 0 & 0 & 0 & \frac{-6EI_{yy}}{l^2} & 0 & \frac{4EI_{yy}}{l} & \\
0 & \frac{6EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{2EI_{zz}}{l} & 0 & \frac{-6EI_{zz}}{l^2} & 0 & 0 & 0 & \frac{4EI_{zz}}{l}
\end{array} \right\} \end{array}
\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{array}
\tag{9.39}
$$

9.4.6 Global Stiffness Matrix

It can be seen that the 12×12 stiffness matrix given in Eq. (9.39) is with respect to the local xyz coordinate system. Since the nodal displacements in the local and global coordinate systems are related by the relation (from Fig. 9.11)

$$
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ q_{12} \end{Bmatrix} =
\begin{bmatrix}
l_{ox} & m_{ox} & n_{ox} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{oy} & m_{oy} & n_{oy} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{oz} & m_{oz} & n_{oz} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_{ox} & m_{ox} & n_{ox} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_{oy} & m_{oy} & n_{oy} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & l_{oz} & m_{oz} & n_{oz} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l_{ox} & m_{ox} & n_{ox} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l_{oy} & m_{oy} & n_{oy} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l_{oz} & m_{oz} & n_{oz} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{ox} & m_{ox} & n_{ox} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{oy} & m_{oy} & n_{oy} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{oz} & m_{oz} & n_{oz}
\end{bmatrix}
\begin{Bmatrix} Q_{6i-5} \\ Q_{6i-4} \\ Q_{6i-3} \\ Q_{6i-2} \\ Q_{6i-1} \\ Q_{6i} \\ Q_{6j-5} \\ Q_{6j-4} \\ Q_{6j-3} \\ Q_{6j-2} \\ Q_{6j-1} \\ Q_{6j} \end{Bmatrix}
\tag{9.40}
$$

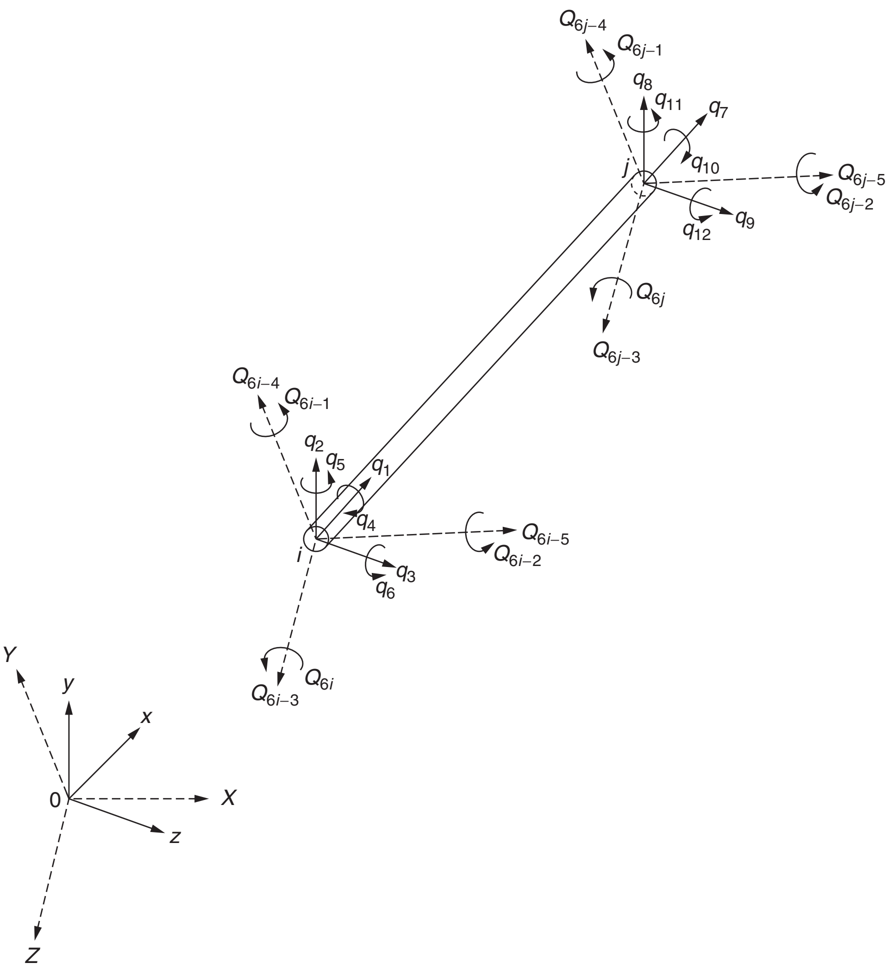


FIGURE 9.11 Local and global degrees of freedom of a space frame element.

the transformation matrix, $[\lambda]$, can be identified as

$$\underset{12\times 12}{[\lambda]} = \begin{bmatrix} [\underset{\sim}{\lambda}] & [0] & [0] & [0] \\ [0] & [\underset{\sim}{\lambda}] & [0] & [0] \\ [0] & [0] & [\underset{\sim}{\lambda}] & [0] \\ [0] & [0] & [0] & [\underset{\sim}{\lambda}] \end{bmatrix} \tag{9.41}$$

where

$$\underset{3\times 3}{[\underset{\sim}{\lambda}]} = \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} \\ l_{oy} & m_{oy} & n_{oy} \\ l_{oz} & m_{oz} & n_{oz} \end{bmatrix} \tag{9.42}$$

and

$$\underset{3\times 3}{[0]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.43}$$

Here l_{ox}, m_{ox}, and n_{ox} denote the direction cosines of the x axis (line ij in Fig. 9.11); l_{oy}, m_{oy}, and n_{oy} represent the direction cosines of the y axis; and l_{oz}, m_{oz}, and n_{oz} indicate the direction cosines of the z axis with respect to the global X, Y, Z axes. It can be seen that finding the direction cosines of the x axis is a straightforward computation since

$$l_{ox} = \frac{X_j - X_i}{l}, \quad m_{ox} = \frac{Y_j - Y_i}{l}, \quad n_{ox} = \frac{Z_j - Z_i}{l} \tag{9.44}$$

where X_k, Y_k, and Z_k indicate the coordinates of node k ($k = i, j$) in the global system. However, the computation of the direction cosines of the y and z axes requires some special effort. Finally, the stiffness matrix of the element with reference to the global coordinate system can be obtained as

$$\left[K^{(e)}\right] = [\lambda]^T \left[k^{(e)}\right] [\lambda] \tag{9.45}$$

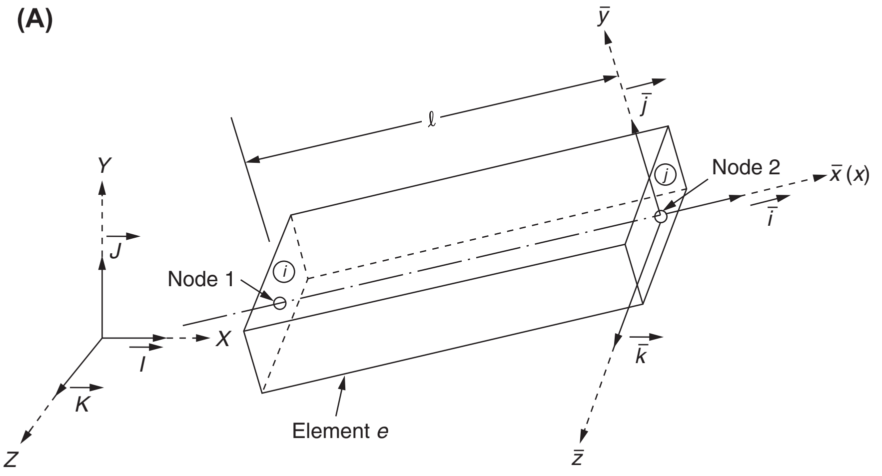
TRANSFORMATION MATRIX

We shall derive the transformation matrix $[\lambda]$ between the local and global coordinate systems in two stages. In the first stage, we derive a transformation matrix $[\lambda_1]$ between the global coordinates XYZ and the coordinates $\bar{x}\,\bar{y}\,\bar{z}$ by assuming the $\bar{z}$ axis to be parallel to the XZ plane (Fig. 9.12A):

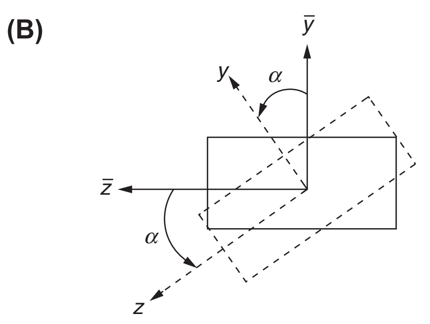
$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} = [\lambda_1] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{9.46}$$

In the second stage, we derive a transformation matrix $[\lambda_2]$ between the local coordinates xyz (principal member axes) and the coordinates $\bar{x}\,\bar{y}\,\bar{z}$ as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [\lambda_2] \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} \tag{9.47}$$



$\bar{z}$-axis parallel to XZ plane (principal cross-sectional axis y and z are assumed to coincide with $\bar{y}$ and $\bar{z}$)



General case (y and z do not coincide with $\bar{y}$ and $\bar{z}$)

FIGURE 9.12 Local and global coordinate systems.

by assuming that the local coordinate system (xyz) can be obtained by rotating the $\bar{x}\,\bar{y}\,\bar{z}$ system about the $\bar{x}$ axis by an angle α as shown in Fig. 9.12B. Thus, the desired transformation between the xyz system and the XYZ system can be obtained as

$$[\underset{\sim}{\lambda}] = [\lambda_2][\lambda_1] \tag{9.48}$$

where

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [\underset{\sim}{\lambda}] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{9.49}$$

EXPRESSION FOR $[\lambda_1]$

From Fig. 9.12A, the direction cosines of the longitudinal axis of the frame element ($\bar{x}$ or x or first local axis) can be obtained as

$$\begin{aligned} l_{o\bar{x}} &= l_{ox} = \frac{X_j - X_i}{l} \\ m_{o\bar{x}} &= m_{ox} = \frac{Y_j - Y_i}{l} \\ n_{o\bar{x}} &= n_{ox} = \frac{Z_j - Z_i}{l} \end{aligned} \tag{9.50}$$

where i and j denote the first and second nodes of the element e in the global system, and l represents the length of the element e:

$$l = \left\{ (X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2 \right\}^{1/2} \tag{9.51}$$

Since the unit vector $\vec{\bar{k}}$ (which is parallel to the $\bar{z}$ axis) is normal to both the unit vectors $\vec{J}$ (parallel to Y axis) and $\vec{\bar{i}}$ (parallel to $\bar{x}$ axis), we have, from vector analysis [9.2],

$$\vec{\bar{k}} = \frac{\vec{\bar{i}} \times \vec{J}}{\| \vec{\bar{i}} \times \vec{J} \|} = \frac{1}{d} \begin{bmatrix} \vec{I} & \vec{J} & \vec{K} \\ l_{ox} & m_{ox} & n_{ox} \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{d}\left(-\vec{I}\, n_{ox} + \vec{K}\, l_{ox} \right) \tag{9.52}$$

where

$$d = \left(l_{ox}^2 + n_{ox}^2 \right)^{1/2} \tag{9.53}$$

Thus, the direction cosines of the $\bar{z}$ axis with respect to the global XYZ system are given by

$$l_{o\bar{z}} = -\frac{n_{ox}}{d}, \quad m_{o\bar{z}} = 0, \quad n_{o\bar{z}} = \frac{l_{ox}}{d} \tag{9.54}$$

To find the direction cosines of the $\bar{y}$ axis, we use the condition that the $\bar{y}$ axis (unit vector $\vec{\bar{j}}$) is normal to the $\bar{x}$ axis $\left(\vec{\bar{i}}\right)$ and $\bar{z}$ axis $\left(\vec{\bar{k}}\right)$. Hence, we can express $\vec{\bar{j}}$ as

$$\begin{aligned} \vec{\bar{j}} = \vec{\bar{k}} \times \vec{\bar{i}} &= \begin{vmatrix} \vec{I} & \vec{J} & \vec{K} \\ l_{o\bar{z}} & -m_{o\bar{z}} & n_{o\bar{z}} \\ l_{ox} & m_{ox} & n_{ox} \end{vmatrix} \\ &= \frac{1}{d}\left[\vec{I}(-l_{ox}m_{ox}) - \vec{J}\left(-n_{ox}^2 - l_{ox}^2\right) + \vec{K}(-m_{ox}n_{ox}) \right] \end{aligned} \tag{9.55}$$

Thus, the direction cosines of the $\bar{y}$ axis are given by

$$\left.\begin{aligned} l_{o\bar{y}} &= -\frac{l_{ox} m_{ox}}{d} \\ m_{o\bar{y}} &= \frac{n_{ox}^2 + l_{ox}^2}{d} \\ n_{o\bar{y}} &= -\frac{m_{ox} n_{ox}}{d} \end{aligned}\right\} \tag{9.56}$$

Thus, the $[\lambda_1]$ matrix is given by

$$\begin{aligned} [\lambda_1] &= \begin{bmatrix} l_{o\bar{x}} & m_{o\bar{x}} & n_{o\bar{x}} \\ l_{o\bar{y}} & m_{o\bar{y}} & n_{o\bar{y}} \\ l_{o\bar{z}} & m_{o\bar{z}} & n_{o\bar{z}} \end{bmatrix} \\ &= \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} \\ -(l_{ox} m_{ox})/d & \left(l_{ox}^2 + n_{ox}^2\right)/d & -(m_{ox} n_{ox})/d \\ -n_{ox}/d & 0 & l_{ox}/d \end{bmatrix} \end{aligned} \tag{9.57}$$

where l_{ox}, m_{ox}, and n_{ox} are given by Eq. (9.50) and d by Eq. (9.53).

EXPRESSION FOR $[\lambda_2]$

When the principal cross-sectional axes of the frame element (xyz axes) are arbitrary, making an angle α with the $\bar{x}\,\bar{y}\,\bar{z}$ axes (x axis is same as $\bar{x}$ axis), the transformation between the two systems can be expressed as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} \equiv [\lambda_2] \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix} \tag{9.58}$$

so that

$$[\lambda_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{9.59}$$

Thus, the transformation between the coordinate axes XYZ and xyz can be found using Eq. (9.48).

Notes

a. When $\alpha = 0$, the matrix $[\lambda_2]$ degenerates to a unit matrix.

b. When the element e lies vertical (i.e., when the x [or $\bar{x}$] axis coincides with the Y axis), $l_{ox} = n_{ox} = 0$ and hence d in Eq. (9.53) becomes zero. This makes some of the terms in the $[\lambda_2]$ matrix indeterminate. Thus, the previous procedure breaks down.

In this case, we can redefine the angle α as the angle in the horizontal (XZ) plane between the axes Z and z, positive when turning from Z to the X axis as shown in Fig. 9.13. In this case, the $[\underset{\sim}{\lambda}]$ matrix can be derived, by going through the same procedure as before, as

$$[\underset{\sim}{\lambda}] = \begin{bmatrix} 0 & m_{ox} & 0 \\ -m_{ox}\cos\alpha & 0 & m_{ox}\sin\alpha \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{9.60}$$

where $m_{ox} = 1$ for this case.

Continued

Notes—cont'd

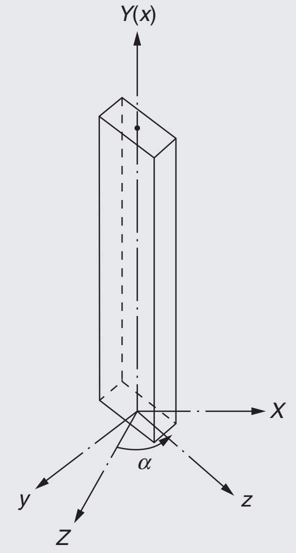


FIGURE 9.13 Transformation for a vertical element.

[λ] MATRIX

Finally, the transformation matrix, [λ], that relates the degrees of freedom in the local and global coordinate systems is given by Eq. (9.41).

9.4.7 Planar Frame Element (as a Special Case of Space Frame Element)

In the case of two-dimensional (planar) frame analysis, we need to use an element having 6 degrees of freedom as shown in Fig. 9.14. This element is assumed to lie in the XZ plane and has two axial and four bending degrees of freedom. By using a linear interpolation model for axial displacement and a cubic model for the transverse displacement, and superimposing the resulting two stiffness matrices, the following stiffness matrix can be obtained (the vector $\vec{q}^{(e)}$ is taken as $\vec{q}^{(e)^T} = \{q_1 \quad q_2 \quad \dots \quad q_6\}$):

$$\left[k^{(e)}\right] = \frac{EI_{yy}}{l^3}\begin{bmatrix} \frac{Al^2}{I_{yy}} & & & \text{Symmetric} & & \\ 0 & 12 & & & & \\ 0 & 6l & 4l^2 & & & \\ -\frac{Al^2}{I_{yy}} & 0 & 0 & \frac{Al^2}{I_{yy}} & & \\ 0 & -12 & -6l & 0 & 12 & \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} \tag{9.61}$$

Note that the bending and axial deformation effects are uncoupled while deriving Eq. (9.61). Eq. (9.61) can also be obtained as a special case of Eq. (9.39) by deleting rows and columns 2, 4, 6, 8, 10, and 12. In this case the stiffness matrix of the element in the global XZ coordinate system can be found as

$$\left[K^{(e)}\right] = [\lambda]^T\left[k^{(e)}\right][\lambda] \tag{9.62}$$

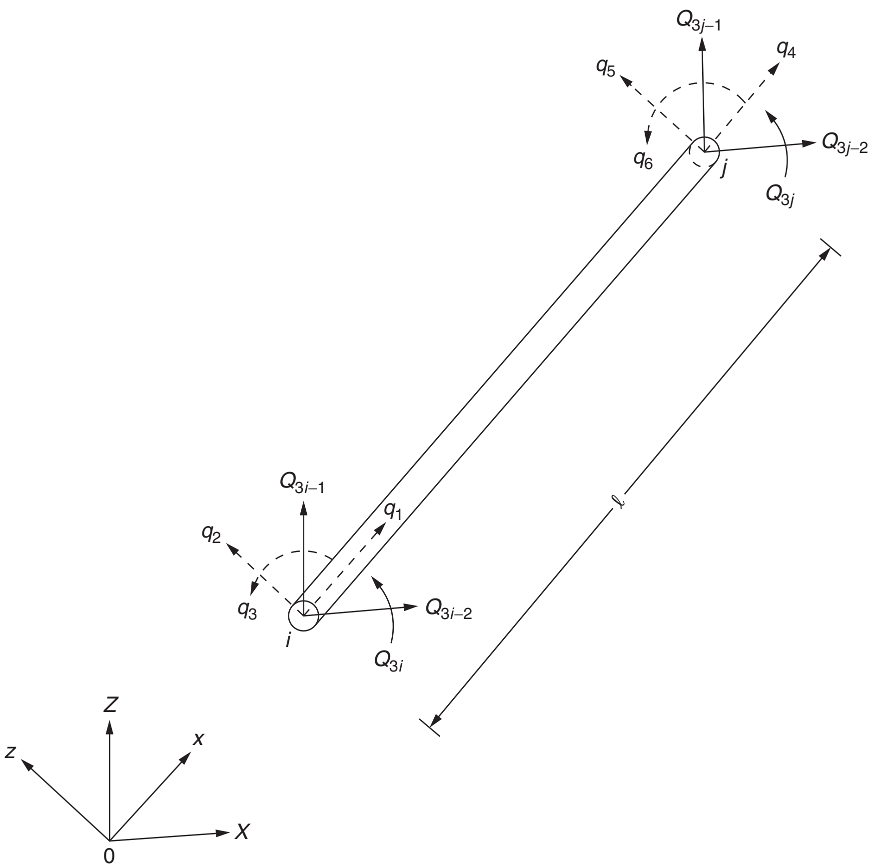


FIGURE 9.14 Planar frame element.

where

$$[\lambda] = \begin{bmatrix} l_{ox} & m_{ox} & 0 & 0 & 0 & 0 \\ l_{oz} & m_{oz} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ox} & m_{ox} & 0 \\ 0 & 0 & 0 & l_{oz} & m_{oz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.63}$$

with (l_{ox}, m_{ox}) denoting the direction cosines of the x axis and (l_{oz}, m_{oz}) indicating the direction cosines of the z axis with respect to the global XZ system.

9.4.8 Beam Element (as a Special Case of Space Frame Element)

For a beam element lying in the local xy plane, the stiffness matrix is given by

$$\left[k^{(e)}\right] = \frac{EI_{zz}}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \tag{9.64}$$

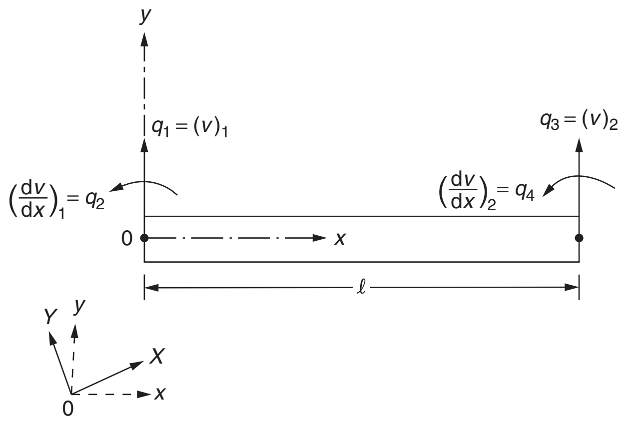


FIGURE 9.15 Degrees of freedom of a beam element.

The stiffness matrix in the global XY plane (Fig. 9.15) can be found as

$$\underset{6\times 6}{\left[K^{(e)}\right]} = \underset{6\times 4}{[\lambda]^T} \underset{4\times 4}{\left[k^{(e)}\right]} \underset{4\times 6}{[\lambda]} \tag{9.65}$$

where the transformation matrix $[\lambda]$ is given by

$$[\lambda] = \begin{bmatrix} l_{oy} & n_{oy} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{oy} & n_{oy} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.66}$$

where (l_{oy}, n_{oy}) are the direction cosines of the y axis with respect to the XY system.

9.5 CHARACTERISTICS OF STIFFNESS MATRICES

The stiffness matrices have the following characteristics and properties:

1. The element and system stiffness matrices are symmetric. This property satisfies the Maxwell—Betti reciprocity theorem of solid and structural mechanics. According to this theorem, when two sets of loads act on a solid body, the work done by the first set of loads due to the displacements produced by the second set of loads is equal to the work done by the second set of loads due to the displacements produced by the first set of loads. For example, if loads $\vec{P}_1$ and $\vec{P}_2$ produce the displacements $\vec{U}_1$ and $\vec{U}_2$, respectively, in the structure or body, then the work done (W_1) by the first load vector $\vec{P}_1$ acting through the displacement vector $\vec{U}_2$ produced by the second load is given by

$$W_1 = \frac{1}{2}\vec{P}_1^T\vec{U}_2 \tag{9.67}$$

Using the relation

$$\vec{U}_2 = [K]^{-1}\vec{P}_2 \tag{9.68}$$

in Eq. (9.67), we obtain

$$W_1 = \frac{1}{2}\vec{P}_1^T[K]^{-1}\vec{P}_2 \tag{9.69}$$

Because the work done is a scalar quantity, Eq. (9.69) can be rewritten by taking the transpose of the right-hand side quantity of Eq. (9.69), as

$$W_1 = \frac{1}{2}\vec{P}_2^T\left([K]^{-1}\right)^T\vec{P}_1 \tag{9.70}$$

Eq. (9.70) can be expressed, if $[K]$ is symmetric, as

$$W_1 = \frac{1}{2}\vec{P}_2^T\left([K]^{-1}\right)\vec{P}_1 = \frac{1}{2}\vec{P}_2^T\vec{U}_1 = W_2 \tag{9.71}$$

where W_2 denotes the work done by the second load vector $\vec{P}_2$ acting through the displacement vector $\vec{U}_1$ produced by the first load vector $\vec{P}_1$. Thus, Eq. (9.71) shows that the Maxwell–Betti reciprocity theorem will be satisfied for symmetric stiffness matrices.

2. The diagonal terms of stiffness matrices, K_{ii}, are positive. The equilibrium equations of a system, composed of one or more finite elements, can be expressed as

$$[K]\vec{U} = \vec{P} \tag{9.72}$$

Assume that all the displacement degrees of freedom of the system, except the ith degree of freedom (U_i), are fixed. For such a system, Eq. (9.72) reduces to

$$K_{ii}U_i = P_i \tag{9.73}$$

where P_i denotes the external load applied along the direction of U_i. For positive values of the load ($P_i > 0$), positive displacement ($U_i > 0$) is expected. For example, when a bar is subjected to a tensile load, we expect the bar to undergo elongation, not contraction. This implies that $K_{ii} > 0$.

3. The element and assembled (system) stiffness matrices are singular; that is, det $[K] = 0$. An individual finite element or the assembly of finite elements (structure or system) is a *free body* in space and can float. It can undergo rigid-body translation or rotation. Consider a bar with displacement degrees of freedom along the x (axial) direction, and an element/system undergoes rigid body translation along the x axis so that all the linear displacement degrees of freedom will have the same value (constant, c). Since no external loads are applied, the load vector is zero and hence the finite element equations become

$$[k]\vec{u} = [k]\begin{Bmatrix} c \\ c \\ \vdots \\ c \end{Bmatrix} = \vec{0} \tag{9.74}$$

Eq. (9.74) needs to be satisfied for infinite possible values of c (even in small displacement theory). The only way to satisfy Eq. (9.74) for arbitrary value of c is to have the matrix $[k]$ singular.

REVIEW QUESTIONS

9.1 Give brief answers to the following questions.

1. Can a truss carry bending moments?
2. How many independent displacement components can a node in a three-dimensional truss have?
3. How many independent displacement components can a node in a three-dimensional frame have?
4. What is the most important difference between a truss and a frame?
5. What is the number of nodal degrees of freedom for a truss element?
6. What is the element stiffness matrix of a truss element in a local (x) coordinate system?
7. Define the direction cosines of a truss element whose nodal coordinates are given by (X_i, Y_i, Z_i) and (X_j, Y_j, Z_j).
8. What is a λ matrix?
9. What is the general expression, in integral form, for generating the element stiffness matrix?
10. What is the difference between a beam element and a planar frame element?
11. How many degrees of freedom are there for a space frame element?

9.2 Fill in the blank with a suitable word.

1. When a structural element is subjected to uniform temperature, only ______________ strains will be developed.
2. The displacement model is a ______________ degree polynomial for a beam element.
3. A space frame element considers bending in two perpendicular planes, axial deformation, and ______________ moment.

9.3 Indicate whether each of the following statements is true or false.

1. Trusses can be considered skeletal types of structures.
2. The finite element method requires all loads to be applied only at node points.
3. All stiffness matrices are symmetric.
4. The diagonal elements of a stiffness matrix can be negative.
5. The assembled stiffness matrix of any structure will be singular.

9.4 Select the most appropriate answer from the multiple choices given.

1. The ends of a truss element are:
   (a) free (b) welded (c) pin-jointed
2. The deformation of a truss element is:
   (a) axial (b) bending (c) twisting
3. The number of nodal displacement components of a planar frame element is:
   (a) 3 (b) 2 (c) 4
4. The number of nodal displacement components for a space frame element is:
   (a) 4 (b) 6 (c) 3
5. The size of the coordinate transformation matrix of an axial bar element lying in the *xy* plane is:
   (a) 2×2 (b) 2×4 (c) 4×4
6. The size of the coordinate transformation matrix of an axial bar element lying in the *xyz* space is:
   (a) 2×6 (b) 4×6 (c) 6×6
7. The size of the coordinate transformation matrix of a beam element lying in the *xy* plane is:
   (a) 4×6 (b) 6×6 (c) 6×4
8. The size of the coordinate transformation matrix of beam element lying in the *xyz* space is:
   (a) 6×6 (b) 6×12 (c) 12×12

PROBLEMS

9.1 Derive the transformation matrices for the members of the frame shown in Fig. 9.16. Indicate clearly the local and global degrees of freedom for each member separately.

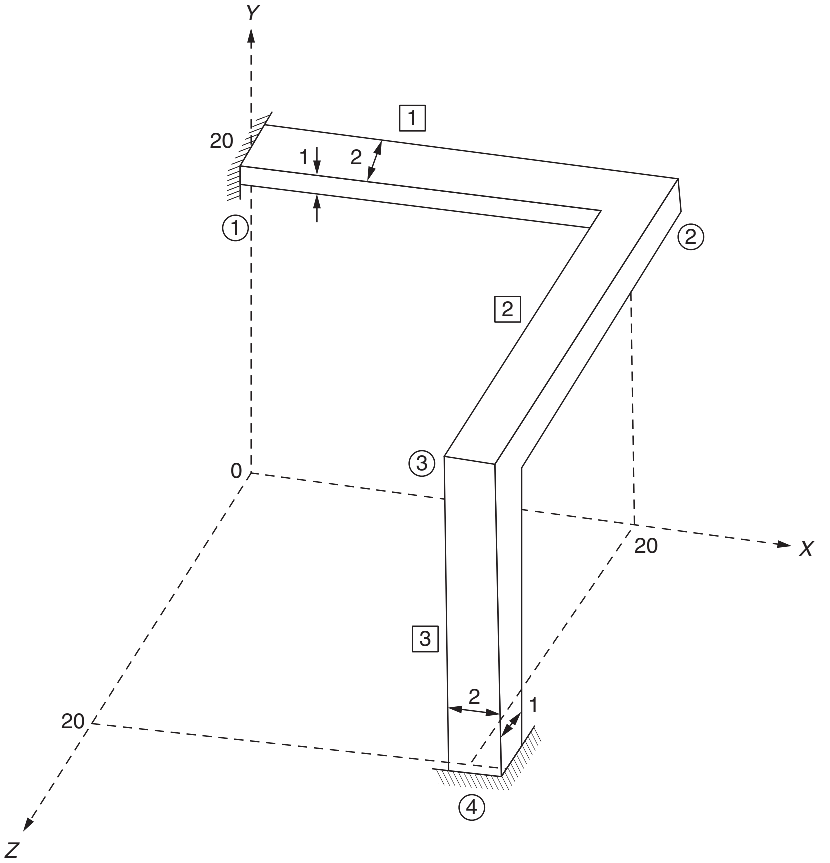


FIGURE 9.16 Space frame.

9.2 Find the deflections of nodes 2 and 3 of the frame shown in Fig. 9.16 under the following load conditions: Assume the material properties as $E = 2 \times 10^7$ N/cm^2 and $G = 0.8 \times 10^7$ N/cm^2.

a. When a load of 100 N acts in the direction of $-Y$ at node 2

b. When a load of 100 N acts at node 3 in the direction of Z

c. When a distributed load of magnitude 1 N per unit length acts on member 2 in the direction of $-Y$

9.3 Derive the stiffness matrix and load vector of a three-dimensional truss element whose area of cross-section varies linearly along its length.

9.4 Derive the transformation relation $[K^{(e)}] = [\lambda]^T[k^{(e)}][\lambda]$ from the equivalence of potential energy in the local and global coordinate systems.

9.5 Find the nodal displacements in the tapered one-dimensional member subjected to an end load of 4000 N (shown in Fig. 9.17). The cross-sectional area decreases linearly from 10 cm^2 at the left end to 5 cm^2 at the right end. Furthermore, the member experiences a temperature increase of 25 °C. Use three 25-cm elements to idealize the member. Assume $E = 2 \times 10^7$ N/cm^2, $v = 0.3$, and $\alpha = 6 \times 10^{-6}$ cm/cm °C.

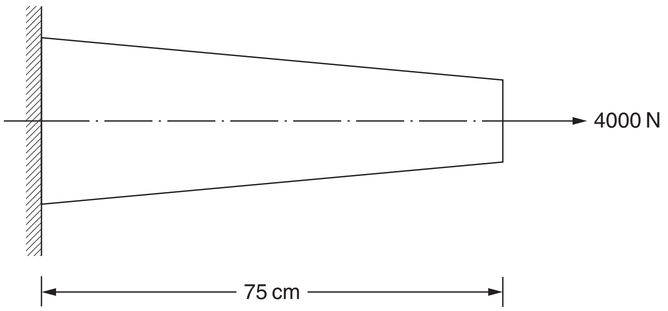


FIGURE 9.17 Tapered bar.

9.6 Derive the equilibrium equations for the beam-spring system shown in Fig. 9.18. Start from the principle of minimum potential energy and indicate briefly the various steps involved in your finite element derivation.

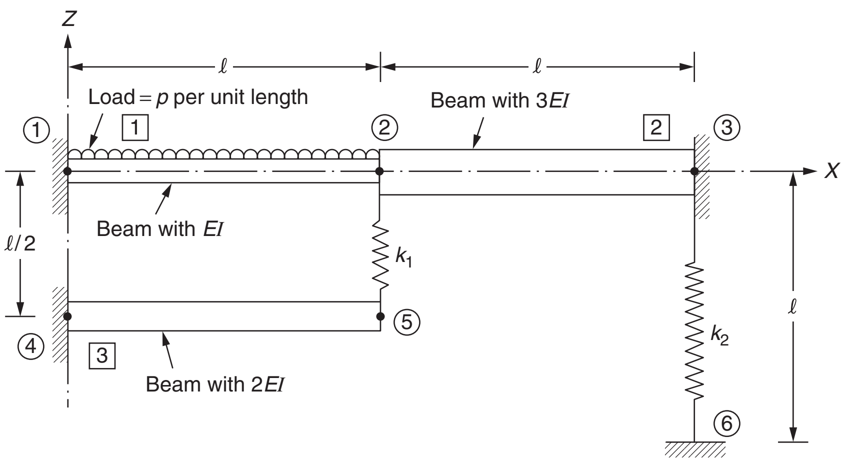


FIGURE 9.18 Beam-spring system.

9.7 Write a program called TRUSS for the displacement and stress analysis of three-dimensional truss structures. Using this subroutine, find the stresses developed in the members of the truss shown in Fig. 9.19 using hand computations.

9.8 Find the stresses developed in the members of the truss shown in Fig. 9.19.

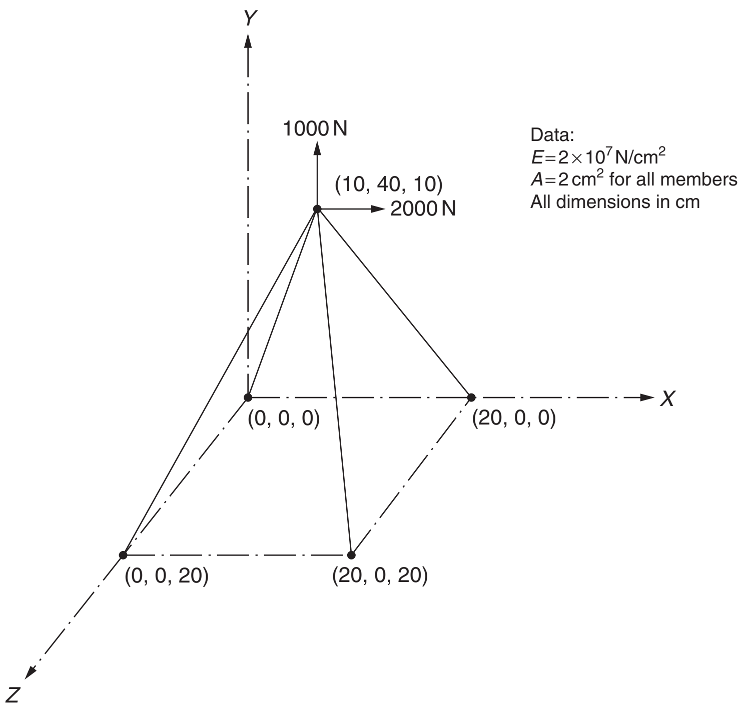


FIGURE 9.19 Three-dimensional truss.

9.9 The stiffness matrix of a spring, of stiffness c, in the local (x) coordinate system is given by (see Fig. 9.20):

$$[k] = c\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Derive the stiffness matrix of the element in the global (XY) coordinate system.

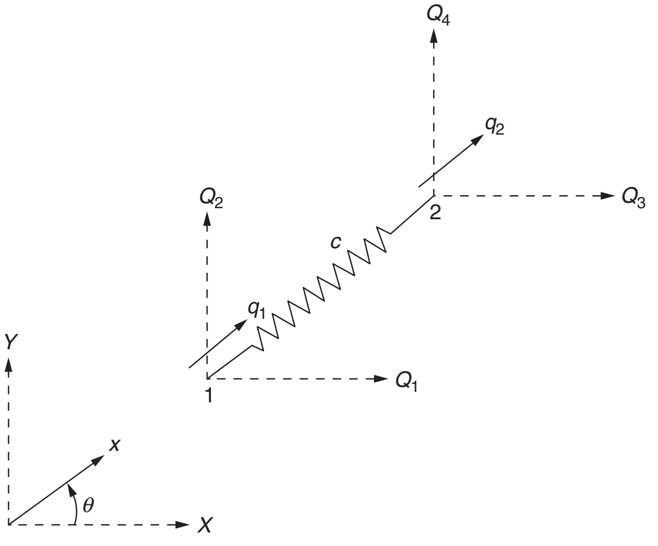


FIGURE 9.20 Spring element.

9.10 Explain why the stiffness matrix given by Eq. (9.7) or Eq. (9.13) is symmetric.
9.11 Explain why the stiffness matrix given by Eq. (9.7) or Eq. (9.13) is singular.
9.12 Explain why the sum of elements in any row of the stiffness matrix given by Eq. (9.7) or Eq. (9.13) is zero.
9.13 The members 1 and 2 of Fig. 9.21 are circular with diameters of 1 and 2 in, respectively. Determine the displacement of node P by assuming the joints to be pin connected.

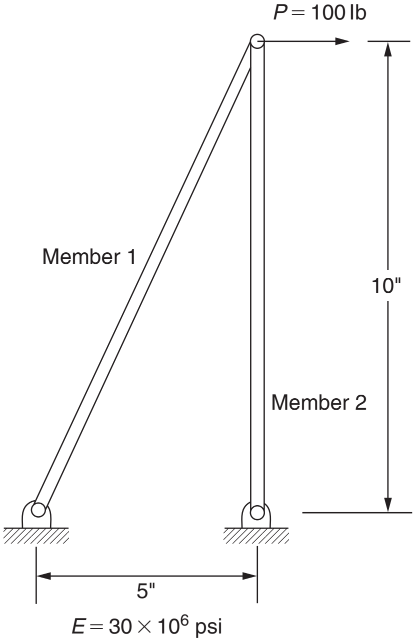


FIGURE 9.21 Two-bar structure.

9.14 The members 1 and 2 of Fig. 9.21 are circular with diameters of 1 and 2 in, respectively. Determine the displacement of node P by assuming the joints to be welded.
9.15 The stepped bar shown in Fig. 9.22 is subjected to an axial load of 100 lb at node 2. The Young's moduli of elements 1, 2, and 3 are given by 30×10^6, 20×10^6, and 10×10^6 psi, respectively. If the cross-sectional areas of elements 1, 2, and 3 are given by 3×3, 2×2, and 1×1 in, respectively, determine the following:
a. Displacements of nodes 2 and 3
b. Stresses in elements 1, 2, and 3
c. Reactions at nodes 1 and 4

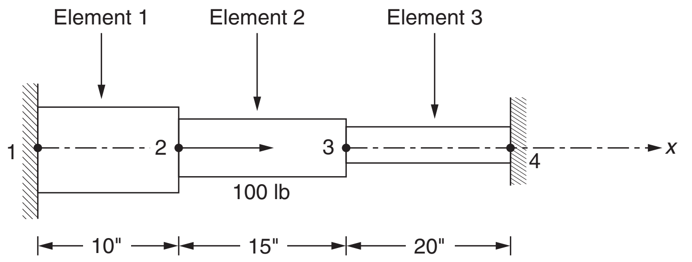


FIGURE 9.22 Stepped bar.

9.16 Loads of magnitude 100 and 200 lb are applied at points C and D of a rigid bar AB that is supported by two cables as shown in Fig. 9.23. If cables 1 and 2 have cross-sectional areas of 1 and 2 in^2 and Young's moduli of 30×10^6 and 20×10^6 psi, respectively, determine the following:

a. The finite element equilibrium equations of the system by modeling each cable as a bar element

b. The boundary conditions of the system

c. The nodal displacements of the system

Hint: A boundary condition involving the degrees of freedom Q_i and Q_j in the form of a linear equation:

$$a_i Q_i + a_j Q_j = a_0$$

where a_i, a_j, and a_o are known constants (also known as a multipoint boundary condition), can be incorporated as follows (see Section 6.5).

Add the quantities, ca_i^2,$ca_i a_j$, $ca_i a_j$, and ca_j^2 to the elements located at (i, i), (i, j), (j, i), and (j, j), respectively, in the assembled stiffness matrix and add the quantities $ca_0 a_i$ and $ca_0 a_j$ to the elements in rows i and j of the load vector. Here c is a large number compared to the magnitude of the elements of the stiffness matrix and the load vector.

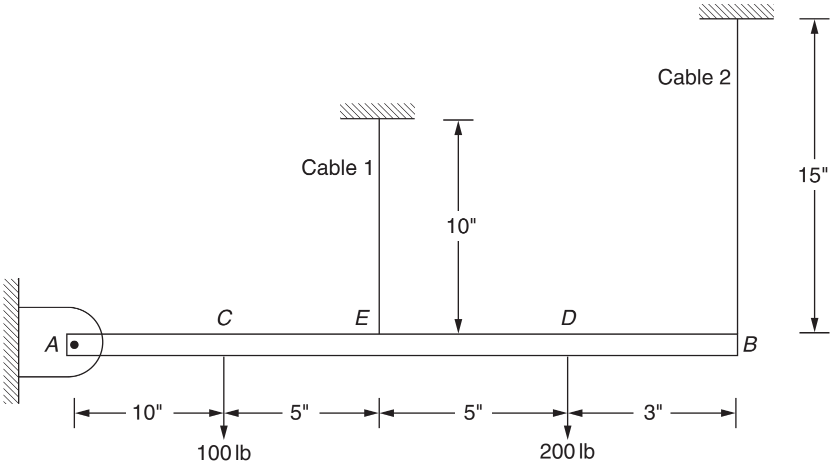


FIGURE 9.23 Rigid bar supported by cables.

9.17 The stepped bar shown in Fig. 9.24 is heated by 100 °F. The cross-sectional areas of elements 1 and 2 are given by 2 and 1 in^2 and the Young's moduli by 30×10^6 and 20×10^6 psi, respectively.

a. Derive the stiffness matrices and the load vectors of the two elements.

b. Derive the assembled equilibrium equations of the system and find the displacement of node C.

c. Find the stresses induced in elements 1 and 2.

Assume the value of α for elements 1 and 2 to be 15×10^{-6} and 10×10^{-6} per °F, respectively.

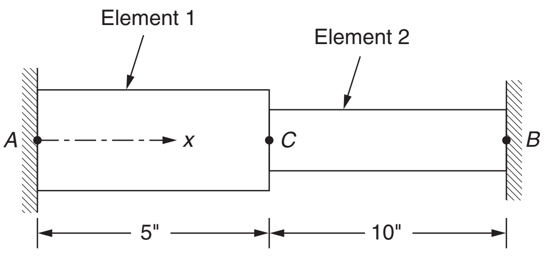


FIGURE 9.24 Stepped bar.

9.18 Consider the two-bar truss shown in Fig. 9.25. The element properties are:
Element 1: $E_1 = 30 \times 10^6$ psi, $A_1 = 1$ in^2
Element 2: $E_2 = 20 \times 10^6$ psi, $A_2 = 0.5$ in^2
The loads acting at node A are given by $P_1 = 100$ and $P_2 = 200$ lb.
a. Derive the assembled equilibrium equations of the truss.
b. Find the displacement of node A.
c. Find the stresses in elements 1 and 2.

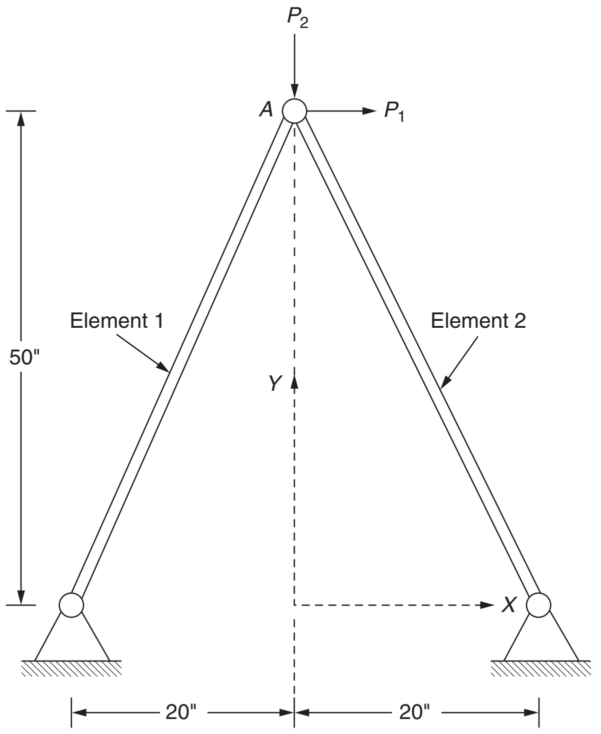


FIGURE 9.25 Two-bar truss.

9.19 A beam is fixed at one end, supported by a cable at the other end, and subjected to a uniformly distributed load of 50 lb/in as shown in Fig. 9.26.
a. Derive the finite element equilibrium equations of the system by using one finite element for the beam and one finite element for the cable.
b. Find the displacement of node 2.

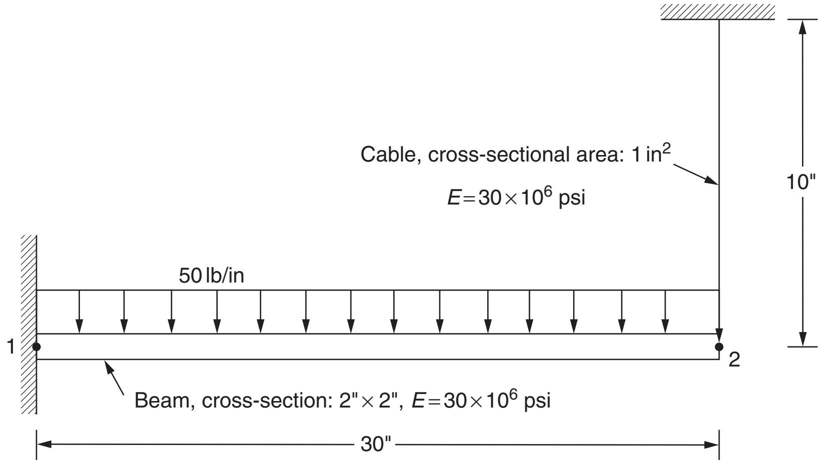


FIGURE 9.26 Beam supported by a cable.

c. Find the stress distribution in the beam.
d. Find the stress distribution in the cable.

9.20 A beam is fixed at one end and is subjected to three forces and three moments at the other end as shown in Fig. 9.27 ($P_x = 100$ N, $M_x = 20$ N-m, $P_y = 200$ N, $M_y = 30$ N-m, $P_z = 300$ N, $M_z = 40$ N-m, $E = 205$ GPa). Find the stress distribution in the beam using a one-beam element idealization.

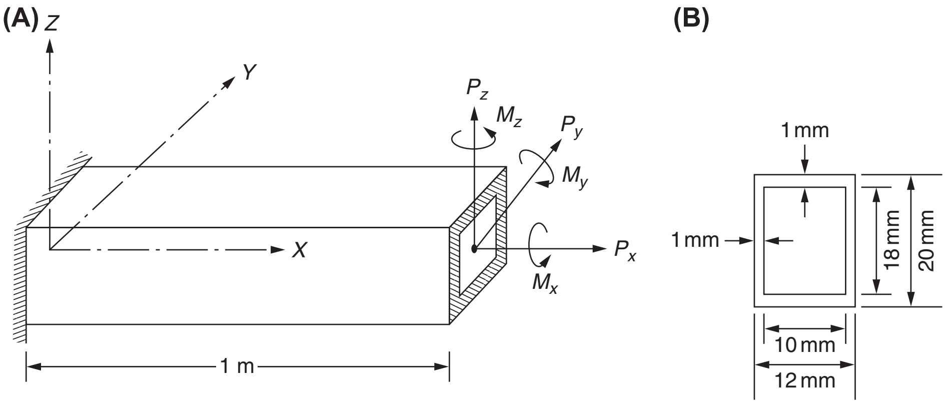


FIGURE 9.27 Beam subjected to loads and moments.

9.21 Determine the stress distribution in the two members of the frame shown in Fig. 9.28. Use one finite element for each member of the frame.

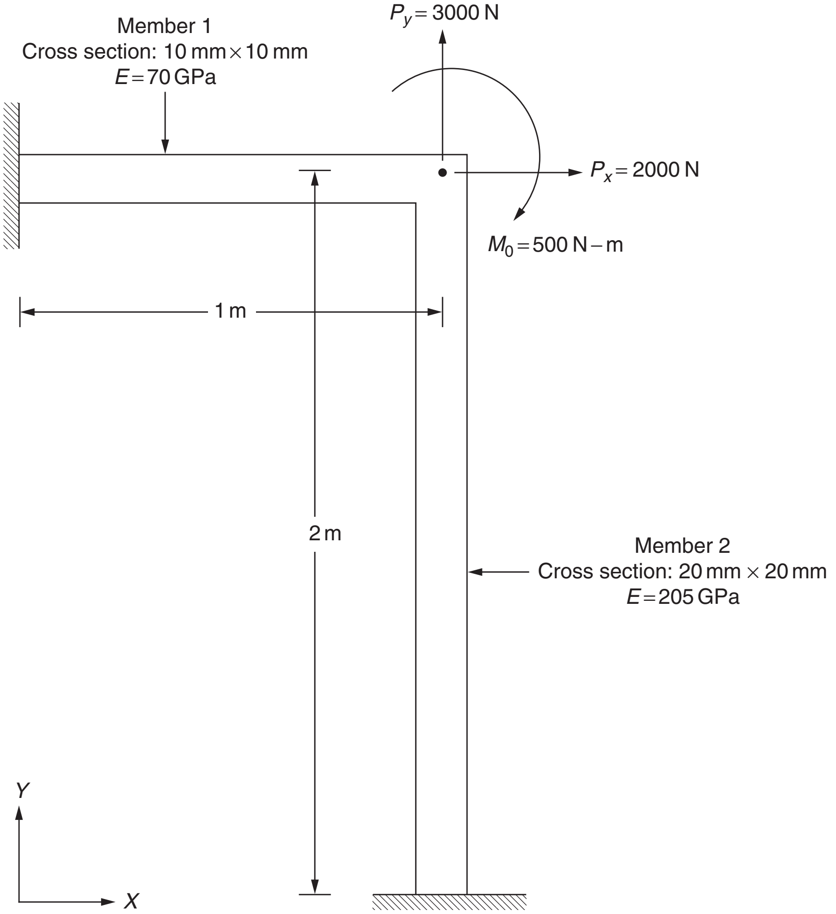


FIGURE 9.28 Two-member planar frame.

9.22 Find the displacement of node 3 and the stresses in the two members of the truss shown in Fig. 9.29. Assume the Young's modulus and the cross-sectional areas of the two members are the same, with $E = 30 \times 10^6$ psi and $A = 1$ in^2.

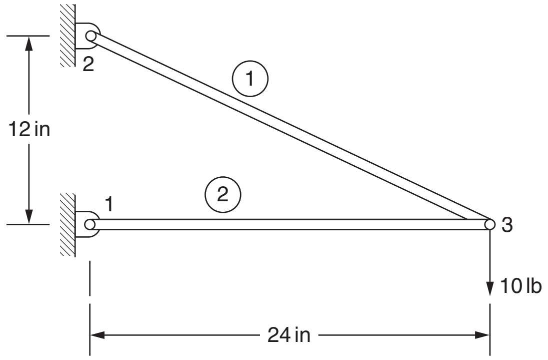


FIGURE 9.29 Two-member truss.

9.23 A simple model of a radial drilling machine structure is shown in Fig. 9.30. Using two beam elements for the column and one beam element for the arm, derive the stiffness matrix of the system. Assume the material of the structure is steel and the foundation is a rigid block. The cross section of the column is tubular with inside diameter 350 mm and outside diameter 400 mm. The cross-section of the arm is hollow rectangular with an overall depth of 400 mm and overall width of 300 mm, with a wall thickness of 10 mm.

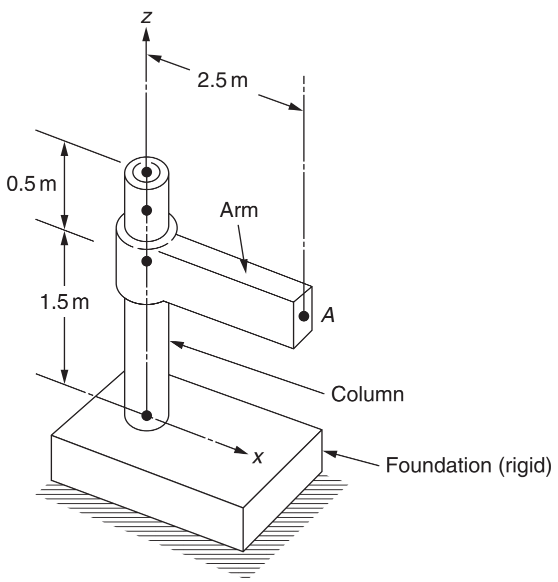


FIGURE 9.30 Drilling machine.

9.24 If a vertical force of 4000 N along the z direction and a bending moment of 1000 N-m in the xz plane are developed at point A during a metal-cutting operation, find the stresses developed in the machine tool structure shown in Fig. 9.30.

9.25 The crank in the slider–crank mechanism shown in Fig. 9.31 rotates at a constant angular speed of 1500 rpm. Find the stresses in the connecting rod and the crank when the pressure acting on the piston is 100 psi and $\theta = 30°$. The diameter of the piston is 10 in, and the material of the mechanism is steel. Model the connecting rod and the crank by one beam element each. The lengths of the crank and the connecting rod are 10 and 45 in, respectively.

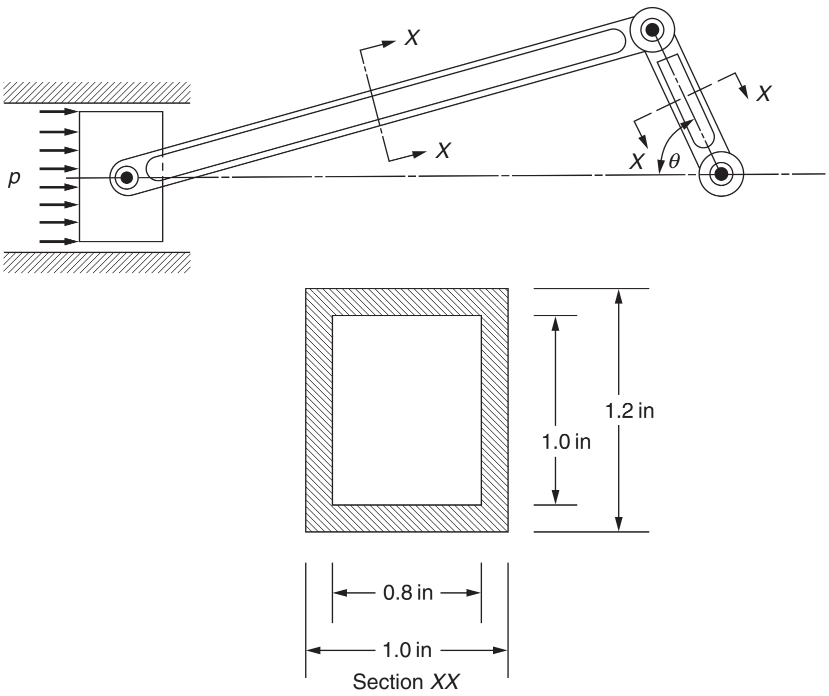


FIGURE 9.31 Slider-crank mechanism.

9.26 A water tank of weight W is supported by a hollow circular steel column of inner diameter d, wall thickness t, and height h. The wind pressure acting on the column can be assumed to vary linearly from 0 to p_{max}, as shown in Fig. 9.32. Find the bending stress induced in the column under the loads using a one-beam element idealization with the following data:

$$W = 15,000 \text{ lb}, \quad h = 30 \text{ ft}, \quad d = 2 \text{ ft}, \quad t = 2 \text{ in}, \quad p_{max} = 200 \text{ psi}$$

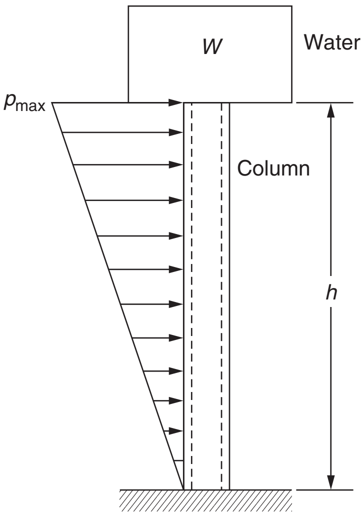


FIGURE 9.32 Water tank.

9.27 Find the nodal displacements and stresses in elements 1, 2, and 3 of the system shown in Fig. 9.33. Use three bar elements and one spring element for modeling.
Data: $A_1 = 3$ in^2, $A_2 = 2$ in^2, $A_3 = 1$ in^2, $E_1 = 30 \times 10^6$ psi, $E_2 = 10 \times 10^6$ psi, $E_3 = 15 \times 10^6$ psi, $l_1 = 10$ in, $l_2 = 20$ in, $l_3 = 30$ in, $k = 10^5$ lb/in.

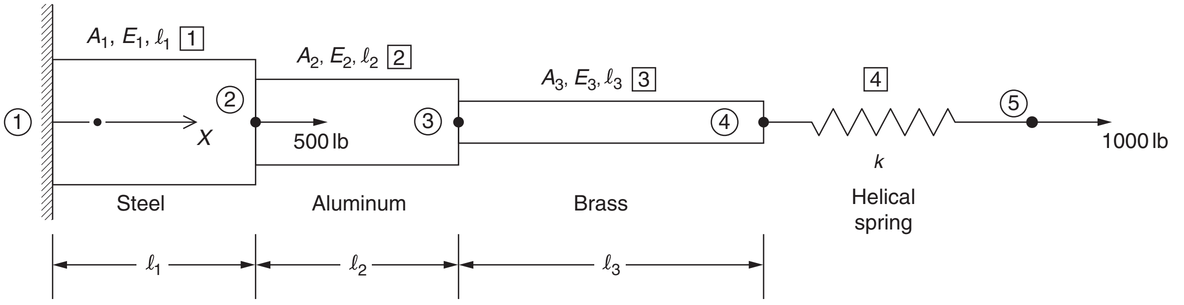


FIGURE 9.33 Stepped-beam-spring system.

9.28 A truss element of length l and cross-sectional area A is subjected to a linearly varying load acting on the surface in the axial direction as shown in Fig. 9.34 (load per unit length varies from 0 to p_0). Derive the consistent load vector of the element using a linear interpolation model. Also indicate the lumped load vector of the element.

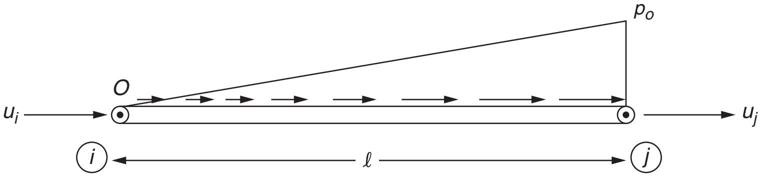


FIGURE 9.34 Bar subjected to linearly varying load.

9.29 A beam of flexural rigidity EI, fixed at the left end, is supported on a spring of stiffness k at the right end as shown in Fig. 9.35, where Q_i, $i = 1, 2, \ldots, 6$ denote the global degrees of freedom.
a. Derive the stiffness matrix of the system before applying the boundary conditions.
b. Find the stiffness matrix of the system after applying the boundary conditions.
c. Find the displacement and slope of the beam at point A for the following data: $EI = 25 \times 10^6$ lb-in^2, $l_1 = 20$ in, $l_2 = 10$ in, $k = 10^4$ lb/in, load at A (acting in a vertically downward direction) $= 100$ lb.

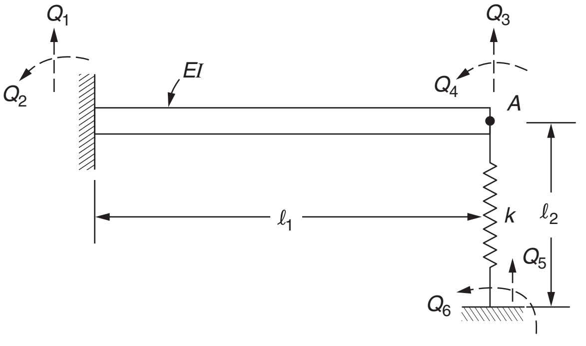


FIGURE 9.35 Beam supported on spring.

9.30 Find the stress in the bar shown in Fig. 9.36 using the finite element method with one bar and one spring element. Data: Cross-sectional area of bar (A) = 2 in^2, Young's modulus of the bar (E) = 30×10^6 psi, spring constant of the spring (k) = 10^5 lb/in.

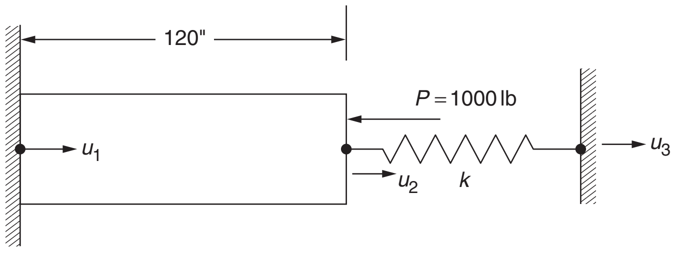


FIGURE 9.36 Bar connected to a spring.

9.31 A two-dimensional frame is shown in Fig. 9.37. Using three degrees of freedom per node, derive the following:

a. Global stiffness and mass matrices of order 9×9 before applying the boundary conditions
b. Global stiffness and mass matrices of order 3×3 after applying the boundary conditions
c. Nodal displacement vector under the given load
d. Natural frequencies and mode shapes of the frame

Data: $E = 30 \times 10^6$ psi, $I = 2$ in^4, $A = 1$ in^2, $l = 30$ in, $\rho = 0.283$ lb/in^3 (weight density), $g = 384$ in/s^2 (gravitational constant), $P_1 = 1000$ lb, $P_2 = 500$ lb.

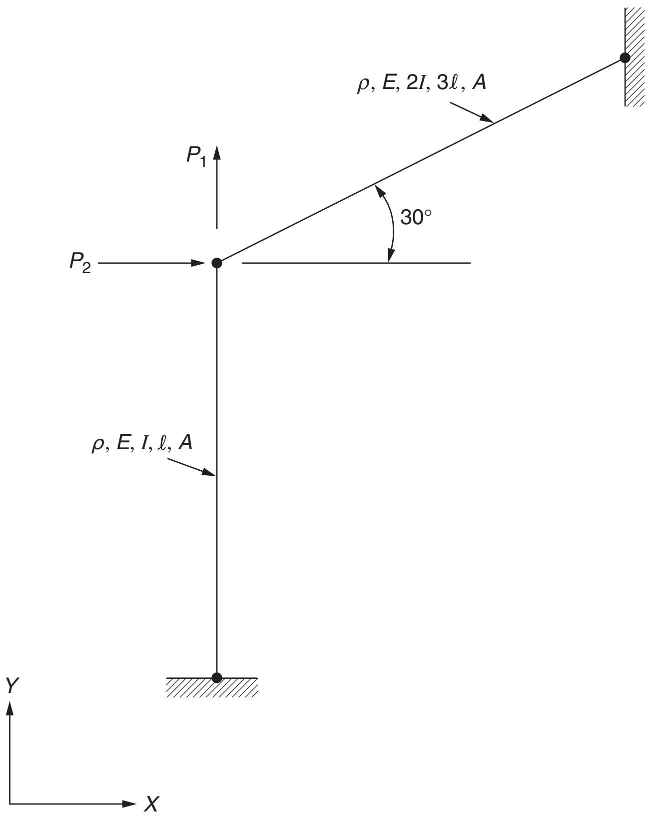


FIGURE 9.37 Two-dimensional frame.

9.32 Derive the stiffness matrix of a beam element in bending using trigonometric functions (instead of a cubic equation) for the interpolation model. Discuss the convergence of the resulting element.

9.33 The beam shown in Fig. 9.38 is subjected to a uniformly distributed load of 50 lb/in. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Determine the stress distribution in the beam using a two-beam element model.

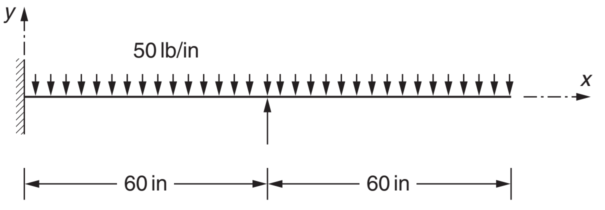


FIGURE 9.38 Fixed beam with simple support at the middle.

9.34 The beam shown in Fig. 9.39 is subjected to a uniformly distributed load of 50 lb/in. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Determine the stress distribution in the beam using a two-beam element model.

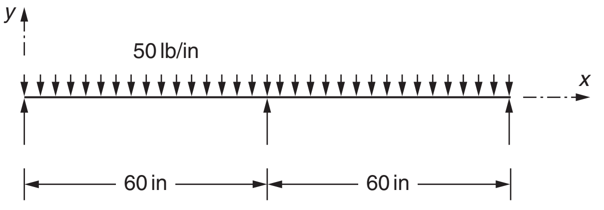


FIGURE 9.39 Beam on three simple supports.

9.35 The beam shown in Fig. 9.40 is subjected to a uniformly distributed load of 50 lb/in. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Determine the stress distribution in the beam using a two-beam element model.

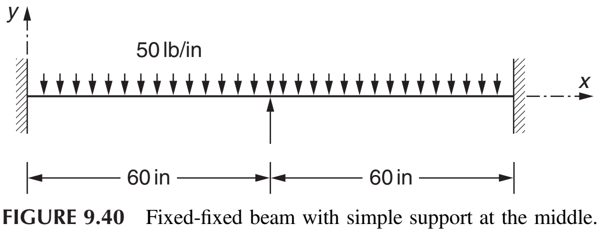


FIGURE 9.40 Fixed-fixed beam with simple support at the middle.

9.36 A fixed-fixed beam is subjected to two concentrated loads as shown in Fig. 9.41. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Using three beam elements to model the beam, find the displacement and stress distributions in the beam.

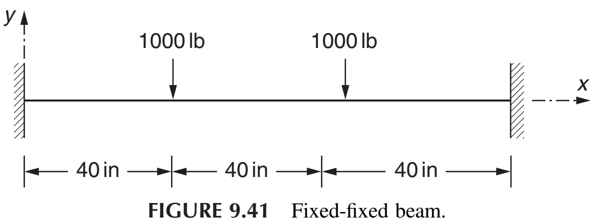


FIGURE 9.41 Fixed-fixed beam.

9.37 The beam shown in Fig. 9.42 is subjected to a uniformly distributed load of 50 lb/in. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Determine the stress distribution in the beam using a two-beam element model.

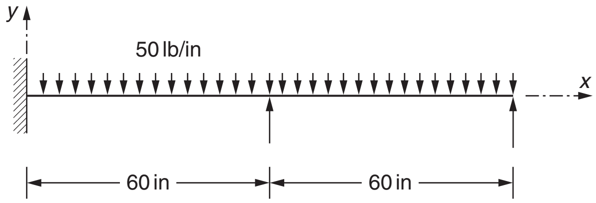


FIGURE 9.42 Fixed beam on two simple supports.

9.38 A beam is simply supported at its left end and the middle point and subjected to a concentrated vertical load at its free end as shown in Fig. 9.43. The beam has a rectangular cross section with a depth of 2 in (in y direction) and width 1 in (in z direction). The Young's modulus of the beam is 30×10^6 psi. Find the stress distribution in the beam.

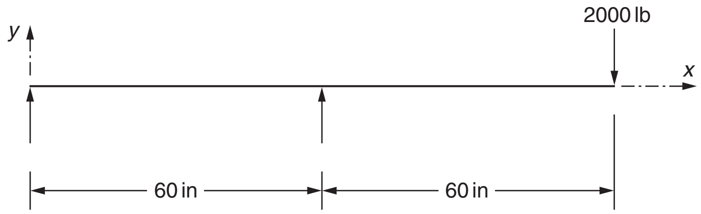


FIGURE 9.43 Beam with simple supports at the end and middle.

9.39 A stepped shaft of length $L = 120$ in has a circular cross section of diameter d_1 over $0 \le x \le \frac{L}{2}$ and a circular cross section of diameter d_2 over $\frac{L}{2} \le x \le L$ as shown in Fig. 9.44. The shear modulus of the shaft is 12×10^6 psi. Determine the shear stress distribution in the shaft when a torque $T = 2000$ lb-in is applied at the free end using two finite elements.

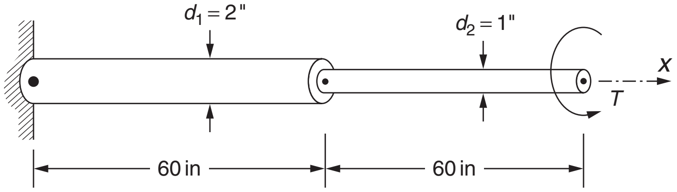


FIGURE 9.44 Two stepped shaft.

9.40 A stepped shaft of length $L = 120$ in has a circular cross section of diameter d_1 over $0 \le x \le \frac{L}{3}$, a circular cross section of diameter d_2 over $\frac{L}{3} \le x \le \frac{2L}{3}$, and a circular cross section of diameter d_3 over $\frac{2L}{3} \le x \le L$ as shown in Fig. 9.45. The shear modulus of the shaft is 12×10^6 psi. Determine the shear stress distribution in the shaft when a torque T is applied at the free end using three finite elements. Assume T = 2000 lb-in.

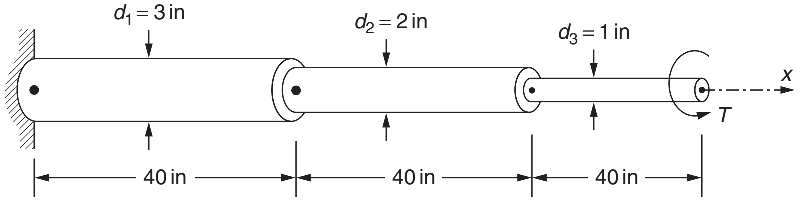


FIGURE 9.45 Three stepped shaft.

9.41 A cantilever beam is subjected to two concentrated loads, $P_1 = 2000$ lb and $P_2 = 1000$ lb, in two different directions at the free end as shown in Fig. 9.46. Find the deflection of the free end and the stress distribution in the beam using one finite element. Assume that the Young's modulus of the beam is 30×10^6 psi.

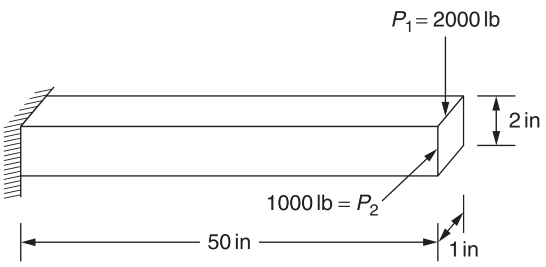


FIGURE 9.46 Cantilever beam with loads in two directions.

9.42 The cantilever beam shown in Fig. 9.47 has two different cross-sections: a 2 in × 2 in square cross section over the length $0 \le x \le \frac{L}{2}$ and a circular cross section of diameter 1 in over the length $\frac{L}{2} \le x \le L$. It is subjected to two concentrated loads P_y and P_z and a bending moment M_z as indicated in Fig. 9.47. If the Young's modulus of the beam is 30×10^6 psi, find the deflections and stresses induced in the beam using two finite elements.

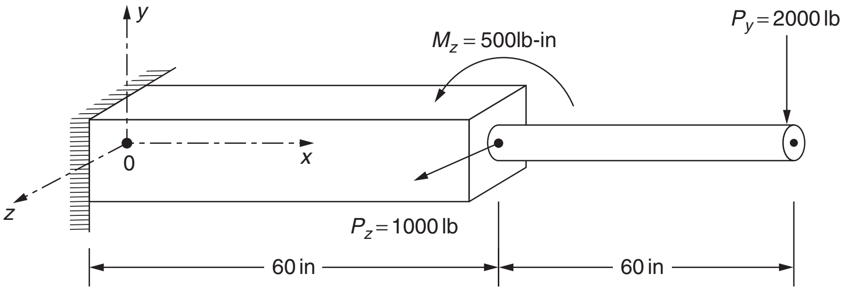


FIGURE 9.47 Cantilever beam with two different sections.

9.43 A three-bar planar truss with its support conditions and loads is shown in Fig. 9.48. The cross-sectional areas of the bars 1, 2, and 3 are 1 in^2, 2 in^2, and 3 in^2, respectively. The material of the bars has a Young's modulus of 30×10^6 psi. Determine the stresses in the various members of the truss.

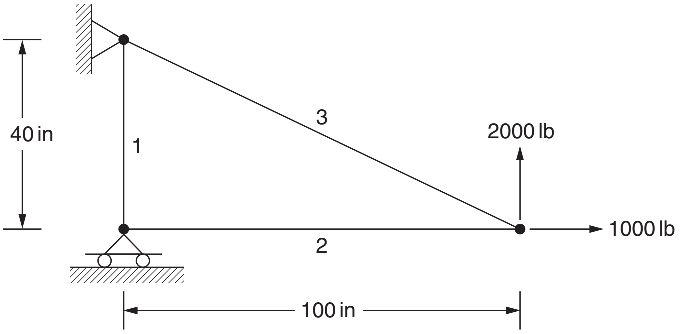


FIGURE 9.48 Three-bar planar truss.

9.44 A four-bar planar truss with its support conditions and loads is shown in Fig. 9.49. Assume that the cross-sectional area of each bar is 1 in^2 and Young's modulus is 30×10^6 psi. Determine the stresses in the various members of the truss.

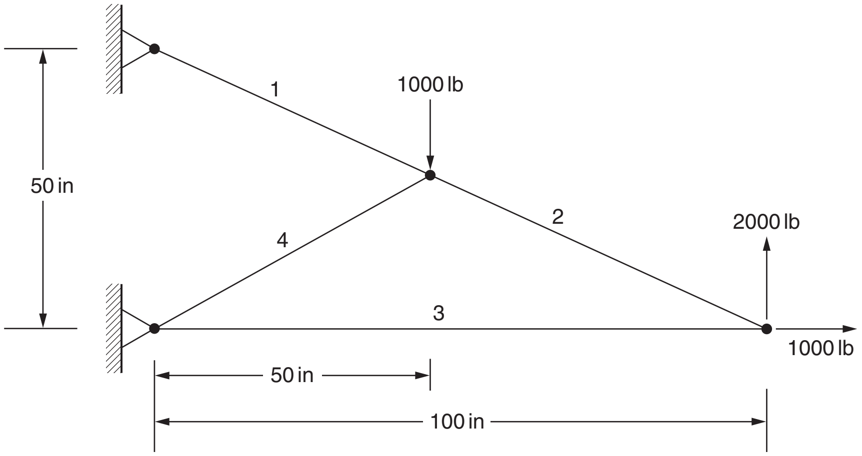


FIGURE 9.49 Four-bar planar truss.

9.45 A six-bar planar truss with its support conditions and loads is shown in Fig. 9.50. Assume that the cross-sectional area of each bar is 1 in^2 and Young's modulus is 30×10^6 psi. Determine the stresses in the various members of the truss.

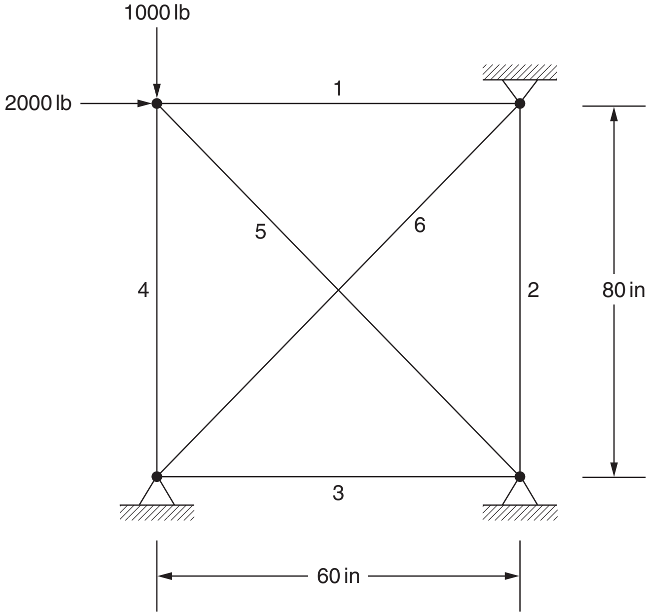


FIGURE 9.50 Six-bar planar truss.

9.46 Consider the planar frame shown in Fig. 9.51. The segments 1, 2, and 3 of the frame have rectangular cross sections with cross-sectional dimensions 2 in × 1 in (in $x \times z$ directions), 3 in × 2 in (in $y \times z$ directions) and 2 in × 1 in (in $x \times z$ directions), respectively. The Young's modulus of the material is 30×10^6 psi. Determine the stresses in the three members of the frame using a three-element idealization.

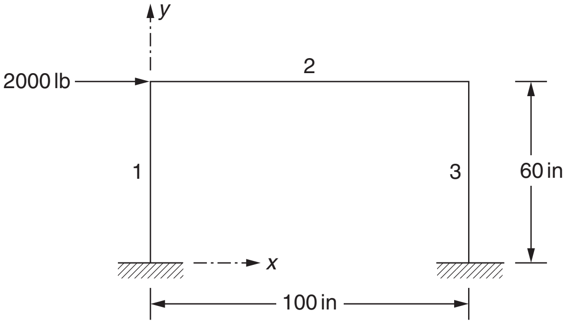


FIGURE 9.51 Three-member planar frame.

9.47 Consider a beam with two perpendicular segments subjected to a concentrated load as shown in Fig. 9.52. Using two beam elements, determine the deflection and stresses in the two segments of the beam. Assume that Young's modulus of the beam is 30×10^6 psi.

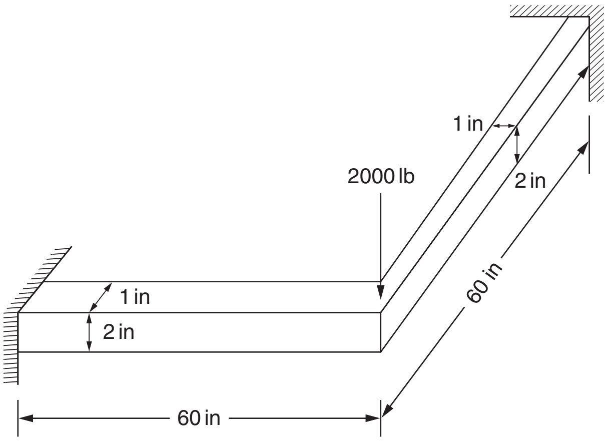


FIGURE 9.52 Two-segmented beam.

9.48 A space frame is composed of three segments and is subjected to three loads in different directions as shown in Fig. 9.53. Each of the three segments of the frame has a circular cross section with diameter 1 in. The material of the frame has a Young's modulus of 30×10^6 psi. All the joints are assumed to be welded. Using three frame elements, determine the deflections and the stresses in the frame.

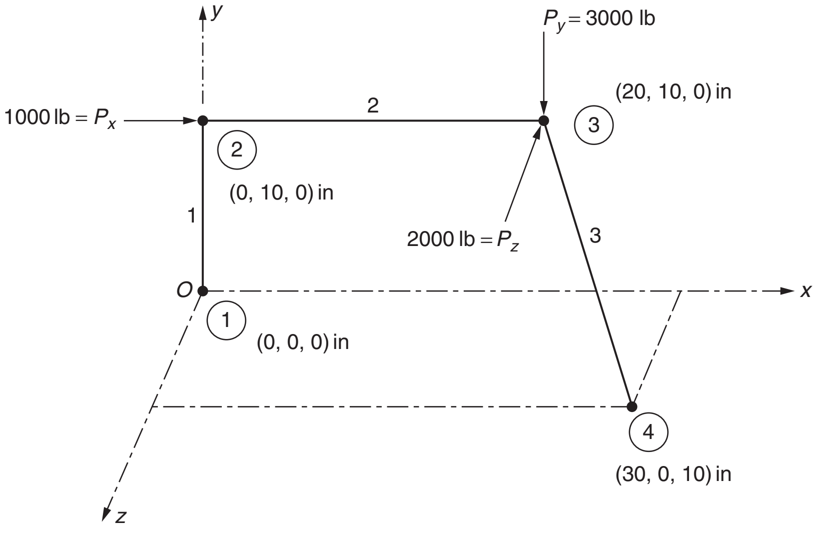


FIGURE 9.53 Three-member space frame.

9.49 A space frame is composed of three segments and is subjected to three loads in different directions as shown in Fig. 9.54. Each of the three segments of the frame has a circular cross section with diameter 1 in. The material of the frame has a Young's modulus of 30×10^6 psi. All the joints are assumed to be welded. Using three frame elements, determine the deflections and stresses in the frame.

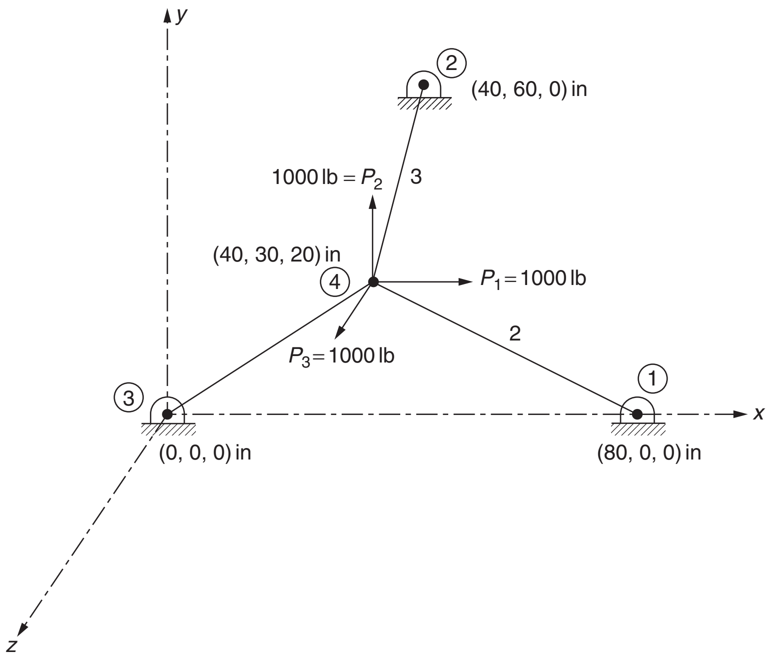


FIGURE 9.54 Space frame with three segments.

9.50 By treating the structure shown in Fig. 9.54 as a space truss (each of the four joints is assumed to be a pin joint), determine the stresses in the truss using a three-truss element model.

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Chapter 10

Analysis of Plates

Chapter Outline

10.1 INTRODUCTION

When a flat plate is subjected to both in-plane and transverse or normal loads as shown in Fig. 10.1A any point inside the plate can have displacement components u, v, and w parallel to x, y, and z axes, respectively. In the small deflection (or linear) theory of thin plates, the transverse deflection w is uncoupled from the in-plane deflections u and v. Consequently, the stiffness matrices for the in-plane and transverse deflections are also uncoupled and they can be calculated independently. Thus, if a plate is subjected to in-plane loads only, it will undergo deformation in its plane only. In this case, the plate is said to be under the action of membrane forces. Similarly, if the plate is subjected to transverse loads (and/or bending moments), any point inside the plate experiences essentially a lateral displacement w (in-plane displacements u and v are also experienced because of the rotation of the plate element). In this case, the plate is said to be under the action of bending forces. The in-plane and bending analysis of plates is considered in this chapter. If the plate elements are used for the analysis of three-dimensional structures, such as folded plate structures, both in-plane and bending actions have to be considered in the development of element properties. This aspect of coupling the membrane and bending actions of a plate element is also considered in this chapter.

10.2 TRIANGULAR MEMBRANE ELEMENT

The triangular membrane element is considered to lie in the xy plane of a local xy coordinate system as shown in Fig. 10.1B. By assuming a linear displacement variation inside the element, the displacement model can be expressed as

$$\begin{aligned} u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \tag{10.1}$$

By considering the displacements u_i and v_i as the local degrees of freedom of node i ($i = 1, 2, 3$), the constants $\alpha_1, \ldots, \alpha_6$ can be evaluated. Thus, by using the conditions

$$\left.\begin{aligned} u(x,y) &= u_1 = q_1 \text{ and } v(x,y) = v_1 = q_2 \text{ at } (x_1,y_1) \\ u(x,y) &= u_2 = q_3 \text{ and } v(x,y) = v_2 = q_4 \text{ at } (x_2,y_2) \\ u(x,y) &= u_3 = q_5 \text{ and } v(x,y) = v_3 = q_6 \text{ at } (x_3,y_3) \end{aligned}\right\} \tag{10.2}$$

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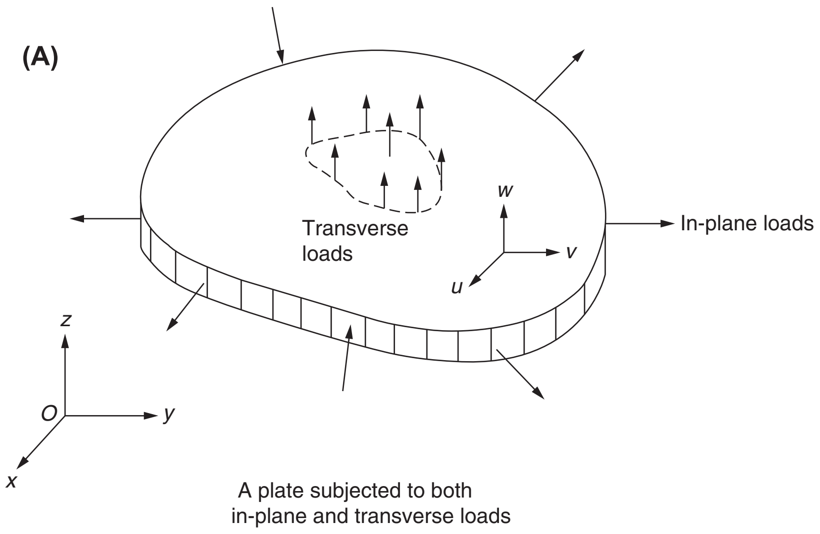


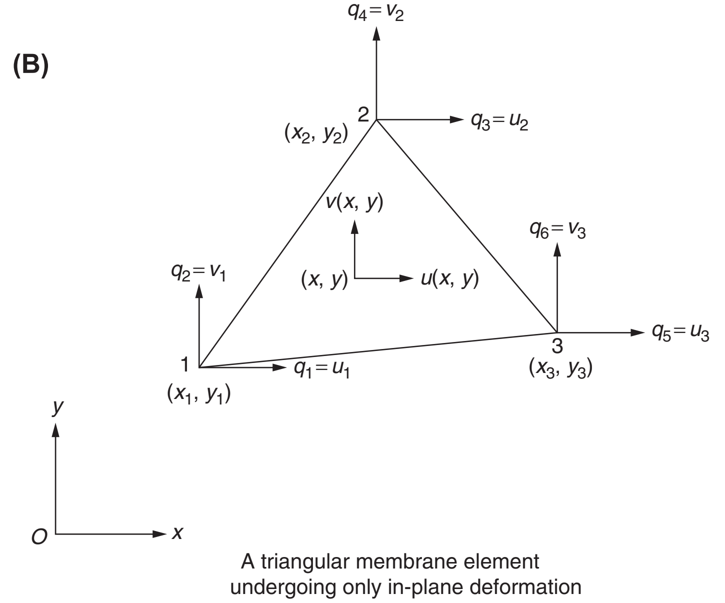


FIGURE 10.1 Loads acting on a plate.

we can express the constants $\alpha_1, \ldots, \alpha_6$ in terms of the nodal degrees of freedom as outlined in Section 3.4. This leads to the displacement model:

$$\vec{U} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N]\vec{q}^{(e)} \tag{10.3}$$

where

$$[N(x,y)] = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \tag{10.4}$$

$$\left.\begin{aligned} N_1(x,y) &= \frac{1}{2A}[y_{32}(x-x_2) - x_{32}(y-y_2)] \\ N_2(x,y) &= \frac{1}{2A}[-y_{31}(x-x_3) + x_{31}(y-y_3)] \\ N_3(x,y) &= \frac{1}{2A}[y_{21}(x-x_1) - x_{21}(y-y_1)] \end{aligned}\right\} \tag{10.5}$$

$$A = \frac{1}{2}(x_{32}y_{21} - x_{21}y_{32}) = \text{area of the triangle } 1\ 2\ 3 \tag{10.6}$$

$$\left.\begin{aligned} x_{ij} &= x_i - x_j \\ y_{ij} &= y_i - y_j \end{aligned}\right\} \tag{10.7}$$

$$\vec{U} = \left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} \tag{10.8}$$

$$\vec{q}^{(e)} = \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{array} \right\}^{(e)} = \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \right\}^{(e)} \tag{10.9}$$

By using the relations

$$\vec{\varepsilon} = \left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \partial u/\partial x \\ \partial v/\partial y \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{array} \right\} \tag{10.10}$$

and Eq. (10.3), the components of strain can be expressed in terms of nodal displacements as

$$\vec{\varepsilon} = [B]\vec{q}^{(e)} \tag{10.11}$$

where

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{32} & 0 & -y_{31} & 0 & y_{21} & 0 \\ 0 & -x_{32} & 0 & x_{31} & 0 & -x_{21} \\ -x_{32} & y_{32} & x_{31} & -y_{31} & -x_{21} & y_{21} \end{bmatrix} \tag{10.12}$$

If the element is in a state of plane stress, the stress–strain relations are given by (Eq. 8.35)

$$\vec{\sigma} = [D]\vec{\varepsilon} \tag{10.13}$$

where

$$\vec{\sigma} = \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} \tag{10.14}$$

and

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix} \tag{10.15}$$

The stiffness matrix of the element $[k^{(e)}]$ can be found by using Eq. (8.87):

$$\left[k^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V \tag{10.16}$$

where $V^{(e)}$ denotes the volume of the element. If the plate thickness is taken as a constant (t), the evaluation of the integral in Eq. (10.16) presents no difficulty since the elements of the matrices $[B]$ and $[D]$ are all constants (not functions of x and y). Hence, Eq. (10.16) can be rewritten as

$$\left[k^{(e)}\right] = [B]^T[D][B]t\iint_A \mathrm{d}A = tA[B]^T[D][B] \tag{10.17}$$

Although the matrix products involved in Eq. (10.17) can be performed conveniently on a computer, the explicit form of the stiffness matrix is given here for convenience:

$$\left[k^{(e)}\right] = \left[k_n^{(e)}\right] + \left[k_s^{(e)}\right] \tag{10.18}$$

where the matrix $\left[k^{(e)}\right]$ is separated into two parts—one due to normal stresses, $\left[k_n^{(e)}\right]$, and the other due to shear stresses, $\left[k_s^{(e)}\right]$. The components of the matrices $\left[k_n^{(e)}\right]$ and $\left[k_s^{(e)}\right]$ are given by

$$\left[k_n^{(e)}\right] = \frac{Et}{4A(1-\nu^2)}\begin{bmatrix} y_{32}^2 & & & & & \\ -\nu y_{32}x_{32} & x_{32}^2 & & \text{Symmetric} & & \\ -y_{32}y_{31} & \nu x_{32}y_{31} & y_{31}^2 & & & \\ \nu y_{32}x_{31} & -x_{32}x_{31} & -\nu y_{31}x_{31} & x_{31}^2 & & \\ y_{32}y_{21} & -\nu x_{32}y_{21} & -y_{31}y_{21} & \nu x_{31}y_{21} & y_{21}^2 & \\ -\nu y_{32}x_{21} & x_{32}x_{21} & \nu y_{31}x_{21} & -x_{31}x_{21} & -\nu y_{21}x_{21} & x_{21}^2 \end{bmatrix} \tag{10.19}$$

and

$$\left[k_s^{(e)}\right] = \frac{Et}{8A(1+\nu)}\begin{bmatrix} x_{32}^2 & & & & & \\ -x_{32}y_{32} & y_{32}^2 & & \text{Symmetric} & & \\ -x_{32}x_{31} & y_{32}x_{31} & x_{31}^2 & & & \\ x_{32}y_{31} & -y_{32}y_{31} & -x_{31}y_{31} & y_{31}^2 & & \\ x_{32}x_{21} & -y_{32}x_{21} & -x_{31}x_{21} & y_{31}x_{21} & x_{21}^2 & \\ -x_{32}y_{21} & y_{32}y_{21} & x_{31}y_{21} & -y_{31}y_{21} & -x_{21}y_{21} & y_{21}^2 \end{bmatrix} \tag{10.20}$$

TRANSFORMATION MATRIX

In actual computations, it is convenient, from the standpoint of calculating the transformation matrix $[\lambda]$, to select the local xy coordinate system as follows. Assuming that the triangular element under consideration is an interior element of a large structure, let node numbers 1, 2, and 3 of the element correspond to node numbers i, j, and k, respectively, of the global system. Then place the origin of the local xy system at node 1 (node i), and take the y axis along the edge 1 2 (edge ij) and the x axis perpendicular to the y axis directed toward node 3 (node k) as shown in Fig. 10.2.

To generate the transformation matrix $[\lambda]$, the direction cosines of lines ox and oy with respect to the global X, Y, and Z axes are required. Since the direction cosines of the line oy are the same as those of line ij, we obtain

$$l_{ij} = \frac{X_j - X_i}{d_{ij}}, \quad m_{ij} = \frac{Y_j - Y_i}{d_{ij}}, \quad n_{ij} = \frac{Z_j - Z_i}{d_{ij}} \tag{10.21}$$

where the distance between the points i and j (d_{ij}) is given by

$$d_{ij} = \left[(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2\right]^{1/2} \tag{10.22}$$

and (X_i, Y_i, Z_i) and (X_j, Y_j, Z_j) denote the (X, Y, Z) coordinates of points i and j, respectively. Since the direction cosines of the line ox cannot be computed unless we know the coordinates of a second point on the line ox (in addition to those of

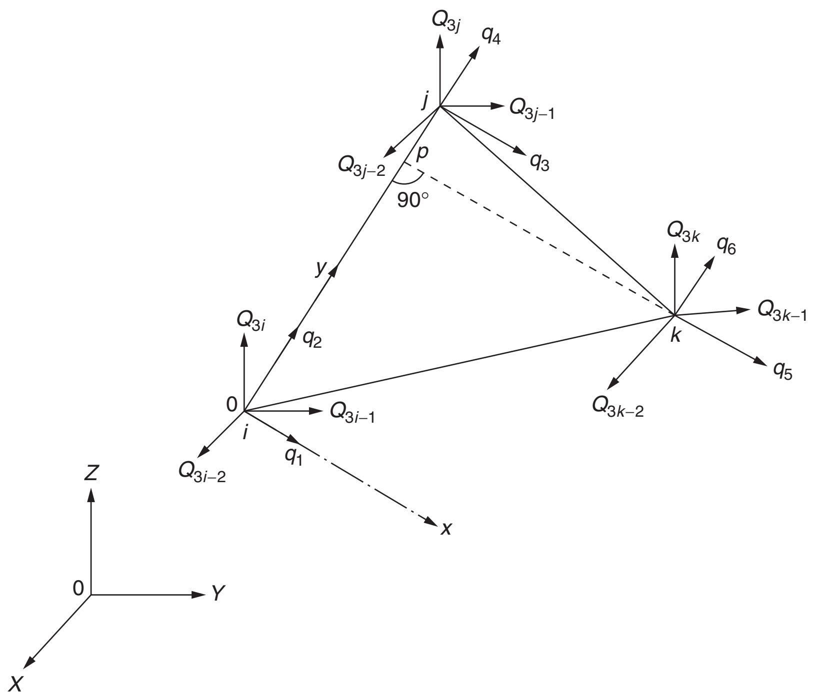


FIGURE 10.2 Local and global coordinates.

point i), we draw a perpendicular line kp from node k onto the line ij as shown in Fig. 10.2. Then the direction cosines of the line ox will be the same as those of the line pk:

$$l_{pk} = \frac{X_k - X_p}{d_{pk}}, \quad m_{pk} = \frac{Y_k - Y_p}{d_{pk}}, \quad n_{pk} = \frac{Z_k - Z_p}{d_{pk}} \tag{10.23}$$

where d_{pk} is the distance between the points p and k. The coordinates (X_p, Y_p, Z_p) of the point p in the global coordinate system can be computed as

$$\begin{aligned} X_p &= X_i + l_{ij} d_{ip} \\ Y_p &= Y_i + m_{ij} d_{ip} \\ Z_p &= Z_i + n_{ij} d_{ip} \end{aligned} \tag{10.24}$$

where d_{ip} is the distance between the points i and p. To find the distance d_{ip}, we use the condition that the lines ij and pk are perpendicular to each other:

$$l_{ij} l_{pk} + m_{ij} m_{pk} + n_{ij} n_{pk} = 0 \tag{10.25}$$

Using Eqs. (10.23) and (10.24), Eq. (10.25) can be rewritten as

$$\frac{1}{d_{pk}} \left[l_{ij}(X_k - X_i - l_{ij} d_{ip}) + m_{ij}(Y_k - Y_i - m_{ij} d_{ip}) + n_{ij}(Z_k - Z_i - n_{ij} d_{ip}) \right] = 0 \tag{10.26}$$

Eq. (10.26) can be solved for d_{ip} as

$$d_{ip} = l_{ij}(X_k - X_i) + m_{ij}(Y_k - Y_i) + n_{ij}(Z_k - Z_i) \tag{10.27}$$

where the condition $l_{ij}^2 + m_{ij}^2 + n_{ij}^2 = 1$ has been used. Finally, the distance d_{pk} can be found by considering the right-angle triangle ikp as

$$d_{pk} = \left(d_{ik}^2 - d_{ip}^2 \right)^{1/2} = \left[(X_k - X_i)^2 + (Y_k - Y_i)^2 + (Z_k - Z_i)^2 - d_{ip}^2 \right]^{1/2} \tag{10.28}$$

The transformation matrix $[\lambda]$ can now be constructed by using the direction cosines of lines ij and pk as

$$[\lambda] = \begin{bmatrix} \vec{\lambda}_{pk} & \vec{0} & \vec{0} \\ \vec{\lambda}_{ij} & \vec{0} & \vec{0} \\ \vec{0} & \vec{\lambda}_{pk} & \vec{0} \\ \vec{0} & \vec{\lambda}_{ij} & \vec{0} \\ \vec{0} & \vec{0} & \vec{\lambda}_{pk} \\ \vec{0} & \vec{0} & \vec{\lambda}_{ij} \end{bmatrix} \tag{10.29}$$

where

$$\underset{1\times 3}{\vec{\lambda}_{pk}} = (\, l_{pk} \quad m_{pk} \quad n_{pk} \,) \tag{10.30}$$

$$\underset{1\times 3}{\vec{\lambda}_{ij}} = (\, l_{ij} \quad m_{ij} \quad n_{ij} \,) \tag{10.31}$$

$$\underset{1\times 3}{\vec{0}} = (\, 0 \quad 0 \quad 0 \,) \tag{10.32}$$

Finally, the stiffness matrix of the element in the global XYZ coordinate system can be computed as

$$\left[K^{(e)}\right] = [\lambda]^T \left[k^{(e)}\right] [\lambda] \tag{10.33}$$

CONSISTENT LOAD VECTOR

The consistent load vectors can be evaluated using Eqs. (8.88) to (8.90):

$$p_i^{(e)} = \text{load vector due to initial strains} = \iiint_{V^{(e)}} [B]^T [D] \vec{\varepsilon}_0 \, dV \tag{10.34}$$

In the case of thermal loading, Eq. (10.34) becomes

$$\vec{p}_i^{(e)} = [B]^T [D] \vec{\varepsilon}_0 tA = \frac{E\alpha t T}{2(1-v)} \begin{bmatrix} y_{32} \\ -x_{32} \\ -y_{31} \\ x_{31} \\ y_{21} \\ -x_{21} \end{bmatrix} \tag{10.35}$$

$$\vec{p}_b^{(e)} = \text{load vector due to constant body forces } \phi_{x0} \text{ and } \phi_{y0} = \iiint_{V^{(e)}} [N]^T \vec{\phi}_0 \, dV \tag{10.36}$$

By using Eq. (10.4), Eq. (10.36) can be rewritten as

$$\vec{p}_b^{(e)} = \iiint_{V^{(e)}} \begin{Bmatrix} N_1 \phi_{x_0} \\ N_1 \phi_{y_0} \\ N_2 \phi_{x_0} \\ N_2 \phi_{y_0} \\ N_3 \phi_{x_0} \\ N_3 \phi_{y_0} \end{Bmatrix} dV \tag{10.37}$$

Substituting the expressions for N_1, N_2, and N_3 from Eq. (10.5) into Eq. (10.37) and carrying out the integration yields

$$\vec{p}_b^{(e)} = \frac{At}{3}\begin{Bmatrix}\phi_{x0}\\ \phi_{y0}\\ \phi_{x0}\\ \phi_{y0}\\ \phi_{x0}\\ \phi_{y0}\end{Bmatrix} \tag{10.38}$$

The following relations have been used in deriving Eq. (10.38):

$$\iint_A x\cdot \,\mathrm{d}A = x_c A \quad \text{and} \quad \iint_A y\cdot \,\mathrm{d}A = y_c A \tag{10.39}$$

where x_c and y_c are the coordinates of the centroid of the triangle 1 2 3 given by

$$x_c = (x_1 + x_2 + x_3)/3 \quad \text{and} \quad y_c = (y_1 + y_2 + y_3)/3 \tag{10.40}$$

The load vector due to the surface stresses $\vec{\vec{\phi}} = \begin{Bmatrix}p_{xo}\\ p_{yo}\end{Bmatrix}$, where p_{xo} and p_{yo} are constants, can be evaluated as

$$\vec{p}_s^{(e)} = \iint_{S_1^{(e)}} [N]^T \begin{Bmatrix}p_{x0}\\ p_{y0}\end{Bmatrix} \mathrm{d}S_1 \tag{10.41}$$

There are three different vectors $\vec{p}_s^{(e)}$ corresponding to the three sides of the element. Let the side between nodes 1 and 2 be subjected to surface stresses of magnitude p_{xo} and p_{yo}. Then

$$\vec{p}_s^{(e)} = \iint_{S_1^{(e)}} \begin{bmatrix}N_1 & 0\\ 0 & N_1\\ N_2 & 0\\ 0 & N_2\\ N_3 & 0\\ 0 & N_3\end{bmatrix} \begin{Bmatrix}p_{x0}\\ p_{y0}\end{Bmatrix} \mathrm{d}S_1 = \frac{S_{12}}{2}\begin{Bmatrix}p_{x0}\\ p_{yo}\\ p_{x0}\\ p_{y0}\\ 0\\ 0\end{Bmatrix} \tag{10.42}$$

where S_{12} is the surface area between nodes 1 and 2 given by

$$S_{12} = t\cdot d_{12} \tag{10.43}$$

with d_{12} denoting the length of side 12. Since the stress components p_{xo} and p_{yo} are parallel to the x and y coordinate directions, Eq. (10.42) shows that the total force in either coordinate direction is $(p_{x0}\bullet S_{12})$ and $(p_{yo}\bullet S_{12})$, respectively. Thus, one-half of the total force in each direction is allotted to each node on the side under consideration. The total load vector in the local coordinate system is thus given by

$$\vec{p}^{(e)} = \vec{p}_i^{(e)} + \vec{p}_b^{(e)} + \vec{p}_s^{(e)} \tag{10.44}$$

This load vector, when referred to the global system, becomes

$$\vec{P}^{(e)} = [\lambda]^T \vec{p}^{(e)} \tag{10.45}$$

CHARACTERISTICS OF THE ELEMENT

1. The constant strain triangle (CST) element was the first finite element developed for the analysis of plane stress problems [1.7]. Because the displacement model is linear (Eq. 10.1), the element is called a linear triangular element. From Eqs. (10.11) and (10.12), we find that the $[B]$ matrix is independent of the position within the element and hence the strains are constant throughout the element. This is the reason why this element is often referred to as a CST element. Obviously, the criterion of constant strain mentioned in the convergence requirements in Section 3.6 is satisfied by the displacement model.
2. The displacement model chosen (Eq. 10.1) guarantees continuity of displacements with adjacent elements because the displacements vary linearly along any side of the triangle (due to linear model).

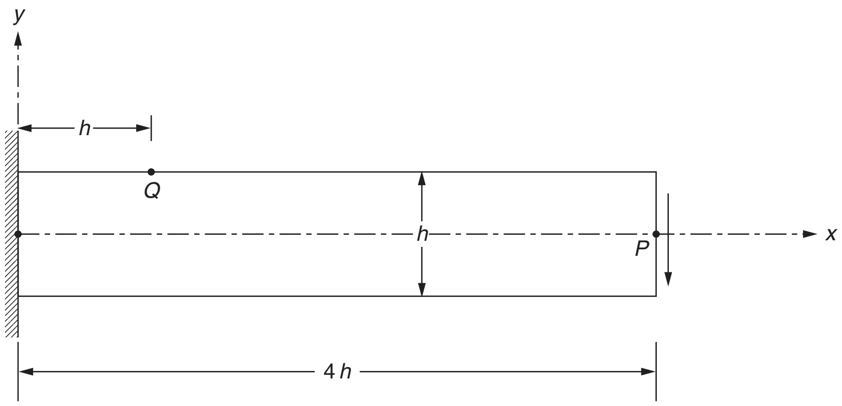


FIGURE 10.3 A uniform plate under tensile load.

3. From Eq. (10.13), we can notice that the stresses are also constant inside an element. Hence, the element is also called a CST element. Since the stresses are independent of x and y, the equilibrium equations (Eq. 8.1) are identically satisfied inside the element since there are no body forces.
4. If the complete plate structure being analyzed lies in a single (e.g., XY) plane, the vector $\vec{Q}^{(e)}$ will also contain six components. In such a case, the matrices $[\lambda]$ and $[K^{(e)}]$ will each be of the order 6×6.
5. In problems involving bending, this element overestimates the bending stiffness (normal stresses). More accurate normal stresses can be obtained by using smaller size elements. However, the convergence to the correct solution will be very slow. To illustrate the numerical performance of the element, a uniform beam with rectangular cross section subjected to nodal forces at the free end shown in Fig. 10.3 is considered [10.2]. The nodal forces indicated will produce bending in the beam. If the beam is modeled using CST elements as indicated in Fig. 10.4A, each element gives a constant value of σ_x in the depth or y direction instead of a linear variation predicted by the exact solution. In fact, the exact solution along the x axis would be $\sigma_x = 0$ while the CST model would predict a constant value of σ_x with alternate signs as we move along the x axis from one element to the next as shown in Fig. 10.4.

In addition, the CST element predicts a spurious shear stress. The element predicts a constant, nonzero, value of the shear stress σ_{xy} in each element. This means that the shear strain ε_{xy} is constant in the beam while it should be zero. Also, in some cases, the CST element can exhibit a phenomenon known as *locking*. The term *locking* denotes excessive stiffness in one or more deformation modes. The numerical results obtained with the CST element are given in Table 10.1. The results indicate that even 512 elements could not predict the bending behavior of the beam (deflection and stress) very accurately.

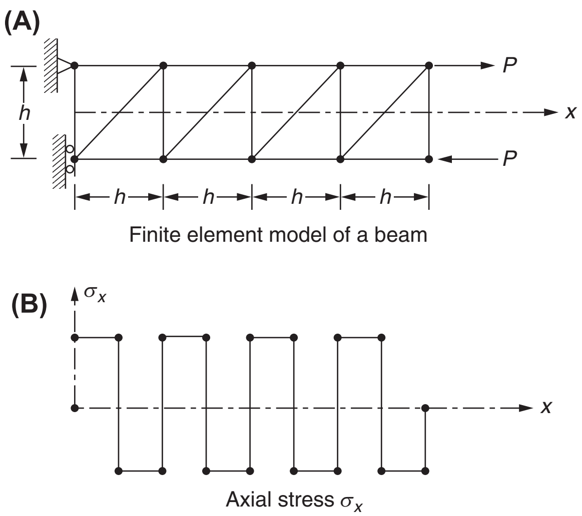


FIGURE 10.4 A cantilever beam modeled with constant strain triangle elements.

TABLE 10.1 Finite Element Results of the Cantilever Beam

Element	Number of Elements	Number of Degrees of Freedom	Tip deflection,* v_P	Stress,* σ_{xQ}
CST	128	160	0.859	0.854
CST	512	576	0.961	0.956
LST	32	160	0.998	0.986

Ratio of finite element result and exact result.

EXAMPLE 10.1

A triangular membrane element of thickness $t = 0.1$ cm, with the (x, y) coordinates of nodes indicated besides the node numbers, is shown in Fig. 10.5. If the material of the element is steel with Young's modulus $E = 207$ GPa and Poisson ratio $v = 0.3$, determine the following:

1. Shape functions of the element, $N_i(x, y)$, $N_j(x, y)$, and $N_k(x, y)$.
2. Matrix $[B]$ that relates the strains to the nodal displacements.
3. Elasticity matrix $[D]$.
4. Element stiffness matrix, $\left[k^{(e)}\right]$.

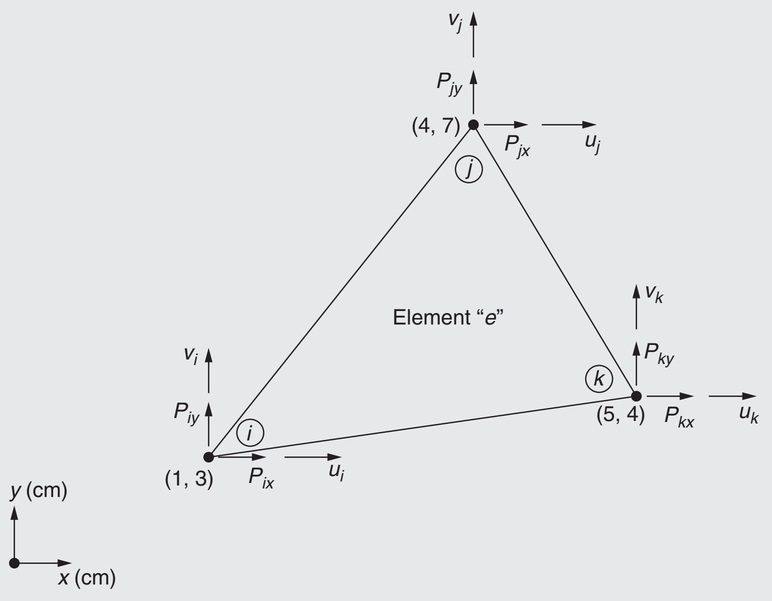


FIGURE 10.5 A triangular membrane element.

Solution

Noting that $x_1 = 1$ cm, $x_2 = 4$ cm, $x_3 = 5$ cm, $y_1 = 3$ cm, $y_2 = 7$ cm, and $y_3 = 4$ cm, we find $x_{32} = x_3 - x_2 = 5 - 4 = 1$ cm, $x_{21} = x_2 - x_1 = 4 - 1 = 3$ cm, $x_{31} = x_3 - x_1 = 5 - 1 = 4$ cm, $y_{32} = y_3 - y_2 = 4 - 7 = -3$ cm, $y_{21} = y_2 - y_1 = 7 - 3 = 4$ cm, and $y_{31} = y_3 - y_1 = 4 - 3 = 1$ cm. The area of the element (A) can be computed as

$$A = \frac{1}{2}\{x_{32}y_{21} - x_{21}y_{32}\} = \frac{1}{2}\{1(4) - 3(-3)\} = 6.5 \text{ cm}^2 = 6.5 \times 10^{-4} \text{ m}^2$$

a. Shape functions of the element:

$$N_i(x, y) = \frac{1}{2A}[y_{32}(x - x_2) - x_{32}(y - y_2)] = \frac{1}{2(6.5)}[(-3)(x - 4) - 1(y - 7)]$$

$$= \frac{1}{13}(-3x - y + 19) \tag{E.1}$$

Continued

EXAMPLE 10.1 —cont'd

$$N_j(x,y) = \frac{1}{2A}[-y_{31}(x-x_3)+x_{31}(y-y_3)] = \frac{1}{2(6.5)}[(-1)(x-5)+1(y-4)]$$
$$= \frac{1}{13}(-x+4y-11) \tag{E.2}$$

$$N_k(x,y) = \frac{1}{2A}[y_{21}(x-x_1)-x_{21}(y-y_1)] = \frac{1}{2(6.5)}[4(x-1)-3(y-3)]$$
$$= \frac{1}{13}(4x-3y+5) \tag{E.3}$$

b. Matrix $[B]$:

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{32} & 0 & -y_{31} & 0 & y_{21} & 0 \\ 0 & -x_{32} & 0 & x_{31} & 0 & -x_{21} \\ -x_{32} & y_{32} & x_{31} & -y_{31} & -x_{21} & y_{21} \end{bmatrix}$$
$$= \frac{100}{13}\begin{bmatrix} -3 & 0 & -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 & 0 & -3 \\ -1 & -3 & 4 & -1 & -3 & 4 \end{bmatrix} \text{l/m} \tag{E.4}$$

c. Elasticity matrix $[D]$:

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \frac{207\times 10^9}{1-0.09}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$
$$= 10^{11}\begin{bmatrix} 2.2747 & 0.6824 & 0 \\ 0.6824 & 2.2747 & 0 \\ 0 & 0 & 0.7961 \end{bmatrix} \text{N/m}^2 \tag{E.5}$$

d. Stiffness matrix of the element $[K^{(e)}]$:

$$\left[K^{(e)}\right] = t\,A[B]^T[D][B] \tag{E.6}$$

where $t = 0.001$ m and $A = 6.5\times 10^{-4}$ m^2. The evaluation of Eq. (E.6) using the known values of t, A, $[B]$, and $[D]$ gives the stiffness matrix as

$$\left[K^{(e)}\right] = 10^8\begin{bmatrix} 0.8180 & 0.1706 & 0.1400 & -0.2843 & -0.9580 & 0.1137 \\ 0.1706 & 0.3631 & -0.3412 & -0.2581 & 0.1706 & -0.1050 \\ 0.1400 & -0.3412 & 0.5774 & -0.2275 & -0.7174 & 0.5686 \\ -0.2843 & -0.2581 & -0.2275 & 1.4304 & 0.5118 & -1.1723 \\ -0.9580 & 0.1706 & -0.7174 & 0.5118 & 1.6754 & -0.6824 \\ 0.1137 & -0.1050 & 0.5686 & -1.1723 & -0.6824 & 1.2773 \end{bmatrix}$$

EXAMPLE 10.2

If the element described in Example 10.1 undergoes a temperature increase of 80 °C and the coefficient of thermal expansion of the material is 10.8×10^{-6} m/m °C, find the nodal load vector due to thermal loading.

Solution

Eq. (10.35) gives the load vector due to initial strain (thermal loading) as

$$\vec{P}_i^{(e)} = \frac{E\alpha t T}{2(1-v)}\begin{Bmatrix} y_{32} \\ -x_{32} \\ -y_{31} \\ x_{31} \\ y_{21} \\ -x_{21} \end{Bmatrix} = \frac{(207\times10^9)(10.8\times10^{-6})(0.001)(80)}{2(1-0.3)}\begin{Bmatrix} -3\times10^{-2} \\ -1\times10^{-2} \\ -1\times10^{-2} \\ 4\times10^{-2} \\ 4\times10^{-2} \\ -3\times10^{-2} \end{Bmatrix}$$

$$= \begin{Bmatrix} -3.8324 \\ -1.2775 \\ -1.2775 \\ 5.1099 \\ 5.1099 \\ -3.8324 \end{Bmatrix} N$$

EXAMPLE 10.3

If distributed body forces of magnitude $\phi_{0x} = 20$ N/m^2 and $\phi_{0y} = 30$ N/m^2 act along x and y directions throughout the element described in Example 10.1, determine the corresponding nodal load vector of the element.

Solution

From Eq. (10.36), we obtain the nodal load vector due to body forces as

$$\vec{P}_b^{(e)} = \frac{At}{3}\begin{Bmatrix} \phi_{0x} \\ \phi_{0y} \\ \phi_{0x} \\ \phi_{0y} \\ \phi_{0x} \\ \phi_{0y} \end{Bmatrix} = \frac{(6.5\times10^{-4})(0.001)}{3}\begin{Bmatrix} 20 \\ 30 \\ 20 \\ 30 \\ 20 \\ 30 \end{Bmatrix} = 10^{-6}\begin{Bmatrix} 4.3333 \\ 6.5 \\ 4.3333 \\ 6.5 \\ 4.3333 \\ 6.5 \end{Bmatrix} \mathrm{N}$$

EXAMPLE 10.4

For the triangular membrane element considered in Example 10.1, distributed surface tractions of magnitude $\Phi_x = p_{0x} = 50$ N/m^2 and $\Phi_y = p_{0y} = -30$ N/m^2 act on the edge (face) connecting nodes 1 and 3. Find the resulting nodal load vector of the element.

Solution

Using an equation similar to Eq. (10.42), we obtain the load vector of the element due to the specified distributed surface tractions as

$$\vec{P}_s^{(e)} = \frac{S_{13}}{2}\begin{Bmatrix} p_{0x} \\ p_{0y} \\ 0 \\ 0 \\ p_{0x} \\ p_{0y} \end{Bmatrix} \tag{E.1}$$

where S_{13} is the surface area of the edge (face) 13 = td_{13} with d_{13} denoting the distance between nodes 1 and 3:

$$d_{13} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} = \sqrt{(5-1)^2 + (4-3)^2} = 4.1231 \text{ cm} = 4.1231 \times 10^{-2} \text{ m}$$

Thus,

$$S_{13} = t\, d_{13} = 10^{-3}\left(4.1231 \times 10^{-2}\right) = 4.1231 \times 10^{-5} \text{ m}^2$$

Hence, Eq. (E.1) gives

$$\vec{P}_s^{(e)} = \frac{4.1231 \times 10^{-5}}{2}\begin{Bmatrix} 50 \\ -30 \\ 0 \\ 0 \\ 50 \\ -30 \end{Bmatrix} = 10^{-5}\begin{Bmatrix} 103.0776 \\ -61.8466 \\ 0 \\ 0 \\ 103.0776 \\ -61.8466 \end{Bmatrix} \text{N} \tag{E.2}$$

EXAMPLE 10.5

A concentrated load, with components $P_{0x} = 1000$ N and $P_{0y} = -500$ N, acts at the point $(x_0,\ y_0) = (3,\ 5)$ cm of the plate described in Example 10.1. Determine the corresponding nodal load vector of the element.

Solution

The equivalent nodal load vector of the element corresponding to the point load, $\vec{p}_0 = \begin{Bmatrix} p_{0x} \\ p_{0y} \end{Bmatrix}$, can be expressed as

$$\vec{P}_{\text{point}}^{(e)} = [N]^T \vec{p}_0 = \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix}_{(x_0,y_0)} \begin{Bmatrix} p_{0x} \\ p_{0y} \end{Bmatrix} = \begin{Bmatrix} N_i(x_0,y_0)p_{0x} \\ N_i(x_0,y_0)p_{0y} \\ N_j(x_0,y_0)p_{0x} \\ N_j(x_0,y_0)p_{0y} \\ N_k(x_0,y_0)p_{0x} \\ N_k(x_0,y_0)p_{0y} \end{Bmatrix} \tag{E.1}$$

At the given point ($x_0 = 3$ cm, $y_0 = 5$ cm), the shape functions, given by Eqs. (E.1) to (E.3) in Example 10.1, assume the values

$$N_i(3,5) = \frac{1}{13}[-3(3) - 5 + 19] = \frac{5}{13}$$

$$N_j(3,5) = \frac{1}{13}[-3 + 4(5) - 11] = \frac{6}{13}$$

$$N_k(3,5) = \frac{1}{13}[4(3) - 3(5) + 5] = \frac{2}{13}$$

EXAMPLE 10.5 —cont'd

Thus, the nodal load vector of Eq. (E.l) becomes

$$\vec{P}^{(e)}_{\text{point}} = \frac{1}{13}\begin{Bmatrix} 5(1000) \\ 5(-500) \\ 6(1000) \\ 6(-500) \\ 2(1000) \\ 2(-500) \end{Bmatrix} = \begin{Bmatrix} 384.6154 \\ -192.3077 \\ 461.5385 \\ -230.7692 \\ 153.8461 \\ -76.9231 \end{Bmatrix} \text{N} \tag{E.2}$$

10.3 NUMERICAL RESULTS WITH MEMBRANE ELEMENT

The following examples are considered to illustrate the application of the membrane element in solving selected problems of linear elasticity.

10.3.1 A Plate Under Tension

The uniform plate under tension, shown in Fig. 10.6A, is analyzed by using the CST elements. Due to symmetry of geometry and loading, only a quadrant is considered for analysis. The finite element modeling is done with eight triangular elements as shown in Fig. 10.6B. The total number of nodes is nine and the displacement unknowns are 18. However, the x components of displacement of nodes 3, 4, and 5 (namely Q_5, Q_7, and Q_9) and the y components of displacement of nodes

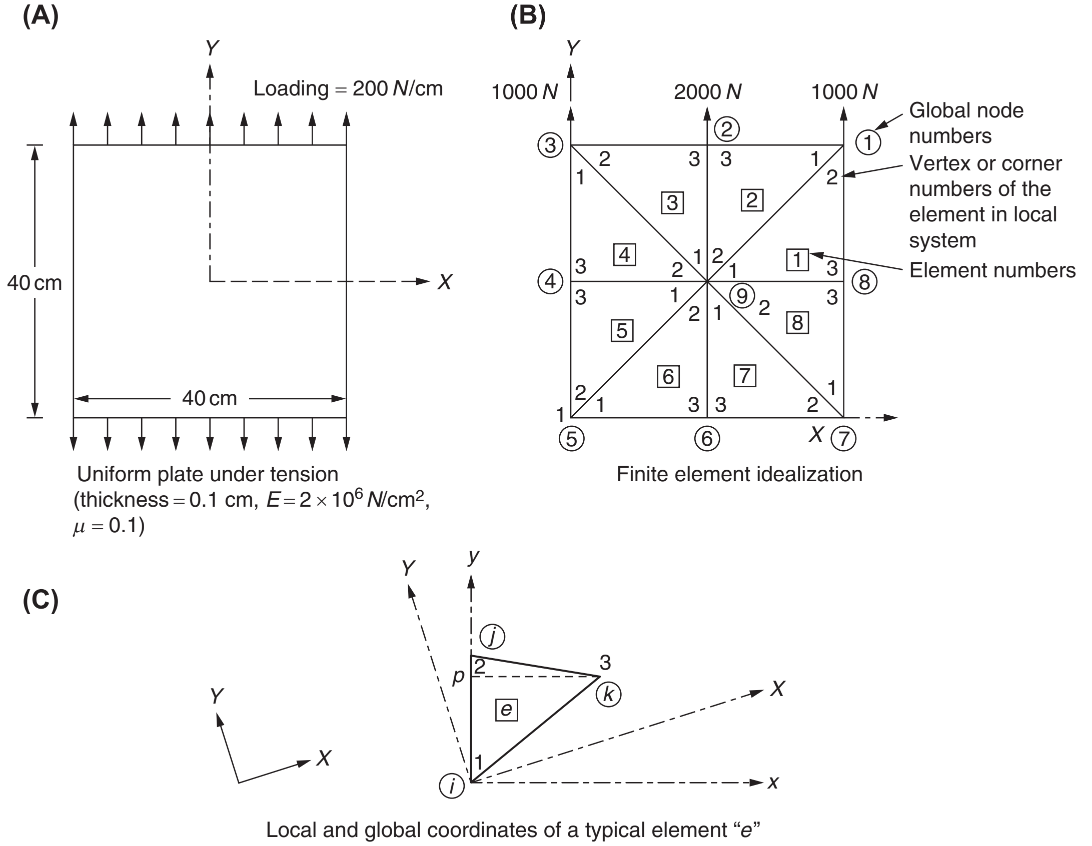


FIGURE 10.6 A uniform plate under tensile load.

5, 6, and 7 (namely Q_{10}, Q_{12}, and Q_{14}) are set equal to zero for maintaining symmetry conditions. After solving the equilibrium equations, the global displacement components can be obtained as

$$Q_i = \begin{cases} 0.020, & i = 2,4,6 \\ 0.010, & i = 8,16,18 \\ -0.002, & i = 1,13,15 \\ -0.001, & i = 3,11,17 \\ 0.000, & i = 5,7,9,10,12,14 \end{cases}$$

COMPUTATION OF STRESSES

For finding the stresses inside any element e, shown in Fig. 10.6C, the following procedure can be adopted:

Step 1: Convert the global displacements of the nodes of element e into local displacements as

$$\underset{6\times 1}{\vec{q}^{(e)}} = \underset{6\times 6}{[\lambda]} \; \underset{6\times 1}{\vec{Q}^{(e)}}$$

where

$$\vec{q}^{(e)} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}, \quad \vec{Q}^{(e)} = \begin{Bmatrix} Q_{2i-1} \\ Q_{2i} \\ Q_{2j-1} \\ Q_{2j} \\ Q_{2k-1} \\ Q_{2k} \end{Bmatrix}$$

and $[\lambda]$ is the transformation matrix of the element given by (two-dimensional specialization of Eq. 10.29)

$$[\lambda] = \begin{bmatrix} l_{pk} & m_{pk} & 0 & 0 & 0 & 0 \\ l_{ij} & m_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & l_{pk} & m_{pk} & 0 & 0 \\ 0 & 0 & l_{ij} & m_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{pk} & m_{pk} \\ 0 & 0 & 0 & 0 & l_{ij} & m_{ij} \end{bmatrix} \tag{10.46}$$

Here (l_{pk}, m_{pk}) and (l_{ij}, m_{ij}) denote the direction cosines of lines pk (x axis) and ij (y axis) with respect to the global (X, Y) system.

Step 2: Using the local displacement vector $\vec{q}^{(e)}$ of element e, find the stresses inside the element in the local system by using Eqs. (10.13) and (10.11) as

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [D][B]\vec{q}^{(e)} \tag{10.47}$$

where $[D]$ and $[B]$ are given by Eqs. (10.15) and (10.12), respectively.

Step 3: Convert the local stresses σ_{xx}, σ_{yy}, and σ_{xy} of the element into global stresses σ_{XX}, σ_{YY}, and σ_{XY} by using the stress transformation relations [10.1]:

$$\sigma_{XX} = \sigma_{xx}l_{pk}^2 + \sigma_{yy}l_{ij}^2 + 2\sigma_{xy}l_{pk}l_{ij}$$

$$\sigma_{YY} = \sigma_{xx}m_{pk}^2 + \sigma_{yy}m_{ij}^2 + 2\sigma_{xy}m_{pk}m_{ij}$$

$$\sigma_{XY} = \sigma_{xx}l_{pk}m_{pk} + \sigma_{yy}l_{ij}m_{ij} + \sigma_{xy}(l_{pk}m_{ij} + m_{pk}l_{ij})$$

The results of computation are shown in Table 10.2. It can be noted that the stresses in the global system exactly match the correct solution given by

$$\sigma_{YY} = \frac{\text{Total tensile load}}{\text{Area of cross} - \text{section}} = \frac{(200 \times 40)}{(40 \times 0.1)} = 2000\ \text{N/cm}^2$$

TABLE 10.2 Computation of Stresses Inside the Elements

Element e	Displacements (cm) In Global System $\vec{Q}^{(e)}$	Displacements (cm) In Local System $\vec{q}^{(e)}$	Stress Vector in Local System $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$ N/cm^2	Stress Vector in Global System $\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{XY} \end{Bmatrix}$ N/cm^2
1	−0.001	0.01556	1000	0
	0.010	−0.01273	1000	2000
	−0.002	0.00778	−1000	0
	0.020	−0.00636		
	−0.002	0.01485		
	0.010	−0.01344		
2	−0.002	0.00636	1000	0
	0.020	0.00778	1000	2000
	−0.001	0.01414	1000	0
	0.010	0.01414		
	−0.001	0.01344		
	0.020	0.01485		
3	−0.001	−0.01414	1000	0
	0.010	−0.01414	1000	2000
	0.000	−0.00636	1000	0
	0.020	−0.00778		
	−0.001	−0.00707		
	0.020	−0.00707		
4	0.000	0.00778	1000	0
	0.020	−0.00636	1000	2000
	−0.001	0.0	−1000	0
	0.010	0.0		
	0.000	0.00707		
	0.010	−0.00707		
5	−0.001	0.0	1000	0
	0.010	0.0	1000	2000
	0.000	−0.00778	−1000	0
	0.000	0.00636		
	0.000	−0.00071		
	0.010	−0.00071		
6	0.000	−0.00636	1000	0
	0.000	−0.00778	1000	2000
	−0.001	0.00141	1000	0
	0.010	−0.00141		
	−0.001	0.00071		
	0.000	−0.00071		

Continued

TABLE 10.2 Computation of Stresses Inside the Elements—cont'd

Element e	Displacements (cm)		Stress Vector in Local System $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$ N/cm^2	Stress Vector in Global System $\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{XY} \end{Bmatrix}$ N/cm^2
	In Global System $\vec{Q}^{(e)}$	In Local System $\vec{q}^{(e)}$		
7	−0.001	−0.00141	1000	0
	0.010	0.00141	1000	2000
	−0.002	0.00636	1000	0
	0.000	0.00778		
	−0.001	0.00566		
	0.000	0.00849		
8	−0.002	−0.00778	1000	0
	0.000	0.00636	1000	2000
	−0.001	−0.01556	−1000	0
	0.010	0.01273		
	−0.002	−0.00849		
	0.010	0.00566		

10.3.2 A Plate With a Circular Hole

The performance of membrane elements for problems of stress concentration due to geometry is studied by considering a tension plate with a circular hole (see Fig. 10.7) [10.2]. Due to the symmetry of geometry and loading, only a quadrant was analyzed using four different finite element idealizations as shown in Fig. 10.8. The results are shown in Table 10.3. The results indicate that the stress concentration is predicted to be smaller than the exact value consistently.

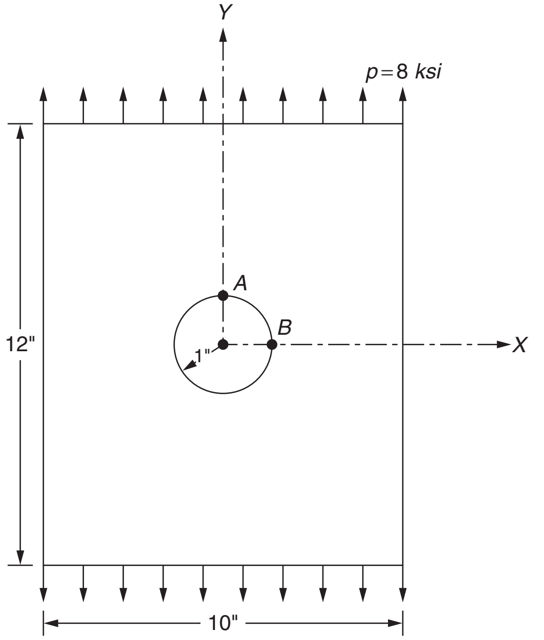


FIGURE 10.7 Plate with a circular hole under uniaxial tension ($E = 30 \times 10^6$ psi, $\nu = 0.25$, t = plate thickness = 1″).

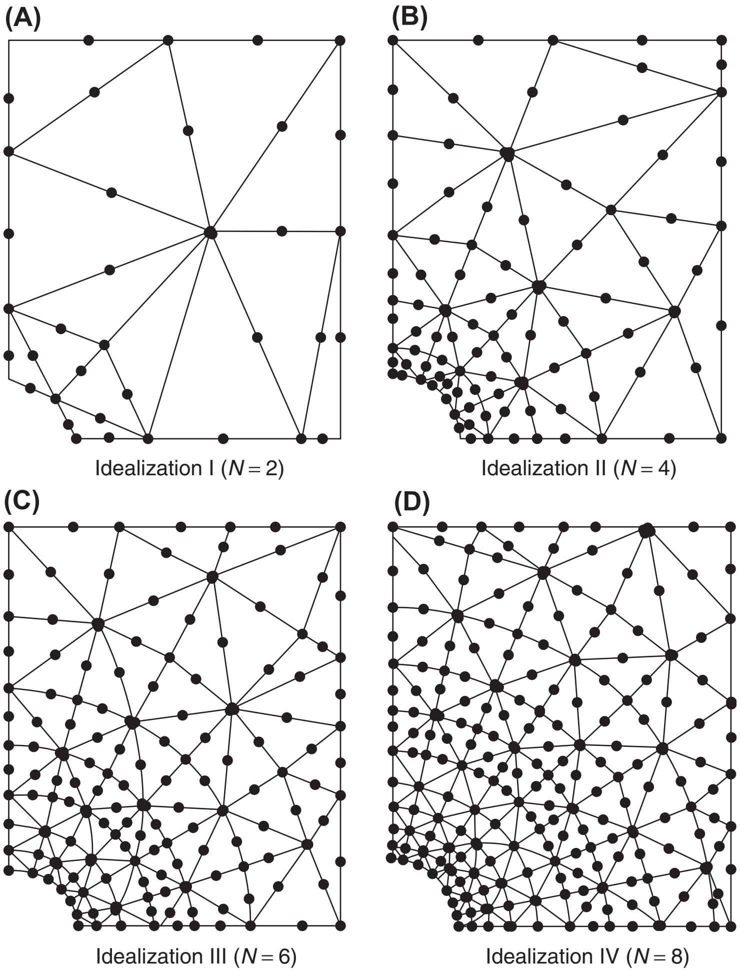


FIGURE 10.8 Finite element idealization of the plate with a circular hole [10.2] (N = number of subdivisions of $\frac{1}{4}$ hole).

TABLE 10.3 Stress Concentration Factors Given by Finite Element Method

Idealization (Fig. 10.7)	Value of ($\sigma_{xx/p}$) at A	Value of (σ_{yy}/p) at B
I	−0.229	1.902
II	−0.610	2.585
III	−0.892	2.903
IV	−1.050	3.049
Exact (theory)	−1.250	3.181

10.3.3 A Cantilevered Box Beam

The cantilevered box beam shown in Fig. 10.9 is analyzed by using CST elements. The finite element idealization consists of 24 nodes, 72 degrees of freedom (in the global XYZ system), and 40 elements as shown in Fig. 10.10. The displacement results obtained for two different load conditions are compared with those given by simple beam theory in Table 10.4. It can be seen that the finite element results compare well with those of simple beam theory.

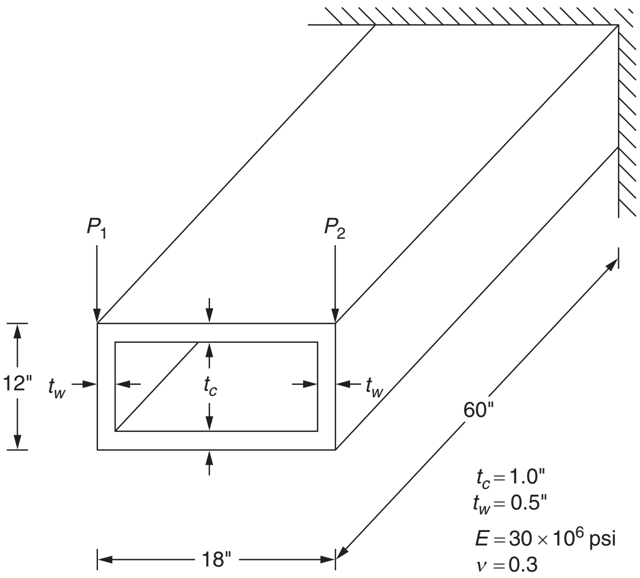


FIGURE 10.9 A cantilevered box beam.

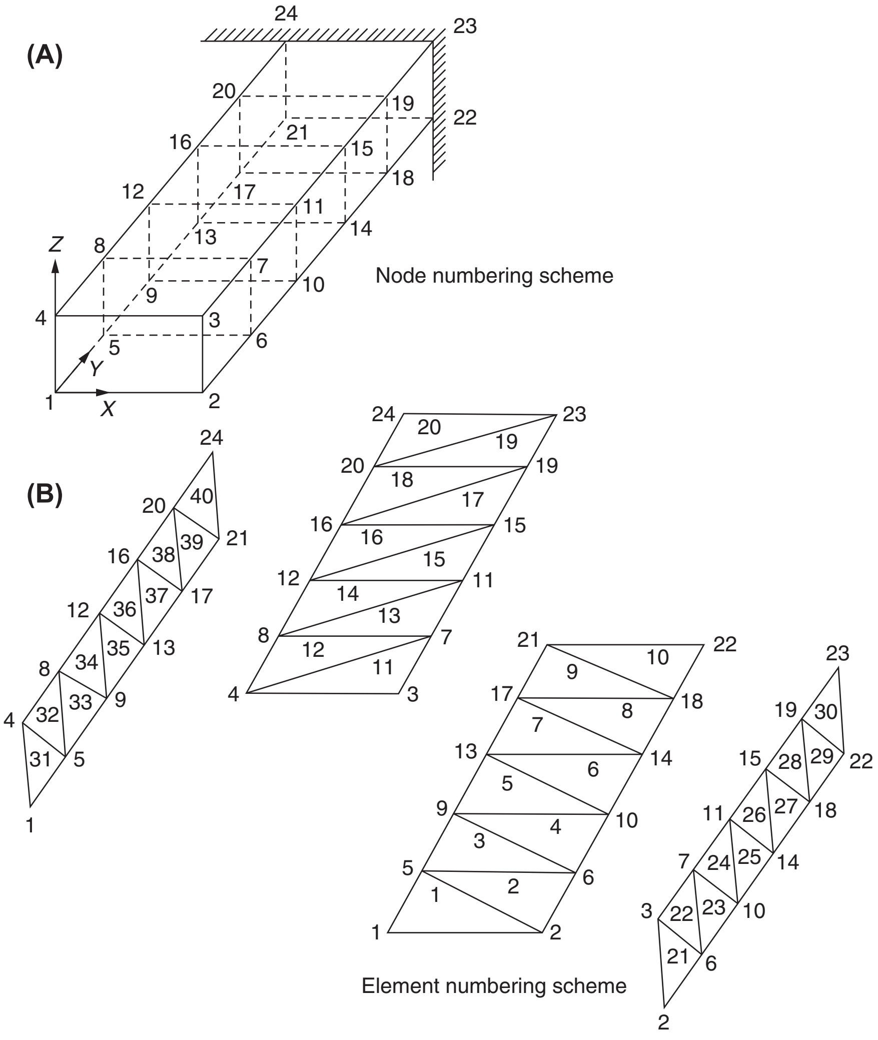


FIGURE 10.10 Finite element idealization of the box beam.

TABLE 10.4 Tip Deflection of Box Beam in Direction of Load

Load Condition (lb)	Finite Element Method (in.)	Simple Beam Theory
$P_1 = P_2 = 5000$	0.0195	0.0204 in
$P_1 = -P_2 = 5000$	0.0175	—

10.4 QUADRATIC TRIANGLE ELEMENT

A triangle element with a quadratic displacement model has six nodes—three at the vertices and three at the mid-points of the sides, as shown in Fig. 10.11. The displacement components of a point in the element parallel to the x and y axes are assumed as

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \tag{10.48}$$

$$v(x, y) = \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2 \tag{10.49}$$

where the constants or generalized degrees of freedom α_i, $i = 1, 2, \ldots, 12$ can be expressed in terms of the vector of nodal displacements of the element

$$\vec{U} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5 \quad u_6 \quad v_6\}^T \tag{10.50}$$

The strains in the element are given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \alpha_2 + 2\alpha_4 x + \alpha_5 y \tag{10.51}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \alpha_9 + \alpha_{11} x + \alpha_{12} y \tag{10.52}$$

$$\begin{aligned} \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_8 + 2\alpha_{10} x + \alpha_{11} y \\ &= (\alpha_3 + \alpha_8) + (\alpha_5 + 2\alpha_{10})x + (2\alpha_6 + \alpha_{11})x \end{aligned} \tag{10.53}$$

It can be seen, from Eqs. (10.51) to (10.53), that the strains vary linearly in the element. Hence, the element is called a linear strain triangle (LST).

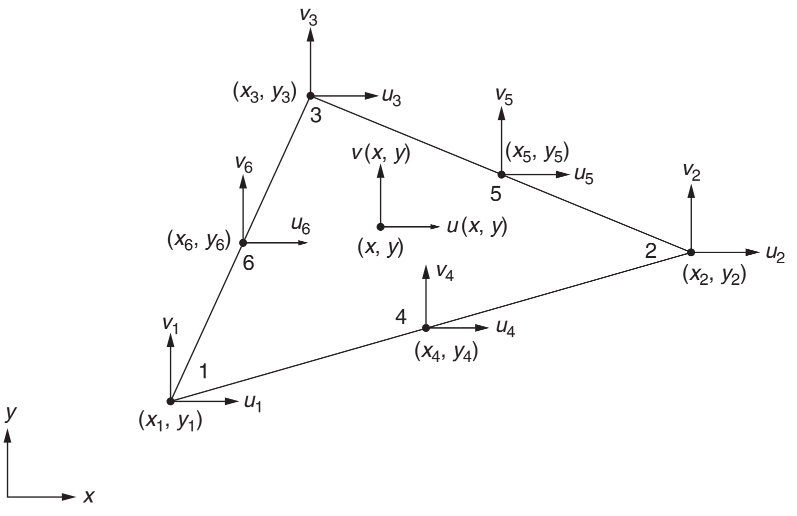


FIGURE 10.11 A quadratic triangular element.

Numerical Accuracy: To study the accuracy of the LST element, Felippa [10.2] considered a cantilever beam of constant thickness (rectangular cross section of the beam) subjected to an end load as shown in Fig. 10.3. The end load is parabolically distributed over the depth of the beam. Both CST and LST elements are used to model the beam. The results of the finite element analysis are shown in Table 10.1. It can be seen that the LST element yielded the values of displacement and axial stress that are more accurate, even with a smaller number of elements, than the CST element. This shows that the LST element is far superior than the CST element. The only limitation of the LST element, when applied to the beam analysis, is that the shear stress ε_{xy} varies linearly within the element while it varies quadratically (parabolically) in the y direction according to the simple beam theory.

10.5 RECTANGULAR PLATE ELEMENT (IN-PLANE FORCES)

Consider a rectangular plate undergoing in-plane displacements due to in-plane forces as shown in Fig. 10.12. The variations of displacements inside the element are assumed as

$$\begin{aligned} u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v(x,y) &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \end{aligned} \tag{10.54}$$

It can be seen that although the displacement distribution is represented by a second-degree surface, the displacement $u(x, y)$, for example, varies linearly along the x (or y) direction for any constant value of y (or x) as shown in Fig. 10.13. By using the nodal values, $u(x_1 = -a,\ y_1 = -b) = u_1$, $u(x_2 = a,\ y_2 = -b) = u_2, \ldots, v(x_4 = -a,\ y_4 = b) = v_4$, Eq. (10.54) can be expressed as

$$\vec{U}(x,y) = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N(x,y)]\vec{q} \tag{10.55}$$

where $\vec{q}$ is the vector of nodal displacements given by

$$\vec{q} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T \tag{10.56}$$

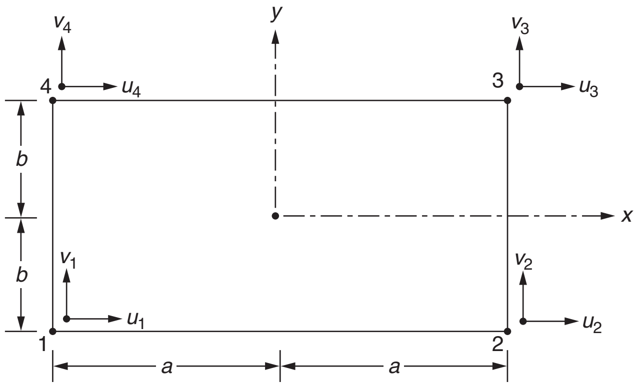


FIGURE 10.12 A rectangular element under in-plane loads.

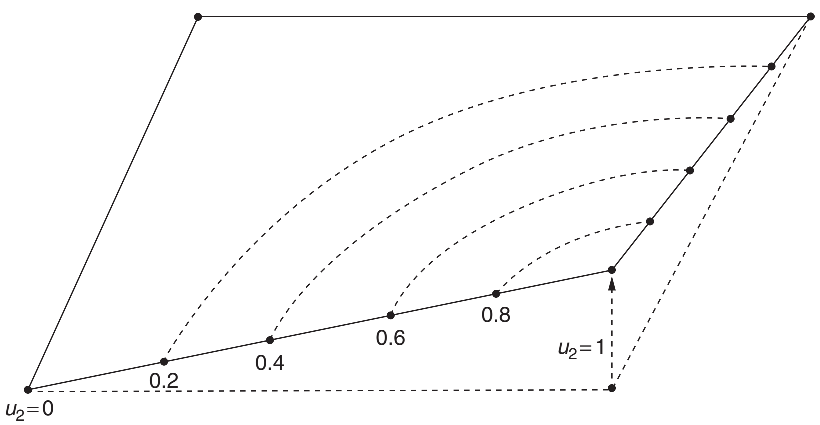


FIGURE 10.13 Distribution of $u(x, y)$ in a rectangular plate.

and $[N(x, y)]$ is the matrix of shape functions given by

$$[N(x,y)] = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 & N_4(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 & N_4(x,y) \end{bmatrix} \tag{10.57}$$

where

$$\begin{aligned} N_1(x,y) &= \frac{(a+x)(b+y)}{4ab}, \quad N_2(x,y) = \frac{(a+x)(b-y)}{4ab}, \\ N_3(x,y) &= \frac{(a+x)(b+y)}{4ab}, \quad N_4(x,y) = \frac{(a-x)(b+y)}{4ab} \end{aligned} \tag{10.58}$$

The local element stiffness matrix, $[k]$, can be generated using Eq. (8.87) as

$$[k] = \int_{-a}^{a} \int_{-b}^{b} [B]^T [D][B] t \, dx \, dy \tag{10.59}$$

where t is the thickness of the plate, $[D]$ is the elasticity matrix given by Eq. (10.15), and $[B]$ is the matrix relating the strains to the nodal displacements that can be obtained by differentiating $u(x, y)$ and $v(x, y)$ of Eq. (10.54) as indicated in Eq. (10.10). The global stiffness matrix of the element $\left[K^{(e)}\right]$ in three-dimensional space can be generated as

$$\left[K^{(e)}\right] = [\lambda]^T [k][\lambda] \tag{10.60}$$

where the coordinate transformation matrix $[\lambda]$ of size 8×12 is given by (see Fig. 10.14)

$$[\lambda] = \begin{bmatrix} [\lambda_{xy}] & [0] & [0] & [0] \\ [0] & [\lambda_{xy}] & [0] & [0] \\ [0] & [0] & [\lambda_{xy}] & [0] \\ [0] & [0] & [0] & [\lambda_{xy}] \end{bmatrix} \tag{10.61}$$

$$[\lambda_{xy}] = \begin{bmatrix} l_{pq} & m_{pq} & n_{pq} \\ l_{ps} & m_{ps} & n_{ps} \end{bmatrix} \tag{10.62}$$

and $[0]$ is a zero matrix of size 2×3. In Eq. (10.62), l_{pq}, m_{pq}, n_{pq} and l_{ps}, m_{ps}, n_{ps} denote, respectively, the direction cosines of the lines pq (x axis) and ps (y axis).

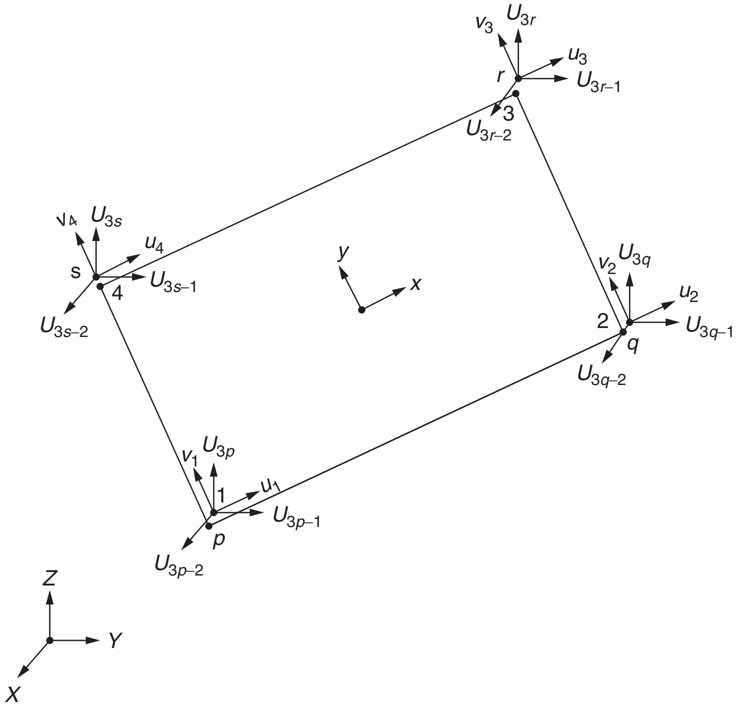


FIGURE 10.14 Rectangular plate element (in-plane forces).

Notes

1. The rectangular element described in this section is called the Q_4 element. It can be observed from the assumed displacement field that the strain ε_{xx} is constant in the x direction and varies linearly in the y direction. Similarly, the strain ε_{yy} is constant in the y direction, but varies linearly in the x direction. On the other hand, the shear strain ε_{xy} varies linearly in both x and y directions.
2. If the element undergoes a constant temperature change, T, the stresses in the element are given by

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [B]\vec{q} - \frac{E\alpha T}{1-\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{10.63}$$

Eq. (10.63) implies that each stress component varies linearly with x and y, which, in general, violates the stress equilibrium equations within the element.

3. This element, as in the case of CST element, cannot exhibit pure bending. In problems involving bending, this Q_4 element displays not only the expected bending (normal) strain, but also spurious shear strain. Thus, the element exhibits *shear locking* behavior under bending deformation.

10.6 BENDING BEHAVIOR OF PLATES

The following assumptions are made in the classical theory of thin plates [10.3]:

1. The thickness of the plate is small compared to its other dimensions.
2. The deflections are small.
3. The middle plane of the plate does not undergo in-plane deformation.
4. The transverse shear deformation is zero.

The stresses induced in an element of a flat plate subjected to bending forces (transverse load and bending moments) are shown in Fig. 10.15A. These stresses are shear stresses σ_{yz}, σ_{xz}, and σ_{xy}, and normal stresses σ_{xx} and σ_{yy}. It can be noted that in beams, which can be considered one-dimensional analogs of plates, the shear stress σ_{xy} will not be present. As in beam theory, the stresses σ_{xx} (and σ_{yy}) and σ_{xz} (and σ_{yz}) are assumed to vary linearly and parabolically, respectively, over the thickness of the plate. The shear stress σ_{xy} is assumed to vary linearly. The stresses σ_{xx}, σ_{yy}, σ_{xy}, σ_{xz}, and σ_{yz} lead to the following force and moment resultants per unit length:

$$\left.\begin{aligned} M_x &= \int_{-t/2}^{t/2} \sigma_{xx} z\,dz, \quad M_y = \int_{-t/2}^{t/2} \sigma_{yy} z\,dz, \\ M_{xy} &= \int_{-t/2}^{t/2} \sigma_{xy} z\,dz, \quad Q_x = \int_{-t/2}^{t/2} \sigma_{xz}\,dz, \quad Q_y = \int_{-t/2}^{t/2} \sigma_{yz}\,dz \end{aligned}\right\} \tag{10.64}$$

These forces and moments are indicated in Fig. 10.15B. By considering an element of the plate, the differential equations of equilibrium in terms of force resultants can be derived. For this, we consider the bending moments and shear forces to be functions of x and y so that if M_x acts on one side of the element, $M'_x = M_x + \frac{\partial M_x}{\partial x} \cdot dx$ acts on the opposite side. The resulting equations can be written as

$$\left.\begin{aligned} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p &= 0 \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} &= Q_x \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} &= Q_y \end{aligned}\right\} \tag{10.65}$$

where p is the distributed surface load. Because the plate is thin in comparison to its length and width, any body force may be converted to an equivalent load p and hence no body force is considered separately in Eq. (10.65).

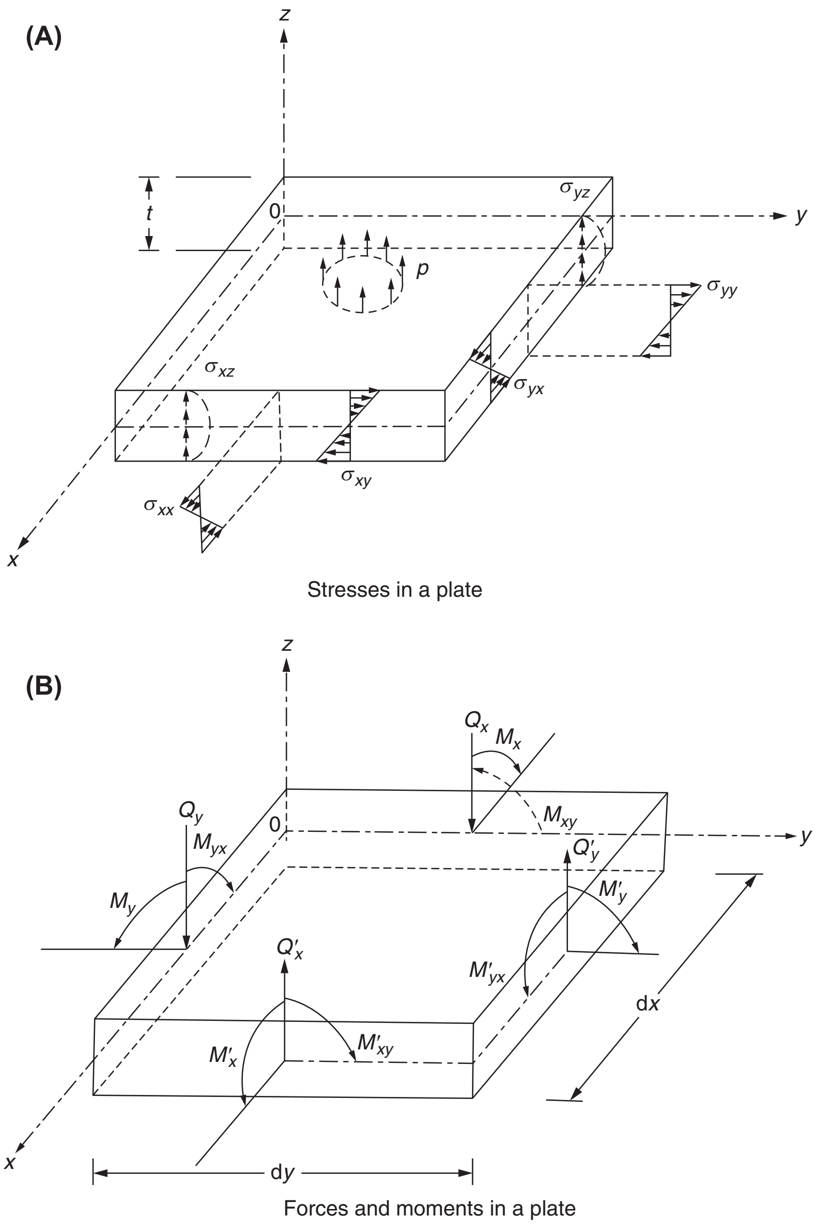


FIGURE 10.15 Plate subjected to bending loads.

To derive the strain–displacement relations for a plate, consider the bending deformation of a small element (by neglecting shear deformation). Any point A in this element experiences both transverse (w) and in-plane (u and v) displacements. The strains can be expressed as

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\ \varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \tag{10.66}$$

Eq. (10.66) show that the transverse displacement w, which is a function of x and y only, completely describes the deformation state.

The moment−displacement relations can also be derived for plates. For this, we assume the plate to be in a state of plane stress by considering the transverse stress σ_{zz} to be negligible in comparison to σ_{xx} and σ_{yy}. Thus, the stress−strain relations are given by (Eq. 8.35)

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [D]\vec{\varepsilon} = [D]\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{10.67}$$

where

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{10.68}$$

By substituting Eq. (10.66) into Eq. (10.67) and the resulting stresses into Eq. (10.64), we obtain the following after integration:

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ M_{yx} &= M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} = -\frac{Gt^3}{6}\cdot\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \tag{10.69}$$

where D is called the flexural rigidity of the plate and is given by

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{10.70}$$

The flexural rigidity D corresponds to the bending stiffness of a beam (EI). In fact, $D = EI$ for a plate of unit width when ν is taken as zero. Eqs. (10.65) and (10.69) give

$$\left.\begin{aligned} Q_x &= -D\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \\ Q_y &= -D\cdot\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \end{aligned}\right\} \tag{10.71}$$

The following boundary conditions have to be satisfied for plates (Fig. 10.16):

1. Simply supported edge (along $y =$ constant):

$$\left.\begin{aligned} w(x,y) &= 0 \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = 0 \end{aligned}\right\} \quad \text{for } y = \text{constant}, \text{ and } 0 \le x \le a \tag{10.72}$$

2. Clamped edge (along $y =$ constant):

$$\left.\begin{aligned} w(x,y) &= 0 \\ \frac{\partial w}{\partial y}(x,y) &= 0 \end{aligned}\right\} \quad \text{for } y = \text{constant}, \text{ and } 0 \le x \le a \tag{10.73}$$

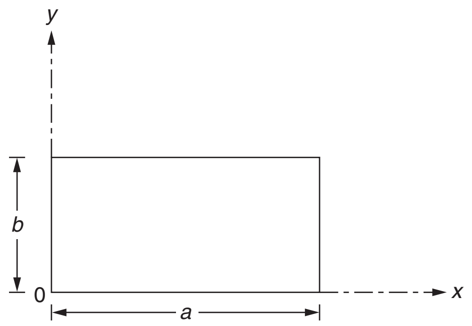


FIGURE 10.16 A rectangular plate.

3. Free edge (along $y = \text{constant}$):

$$\left.\begin{aligned} M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) = 0 \\ Q_y + \frac{\partial M_{yx}}{\partial x} &= \text{vertical shear} \\ &= -(2-v)D\frac{\partial^3 w}{\partial x^2 \partial y} - D\frac{\partial^3 w}{\partial y^3} = 0 \end{aligned}\right\} \text{for } y = \text{constant}, \ \leq x \leq a \tag{10.74}$$

In the classical theory of plates, first the displacement $w(x, y)$ is found by solving the equilibrium Eq. (10.65) under the prescribed loading condition $p(x, y)$. By substituting Eq. (10.71) into Eq. (10.65), we notice that the second and third equilibrium equations are automatically satisfied and the first one gives

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{10.75}$$

Thus, the problem is to solve the fourth-order partial differential Eq. (10.75) by using appropriate boundary conditions. Once $w(x,y)$ is found, the strains, stresses, and moments developed in the plate can be determined by using Eqs. (10.66), (10.67), and (10.69), respectively. Note that the closed-form solution of Eq. (10.75) cannot be obtained except for plates having simple configuration (e.g., rectangular and circular plates) and simple loading and boundary conditions. However, the finite element method can be used for analyzing problems involving plates of arbitrary plane form and loading conditions that may sometimes have cutouts or cracks.

10.7 FINITE ELEMENT ANALYSIS OF PLATES IN BENDING

A large number of plate bending elements have been developed and reported in the literature [10.4,10.5]. In the classical theory of thin plates discussed in this section, certain simplifying approximations are made. One of the important assumptions made is that shear deformation is negligible. Some elements have also been developed by including the effect of transverse shear deformation.

According to thin plate theory, the deformation is completely described by the transverse deflection of the middle surface of the plate (w) only. Thus, if a displacement model is assumed for w, the continuity of not only w but also its derivatives has to be maintained between adjacent elements. According to the convergence requirements stated in Section 3.6, the polynomial for w must be able to represent constant strain states. This means, from Eq. (10.66), that the assumed displacement model must contain constant curvature states ($\partial^2 w/\partial x^2$) and ($\partial^2 w/\partial y^2$) and constant twist ($\partial^2 w/\partial x \partial y$). Also, the polynomial for w should have geometric isotropy.

Thus, it becomes evident that choosing a displacement model to satisfy all these requirements is much more difficult. In surmounting these difficulties, especially for triangular and general quadrilateral elements, different investigators have developed different elements, some of them quite complicated. In the following section, a simple triangular plate bending element is described along with its characteristics.

10.8 TRIANGULAR PLATE BENDING ELEMENT

At each node of the triangular plate element shown in Fig. 10.17, the transverse displacement w and slopes (rotations) about the x and y axes ($(\partial w/\partial y)$ and $-(\partial w/\partial x)$) are taken as the degrees of freedom. The minus sign for the third degree of freedom indicates that if we take a positive displacement dw at a distance dx from node 1, the rotation (dw/dx) about the y axis at node 1 will be opposite to the direction of the degree of freedom q_3 indicated in Fig. 10.17. Since there are nine displacement degrees of freedom in the element, the assumed polynomial for $w\,(x, y)$ must also contain nine constant terms. To maintain geometric isotropy, the displacement model is taken as

$$\begin{aligned} w(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8\left(x^2 y + xy^2\right) + \alpha_9 y^3 \\ &= [\eta]\vec{\alpha} \end{aligned} \tag{10.76}$$

where

$$[\eta] = \left[1 \;\; x \;\; y \;\; x^2 \;\; xy \;\; y^2 \;\; x^3 \left(x^2 y + xy^2\right) \;\; y^3\right] \tag{10.77}$$

and

$$\vec{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_9 \end{Bmatrix} \tag{10.78}$$

The constants $\alpha_1, \alpha_2, \ldots, \alpha_9$ have to be determined from the nodal conditions

$$\left.\begin{array}{llll} w(x,y) = q_1, & \dfrac{\partial w}{\partial y}(x,y) = q_2, & -\dfrac{\partial w}{\partial x}(x,y) = q_3 & \text{at} \quad (x_1, y_1) = (0,0) \\ w(x,y) = q_4, & \dfrac{\partial w}{\partial y}(x,y) = q_5, & -\dfrac{\partial w}{\partial x}(x,y) = q_6 & \text{at} \quad (x_2, y_2) = (0, y_2) \\ w(x,y) = q_7, & \dfrac{\partial w}{\partial y}(x,y) = q_8, & -\dfrac{\partial w}{\partial x}(x,y) = q_9 & \text{at} \quad (x_3, y_3) \end{array}\right\} \tag{10.79}$$

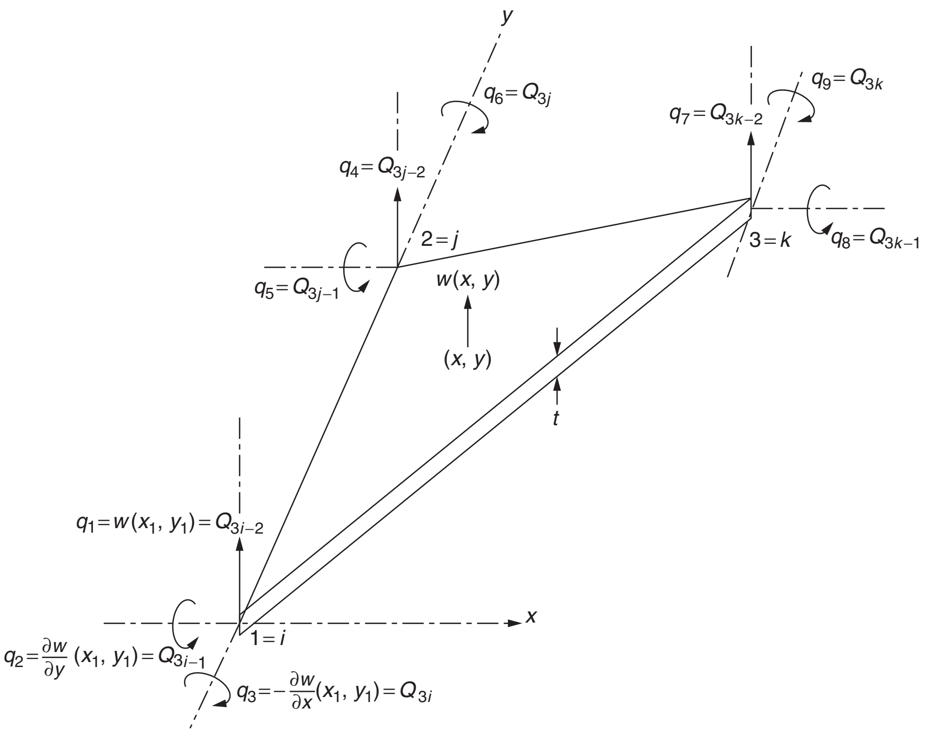


FIGURE 10.17 Nodal degrees of freedom of a triangular plate in bending.

Note that the local y axis is taken to be the same as the line connecting nodes 1 and 2 with the origin placed at node 1. The local x axis is taken toward node 3 as shown in Fig. 10.17. The local node numbers 1, 2, and 3 are assumed to correspond to nodes i, j, and k, respectively, in the global system. By using Eq. (10.76), Eq. (10.79) can be stated in matrix form as

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_9 \end{Bmatrix}^{(e)} = \left[\underset{\sim}{\eta}\right] \vec{\alpha} \tag{10.80}$$

where

$$\left[\underset{\sim}{\eta}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & y_2 & 0 & 0 & y_2^2 & 0 & 0 & y_2^3 \\ 0 & 0 & 1 & 0 & 0 & 2y_2 & 0 & 0 & 3y_2^2 \\ 0 & -1 & 0 & 0 & -y_2 & 0 & 0 & -y_2^2 & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 & x_3^3 & \left(x_3^2y_3 + x_3y_3^2\right) & y_3^3 \\ 0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & \left(2x_3y_3 + x_3^2\right) & 3y_3^2 \\ 0 & -1 & 0 & -2x_3 & -y_3 & 0 & -3x_3^2 & \left(-y_3^2 + 2x_3y_3\right) & 0 \end{bmatrix} \tag{10.81}$$

By using Eqs. (10.76) and (10.80), Eq. (10.66) can be expressed as

$$\vec{\varepsilon} = \left[\underset{\sim}{B}\right] \vec{\alpha} = [B]\vec{q}^{(e)} \tag{10.82}$$

where

$$\left[\underset{\sim}{B}\right] = -z \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix} \tag{10.83}$$

and

$$[B] = \left[\underset{\sim}{B}\right]\left[\underset{\sim}{\eta}\right]^{-1} \tag{10.84}$$

Finally, the element stiffness matrix in the local (xy) coordinate system can be derived as

$$\left[k^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V \tag{10.85}$$

where $V^{(e)}$ indicates the volume of the element, and the matrix $[D]$ is given by Eq. (10.68). By substituting for $[B]$ from Eq. (10.84), Eq. (10.85) can be expressed as

$$\left[k^{(e)}\right] = \left(\left[\underset{\sim}{\eta}\right]^{-1}\right)^T \left\{\iint_{\text{area}} \mathrm{d}A \left(\int_{-t/2}^{t/2} \left[\underset{\sim}{B}\right]^T [D] \left[\underset{\sim}{B}\right] \mathrm{d}z\right)\right\} \left[\underset{\sim}{\eta}\right]^{-1} \tag{10.86}$$

where t denotes the thickness of the plate. The integrals within the curved brackets of Eq. (10.86) can be rewritten as

$$\iint_{\text{area}} \mathrm{d}A \int_{-t/2}^{t/2} [\underset{\sim}{B}]^T [D] [\underset{\sim}{B}] \mathrm{d}z = \frac{Et^3}{(12(1-\nu^2))} \iint_{\text{area}} \mathrm{d}x\mathrm{d}y$$

$$\times \begin{bmatrix} 0 & & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & \text{Symmetric} & & & \\ 0 & 0 & 0 & 4 & & & & & \\ 0 & 0 & 0 & 0 & 2(1-\nu) & & & & \\ 0 & 0 & 0 & 4\nu & 0 & 4 & & & \\ 0 & 0 & 0 & 12x & 0 & 12\nu x & 36x^2 & & \\ 0 & 0 & 0 & 4(\nu x+y) & 4(1-\nu)(x+y) & 4(x+\nu y) & 12x(\nu x+y) & \{(12-8\nu)(x+y)^2 - 8(1-\nu)xy\} & \\ 0 & 0 & 0 & 12\nu y & 0 & 12y & 36\nu xy & 12(x+\nu y)y & 36y^2 \end{bmatrix} \quad (10.87)$$

The area integrals appearing on the right-hand side of Eq. (10.87) can be evaluated in the general XY coordinate system as well as in the particular local xy system chosen in Fig. 10.17 using the following relations:

$$\iint_{\text{area}} \mathrm{d}x\,\mathrm{d}y = A = \frac{1}{2}x_3y_2 \quad (10.88)$$

$$\iint_{\text{area}} x\,\mathrm{d}x\,\mathrm{d}y = X_cA = \frac{1}{6}x_3^2y_2 \quad (10.89)$$

$$\iint_{\text{area}} y\,\mathrm{d}x\,\mathrm{d}y = Y_cA = \frac{1}{6}x_3y_2(y_2+y_3) \quad (10.90)$$

$$\begin{aligned}\iint_{\text{area}} x^2\mathrm{d}x\,\mathrm{d}y &= X_c^2A + \frac{A}{12}\left[(X_i-X_c)^2 + (X_j-X_c)^2 + (X_k-X_c)^2\right] \\ &= \frac{1}{12}x_3^3y_2\end{aligned} \quad (10.91)$$

$$\begin{aligned}\iint_{\text{area}} xy\,\mathrm{d}x\cdot\,\mathrm{d}y &= X_cY_cA + \frac{A}{12}[(X_i-X_c)(Y_i-Y_c) + (X_j-X_c)(Y_j-Y_c) + (X_k-X_c)(Y_k-Y_c)] \\ &= \frac{1}{24}x_3^2y_2(y_2+2y_3)\end{aligned} \quad (10.92)$$

$$\begin{aligned}\iint_{\text{area}} y^2\mathrm{d}x\cdot\,\mathrm{d}y &= Y_c^2A + \frac{A}{12}\left[(Y_i-Y_c)^2 + (Y_j-Y_c)^2 + (Y_k-Y_c)^2\right] \\ &= \frac{1}{12}x_3y_2\left(y_2^2 + y_2y_3 + y_3^2\right)\end{aligned} \quad (10.93)$$

where

$$X_c = (X_i + X_j + X_k)/3 \quad (10.94)$$

and

$$Y_c = (Y_i + Y_j + Y_k)/3 \quad (10.95)$$

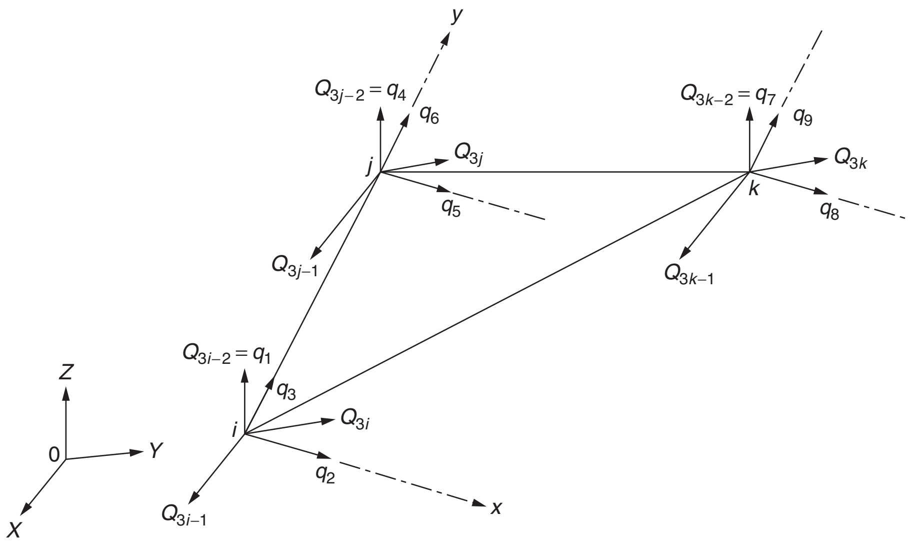


FIGURE 10.18 Local and global degrees of freedom.

It can be seen that the evaluation of the element stiffness matrix from Eqs. (10.86) and (10.87) involves the numerical determination of the inverse of the 9 × 9 matrix, $\left[\underset{\sim}{\eta}\right]$, for each element separately. Finally, the element stiffness matrix in the global coordinate system (whose XY plane is assumed to be the same as the local xy plane) can be obtained from

$$\left[K^{(e)}\right] = [\lambda]^T \left[k^{(e)}\right][\lambda] \tag{10.96}$$

where the transformation matrix $[\lambda]$ is given by

$$\underset{9\times 9}{[\lambda]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{0x} & m_{0x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{0y} & m_{0y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{0x} & m_{0x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{0y} & m_{0y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{0x} & m_{0x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_{0y} & m_{0y} \end{bmatrix} \tag{10.97}$$

where (l_{0x}, m_{0x}) and (l_{0y}, m_{0y}) represent the direction cosines of the lines $0x$ and $0y$, respectively (Fig. 10.18).

10.9 NUMERICAL RESULTS WITH BENDING ELEMENTS

The triangular plate bending element considered in Section 10.8 is one of the simplest elements. Several other elements were developed for the analysis of plates. Since the strains developed in a plate under bending involve second derivatives of the transverse displacement w, the expression for w must contain a complete second-degree polynomial in x and y. Furthermore, the interelement compatibility requires the continuity of w as well as of the normal derivative $(\partial w/\partial n)$ across the boundaries of two elements.

For a rectangular element (Fig. 10.16), the simplest thing to do is to take the values of w, $(\partial w/\partial x)$, and $(\partial w/\partial y)$ at each of the four corners as nodal degrees of freedom. This gives a total of 12 degrees of freedom for the element. Thus, the polynomial for w must also contain 12 constants α_i. Since a complete polynomial of degree three in x and y contains 10 terms, we need to include two additional terms. These terms can be selected arbitrarily, but we should preserve the symmetry of the expansion to ensure geometric isotropy. Thus, we have three possibilities, namely to take x^3y and xy^3, x^3y^2

and x^2y^3, or x^2y^2 and x^3y^3 in the expression of w. All these choices satisfy the condition that along any edge of the element w varies as a cubic. This can be verified by setting $x = 0$ or a (or $y = 0$ or b) in the expression of w. Since there are four nodal unknowns for any edge (e.g., along the edge $x = 0$, we have w and $(\partial w/\partial y)$ at the two corners as degrees of freedom), w is uniquely specified along that edge. This satisfies the continuity condition of w across the boundaries. For the continuity of $(\partial w/\partial n)$, we need to have $(\partial w/\partial n)$ vary linearly on a side since it is specified only at the node points. Irrespective of what combination of 12 polynomial terms we choose for w, we cannot avoid ending up with a cubic variation for $(\partial w/\partial n)$ ($n = x$ for the sides defined by $x = 0$ and a and $n = y$ for the edges defined by $y = 0$ and b). Therefore, it is not possible to satisfy the interelement compatibility conditions (continuity of both w and $(\partial w/\partial n)$) with 12 degrees of freedom only. A similar reasoning will reveal that the triangular element considered in Section 10.8 is also nonconforming.

The displacement models of some of the plate bending elements available in the literature are given next.

10.9.1 Rectangular Elements

1. Nonconforming element due to Adini–Clough–Melosh (ACM):

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_3 y^3 + \alpha_9 x^2 y + \alpha_{10} xy^2 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \tag{10.98}$$

Degrees of freedom at each node: w, $(\partial w/\partial x)$, $(\partial w/\partial y)$ [10.6].

2. Conforming element due to Bogner–Fox–Schmit (BFS)-16:

$$\begin{aligned} w(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg[& H_{0i}^{(1)}(x)H_{0j}^{(1)}(y)w_{ij} + H_{1i}^{(1)}(x)H_{0j}^{(1)}(y)\left(\frac{\partial w}{\partial x}\right)_{ij} \\ & + H_{0i}^{(1)}(x)H_{1j}^{(1)}(y)\left(\frac{\partial w}{\partial y}\right)_{ij} + H_{1i}^{(1)}(x)H_{1j}^{(1)}(y)\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{ij}\Bigg] \end{aligned} \tag{10.99}$$

Degrees of freedom at each node: w_{ij}, $(\partial w/\partial x)_{ij}$, $(\partial w/\partial y)_{ij}$, $(\partial^2 w/\partial x\partial y)_{ij}$ (node numbering scheme shown in Fig. 4.16) [10.7].

3. More accurate conforming element due to BFS-24:

$$\begin{aligned} ptw(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2}\Bigg[& H_{0i}^{(2)}(x)H_{0j}^{(2)}(y)w_{ij} + H_{1i}^{(2)}(x)H_{0j}^{(2)}(y)\left(\frac{\partial w}{\partial x}\right)_{ij} \\ & + H_{0i}^{(2)}(x)H_{1j}^{(2)}(y)\left(\frac{\partial w}{\partial y}\right)_{ij} + H_{2i}^{(2)}(x)H_{0j}^{(2)}(y)\left(\frac{\partial^2 w}{\partial x^2}\right)_{ij} \\ & + H_{0i}^{(2)}(x)H_{2j}^{(2)}(y)\left(\frac{\partial^2 w}{\partial y^2}\right)_{ij} + H_{1i}^{(2)}(x)H_{1j}^{(2)}(y)\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{ij}\Bigg] \end{aligned} \tag{10.100}$$

Degrees of freedom at each node: w_{ij}, $(\partial w/\partial x)_{ij}$, $(\partial w/\partial y)_{ij}$, $\left(\frac{\partial^2 w}{\partial x^2}\right)_{ij}$, $\left(\frac{\partial^2 w}{\partial y^2}\right)_{ij}$, $\left(\frac{\partial^2 w}{\partial x \partial y}\right)_{ij}$

(node numbering scheme shown in Fig. 4.16) [10.7].

10.9.2 Triangular Elements

1. Nonconforming element due to Tocher (T-9):

$$w(x,y) = \text{same as Eq.}(10.76)$$

Degrees of freedom at each node: w, $\partial w/\partial x$, $\partial w/\partial y$ [10.8].

2. Nonconforming element due to Tocher (T-10):

$$w(y,x) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_3 y^3 + \alpha_9 x^2 y + x_{10} xy^2 \tag{10.101}$$

Degrees of freedom at each node: w, $\partial w/\partial x$, $\partial w/\partial y$. (The 10th constant was suppressed using the Ritz method [10.8].)

3. Nonconforming element due to Adini (A):

$$w(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 y^3 + \alpha_8 x^2 y + \alpha_9 xy^2 \tag{10.102}$$

(The uniform twist term xy was neglected.)
Degrees of freedom at each node: w, $\partial w/\partial x$, $\partial w/\partial y$ [10.9].

4. Conforming element due to Cowper et al. (C) [10.10]:

$$\begin{aligned} w(x,y) = {} & \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_8 y^3 + \alpha_9 x^2 y \\ & + \alpha_{10} xy^2 + \alpha_{11} x^4 + \alpha_{12} y^4 + \alpha_{13} x^3 y + \alpha_{14} xy^3 + \alpha_{15} x^2 y^2 \\ & + \alpha_{16} x^5 + \alpha_{17} y^5 + \alpha_{18} x^4 y + \alpha_{19} xy^4 + \alpha_{20} x^3 y^2 + \alpha_{21} x^2 y^3 \end{aligned} \tag{10.103}$$

(Three constraints are imposed to reduce the number of unknowns from 21 to 18. These determine that the normal slope $\partial w/\partial n$ along any edge must have a cubic variation.) Degrees of freedom at each node: w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, $\partial^2 w/\partial x \partial y$ [10.10].

10.9.3 Numerical Results

Typical numerical results obtained for a clamped square plate subjected to uniformly distributed load with nonconforming and conforming bending elements are shown in Fig. 10.19 and Table 10.5, respectively. The finite element idealizations considered are shown in Fig. 10.20. Due to symmetry of geometry and load condition, only a quarter of the plate is considered for analysis. Of course, the symmetry conditions have to be imposed before solving the problem. For example, if the quarter plate 1, 2, 3, 4 shown in Fig. 10.20 is to be analyzed, $\partial w/\partial x$ has to be set equal to zero along line 2, 4, and $\partial w/\partial y$

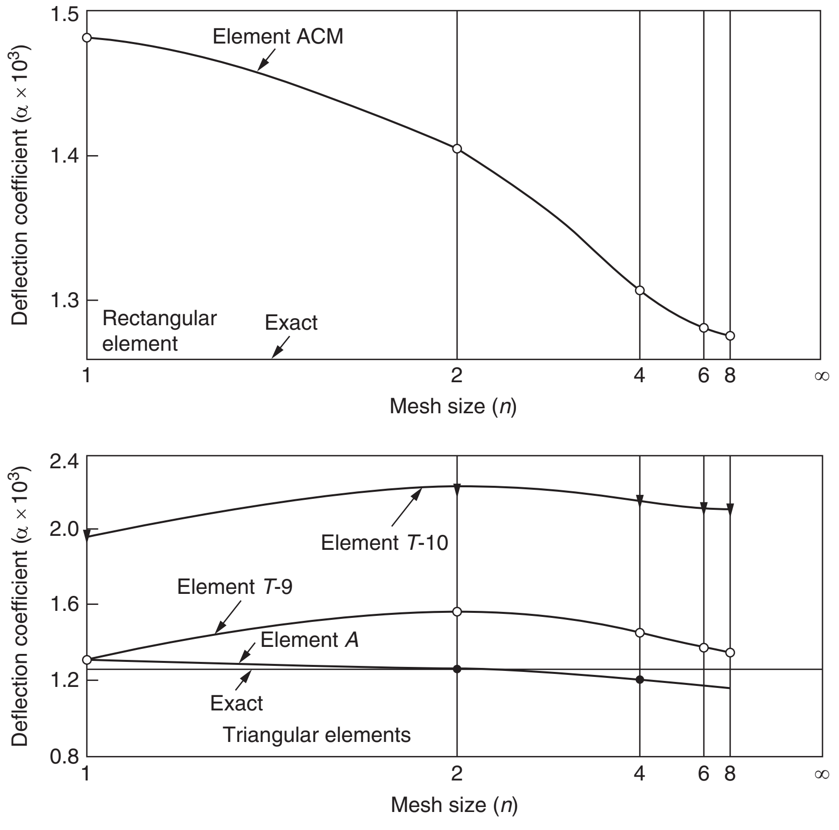


FIGURE 10.19 Central deflection of a clamped plate under uniformly distributed load.

TABLE 10.5 Central Deflection of a Square Clamped Plate Under Uniformly Distributed Load ($\nu = 0.3$)

Results Given by the Triangular Element in Cowper et al. [10.10]:

Idealization (Fig. 10.20)	Number of dof for One-Quarter Plate	Value of w_c (10^3 D/qa^4)
n = 1	5	1.14850
n = 2	21	1.26431
n = 3 (not shown in Fig. 10.20)	49	1.26530
Exact [10.3]	—	1.26

Results Given by the Rectangular Elements in Bogner et al. [10.7]:

	16 dof Element (BFS-16)		24 dof Element (BFS-24)	
Number of Elements in a Quadrant	Number of dof	Value of w_c*	Number of dof	Value of w_c*
1	1	0.042393″	5	0.0405″
4 (2 × 2 grid)	9	0.040475″	21	0.0402″
9 (3 × 3 grid)	25	0.040482″	—	—
16 (4 × 4 grid)	49	0.040487″	—	—
Exact [10.3]	—	0.0403″	—	0.0403″

dof, degrees of freedom.
*For $a = 20''$, $q = 0.2$ psi, $E = 10.92 \times 10^6$ psi, $t = 0.1''$.

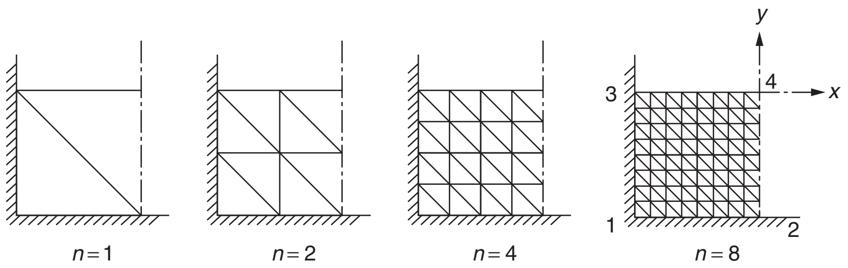


FIGURE 10.20 Typical finite element idealizations considered in analysis of square plate.

has to be set equal to zero along line 3, 4. The deflection of the center of the clamped plate (w_c) is taken as the measure of the quality of the approximation, and the deflection coefficient α of Fig. 10.19 is defined by

$$w_c = \frac{\alpha q a^4}{D}$$

where q denotes the intensity of the uniformly distributed load, a is the side of the plate, and D is the flexural rigidity. An important conclusion that can be drawn from the results of Fig. 10.19 is that monotonic convergence of deflection cannot always be expected from any of the nonconforming elements considered.

10.10 ANALYSIS OF THREE-DIMENSIONAL STRUCTURES USING PLATE ELEMENTS

If three-dimensional structures under arbitrary load conditions are to be analyzed using plate elements, we have to provide both in-plane and bending load-carrying capacity for the elements. The procedure to be adopted will be illustrated with reference to a triangular element. If a linear displacement field is assumed under in-plane loads (as in Eq. 10.1), the resulting 6×6 in-plane stiffness matrix (in local coordinate system) can be expressed as

$$\underset{6\times 6}{\left[k^{(e)}\right]_m} = \begin{bmatrix} \underset{2\times 2}{[k_{11}]_m} & \underset{2\times 2}{[k_{12}]_m} & \underset{2\times 2}{[k_{13}]_m} \\ \underset{2\times 2}{[k_{21}]_m} & \underset{2\times 2}{[k_{22}]_m} & \underset{2\times 2}{[k_{23}]_m} \\ \underset{2\times 2}{[k_{31}]_m} & \underset{2\times 2}{[k_{32}]_m} & \underset{2\times 2}{[k_{33}]_m} \end{bmatrix} \tag{10.104}$$

where the submatrices $[k_{ij}]_m$ correspond to the stiffness coefficients associated with nodes i and j, and the subscript m is used to indicate membrane action. In this case, the relationship between the nodal displacements and nodal forces can be written as

$$\begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix} = \left[k^{(e)}\right]_m \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{10.105}$$

where u_i and v_i denote the components of displacement of node $i(i = 1, 2, 3)$ parallel to the local x and y axes, respectively. Similarly, P_{xi} and P_{yi} indicate the components of force at node $i(i = 1, 2, 3)$ parallel to the x and y axes, respectively.

Similarly, the relation between the forces and displacements corresponding to the bending of the plate (obtained from Eq. 10.76) can be written as

$$\begin{Bmatrix} P_{z1} \\ M_{y1} \\ -M_{x1} \\ p_{z2} \\ M_{y2} \\ -M_{x2} \\ P_{z3} \\ M_{y3} \\ -M_{x3} \end{Bmatrix} = \left[k^{(e)}\right]_b \begin{Bmatrix} w_1 \\ w_{y1} \\ -w_{x1} \\ w_2 \\ w_{y2} \\ -w_{x2} \\ w_3 \\ w_{y3} \\ -w_{x3} \end{Bmatrix} \tag{10.106}$$

where w_i and P_{zi} indicate the components of displacement and force parallel to the z axis at node i, M_{yi} and M_{xi} represent the generalized forces corresponding to the rotations (generalized displacements) $w_{yi}(\theta_{xi})$ and w_{xi} (θ_{yi}) at node i $(i = 1, 2, 3)$, respectively, and the subscript b has been used to denote the bending stiffness matrix. The 9×9 bending stiffness matrix (in the local coordinate system) can be written as

$$\left[k^{(e)}\right]_b = \begin{bmatrix} \underset{3\times 3}{[k_{11}]_b} & \underset{3\times 3}{[k_{12}]_b} & \underset{3\times 3}{[k_{13}]_b} \\ \underset{3\times 3}{[k_{21}]_b} & \underset{3\times 3}{[k_{22}]_b} & \underset{3\times 3}{[k_{23}]_b} \\ \underset{3\times 3}{[k_{31}]_b} & \underset{3\times 3}{[k_{32}]_b} & \underset{3\times 3}{[k_{33}]_b} \end{bmatrix} \tag{10.107}$$

In the analysis of three-dimensional structures, the in-plane and bending stiffnesses have to be combined in accordance with the following observations:

1. For small displacements, the in-plane (membrane) and bending stiffnesses are uncoupled.
2. The in-plane rotation θ_Z (rotation about the local z axis) is not necessary for a single element. However, θ_Z and its conjugate force M_z have to be considered in the analysis by including the appropriate number of zeroes to obtain the element stiffness matrix for the purpose of assembling several elements.

Therefore, to obtain the total element stiffness matrix $\left[K^{(e)}\right]$, the in-plane and bending stiffnesses are combined as shown next.

$$\underset{18\times18}{\left[k^{(e)}\right]} = \left[\begin{array}{cccccccccccccccccc}
\underset{2\times2}{[k_{11}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{12}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{13}]_m} & & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & & \underset{3\times3}{[k_{11}]_b} & & 0 & 0 & 0 & & \underset{3\times3}{[k_{12}]_b} & & 0 & 0 & 0 & & \underset{3\times3}{[k_{13}]_b} & & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\
\underset{2\times2}{[k_{21}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{22}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{23}]_m} & & 0 & 0 & 0 & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & & \underset{3\times3}{[k_{21}]_b} & & 0 & 0 & 0 & & \underset{3\times3}{[k_{22}]_b} & & 0 & 0 & 0 & & \underset{3\times3}{[k_{23}]_b} & & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\
\underset{2\times2}{[k_{31}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{32}]_m} & & 0 & 0 & 0 & 0 & \underset{2\times2}{[k_{33}]_m} & & 0 & 0 & 0 & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & & \underset{3\times3}{[k_{31}]_b} & & 0 & 0 & 0 & & \underset{3\times3}{[k_{32}]_b} & & 0 & 0 & 0 & & [k_{33}]_b & & 0 \\
0 & 0 & & & & 0 & 0 & 0 & & & & 0 & 0 & 0 & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{10.108}$$

The stiffness matrix given by Eq. (10.108) is with reference to the local xyz coordinate system shown in Fig. 10.21. In the analysis of three-dimensional structures in which different finite elements have different orientations, it is necessary to transform the local stiffness matrices to a common set of global coordinates. In this case, the global stiffness matrix of the element can be obtained as

$$\left[K^{(e)}\right] = [\lambda]^T\left[k^{(e)}\right][\lambda] \tag{10.109}$$

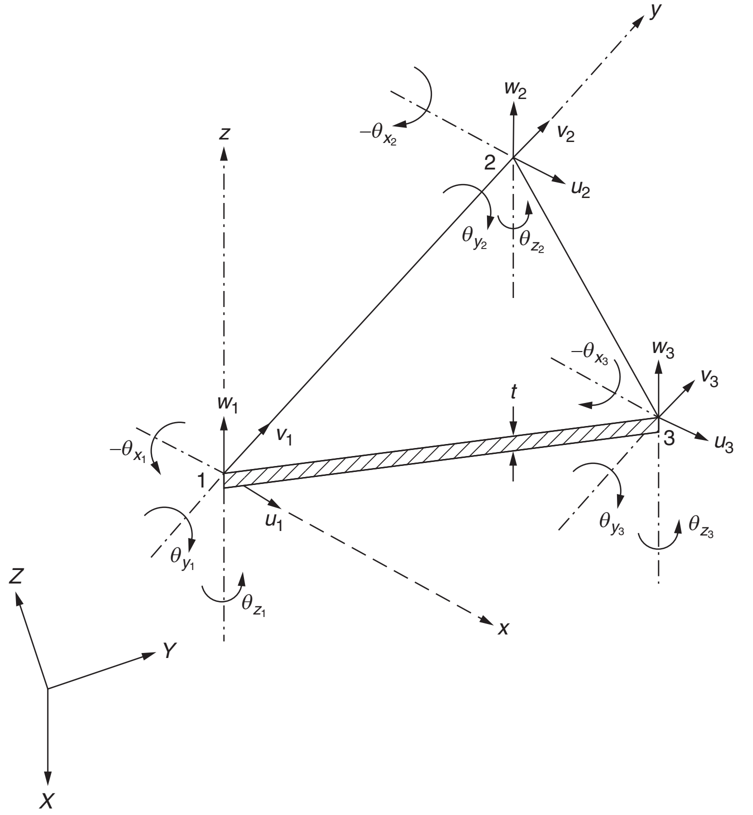


FIGURE 10.21 In-plane and bending displacements in a local xyz coordinate system.

where the transformation matrix, $[\lambda]$, is given by

$$\underset{18\times 18}{[\lambda]} = \begin{bmatrix} [\underset{\sim}{\lambda}] & [0] & [0] \\ [0] & [\underset{\sim}{\lambda}] & [0] \\ [0] & [0] & [\underset{\sim}{\lambda}] \end{bmatrix} \tag{10.110}$$

and

$$\underset{6\times 6}{[\underset{\sim}{\lambda}]} = \begin{bmatrix} l_{0x} & m_{0x} & n_{0x} & 0 & 0 & 0 \\ l_{0y} & m_{0y} & n_{0y} & 0 & 0 & 0 \\ l_{0z} & m_{0z} & n_{0z} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{0x} & m_{0x} & n_{0x} \\ 0 & 0 & 0 & l_{0y} & m_{0y} & n_{0y} \\ 0 & 0 & 0 & l_{0z} & m_{0z} & n_{0z} \end{bmatrix} \tag{10.111}$$

Here, (l_{0x}, m_{0x}, n_{0x}), for example, denotes the set of direction cosines of the x axis, and [0] represents a null square matrix of order six.

REVIEW QUESTIONS

10.1. Give brief answers to the following questions.

1. What is the difference between a membrane and a plate?
2. Why is the transverse displacement uncoupled from the in-plane displacements for a plate?
3. What is a CST element?
4. What is a quadratic triangular membrane element?
5. What is an LST element?
6. What is the order of the coordinate transformation matrix of a rectangular plate element under in-plane forces lying in three-dimensional space?
7. What is a conforming element?

10.2 Fill in the blank space with a suitable word.

1. For plates used in three-dimensional applications such as folded plates, both ____________ and bending actions are to be considered.
2. The triangular membrane element assumes a ____________ model for each of the in-plane displacement components.
3. The order of the coordinate transformation matrix of a triangular membrane element is ____________.
4. The number of degrees of freedom of the simplest triangular bending element is ____________.
5. The order of the coordinate transformation matrix of the simple triangular bending element in its own plane is ____________

10.3 Indicate whether each of the following statements is true or false.

1. A triangular membrane element has nine degrees of freedom in the local xy-coordinate system.
2. For a constant thickness triangular membrane element, the computation of the element stiffness matrix does not need any integration.
3. The displacement model of a CST element ensures displacement continuity along element boundaries.
4. The stresses vary linearly in a triangular membrane element.
5. The triangular membrane element can be used to find the stresses in a gear tooth.

10.4 Select the most appropriate answer among the multiple choices given.

1. The state of stress in a planar membrane element is:
   (a) plane strain (b) plane stress (c) three-dimensional
2. The order of the coordinate transformation matrix within the plane of the triangular membrane element is:
   (a) 2×6 (b) 5×6 (c) 5×9

3. The number of nodal degrees of freedom for a quadratic triangular element is:
(a) 6 (b) 9 (c) 12
4. The nature of the displacement model for a rectangular plate under in-plane forces is:
(a) linear (b) quadratic (c) cubic
5. The size of a triangular element that considers both the in-plane and bending displacements in three-dimensional space is:
(a) 18×18 (b) 9×9 (c) 12×12

10.5 Match the elements given with their corresponding displacement models.

1. CST element	(a) Triangular element with cubic displacement model
2. LST element	(b) Triangular element with linear displacement model
3. Q4 element	(c) Triangular element with quadratic displacement model
4. Simple bending element	(d) Rectangular element with quadratic displacement model

PROBLEMS

10.1 Find the stresses in the plate shown in Fig. 10.22 using one triangular membrane element.

10.2 Find the stresses in the plate shown in Fig. 10.23 using two triangular membrane elements.

10.3 Find the coordinate transformation matrix for the triangular membrane element shown in Fig. 10.24.

10.4 The plate shown in Fig. 10.22 is heated by 50 °C. Determine the load vector. Assume the coefficient of expansion of the material as $\alpha = 12 \times 10^{-6}$ per °C.

10.5 The nodal coordinates and the nodal displacements of a triangular element, under a specific load condition, are:
$X_i = 0$, $Y_i = 0$, $X_j = 1$ in, $Y_j = 3$ in, $X_k = 4$ in, $Y_k = 1$ in
$Q_{2i-1} = 0.001$ in, $Q_{2i} = 0.0005$ in, $Q_{2j-i} = -0.0005$ in, $Q_{2j} = 0.0015$ in
$Q_{2k-1} = 0.002$ in, $Q_{2k} = -0.001$ in
If $E = 30 \times 10^6$ psi and $\nu = 0.3$, find the stresses in the element.

10.6 For a triangular element in a state of plane stress, it is proposed to consider three corner and three mid-side nodes. Suggest a suitable displacement model and discuss its convergence and other properties.

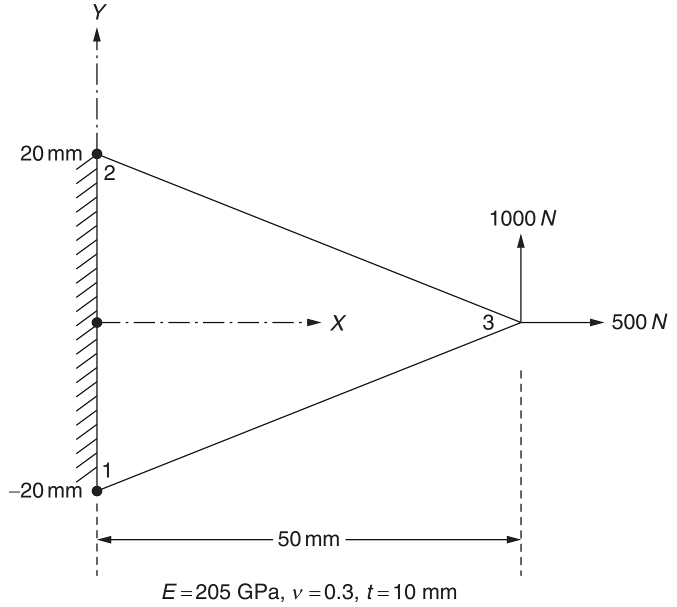
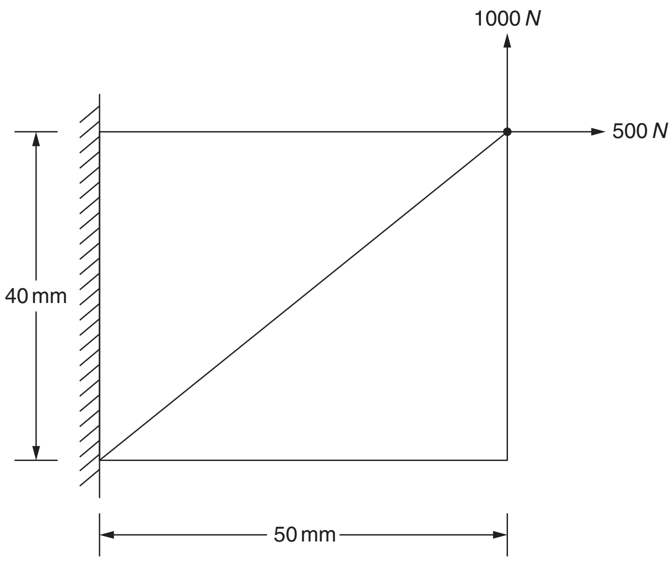


FIGURE 10.22 Triangular plate.



$E = 205$ GPa, $\nu = 0.3$, $t = 10$ mm

FIGURE 10.23 Rectangular plate.

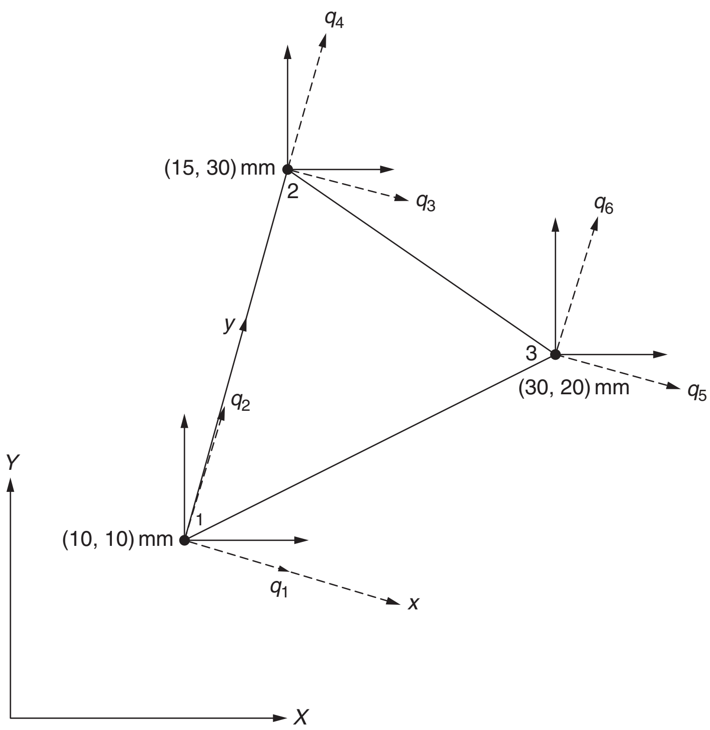


FIGURE 10.24 Membrane element in two dimensions.

10.7 Explain why the sum of coefficients of the stiffness matrix in any row for triangular plates with only in-plane loads is equal to zero; that is, $\Sigma_j k_{ij} = 0$ for any row i.

10.8 Consider two rectangular plate elements joined as shown in Fig. 10.25. If both in-plane and bending actions are considered, what conditions do you impose on the nodal displacements of the two elements if the edge AB is (a) hinged and (b) welded?

10.9 A triangular plate is subjected to a transverse load of 1000 N as shown in Fig. 10.26. Find the transverse displacement and the stresses induced in the plate using a one-element idealization. Assume $E = 205$ GPa, $\nu = 0.33$, and $t = 10$ mm.

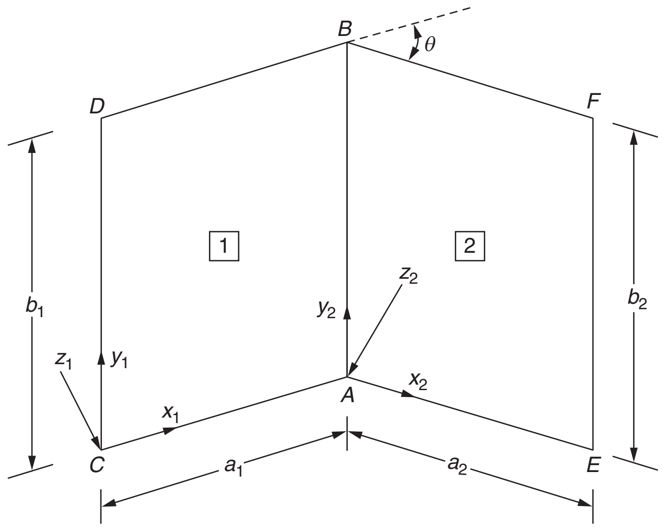


FIGURE 10.25 Two plates joined at an angle.

10.10 A rectangular plate, simply supported on all the edges, is subjected to a distributed transverse load of

$$p(x,y) = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

where a and b are the dimensions of the plate (Fig. 10.27).

a. Verify that the displacement solution

$$w(x,y) = c \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{P.1}$$

where

$$c = -\frac{p_0}{\pi^4 D\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^2}$$

satisfies the equilibrium equation and the boundary conditions.

b. Using the solution of Eq. (P.1), find expressions for the moments and reactions in the plate.

10.11 A gear tooth is shown in Fig. 10.28. Show a suitable finite element mesh of this gear tooth using approximately 50 CST elements. Label the nodes to minimize the bandwidth of the resulting stiffness matrix of the assembly (system).

10.12 For the rectangular plane stress element described in Section 10.5, show that the strains can be expressed as

$$\vec{\varepsilon} = [B]\vec{U} \tag{P.2}$$

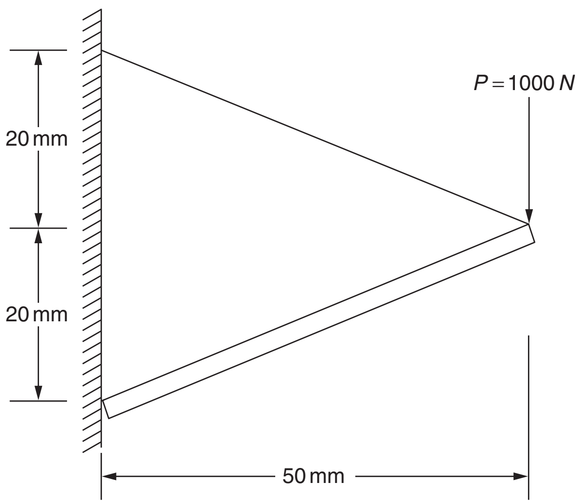


FIGURE 10.26 Triangular plate.

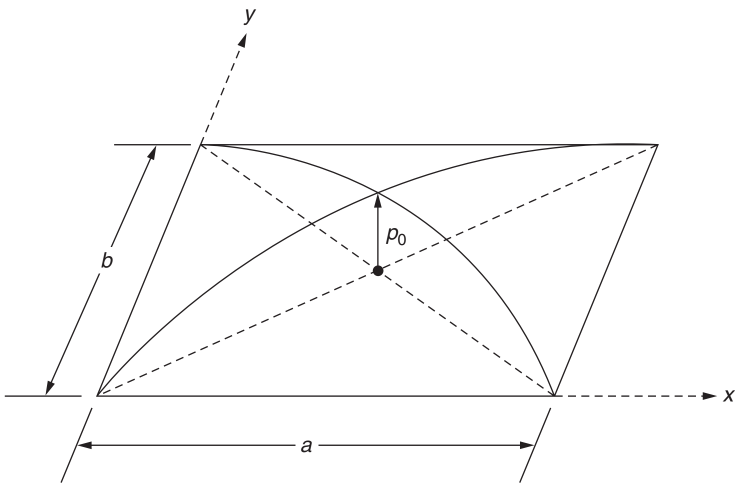
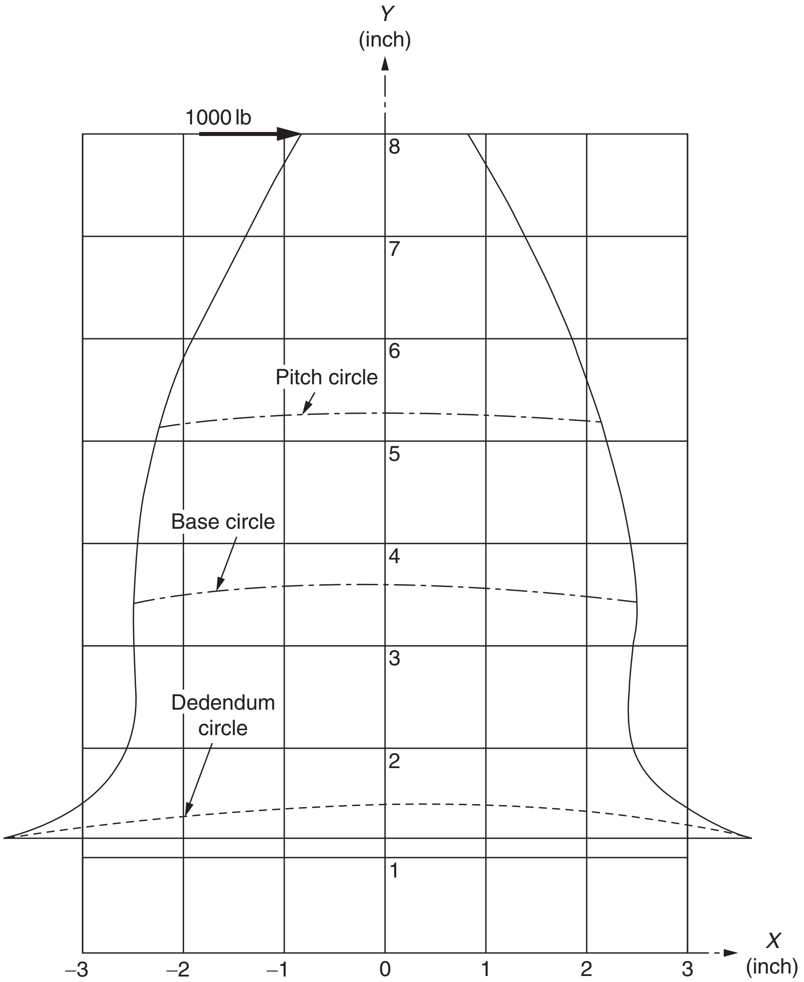


FIGURE 10.27 Rectangular plate under distributed load.



Material: Steel, $E = 30 \times 10^6$ psi, $\nu = 0.3$, face width = 1 in

FIGURE 10.28 Gear tooth.

where

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

and

$$[B] = \begin{bmatrix} -\frac{1}{a}\left(1-\frac{y}{b}\right) & 0 & \frac{1}{a}\left(1-\frac{y}{b}\right) & 0 & \frac{y}{a\,b} & 0 & -\frac{y}{a\,b} & 0 \\ 0 & -\frac{1}{b}\left(1-\frac{x}{a}\right) & 0 & -\frac{x}{ab} & 0 & \frac{x}{a\,b} & 0 & \frac{1}{b}\left(1-\frac{x}{a}\right) \\ -\frac{1}{b}\left(1-\frac{x}{a}\right) & -\frac{1}{a}\left(1-\frac{y}{b}\right) & -\frac{x}{ab} & \frac{1}{a}\left(1-\frac{y}{b}\right) & \frac{x}{a\,b} & \frac{y}{a\,b} & \frac{1}{b}\left(1-\frac{x}{a}\right) & -\frac{y}{a\,b} \end{bmatrix} \tag{P.3}$$

10.13 The stiffness matrix of the rectangular plane stress element considered in Problem 10.12 can be derived as

$$\left[k^{(e)}\right] = t\iint_{A^{(e)}} [B]^T[D][B]\,\mathrm{d}A \tag{P.4}$$

where $A^{(e)}$ is the area of the rectangular element, $[B]$ is given by Eq. (P.3) of Problem 10.12, and $[D]$ is given by Eq. (10.15). Show that the elements of the 8×8 element stiffness matrix, k_{ij}, can be expressed as

$$k_{11} = k_{33} = k_{55} = k_{77} = c\left(\frac{b}{3a} + \frac{1-v}{6}\frac{a}{b}\right)$$

$$k_{12} = -k_{34} = -k_{25} = -k_{16} = k_{56} = k_{47} = k_{38} = k_{78} = c\left(\frac{v}{4} + \frac{1-v}{8}\right)$$

$$k_{22} = k_{44} = k_{66} = k_{88} = c\left(\frac{a}{3b} + \frac{1-v}{6}\frac{b}{a}\right)$$

$$k_{13} = k_{57} = c\left(-\frac{b}{3a} + \frac{1-v}{12}\frac{a}{b}\right)$$

$$k_{23} = -k_{14} = k_{45} = -k_{36} = -k_{27} = k_{67} = k_{18} = -k_{58} = -c\left(\frac{v}{4} - \frac{1-v}{8}\right)$$

$$k_{24} = k_{68} = c\left(\frac{a}{6b} - \frac{1-v}{6}\frac{b}{a}\right)$$

$$k_{35} = k_{17} = c\left(\frac{b}{6a} - \frac{1-v}{6}\frac{a}{b}\right)$$

$$k_{46} = k_{28} = c\left(-\frac{a}{3b} + \frac{1-v}{12}\frac{b}{a}\right)$$

$$c = \frac{Et}{1-v^2}$$

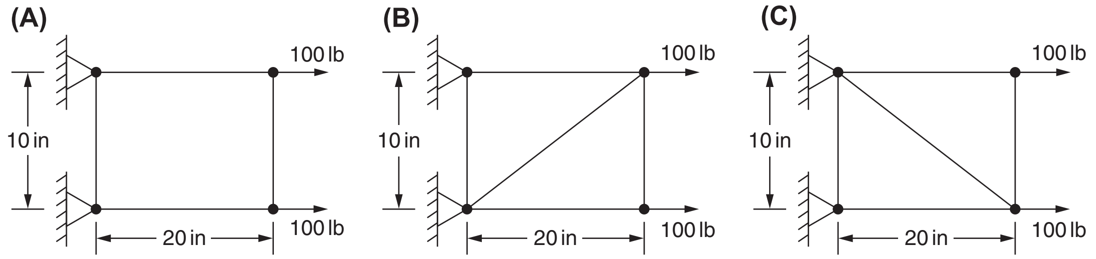


FIGURE 10.29 Rectangular plate.

10.14 Consider a rectangular steel plate subjected to in-plane loads as shown in Fig. 10.29A. Using a single rectangular plane stress element model (with the stiffness matrix given in Problem 10.13), find the stresses induced in the plate.

10.15 Find the stresses induced in the rectangular plate described in Problem 10.14 using two constant stress triangular elements (described in Section 10.2) as shown in Fig. 10.29B.

10.16 Find the stresses induced in the rectangular plate described in Problem 10.14 using two constant stress triangular elements (described in Section 10.2) as shown in Fig. 10.29C.

10.17 Consider a rectangular plate element under plane stress with nodal displacement degrees of freedom in the local (x, y) and global (X, Y, Z) coordinate systems as indicated in Fig. 10.30. Find the coordinate transformation matrix $[\lambda]$ that relates the two sets of degrees of freedom,

$$\underset{8\times 1}{\vec{u}} = \underset{8\times 12}{[\lambda]} \; \underset{12\times 1}{\vec{U}}$$

in terms of the global coordinates of the four nodes of the element.

10.18 Find the stresses induced in the steel plate shown in Fig. 10.31 using a single triangular membrane element.

10.19 Consider a rectangular element in plane stress with four nodes and eight displacement degrees of freedom as shown in Fig. 10.12. Instead of assuming a displacement variation model directly, the variations of stresses inside the element can be assumed as

$$\sigma_{xx} = \alpha_1 + \alpha_2 y, \quad \sigma_{yy} = \alpha_3 + \alpha_4 x, \quad \alpha_{xy} = \alpha_5$$

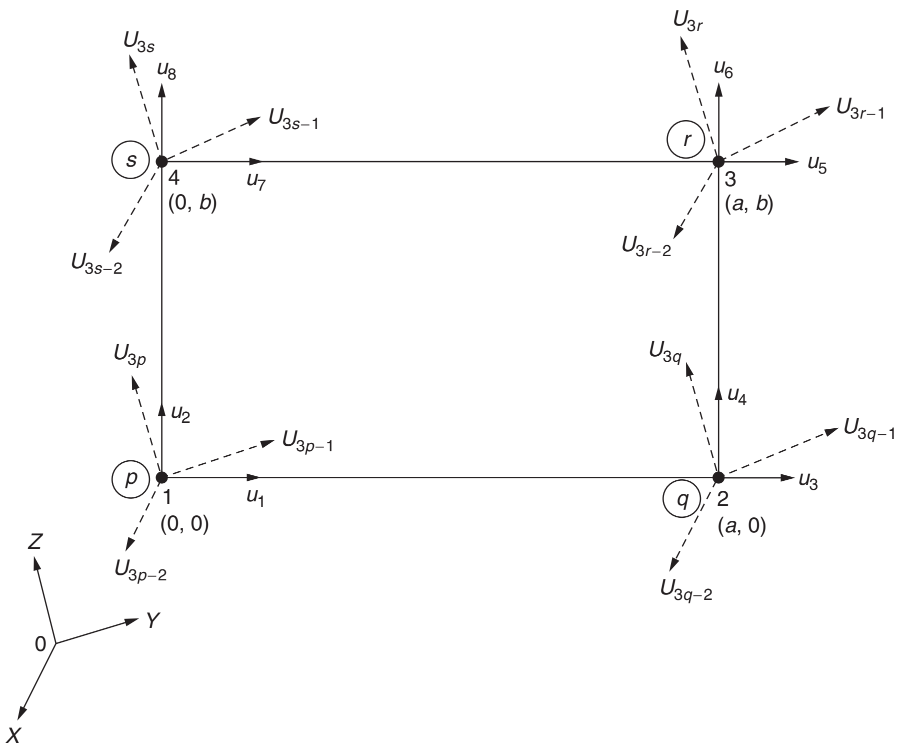


FIGURE 10.30 Rectangular plate element in global XYZ system.

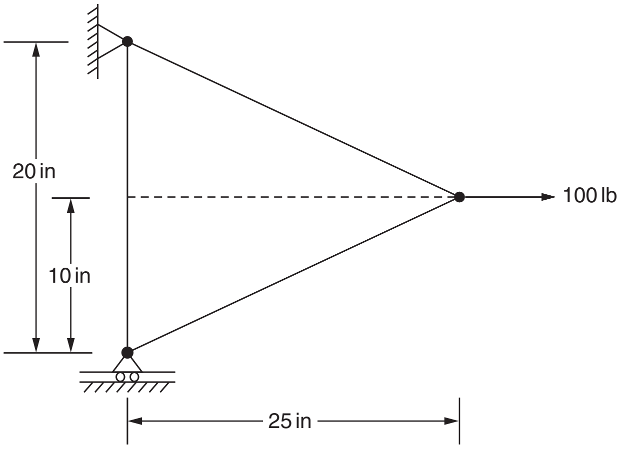


FIGURE 10.31 Triangular plate.

where α_i, $i = 1, 2, \ldots, 5$ are constants. Find the corresponding displacement distributions, $u(x, y)$ and $v(x, y)$, inside the element by integrating the strain−displacement relations.

10.20 The triangular membrane element shown in Fig. 10.32 has a thickness of $t = 0.2$ cm. The (x, y) coordinates of the nodes of the element are indicated next to the nodes in Fig. 10.32. The material of the element is aluminum with $E = 71.0$ GPa and $v = 0.33$. If the element is in a state of plane stress, determine the following:

a. Shape functions of the element, $N_i\,(x, y)$, $N_j(x, y)$, and $N_k(x, y)$
b. Matrix $[B]$ that relates the strains to the nodal displacements
c. Elasticity matrix $[D]$ for plane stress condition
d. Element stiffness matrix $\left[K^{(e)}\right]$

10.21 Assuming that the triangular membrane element described in Problem 10.20 is in a state of plane strain, determine the following:

a. Shape functions of the element, $N_i\,(x, y)$, $N_j(x, y)$, and $N_k(x, y)$
b. Matrix $[B]$ that relates the strains to the nodal displacements
c. Elasticity matrix $[D]$ for plane strain condition (see Eq. (8.24))
d. Element stiffness matrix $\left[K^{(e)}\right]$

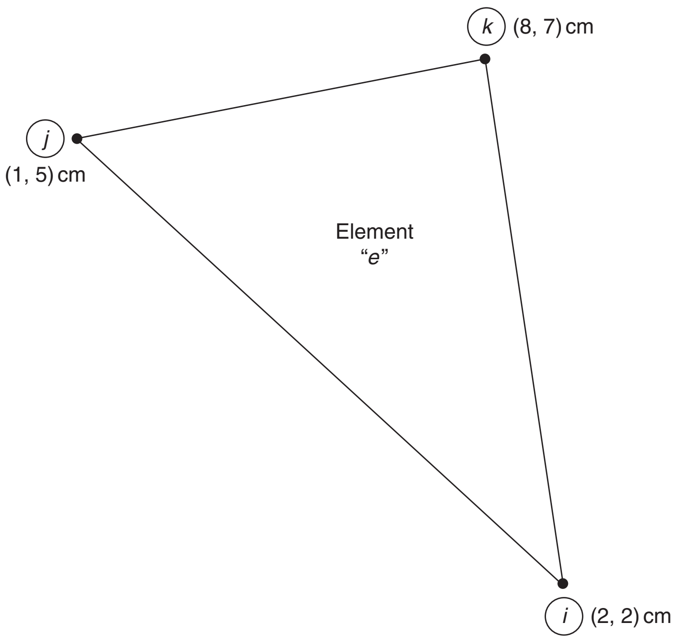


FIGURE 10.32 Triangular plate.

10.22 For the element considered in Problem 10.20, determine the element nodal force vector due to a temperature increase of 50 °C. Assume that the element is in a state of plane stress.

10.23 For the element considered in Problem 10.20, determine the element nodal force vector due to a temperature increase of 50 °C. Assume that the element is in a state of plane strain.

10.24 For the element described in Problem 10.20, determine the element nodal force vector as a result of the following prestress: $\sigma_{xx0} = 1000$ N/m^2, $\sigma_{yy0} = -1500$ N/m^2, and $\sigma_{xy0} = 500$ N/m^2. Assume the element to be in a state of plane stress.

10.25 For the element described in Problem 10.20, determine the element nodal force vector as a result of the following prestress: $\sigma_{xx0} = 1000$ N/m^2, $\sigma_{yy0} = -1500$ N/m^2, and $\sigma_{xy0} = 500$ N/m^2. Assume the element to be in a state of plane strain.

10.26 If the element described in Problem 10.20 is subjected to the distributed body forces $\phi_{x0} = 1000$ N/m^3 and $\phi_{y0} = 2000$ N/m^3, determine the corresponding nodal force vector of the element.

10.27 For the element described in Problem 10.20, uniform surface tractions with magnitudes $\Phi_{x0} = 500$ N/m^2 and $\Phi_{y0} = -1000$ N/m^2 act on the edge (face) *ij*. Determine the resulting nodal force vector of the element.

10.28 For the element described in Problem 10.20, uniform surface tractions with magnitudes $\Phi_{x0} = 500$ N/m^2 and $\Phi_{y0} = -1000$ N/m^2 act on the edge (face) *jk*. Determine the resulting nodal force vector of the element.

10.29 For the element described in Problem 10.20, uniform surface tractions with magnitudes $\Phi_{x0} = 500$ N/m^2 and $\Phi_{y0} = -1000$ N/m^2 act on the edge (face) *ki*. Determine the resulting nodal force vector of the element.

10.30 For the element considered in Problem 10.20, find the nodal force vector when a concentrated or point load $\vec{P}_0 = \begin{Bmatrix} P_{x0} \\ P_{y0} \end{Bmatrix} = \begin{Bmatrix} 1000 \\ -500 \end{Bmatrix}$ N acts at a point located at ($x_0 = 4$ cm, $y_0 = 4$ cm) in the element.

10.31 For the element considered in Problem 10.20, linearly varying surface tractions act on the edge (face) *ij*. The magnitude of Φ_{x0} varies linearly from $\Phi_{x0i} = 500$ Pa at node *i* to $\Phi_{x0j} = 1500$ Pa at node *j* while the magnitude of Φ_{y0} varies linearly from $\Phi_{y0i} = 500$ Pa at node *i* to $\Phi y_{0j} = 2000$ Pa at node *j* as shown in Fig. 10.33. Determine the corresponding nodal force vector of the element.
Hint: The magnitudes of the components of the surface tractions at any point on the edge *ij* can be expressed as $\Phi_{x0} = \Phi_{x0i} N_i + \Phi_{x0j} N_j$ and $\Phi_{y0} = \Phi_{y0i} N_i + \Phi_{y0j} N_j$.

10.32 For the element considered in Problem 10.20, linearly varying surface tractions act on the edge (face) *jk*. The magnitude of Φ_{x0} varies linearly from $\Phi_{x0j} = 500$ Pa at node *j* to $\Phi_{x0k} = 1500$ Pa at node *k* while the magnitude of Φ_{y0} varies linearly from $\Phi_{y0j} = 500$ Pa at node *j* to $\Phi_{y0k} = 2000$ Pa at node *k*. Determine the corresponding nodal force vector of the element.

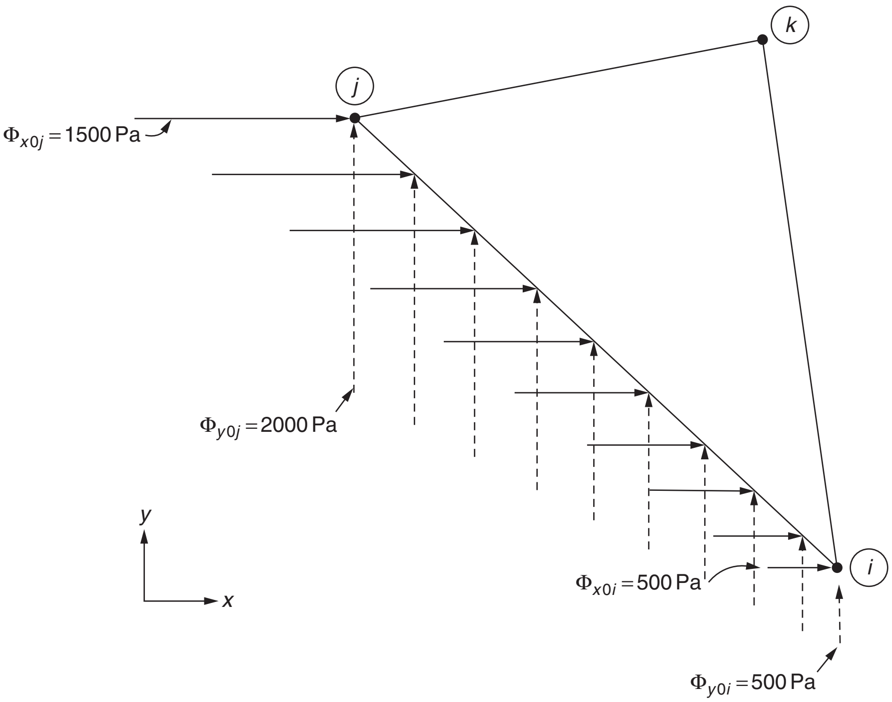


FIGURE 10.33 Surface traction on edge *ij*.

Hint: The magnitudes of the components of the surface tractions at any point on the edge *jk* can be expressed as $\Phi_{x0} = \Phi_{x0j}N_j + \Phi_{x0k}N_k$ and $\Phi_{y0} = \Phi_{y0j}N_j + \Phi_{y0k}N_k$.

10.33 For the element considered in Problem 10.20, linearly varying surface tractions act on the edge (face) *ki*. The magnitude of Φ_{x0} varies linearly from $\Phi_{x0k} = 500$ Pa at node k to $\Phi_{x0i} = 1500$ Pa at node i while the magnitude of Φ_{y0} varies linearly from $\Phi_{y0k} = 500$ Pa at node k to $\Phi_{y0i} = 2000$ Pa at node i. Determine the corresponding nodal force vector of the element.
Hint: The magnitudes of the components of the surface tractions at any point on the edge *ki* can be expressed as $\Phi_{x0} = \Phi_{x0k}N_k + \Phi_{x0i}N_i$ and $\Phi_{y0} = \Phi_{y0k}N_k + \Phi_{y0i}N_i$.

10.34 Under a specific set of applied loads, the nodes of the element considered in Problem 10.20 undergo the following displacements:
$u_i = 0.00225$ cm, $v_i = -0.00350$ cm, $u_j = 0.00050$ cm, $v_j = 0.00200$ cm, $u_k = -0.00550$ cm, and $v_k = 0.00600$ cm
Find the strains and stresses induced in the element due to this displacement field. Assume the element to be in a state of plane stress.

10.35 Under a specific set of applied loads, the nodes of the element considered in Problem 10.20 undergo the following displacements:
$u_i = 0.00225$ cm, $v_i = -0.00350$ cm, $u_j = 0.00050$ cm, $v_j = 0.00200$ cm, $u_k = -0.00550$ cm, and $v_k = 0.00600$ cm
Find the strains and stresses induced in the element due to this displacement field. Assume the element to be in a state of plane strain.

10.36 The triangular membrane element shown in Fig. 10.34 has a thickness of $t = 0.1$ in. The (x, y) coordinates of the nodes of the element are indicated next to the nodes in Fig. 10.34. The material of the element is brass with $E = 15.4 \times 10^6$ psi and $\nu = 0.32$. If the element is in a state of plane stress, determine the following:
a. Shape functions of the element, $N_i(x, y)$, $N_j(x, y)$, and $N_k(x, y)$
b. Matrix $[B]$ that relates the strains to the nodal displacements
c. Elasticity matrix $[D]$ for plane stress condition
d. Element stiffness matrix $\left[K^{(e)}\right]$

10.37 Assuming that the triangular membrane element described in Problem 10.36 is in a state of plane strain, determine the following:
a. Shape functions of the element, $N_i(x, y)$, $N_j(x, y)$, and $N_k(x, y)$
b. Matrix $[B]$ that relates the strains to the nodal displacements
c. Elasticity matrix $[D]$ for plane strain condition (see Eq. (8.24))
d. Element stiffness matrix $\left[K^{(e)}\right]$

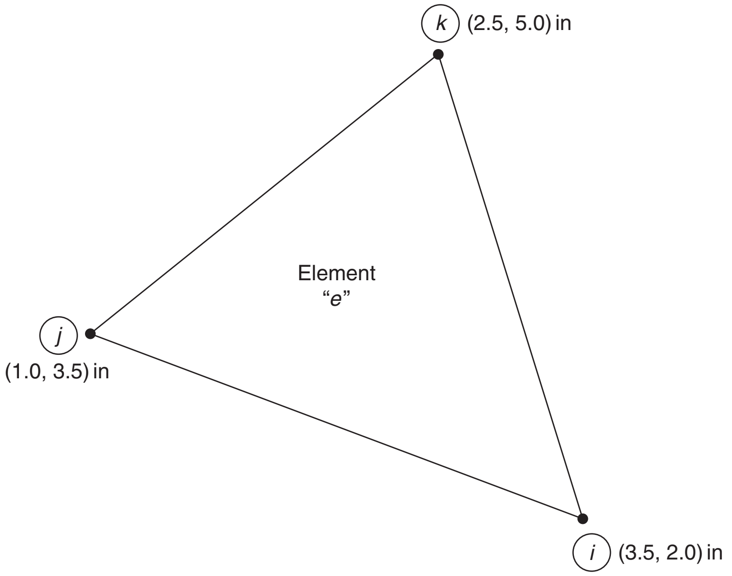


FIGURE 10.34 Triangular membrane element.

10.38 For the element considered in Problem 10.36, determine the element nodal force vector due to a temperature increase of 100 °F. Assume that the element is in a state of plane stress.

10.39 For the element considered in Problem 10.36, determine the element nodal force vector due to a temperature increase of 100 °F. Assume that the element is in a state of plane strain.

10.40 For the element described in Problem 10.36, determine the element nodal force vector as a result of the following prestress: $\sigma_{xx0} = -800$ psi, $\sigma_{yy0} = 500$ psi, and $\sigma_{xy0} = 750$ psi. Assume the element to be in a state of plane stress.

10.41 For the element described in Problem 10.36, determine the element nodal force vector as a result of the following prestress: $\sigma_{xx0} = -800$ psi, $\sigma_{yy0} = 500$ psi, and $\sigma_{xy0} = 750$ psi. Assume the element to be in a state of plane strain.

10.42 If the element described in Problem 10.36 is subjected to the distributed body forces $\phi_{x0} = 200$ lb$_f$/in^3 and $\phi_{y0} = -100$ lb$_f$/in^3, determine the corresponding nodal force vector of the element.

10.43 For the element described in Problem 10.36, uniform surface tractions with magnitudes $\Phi_{x0} = 750$ psi and $\Phi_{y0} = -750$ psi act on the edge (face) *ij*. Determine the resulting nodal force vector of the element.

10.44 For the element described in Problem 10.36, uniform surface tractions with magnitudes $\Phi_{x0} = 750$ psi and $\Phi_{y0} = -750$ psi act on the edge (face) *jk*. Determine the resulting nodal force vector of the element.

10.45 For the element described in Problem 10.36, uniform surface tractions with magnitudes $\Phi_{x0} = 750$ psi and $\Phi_{y0} = -750$ psi act on the edge (face) *ki*. Determine the resulting nodal force vector of the element.

10.46 For the element considered in Problem 10.36, find the nodal force vector when a concentrated or point load $\vec{P}_0 = \begin{Bmatrix} P_{x0} \\ P_{y0} \end{Bmatrix} = \begin{Bmatrix} 250 \\ -750 \end{Bmatrix}$ lb$_f$ acts at a point located at ($x_0 = 20$in, $y_0 = 4$ in) in the element.

10.47 For the element considered in Problem 10.36, linearly varying surface tractions act on the edge (face) *ij*. The magnitude of Φ_{x0} varies linearly from $\Phi_{x0i} = 200$ psi at node *i* to $\Phi_{x0j} = 1000$ psi at node *j* while the magnitude of Φ_{y0} varies from $\Phi_{y0i} = -500$ psi at node *i* to $\Phi_{y0j} = 500$ psi at node *j*. Determine the corresponding nodal force vector of the element.

Hint: The magnitudes of the components of the surface tractions at any point on the edge *ij* can be expressed as $\Phi_{x0} = \Phi_{x0i}N_i + \Phi_{x0j}N_j$ and $\Phi_{y0} = \Phi_{y0i}N_i + \Phi_{y0j}N_j$.

10.48 For the element considered in Problem 10.36, linearly varying surface tractions act on the edge (face) *jk*. The magnitude of Φ_{x0} varies from $\Phi_{x0j} = 200$ psi at node *j* to $\Phi_{x0k} = 1000$ psi at node *k* while the magnitude of Φ_{y0} varies from $\Phi_{y0j} = -500$ psi at node *j* to $\Phi_{y0k} = 500$ psi at node *k*. Determine the corresponding nodal force vector of the element.

Hint: The magnitudes of the components of the surface tractions at any point on the edge *jk* can be expressed as $\Phi_{x0} = \Phi_{x0j}N_j + \Phi_{x0k}N_k$ and $\Phi_{y0} = \Phi_{y0j}N_j + \Phi_{y0k}N_k$.

10.49 For the element considered in Problem 10.36, linearly varying surface tractions act on the edge (face) *ki*. The magnitude of Φ_{x0} varies linearly from $\Phi_{x0k} = 200$ psi at node *k* to $\Phi_{x0i} = 1000$ psi at node *i* while the magnitude of Φ_{y0} varies from $\Phi_{y0k} = -500$ psi at node *k* to $\Phi_{y0\mathrm{i}} = 500$ psi at node *i*. Determine the corresponding nodal force vector of the element.

Hint: The magnitudes of the components of the surface tractions at any point on the edge *ki* can be expressed as $\Phi_{x0} = \Phi_{x0k}N_k + \Phi_{x0i}N_i$ and $\Phi_{y0} = \Phi_{y0k}N_k + \Phi_{y0i}N_i$..

10.50 Under a specific set of applied loads, the nodes of the element considered in Problem 10.36 undergo the following displacements:

$u_i = 0.00125$ in, $v_i = -0.00250$ in, $u_j = -0.00050$ in, $v_j = 0.00100$ in, $u_k = -0.00450$ in, and $v_k = 0.00500$ in

Find the strains and stresses induced in the element due to this displacement field. Assume the element to be in a state of plane stress.

10.51 Under a specific set of applied loads, the nodes of the element considered in Problem 10.36 undergo the following displacements:

$u_i = 0.00125$ in, $v_i = -0.00250$ in, $u_j = -0.00050$ in, $v_j = 0.00100$ in, $u_k = -0.00450$ in, and $v_k = 0.00500$ in

Find the strains and stresses induced in the element due to this displacement field. Assume the element to be in a state of plane strain.

10.52 Consider a rectangular element in plane stress state with the geometry shown in Fig. 10.35. The element is made of aluminum with E = 71.0 GPa and $v = 0.33$ and has a thickness of 0.2 cm. Using the [*B*] matrix given in Problem 10.12 and the [*D*] matrix given in Eq. (10.15), find the element stiffness matrix using the relation

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V \tag{P.5}$$

FIGURE 10.35 Rectangular plate element.

Note

Perform the integration indicated in Eq. (P.5) by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

10.53 Consider a rectangular element in plane strain state with the geometry shown in Fig. 10.35. The element is made of aluminum with $E = 71.0$ GPa and $v = 0.33$ and has a thickness of 0.2 cm. Using the $[B]$ matrix given in Problem 10.12 and the $[D]$ matrix given in Eq. (8.24), find the element stiffness matrix using the relation

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V \tag{P.6}$$

Note

Perform the integration indicated in Eq. (P.6) by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

10.54 For the rectangular element considered in Problem 10.52 and Fig. 10.35, find the element nodal force vector due to an increase in the temperature of the element by 50 °C. Perform the needed integration by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element. Assume a plane stress condition for the element.

10.55 For the rectangular element considered in Problem 10.52 and Fig. 10.35, find the element nodal force vector due to an increase in the temperature of the element by 50 °C. Perform the needed integration by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element. Assume a plane strain condition for the element.

10.56 For the rectangular element considered in Problem 10.52 and Fig. 10.35, find the element nodal force vector due to distributed body force given by $\phi_{x0} = 0$ and $\phi_{y0} = -\rho g$ where ρ is the density of the material and g is the acceleration due to gravity. Assume the value of ρ as 2800 kg/m^3 and g = 981 m/s^2. Perform the needed integration by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

10.57 For the rectangular element considered in Problem 10.52 and Fig. 10.35, find the element nodal force vector due to uniform surface tractions, with $\Phi_{x0} = 1000$ Pa and $\Phi_{y0} = -500$ Pa, applied on the edge (face) ij. Perform the needed integration by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

10.58 For the rectangular element considered in Problem 10.52 and Fig. 10.35, find the element nodal force vector when a concentrated (point) load, with $P_{x0} = 100$ N and $P_{y0} = 500$ N, act at the point ($x_0 = 4$ cm, $y_0 = 5$ cm). Perform the needed integration by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

FIGURE 10.36 Rectangular plate element.

10.59 Consider a rectangular element in plane stress state with geometry shown in Fig. 10.36. The element is made of brass with $E = 15.4 \times 10^6$ psi and $v = 0.32$ and has a thickness of $t = 0.1$ in. Using the $[B]$ matrix given in Problem 10.12 and the $[D]$ matrix given in Eq. (10.15), find the element stiffness matrix using the relation

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V \tag{P.7}$$

Note

Perform the integration indicated in Eq. (P.7) by evaluating the integrand at the centroid of the element and treating the integrand as a constant throughout the element.

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Chapter 11

Analysis of Three-Dimensional Problems

Chapter Outline

11.1 INTRODUCTION

For the realistic analysis of certain problems such as thick short beams, thick pressure vessels, elastic half space acted on by a concentrated load, and machine foundations, we have to use three-dimensional finite elements. Just like a triangular element is a basic element for analyzing two-dimensional problems, the tetrahedron element, with four corner nodes, is the basic element for modeling three-dimensional problems. One of the major difficulties associated with the use of three-dimensional elements (e.g., tetrahedra, hexahedra, and rectangular parallelepiped elements) is that a large number of elements have to be used for obtaining reasonably accurate results. This will result in a very large number of simultaneous equations to be solved in static analyses. Despite this difficulty, we may not have any choice except to use three-dimensional elements in certain situations. Hence, the tetrahedron and hexahedron elements are considered in this chapter [11.1–11.3].

11.2 TETRAHEDRON ELEMENT

The tetrahedron element, with three translational degrees of freedom per node, is shown in the global *xyz* coordinate system in Fig. 11.1 (the global coordinates are denoted as x, y, z instead of X, Y, Z, for simplicity). For this element, there will be no advantage in setting up a local coordinate system, and hence we shall derive all the elemental equations in the global system. Since there are 12 nodal degrees of freedom Q_{3i-2}, Q_{3i-1}, Q_{3i}, Q_{3j-2} …, Q_{3l} and three displacement components u, v, and w, we choose the displacement variation to be linear as

$$\left.\begin{aligned} u(x,y,z) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \\ v(x,y,z) &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z \\ w(x,y,z) &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z \end{aligned}\right\} \tag{11.1}$$

where α_1, α_2, …, α_{12} are constants. By using the nodal conditions

$$\begin{aligned} u &= Q_{3i-2}, & v &= Q_{3i-1}, & w &= Q_{3i} & &\text{at} & &(x_i, y_i, z_i) \\ u &= Q_{3j-2}, & v &= Q_{3j-1}, & w &= Q_{3j} & &\text{at} & &(x_j, y_j, z_j) \\ u &= Q_{3k-2}, & v &= Q_{3k-1}, & w &= Q_{3k} & &\text{at} & &(x_k, y_k, z_k) \\ u &= Q_{3l-2}, & v &= Q_{3l-1}, & w &= Q_{3l} & &\text{at} & &(x_l, y_l, z_l) \end{aligned} \tag{11.2}$$

we can obtain

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00011-0

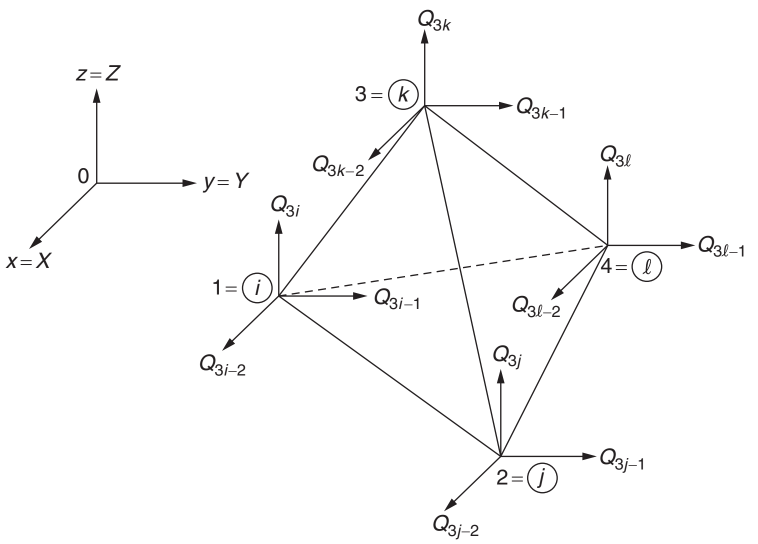

FIGURE 11.1 A Tetrahedron element in a global *xyz* system.

$$u(x,y,z) = N_i(x,y,z)Q_{3i-2} + N_j(x,y,z)Q_{3j-2} + N_k(x,y,z)Q_{3k-2} + N_l(x,y,z)Q_{3l-2} \tag{11.3}$$

where N_i, N_j, N_k, and N_l are given by Eq. (3.48), and similar expressions for $v(x, y, z)$ and $w(x, y, z)$. Thus, the displacement field can be expressed in matrix form as

$$\underset{3\times 1}{\vec{U}} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \underset{3\times 12}{[N]} \; \underset{12\times 1}{\vec{Q}^{(e)}} \tag{11.4}$$

where

$$[N] = \begin{bmatrix} N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 & 0 \\ 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l & 0 \\ 0 & 0 & N_i & 0 & 0 & N_j & 0 & 0 & N_k & 0 & 0 & N_l \end{bmatrix} \tag{11.5}$$

and

$$\vec{Q}^{(e)} = \begin{Bmatrix} Q_{3i-2} \\ Q_{3i-1} \\ Q_{3i} \\ \vdots \\ Q_{3l} \end{Bmatrix} \tag{11.6}$$

Noting that all six strain components are relevant in three-dimensional analysis, the strain–displacement relations can be expressed using Eq. (11.4) as follows:

$$\underset{6\times 1}{\vec{\varepsilon}} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{Bmatrix} = \underset{6\times 12}{[B]} \; \underset{12\times 1}{\vec{Q}^{(e)}} \tag{11.7}$$

where

$$[B] = \frac{1}{6V}\begin{bmatrix} b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & b_l & 0 & 0 \\ 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & c_l & 0 \\ 0 & 0 & d_i & 0 & 0 & d_j & 0 & 0 & d_k & 0 & 0 & d_l \\ c_i & b_i & 0 & c_j & b_j & 0 & c_k & b_k & 0 & c_l & b_l & 0 \\ 0 & d_i & c_i & 0 & d_j & c_j & 0 & d_k & c_k & 0 & d_l & c_l \\ d_i & 0 & b_i & d_j & 0 & b_j & d_k & 0 & b_k & d_l & 0 & b_l \end{bmatrix} \tag{11.8}$$

The stress–strain relations, in the case of three-dimensional analysis, are given by Eq. (8.10) as

$$\vec{\sigma} = [D]\vec{\varepsilon} \tag{11.9}$$

where

$$\vec{\sigma}^T = \{\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \sigma_{xy} \;\; \sigma_{yz} \;\; \sigma_{zx}\}$$

and

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1-2v}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1-2v}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{1-2v}{2}\right) \end{bmatrix} \tag{11.10}$$

The stiffness matrix of the element (in the global system) can be obtained as

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V \tag{11.11}$$

Since the matrices $[B]$ and $[D]$ are independent of x, y, and z, the stiffness matrix can be obtained by carrying out matrix multiplications as

$$\left[K^{(e)}\right] = V^{(e)}[B]^T[D][B] \tag{11.12}$$

In this case, since the assumed displacement model is linear, the continuity of displacement along the interface between neighboring elements will be satisfied automatically.

11.2.1 Consistent Load Vector

The total load vector due to initial (thermal) strains, body forces $\vec{\vec{\phi}} = \begin{Bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{Bmatrix}$, and surface (distributed) forces $\vec{\vec{\Phi}} = \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix}$ can be computed using Eqs. (8.88), (8.90), and (8.89) as

$$\vec{P}^{(e)} = \iiint_{V^{(e)}} [B]^T [D] \cdot \alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \mathrm{d}V + \iiint_{V^{(e)}} [N]^T \begin{Bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{Bmatrix} \cdot \mathrm{d}V + \iint_{S_1^{(e)}} [N]^T \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} \cdot \mathrm{d}S_1$$

$$= \frac{E \cdot \alpha \cdot T \cdot V^{(e)}}{(1-2v)} [B]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{V^{(e)}}{4} \begin{Bmatrix} \phi_x \\ \phi_y \\ \phi_z \\ \phi_x \\ \phi_y \\ \phi_z \\ \phi_x \\ \phi_y \\ \phi_z \\ \phi_x \\ \phi_y \\ \phi_z \end{Bmatrix} + \frac{S_{ijk}^{(e)}}{3} \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \\ p_{x0} \\ p_{y0} \\ p_{z0} \\ p_{x0} \\ p_{y0} \\ p_{z0} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{11.13}$$

Eq. (11.13) shows that the body force is distributed equally between the four nodes of the element. It is assumed in deriving Eq. (11.13) that the surface forces are distributed only on the face *ijk* of the element *e*. These surface forces can be seen to be equally distributed between the three nodes *i*, *j*, and *k*, which define the loaded face. $S_{ijk}^{(e)}$ denotes the area of the face *ijk* of element *e*. The last three components of the surface load vector are zero since they are related to the term $\iint N_l \cdot \mathrm{d}S_1$ and N_l is zero on the face ijk. Note that the location of the zero terms changes in the last column of Eq. (11.13), and their location depends upon which face the surface forces are acting. If more than one face of the element e is subjected to the surface forces, then there will be additional surface load vectors in Eq. (11.13).

EXAMPLE 11.1

Consider a tetrahedron element with the (x, y, z) coordinates as indicated in Fig. 11.2. If the Young's modulus (E) and Poisson's ratio (v) are given by $E = 207$ GPa and $v = 0.3$, find the stiffness matrix of the element.

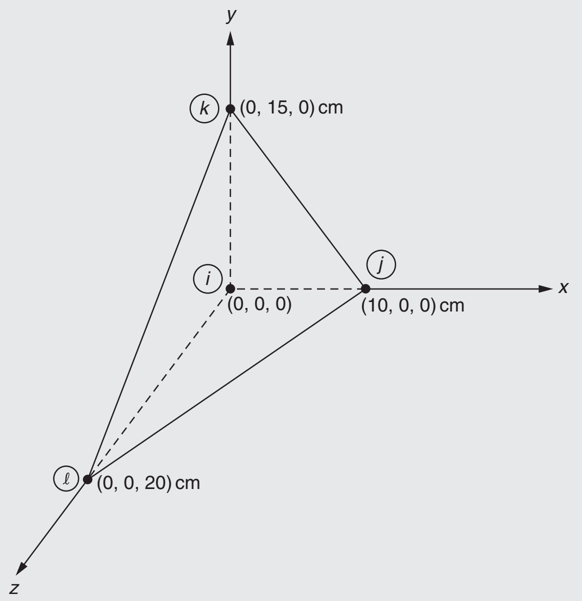

FIGURE 11.2 Tetrahedron element.

Solution

Using the nodal coordinates indicated in Fig. 11.2, the volume (V) and the shape functions of the element as well as the constants in the shape functions can be determined as follows (see Problem 3.13):

$$V = 500\ \text{cm}^3 = 5 \times 10^{-4}\ \text{m}^3$$

$$N_i(x, y, z) = \frac{1}{60}(60 - 5x - 4y - 5z)$$

$$N_j(x, y, z) = \frac{x}{10}, \quad N_k(x, y, z) = \frac{y}{15}, \quad N_l(x, y, z) = \frac{z}{20}$$

$$b_i = -300, b_j = 300, b_k = 0, b_l = 0, c_i = -200, c_j = 0, c_k = 200, c_l = 0,$$

$$d_i = -150, d_j = 0, d_k = 0, d_l = 150$$

The matrix $[B]$ given by Eq. (11.8) becomes

Continued

EXAMPLE 11.1 —cont'd

$$[B] = \frac{1}{6V}\begin{bmatrix} -300 & 0 & 0 & 300 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -200 & 0 & 0 & 0 & 0 & 0 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & -150 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 150 \\ -200 & -300 & 0 & 0 & 300 & 0 & 200 & 0 & 0 & 0 & 0 & 0 \\ 0 & -150 & -200 & 0 & 0 & 0 & 0 & 0 & 200 & 0 & 150 & 0 \\ -150 & 0 & -300 & 0 & 0 & 300 & 0 & 0 & 0 & 150 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -10.0000 & 0 & 0 & 10.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6.6667 & 0 & 0 & 0 & 0 & 0 & 6.6667 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.0000 \\ -6.6667 & -10.0000 & 0 & 0 & 10.0000 & 0 & 6.6667 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5.0000 & -6.6667 & 0 & 0 & 0 & 0 & 0 & 6.6667 & 0 & 5.0000 & 0 \\ -5.0000 & 0 & -10.0000 & 0 & 0 & 10.0000 & 0 & 0 & 0 & 5.0000 & 0 & 0 \end{bmatrix}\frac{1}{\mathrm{m}} \tag{E.1}$$

Noting that $\frac{E}{(1+v)(1-2v)} = \frac{207(10^9)}{(1+0.3)(1-2\times0.3)} = 3.9808(10^{11})$ Pa, the elasticity matrix $[D]$ of Eq. (11.10) can be expressed as

$$[D] = 10^{11}\begin{bmatrix} 2.7865 & 1.1942 & 1.1942 & 0 & 0 & 0 \\ 1.1942 & 2.7865 & 1.1942 & 0 & 0 & 0 \\ 1.1942 & 1.1942 & 2.7865 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7962 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7962 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7962 \end{bmatrix}\mathrm{Pa} \tag{E.2}$$

The stiffness matrix of the element can be found, using Eq. (11.12), as

EXAMPLE 11.1 —cont'd

$$\left[K^{(e)}\right] = V^{(e)}[B]^T[D][B]$$

$$= 10^{10}\begin{bmatrix} 1.6697 & 0.6635 & 0.4976 & -1.3933 & -0.2654 & -0.1990 & -0.1769 & -0.3981 & 0 & -0.0995 & 0 & -0.2986 \\ 0.6635 & 1.1168 & 0.3317 & -0.3981 & -0.3981 & 0 & -0.2654 & -0.6192 & -0.1327 & 0 & -0.0995 & -0.1990 \\ 0.4976 & 0.3317 & 0.9233 & -0.2986 & 0 & -0.3981 & 0 & -0.1990 & -0.1769 & -0.1990 & -0.1327 & -0.3483 \\ -1.3933 & -0.3981 & -0.2986 & 1.3933 & 0 & 0 & 0 & 0.3981 & 0 & 0 & 0 & 0.2986 \\ -0.2654 & -0.3981 & 0 & 0 & 0.3981 & 0 & 0.2654 & 0 & 0 & 0 & 0 & 0 \\ -0.1990 & 0 & -0.3981 & 0 & 0 & 0.3981 & 0 & 0 & 0 & 0.1990 & 0 & 0 \\ -0.1769 & -0.2654 & 0 & 0 & 0.2654 & 0 & 0.1769 & 0 & 0 & 0 & 0 & 0 \\ -0.3981 & -0.6192 & -0.1990 & 0.3981 & 0 & 0 & 0 & 0.6192 & 0 & 0 & 0 & 0.1990 \\ 0 & -0.1327 & -0.1769 & 0 & 0 & 0 & 0 & 0 & 0.1769 & 0 & 0.1327 & 0 \\ -0.0995 & 0 & -0.1990 & 0 & 0 & 0.1990 & 0 & 0 & 0 & 0.0995 & 0 & 0 \\ 0 & -0.0995 & -0.1327 & 0 & 0 & 0 & 0 & 0 & 0.1327 & 0 & 0.0995 & 0 \\ -0.2986 & -0.1990 & -0.3483 & 0.2986 & 0 & 0 & 0 & 0.1990 & 0 & 0 & 0 & 0.3483 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{E.3}$$

EXAMPLE 11.2

Determine the nodal load vector of the tetrahedron element described in Example 11.1 corresponding to the gravity force acting in the negative y direction. Assume the density of the material as $\rho = 7850\ \text{kg/m}^3$.

Solution

The body forces, distributed uniformly throughout the volume of the element, are given by the vector

$$\vec{\vec{\phi}} = \begin{Bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\rho g \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -7850(9.81) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -77008.5 \\ 0 \end{Bmatrix} \text{N/m}^3$$

The nodal load vector of the tetrahedron element is given by (middle term on the right-hand side of Eq. 11.13):

$$\vec{P}^{(e)} = \frac{V^{(e)}}{4}\begin{Bmatrix} 0 \\ \phi_y \\ 0 \\ 0 \\ \phi_y \\ 0 \\ 0 \\ \phi_y \\ 0 \\ 0 \\ \phi_y \\ 0 \end{Bmatrix} = \frac{5(10^{-4})}{4}\begin{Bmatrix} 0 \\ -77008.5 \\ 0 \\ 0 \\ -77008.5 \\ 0 \\ 0 \\ -77008.5 \\ 0 \\ 0 \\ -77008.5 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 9.6261 \\ 0 \\ 0 \\ 9.6261 \\ 0 \\ 0 \\ 9.6261 \\ 0 \\ 0 \\ 9.6261 \\ 0 \end{Bmatrix} \text{N}$$

EXAMPLE 11.3

If uniformly distributed surface tractions of magnitude

$$\vec{\vec{\Phi}} = \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} = \begin{Bmatrix} 10(10^4) \\ 5(10^4) \\ -5(10^4) \end{Bmatrix} \text{N/m}^2 \tag{E.1}$$

act on the face jkl of the tetrahedron element considered in Example 11.1, determine the corresponding nodal load vector of the element.

Solution

The nodal load vector corresponding to the surface tractions acting on the surface jkl of the tetrahedron element is given by (see Problem 11.12):

$$\vec{P}^{(e)} = \frac{S_{jkl}^{(e)}}{3}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ p_{x0} \\ p_{y0} \\ p_{z0} \\ p_{x0} \\ p_{y0} \\ p_{z0} \\ p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix} \tag{E.2}$$

EXAMPLE 11.3 —cont'd

where $S_{jkl}^{(e)}$ is the area of the triangular surface *jkl*. The area of the triangle with known nodal coordinates (x_j, y_j, z_j), (x_k, y_k, z_k), and (x_l, y_l, z_l) is the Pythagorean sum of the areas of the respective projections on the three principal planes (i.e., $x=0$, $y=0$, and $z=0$):

$$A = S_{jkl}^{(e)} = \frac{1}{2}\left[\left(\det\begin{vmatrix} x_j & x_k & x_l \\ y_j & y_k & y_l \\ 1 & 1 & 1 \end{vmatrix}\right)^2 + \left(\det\begin{vmatrix} y_j & y_k & y_l \\ z_j & z_k & z_l \\ 1 & 1 & 1 \end{vmatrix}\right)^2 + \left(\det\begin{vmatrix} z_j & z_k & z_l \\ x_j & x_k & x_l \\ 1 & 1 & 1 \end{vmatrix}\right)^2\right]^{\frac{1}{2}} \tag{E.3}$$

Using the known coordinates (shown in Fig. 11.2 of Example 11.1), the determinants in Eq. (E.3) can be evaluated as

$$\det\begin{vmatrix} x_j & x_k & x_l \\ y_j & y_k & y_l \\ 1 & 1 & 1 \end{vmatrix} = \det\begin{vmatrix} 10 & 0 & 0 \\ 0 & 15 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 150$$

$$\det\begin{vmatrix} y_j & y_k & y_l \\ z_j & z_k & z_l \\ 1 & 1 & 1 \end{vmatrix} = \det\begin{vmatrix} 0 & 15 & 0 \\ 0 & 0 & 20 \\ 1 & 1 & 1 \end{vmatrix} = 300$$

$$\det\begin{vmatrix} z_j & z_k & z_l \\ x_j & x_k & x_l \\ 1 & 1 & 1 \end{vmatrix} = \det\begin{vmatrix} 0 & 0 & 20 \\ 10 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 200$$

Thus, the area $S_{jkl}^{(e)}$ can be computed as

$$S_{jkl}^{(e)} = \frac{1}{2}\sqrt{(150)^2 + (300)^2 + (200)^2} = 195.2562 \text{ cm}^2$$

$$= 195.2562 \times 10^{-4} \text{ m}^2$$

The nodal load vector of the element, then, is given by Eq. (E.2) as

$$\vec{P}^{(e)} = \frac{195.2562(10^{-4})}{3}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 5 \\ -5 \\ 10 \\ 5 \\ -5 \\ 10 \\ 5 \\ -5 \end{Bmatrix}(10^4) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 650.854 \\ 325.427 \\ -325.427 \\ 650.854 \\ 325.427 \\ -325.427 \\ 650.854 \\ 325.427 \\ -325.427 \end{Bmatrix} \text{N} \tag{E.4}$$

EXAMPLE 11.4

Assuming the coefficient of thermal expansion of the material to be $\alpha = 10.8 \times 10^{-6}$ m/m °C, find the nodal load vector of the tetrahedron element considered in Example 11.1 if the element experiences a temperature rise of $T = 50$ °C.

Solution

The nodal load vector of the element, resulting from a temperature rise, is given by (see Eq. 11.13):

$$\vec{P}^{(e)} = \frac{E\alpha\, T V^{(e)}}{1-2v}[B]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{(207 \times 10^9)(10.8 \times 10^{-6})(50)(500 \times 10^{-6})}{1-2(0.3)}[B]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

$$= 139.725 \times 10^3 [B]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{N}$$

Using the $[B]$ matrix given by Eq. (E.1) of Example 11.1, the load vector can be found as

$$\vec{P}^{(e)} = \begin{Bmatrix} -1397250 \\ -931500 \\ -698625 \\ 1397250 \\ 0 \\ 0 \\ 0 \\ 931500 \\ 0 \\ 0 \\ 0 \\ 698625 \end{Bmatrix} \text{N} \tag{E.2}$$

11.3 HEXAHEDRON ELEMENT

In this section, we consider the simplest hexahedron element having eight corner nodes with three degrees of freedom per node. For convenience, we derive the element matrices by treating the element as isoparametric. This element is also known as Zienkiewicz–Irons–Brick with eight nodes (ZIB 8) and is shown in Fig. 11.3A.

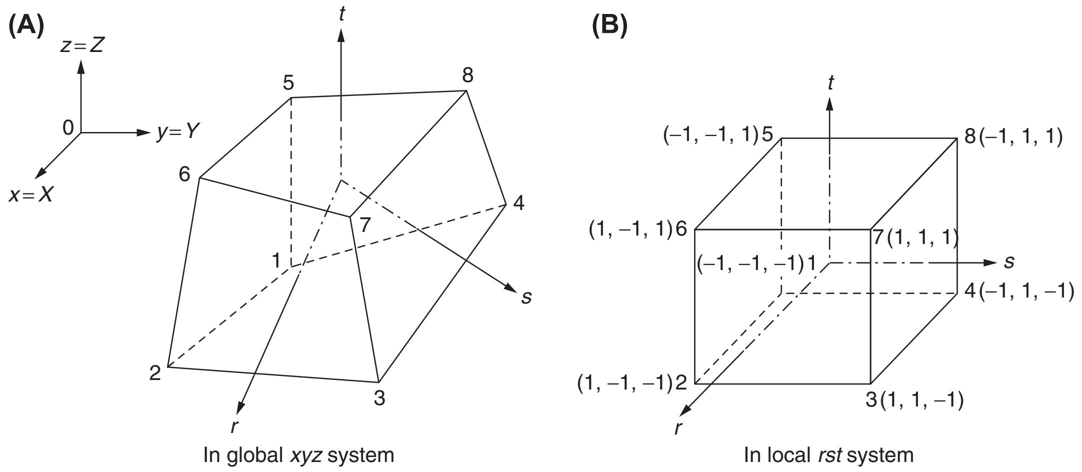

FIGURE 11.3 A hexahedron element with eight nodes.

11.3.1 Natural Coordinate System

As shown in Fig. 11.3A, the natural coordinates are r, s, and t with the origin of the system taken at the centroid of the element. It can be seen that each of the coordinate axes r, s, and t is associated with a pair of opposite faces, which are given by the coordinate values ± 1. Thus, in the local (natural) coordinates, the element is a cube as shown in Fig. 11.3B, although in the global Cartesian coordinate system it may be an arbitrarily warped and distorted six-sided solid as shown in Fig. 11.3A. The relationship between the local and global coordinates can be expressed as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [N]\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ \vdots \\ z_8 \end{Bmatrix} \tag{11.14}$$

where

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & \dots & 0 \\ 0 & N_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & N_1 & 0 & \dots & N_8 \end{bmatrix} \tag{11.15}$$

and

$$N_i(r,s,t) = \frac{1}{8}(1+rr_i)(1+ss_i)(1+tt_i); \quad i = 1,2,\dots,8 \tag{11.16}$$

or

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{8} N_i x_i \\ \sum_{i=1}^{8} N_i y_i \\ \sum_{i=1}^{8} N_i z_i \end{Bmatrix} \tag{11.17}$$

11.3.2 Displacement Model

By assuming the variations of the displacements in between the nodes to be linear, the displacements can be expressed by the same interpolation functions used to describe the geometry as (analogous to Eq. 11.14)

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N] \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ \vdots \\ w_8 \end{Bmatrix} = [N] \vec{Q}^{(e)} \tag{11.18}$$

where $\vec{Q}^{(e)}$ is the vector of nodal displacement degrees of freedom, and (u_i, v_i, w_i) denote the displacements of node i, $i = 1-8$.

11.3.3 Strain–Displacement and Stress–Strain Relations

Using Eq. (11.18), the three-dimensional strain–displacement relations can be expressed as

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{Bmatrix} = \underset{6\times 24}{[B]} \underset{24\times 1}{\vec{Q}^{(e)}} \tag{11.19}$$

where

$$\underset{6\times 24}{[B]} = [[B_1][B_2]\ldots[B_8]] \tag{11.20}$$

and

$$\underset{6\times 3}{[B_i]} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial N_i}{\partial z} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial z} & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial z} & 0 & \dfrac{\partial N_i}{\partial x} \end{bmatrix}, \quad i = 1-8 \tag{11.21}$$

The derivatives in the matrix $[B_i]$ may be evaluated by applying the chain rule of differentiation as follows:

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\[2ex] \dfrac{\partial N_i}{\partial s} \\[2ex] \dfrac{\partial N_i}{\partial t} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial N_i}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial N_i}{\partial z}\dfrac{\partial z}{\partial r} \\[2ex] \dfrac{\partial N_i}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial N_i}{\partial z}\dfrac{\partial z}{\partial s} \\[2ex] \dfrac{\partial N_i}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial N_i}{\partial z}\dfrac{\partial z}{\partial t} \end{Bmatrix} \tag{11.22}$$

$$= \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \\[2ex] \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\[2ex] \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\[2ex] \dfrac{\partial N_i}{\partial y} \\[2ex] \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\[2ex] \dfrac{\partial N_i}{\partial y} \\[2ex] \dfrac{\partial N_i}{\partial z} \end{Bmatrix}$$

where $[J]$ is the Jacobian matrix, which can be expressed using Eq. (11.17) as

$$\underset{3\times 3}{[J]} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \\[1ex] \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\[1ex] \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial r}x_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial r}y_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial r}z_i\right) \\ \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial s}x_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial s}y_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial s}z_i\right) \\ \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial t}x_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial t}y_i\right) & \sum_{i=1}^{8}\left(\dfrac{\partial N_i}{\partial t}z_i\right) \end{bmatrix} \tag{11.23}$$

The derivatives of the interpolation functions can be obtained from Eq. (11.16) as

$$\left.\begin{aligned} \frac{\partial N_i}{\partial r} &= \frac{1}{8}r_i(1+ss_i)(1+tt_i) \\ \frac{\partial N_i}{\partial s} &= \frac{1}{8}s_i(1+rr_i)(1+tt_i) \\ \frac{\partial N_i}{\partial t} &= \frac{1}{8}t_i(1+rr_i)(1+ss_i) \end{aligned}\right\}, \quad i = 1-8 \tag{11.24}$$

and the coordinates of the nodes in the local system (r_i, s_i, t_i) are shown in Fig. 11.3. By inverting Eq. (11.22), we obtain

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\[1ex] \dfrac{\partial N_i}{\partial y} \\[1ex] \dfrac{\partial N_i}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial r} \\[1ex] \dfrac{\partial N_i}{\partial s} \\[1ex] \dfrac{\partial N_i}{\partial t} \end{Bmatrix} \tag{11.25}$$

from which the matrix $[B_i]$ can be evaluated. The stress–strain relations are the same as those given in Eqs. (11.9) and (11.10).

11.3.4 Element Stiffness Matrix

The element stiffness matrix is given by

$$\left[K^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V \tag{11.26}$$

Since the matrix $[B]$ is expressed in natural coordinates (evident from Eqs. 11.20, 11.21, and 11.25), it is necessary to carry out the integration in Eq. (11.26) in natural coordinates too, using the relationship

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \det[J] \cdot \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \tag{11.27}$$

Thus, Eq. (11.26) can be rewritten as

$$\left[K^{(e)}\right] = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] \det[J]\, \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \tag{11.28}$$

11.3.5 Numerical Computation

Since the matrix $[B]$ is an implicit (not explicit!) function of r, s, and t, a numerical method has to be used to evaluate the multiple integral of Eq. (11.28). The Gaussian quadrature has been proven to be the most efficient method of numerical integration for this class of problems. By using the two-point Gaussian quadrature, which yields sufficiently accurate results, Eq. (11.28) can be evaluated [11.4] as follows:

$$\left[K^{(e)}\right] = \sum_{r=R_i=R_1}^{R_2} \sum_{s=S_j=S_1}^{S_2} \sum_{t=T_k=T_1}^{T_2} \left[\left([B]^T [D][B] \cdot \det[J] \right)\Big|_{(R_i, S_j, T_k)} \right] \tag{11.29}$$

where

$$\left[\left([B]^T [D][B] \cdot \det[J] \right)\Big|_{(R_i, S_j, T_k)} \right] \tag{11.30}$$

indicates the value of

$$\left([B]^T [D] [B] \det[J] \right)$$

evaluated at $r = R_i$, $s = S_j$, and $t = T_k$, and $R_1 = S_1 = T_1 = -0.57735$ and $R_2 = S_2 = T_2 = +0.57735$.

11.3.6 Numerical Results

The performance of the three-dimensional elements considered in Sections 11.2 and 11.3, namely the tetrahedron and hexahedron elements, is studied by taking the short cantilever beam shown in Fig. 11.4 as the test case. This cantilever is

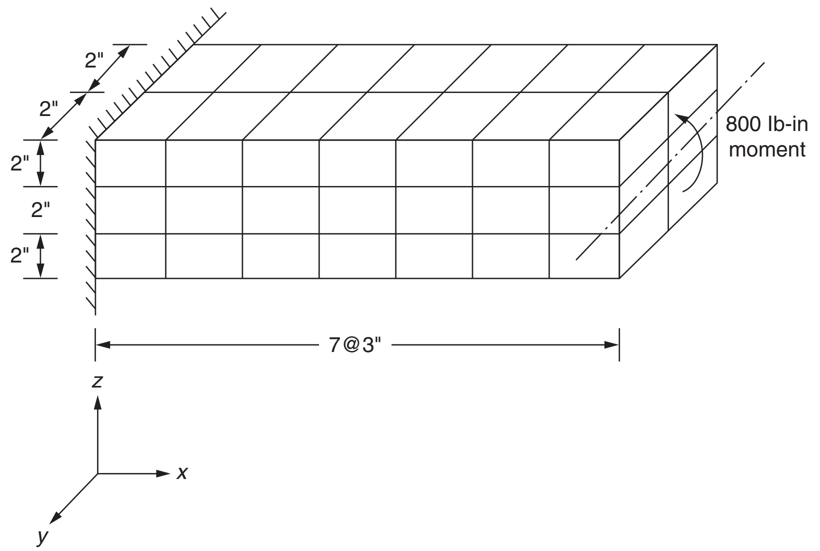

FIGURE 11.4 A cantilever beam subjected to tip moment.

modeled as an assemblage of 42 identical hexahedra, each 2 × 2 × 3 in. In the case of the tetrahedron element, each of the 42 hexahedra is considered to be composed of 5 tetrahedron elements. The cantilever beam is subjected to a tip moment of 800 lb-in as indicated in Fig. 11.4. The numerical results obtained are indicated in the following table [11.4]:

Element Type	Maximum Stress σ_{xx}	σ_{zz}	Maximum Deflection at Center of Gravity of Tip (in)
Tetrahedron	—	—	0.606×10^{-4}
ZIB 8	31.6 psi	1.4 psi	0.734×10^{-4}
Beam theory	33.3 psi	0.0 psi	0.817×10^{-4}

It can be seen that ZIB 8 is superior to the tetrahedron element.

11.4 ANALYSIS OF SOLIDS OF REVOLUTION

11.4.1 Introduction

The problem of stress analysis of solids of revolution (axisymmetric solids) under axisymmetric loads is of considerable practical interest. This problem is similar to those of plane stress and plane strain since the displacements are confined to only two directions (radial and axial) [11.5,11.6]. The basic element that can be used for modeling solids of revolution is the axisymmetric ring element having triangular cross section. This element was originally developed by Wilson [11.7]. This element is useful for analyzing thick axisymmetric shells, solid bodies of revolution, turbine disks (Fig. 11.5), and circular footings on a soil mass. In this section, the derivation of the element stiffness matrix and load vectors for the axisymmetric ring element is presented.

11.4.2 Formulation of Elemental Equations for an Axisymmetric Ring Element

An axisymmetric ring element with a triangular cross section is shown in cylindrical coordinates in Fig. 11.6. For axisymmetric deformation, since the displacement v along θ direction is zero (due to symmetry), the relevant displacement components are only u and w in the r and z directions, respectively. By taking the nodal values of u and w as the degrees of freedom, a linear displacement model can be assumed in terms of triangular coordinates L_i, L_j, and L_k as

$$\vec{U} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = [N]\vec{Q}^{(e)} \tag{11.31}$$

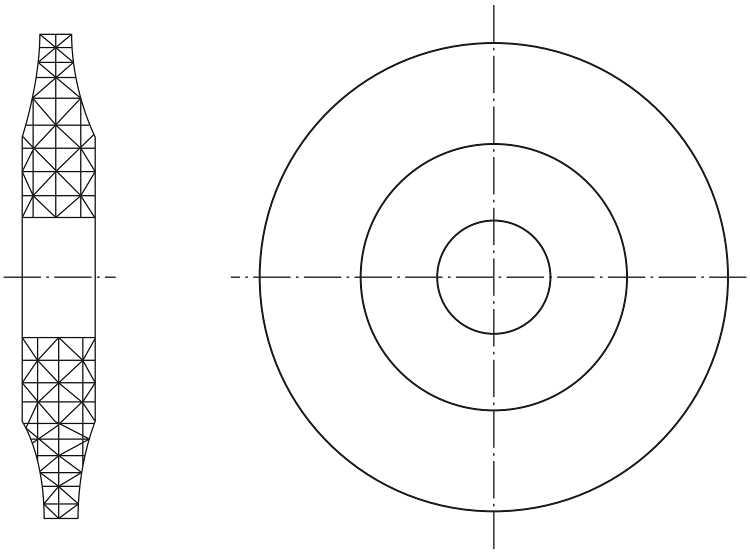

FIGURE 11.5 Turbine disk modeled by triangular ring elements.

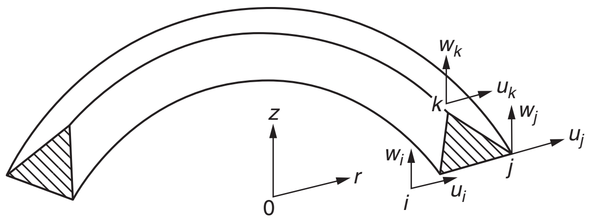

FIGURE 11.6 An axisymmetric ring element with triangular cross section.

where

$$[N] = \begin{bmatrix} N_i & O & N_j & O & N_k & O \\ O & N_i & O & N_j & O & N_k \end{bmatrix} \tag{11.32}$$

$$\vec{Q}^{(e)} = \begin{Bmatrix} Q_{2i-1} \\ Q_{2i} \\ Q_{2j-1} \\ Q_{2j} \\ Q_{2k-1} \\ Q_{2k} \end{Bmatrix}^{(e)} \equiv \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_k \\ w_k \end{Bmatrix}^{(e)} \tag{11.33}$$

$$\begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} = \begin{Bmatrix} L_i \\ L_j \\ L_k \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_i r + c_i z \\ a_j + b_j r + c_j z \\ a_k + b_k r + c_k z \end{Bmatrix} \tag{11.34}$$

$$A = \frac{1}{2}(r_i z_j + r_j z_k + r_k z_i - r_i z_k - r_j z_i - r_k z_j) \tag{11.35}$$

(r_i, z_i) are the (r, z) coordinates of node i, and a_i, a_j, $a_k, \ldots, c_k$ can be obtained from Eq. (3.32) by substituting r and z in place of x and y, respectively. In this case, there are four relevant strains, namely $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$, and ε_{rz}, for the axisymmetric case.

The strain–displacement relations can be expressed as

$$\vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = [B]\vec{Q}^{(e)} \tag{11.36}$$

where

$$[B] = \frac{1}{2A} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ (N_i/r) & 0 & (N_j/r) & 0 & (N_k/r) & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \tag{11.37}$$

The stress–strain relations are given by

$$\vec{\sigma} = [D]\vec{\varepsilon} \tag{11.38}$$

where $\vec{\sigma} = \{\sigma_{rr}\ \sigma_{\theta\theta}\ \sigma_{zz}\ \sigma_{rz}\}^T$ and

$$[D] = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \left(\frac{1-2v}{2}\right) \end{bmatrix} \tag{11.39}$$

Since the matrix $[B]$ contains terms that are functions of the coordinates r and z, the product $[B]^T$ $[D]$ $[B]$ cannot be removed from under the integral sign in the expression of the element stiffness matrix $[K^{(e)}]$, Eq. (8.87). However, we can adopt an approximate procedure for evaluating the integral involved in the expression of $[K^{(e)}]$. If we evaluate the matrix $[B]$ using the r and z values at the centroid of the element, the product $[B]^T[D][B]$ can be removed from under the integral sign as

$$\left[K^{(e)}\right] \simeq [\underline{B}]^T[D][\underline{B}] \iiint_{V^{(e)}} \mathrm{d}V \tag{11.40}$$

where the bar below $[B]$ denotes that the matrix $[B]$ is evaluated at the point $(\underline{r}, \underline{z})$ with

$$\underline{r} = (r_i + r_j + r_k)/3 \quad \text{and} \quad \underline{z} = (z_i + z_j + z_k)/3 \tag{11.41}$$

By using the relation

$$\iiint_{V^{(e)}} \mathrm{d}V = V^{(e)} = 2\pi \underline{r} A \tag{11.42}$$

Eq. (11.40) can be expressed as

$$\left[K^{(e)}\right] = [\underline{B}]^T[D][\underline{B}]2\pi\underline{r}A \tag{11.43}$$

Although Eq. (11.43) is approximate, it yields reasonably accurate results. The components of the load vector of the element are given by Eqs. (8.88) to (8.90). The load vector due to initial strains (caused by the temperature change T) can be handled as in the case of $[K^{(e)}]$ since $[B]$ occurs in the integral. Thus,

$$\begin{aligned} \vec{P}_i^{(e)} &= \iiint_{V^{(e)}} [B]^T\left[D\right]\vec{\varepsilon}_0 \mathrm{d}V = \iiint_{V^{(e)}} [B]^T[D]E\alpha T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \mathrm{d}V \\ &\simeq \frac{E\alpha T}{(1-2v)}[\underline{B}]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} 2\pi\underline{r}A \end{aligned} \tag{11.44}$$

If $\overline{\phi}_r$ and $\overline{\phi}_z$ denote the components of the body force in the directions of r and z, respectively, the load vector $\vec{P}_b^{(e)}$ can be evaluated either exactly using the area coordinates or approximately using the procedure adopted earlier. If we use the area coordinates, Eq. (8.90) can be expressed as

$$\vec{P}_b^{(e)} = \iiint_{V^{(e)}} [N]^T \vec{\overline{\phi}}\, \mathrm{d}V = \iint_A \begin{bmatrix} L_i & 0 \\ 0 & L_i \\ L_j & 0 \\ 0 & L_j \\ L_k & 0 \\ 0 & L_k \end{bmatrix} \begin{Bmatrix} \overline{\phi}_r \\ \overline{\phi}_z \end{Bmatrix} 2\pi r\, \mathrm{d}A \tag{11.45}$$

The radial distance r can be written in terms of the area coordinates as

$$r = r_i L_i + r_j L_j + r_k L_k \tag{11.46}$$

By substituting Eq. (11.46) into Eq. (11.45) and evaluating the resulting area integrals using Eq. (3.78), we obtain

$$\vec{P}_b^{(e)} = \frac{2\pi A}{12} \begin{Bmatrix} (2r_i + r_j + r_k)\overline{\phi}_r \\ (2r_i + r_j + r_k)\overline{\phi}_z \\ (r_i + 2r_j + r_k)\overline{\phi}_r \\ (r_i + 2r_j + r_k)\overline{\phi}_z \\ (r_i + r_j + 2r_k)\overline{\phi}_r \\ (r_i + r_j + 2r_k)\overline{\phi}_z \end{Bmatrix} \tag{11.47}$$

It can be seen from Eq. (11.47) that the body forces are not distributed equally between the three nodes i, j, and k. If $\overline{\Phi}_r$ and $\overline{\Phi}_z$ denote the applied stresses in the r and z directions, the load vector $\vec{P}_s^{(e)}$ can be evaluated using the area coordinates as in the case of $\vec{P}_b^{(e)}$. If we assume that only the edge ij lies on the surface $S_1^{(e)}$ on which the stresses $\overline{\phi}_r$ and $\overline{\Phi}_z$ are acting (this implies that $L_k = 0$), we can write

$$\vec{P}_s^{(e)} = \iint_{S_1^{(e)}} [N]^T \begin{Bmatrix} \overline{\Phi}_r \\ \overline{\Phi}_z \end{Bmatrix} \mathrm{d}S_1 = \int_{s_{ij}} \begin{bmatrix} L_i & 0 \\ 0 & L_i \\ L_j & 0 \\ 0 & L_j \\ L_k & 0 \\ 0 & L_k \end{bmatrix} \begin{Bmatrix} \overline{\Phi}_r \\ \overline{\Phi}_z \end{Bmatrix} 2\pi r \cdot \mathrm{d}s \tag{11.48}$$

where $\mathrm{d}S_1 = 2\pi r\, \mathrm{d}s$, and s_{ij} denotes the length of the edge ij. By substituting Eq. (11.46) into Eq. (11.48), Eq. (3.77) can be used to evaluate the line integral of Eq. (11.48). This results in

$$\vec{P}_s^{(e)} = \frac{\pi s_{ij}}{3} \begin{bmatrix} (2r_i + r_j)\overline{\Phi}_r \\ (2r_i + r_j)\overline{\Phi}_z \\ (r_i + 2r_j)\overline{\Phi}_r \\ (r_i + 2r_j)\overline{\Phi}_z \\ 0 \\ 0 \end{bmatrix} \tag{11.49}$$

Note

If the edge, for example, ij, is vertical, we have $r = r_i = r_j$ along this edge, and hence Eq. (11.48) leads to

$$\vec{P}_s^{(e)} = \pi r_i s_{ij} \begin{Bmatrix} \overline{\Phi}_r \\ \overline{\Phi}_z \\ \overline{\Phi}_r \\ \overline{\Phi}_z \\ 0 \\ 0 \end{Bmatrix} \tag{11.50}$$

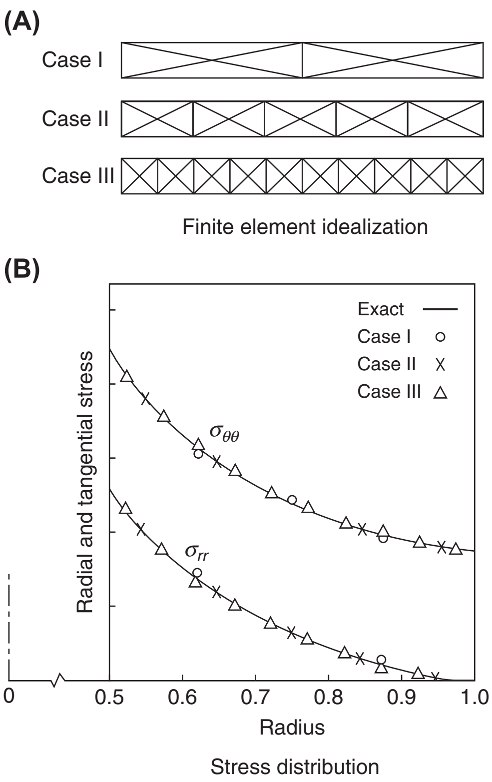

FIGURE 11.7 Infinite cylinder under internal pressure.

11.4.3 Numerical Results

An infinite cylinder subjected to an internal pressure, for which an exact solution is known, is selected as a means of demonstrating the accuracy of the finite element considered. Fig. 11.7A shows three finite element meshes [11.7]. The resulting radial and hoop stresses are plotted in Fig. 11.7B. Except for the very coarse mesh, agreement with the exact solution is excellent. In this figure, stresses are plotted at the center of the quadrilaterals and are obtained by averaging the stresses in the four connecting triangles. In general, good boundary stresses are estimated by plotting the interior stresses and extrapolating to the boundary. This type of engineering judgment is always necessary in evaluating results from a finite element analysis.

EXAMPLE 11.5

A triangular axisymmetric ring element with nodes i, j, and k is shown in Fig. 11.8. The (r, z) coordinates of the nodes in centimeters are also indicated in Fig. 11.8. Find the shape functions corresponding to the nodal degrees of freedom of the element.

Solution

The matrix of shape functions corresponding to the six degrees of freedom of the element is given by Eq. (11.32) with the shape functions defined in Eq. (11.34). The constants a_i, a_j, a_k, ..., c_k of Eq. (11.34) can be found by replacing x and y by r and z, respectively, in Eq. (3.32):

$$\begin{aligned}
a_i &= r_j z_k - r_k z_j = 70(20) - 50(60) = -1600 \\
a_j &= r_k z_i - r_i z_k = 50(40) - 80(20) = 400 \\
a_k &= r_i z_j - r_j z_i = 80(60) - 70(40) = 2000 \\
b_i &= z_j - z_k = 60 - 20 = 40 \\
b_j &= z_k - z_i = 20 - 40 = -20 \\
b_k &= z_i - z_j = 40 - 60 = -20 \\
c_i &= r_k - r_j = 50 - 70 = -20 \\
c_j &= r_i - r_k = 80 - 50 = 30 \\
c_k &= r_j - r_i = 70 - 80 = -10
\end{aligned}$$

Continued

EXAMPLE 11.5 —cont'd

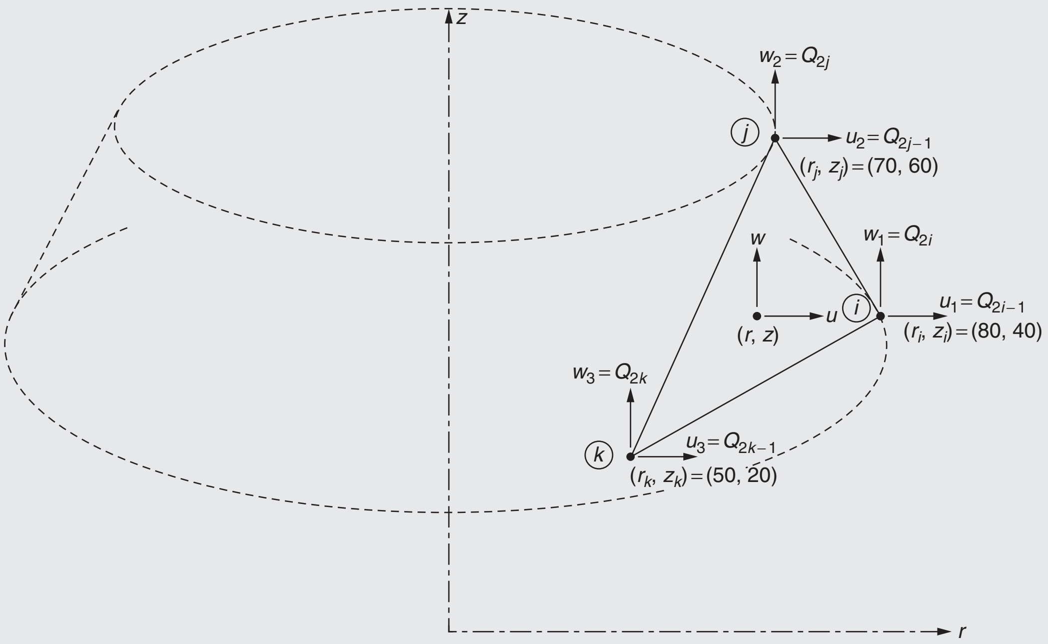

FIGURE 11.8 Triangular axisymmetric ring element.

The area of the triangle *ijk* is given by Eq. (11.35):

$$A = \frac{1}{2}(r_i z_j + r_j z_k + r_k z_i - r_i z_k - r_j z_i - r_k z_j)$$

$$= \frac{1}{2}[80(60) + 70(20) + 50(40) - 80(20) - 70(40) - 50(60)] = 400\ \text{cm}^2 = 0.04\ \text{m}^2$$

Thus, the shape functions are defined as

$$N_i(r,z) = \frac{1}{2A}(a_i + b_i r + c_i z) = -2.0 + 0.05r - 0.025z$$

$$N_j(r,z) = \frac{1}{2A}(a_j + b_j r + c_j z) = 0.5 - 0.025r + 0.0375z$$

$$N_k(r,z) = \frac{1}{2A}(a_k + b_k r + c_k z) = 2.5 - 0.025r + 0.0125z$$

EXAMPLE 11.6

Find the matrix $[\underline{B}]$ for the triangular ring element considered in Example 11.5.

Solution

The matrix $[B]$ is given by Eq. (11.37). Using the solution of Example 11.5, the $[B]$ matrix can be expressed as

$$[B] = \begin{bmatrix} 40 & 0 & -20 & 0 & -20 & 0 \\ \frac{1}{r}(-2 + 0.05r - 0.025z) & 0 & \frac{1}{r}(0.5 - 0.025r + 0.0375z) & 0 & \frac{1}{r}(2.5 - 0.025r - 0.0125z) & 0 \\ 0 & -20 & 0 & 30 & 0 & -10 \\ -20 & 40 & 30 & -20 & -10 & -20 \end{bmatrix} \quad \text{(E.1)}$$

EXAMPLE 11.6 —cont'd

The centroid of the element is given by

$$\{\underline{r}, \underline{z}\} = \left\{\frac{1}{3}(r_i + r_j + r_k), \frac{1}{3}(z_i + z_j + z_k)\right\} = \left\{\frac{1}{3}(80 + 70 + 50), \frac{1}{3}(40 + 60 + 20)\right\} = \{66.6667, 40.0\} \text{ cm} \quad \text{(E.2)}$$

The matrix $[\underline{B}]$ can be determined by evaluating $[B]$ at the centroid, $(r, z) = (\underline{r}, \underline{z})$. Noting that

$$\frac{1}{\underline{r}}(-2 + 0.05\underline{r} - 0.0125\underline{z}) = \frac{1}{66.6667}\{-2 + 0.05(66.6667) - 0.025(40)\} = 0.005 \quad \text{(E.3)}$$

$$\frac{1}{\underline{r}}(0.5 - 0.025\underline{r} + 0.0375\underline{z}) = \frac{1}{66.6667}\{0.5 - 0.025(66.6667) + 0.0375(40)\} = 0.005 \quad \text{(E.4)}$$

$$\frac{1}{\underline{r}}(2.5 - 0.025\underline{r} - 0.0125\underline{z}) = \frac{1}{66.6667}\{2.5 - 0.025(66.6667) - 0.0125(40)\} = 0.005 \quad \text{(E.5)}$$

the matrix $[\underline{B}]$ can be found as

$$[\underline{B}] = [B(\underline{r}, \underline{z})] = \begin{bmatrix} 0.05 & 0 & -0.025 & 0 & -0.025 & 0 \\ 0.005 & 0 & 0.005 & 0 & 0.005 & 0 \\ 0 & -0.025 & 0 & 0.0375 & 0 & -0.0125 \\ -0.025 & 0.05 & 0.0375 & -0.025 & -0.0125 & -0.025 \end{bmatrix} \quad \text{(E.6)}$$

EXAMPLE 11.7

If the Young's modulus and the Poisson's ratio of the material of the triangular ring element *ijk* considered in Example 11.5 are given by 207 GPa and 0.3, respectively, find the stiffness matrix of the element.

Solution

The stiffness matrix of the element is given by Eq. (11.43):

$$\left[K^{(e)}\right] = [\underline{B}]^T[D][\underline{B}]2\pi\underline{r}A \quad \text{(E.1)}$$

where $[\underline{B}]$ is given by Eq. (E.6) of Example 11.6 and $[D]$ by Eq. (11.39). For the given values of E and v, the matrix $[D]$ becomes

$$[D] = \frac{207(10^9)}{(1.3)(0.4)}\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} = 10^{11}\begin{bmatrix} 2.7865 & 1.1942 & 1.1942 & 0 \\ 1.1942 & 2.7865 & 1.1942 & 0 \\ 1.1942 & 1.1942 & 2.7865 & 0 \\ 0 & 0 & 0 & 0.7962 \end{bmatrix} \text{N/m}^2 \quad \text{(E.2)}$$

Noting that

$$2\pi\underline{r}A = 2\pi(66.6667)(400) = 1675.5208(10^2) \text{ cm}^3 = 0.167552 \text{ m}^3 \quad \text{(E.3)}$$

the stiffness matrix of the triangular ring element *ijk* can be found from Eq. (E.1) as

$$\left[K^{(e)}\right] = 10^8\begin{bmatrix} 1.3623 & -0.4419 & -0.6720 & 0.4961 & -0.5052 & -0.0542 \\ -0.4419 & 0.6253 & 0.3502 & -0.6045 & 0.0167 & -0.0208 \\ -0.6720 & 0.3502 & 0.4410 & -0.2751 & 0.1909 & -0.0750 \\ 0.4961 & -0.6045 & -0.2751 & 0.7399 & -0.1084 & -0.1355 \\ -0.5052 & 0.0167 & 0.1909 & -0.1084 & 0.2743 & 0.0917 \\ -0.0542 & -0.0208 & -0.0750 & -0.1355 & 0.0917 & 0.1563 \end{bmatrix} \frac{\text{N}}{\text{m}} \quad \text{(E.4)}$$

EXAMPLE 11.8

If the triangular ring element *ijk* considered in Examples 11.5 to 11.7 is heated by 20 °C, determine the resulting nodal load vector of the element. Assume the coefficient of thermal expansion of the material as $\alpha = 10.8 \times 10^{-6}$ m/m °C.

Solution

The nodal load vector of the element *ijk* due to a temperature change *T* (initial strain) is given by Eq. (11.44):

$$\vec{P}^{(e)}_{\text{initial}} = \frac{E\alpha T}{1-2v}[\underline{B}]^T \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} 2\pi \underline{r} A \tag{E.1}$$

Using the values of $E = 207 \times 10^9$ Pa, $v = 0.3$, $\alpha = 10.8 \times 10^{-6}$ m/m °C, $T = 20$ °C, $2\pi \underline{r} A = 0.167552$ m^3 (from Eq. (E.3) of Example 11.7), and the matrix $[\underline{B}]$ given by Eq. (E.3) of Example 11.6, the nodal load vector of Eq. (E.1) can be found as

$$\vec{P}^{(e)}_{\text{initial}} = 10^6 \begin{Bmatrix} 1.0301 \\ -0.4682 \\ -0.3746 \\ 0.7023 \\ -0.3746 \\ -0.2341 \end{Bmatrix} \text{N} \tag{E.2}$$

EXAMPLE 11.9

Find the nodal load vector of the triangular ring element *ijk* considered in Example 11.5 corresponding to the gravity force acting in the negative *z* direction. Assume the density of the material as $\rho = 7850$ kg/m^3.

Solution

The body forces, distributed uniformly over the volume of the element, are given by

$$\overline{\phi}_r = 0, \quad \overline{\phi}_z = -\rho g = -7850(9.81) = -77{,}008.5 \text{ N/m}^3 \tag{E.1}$$

Noting that the area of the triangle *ijk* is equal to $A = 0.04$ m^2 (from Example 11.5), $r_i = 0.8$ m, $r_j = 0.7$ m, $r_k = 0.5$ m, $2\, r_i + r_j + r_k = 2.8$ m, $r_i + 2\, r_j + r_k = 2.7$ m, $r_i + r_j + 2\, r_k = 2.5$ m, the nodal load vector of the element, given by Eq. (11.47), can be evaluated as

$$\vec{P}^{(e)}_{b} = \frac{2\pi(0.04)}{12} \begin{Bmatrix} 0 \\ 2.8(-77{,}008.5) \\ 0 \\ 2.7(-77{,}008.5) \\ 0 \\ 2.5(-77{,}008.5) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4516.0249 \\ 0 \\ 4354.7383 \\ 0 \\ 4032.1651 \end{Bmatrix} \text{N} \tag{E.2}$$

EXAMPLE 11.10

If uniformly distributed surface tractions (pressure loads) of magnitudes

$$\overline{\Phi}_r = 10000 \text{ N/m}^2, \quad \overline{\Phi}_z = 2000 \text{ N/m}^2 \tag{E.1}$$

act on the surface *ij* of the triangular ring element considered in Example 11.5, determine the corresponding nodal load vector of the element.

EXAMPLE 11.10 —cont'd

Solution

The nodal load vector corresponding to the surface tractions acting on the surface ij of the triangular ring element is given by Eq. (11.49):

$$\vec{P}_s^{(e)} = \frac{\pi\ s_{ij}}{3}\begin{Bmatrix}(2\ r_i + r_j)\ \overline{\Phi}_r \\ (2\ r_i + r_j)\ \overline{\Phi}_z \\ (r_i + 2\ r_j)\ \overline{\Phi}_r \\ (r_i + 2\ r_j)\ \overline{\Phi}_z \\ 0 \\ 0\end{Bmatrix} \tag{E.2}$$

where $r_i = 0.8$ m, $r_j = 0.7$ m, $z_i = 0.4$ m, $z_j = 0.6$ m, $2\ r_i + r_j = 2.3$ m, $r_i + 2\ r_j = 2.2$ m, and the length of the edge ij can be computed as

$$s_{ij} = \sqrt{(r_j - r_i)^2 + (z_j - z_i)^2} = \sqrt{(0.7 - 0.8)^2 + (0.6 - 0.4)^2} = 0.22361 \text{ m} \tag{E.3}$$

Using the values of the surface tractions in Eq. (E.1), the nodal load vector can be found as

$$\vec{P}_s^{(e)} = \frac{\pi(0.22361)}{3}\begin{Bmatrix}2.3(10,000) \\ 2.3(2000) \\ 2.2(10,000) \\ 2.2(2000) \\ 0 \\ 0\end{Bmatrix} = \begin{Bmatrix}5385.781 \\ 1077.154 \\ 5151.608 \\ 1030.322 \\ 0 \\ 0\end{Bmatrix} \text{N} \tag{E.4}$$

REVIEW QUESTIONS

11.1 Give brief answers to the following questions.

1. State two practical problems that require three-dimensional finite element modeling.
2. What is the basic or simplest element for three-dimensional problems?
3. Show a typical hexahedron element through a sketch.
4. What are the directions of displacements in axisymmetric solid body subjected to axisymmetric loads?
5. What is the basic element that can be used for the analysis of axisymmetric problems?
6. Show the nodal degrees of freedom for an axisymmetric ring element with triangular cross section.

11.2 Fill in the blank with a suitable word.

1. For the stress analysis of a three-dimensional problem using a four-noded tetrahedron element, the displacement components u, v, and w are assumed to vary ________________ inside the element.
2. ZIB 8 is the abbreviation of ________________ ________________ Brick with 8 nodes.
3. The value of each of the natural coordinates r, s, and t at every node in a hexahedron element is either — one or ________________ one.
4. The triple integrals involved in the generation of the stiffness matrix of a hexahedron element are usually evaluated using ________________ quadrature.
5. The size of the stiffness matrix of an axisymmetric element with triangular cross section is ________________.

11.3 Indicate whether the following statement is true or false.

1. Two coordinate systems—a local one and a global one—are necessary for a tetrahedron element.
2. The evaluation of the stiffness matrix of a four-noded tetrahedron does not require any integrations.
3. A solid of revolution can be called an axisymmetric solid.
4. The ZIB 8 and the eight-noded hexahedron element are two different elements.

11.4 Select the most appropriate answer among the multiple choices given.

1. The number of strain/stress components need to be considered in a four-noded tetrahedron element is:
   (a) 3 (b) 6 (c) 4
2. The size of the stiffness matrix of a four-noded tetrahedron element is:
   (a) 4×4 (b) 8×8 (c) 12×12
3. The natural coordinates r, s, and t for a hexahedron element is more convenient because these coordinates are associated with pairs of opposite:
   (a) n faces (b) nodes (c) diagonals
4. The size of the stiffness matrix of the ZIB 8 element is:
   (a) 8×8 (b) 24×24 (c) 12×12
5. The coordinates that are convenient for modeling axisymmetric problems are:
   (a) Cartesian (b) cylindrical (c) spherical
6. The number of nonzero strains in an axisymmetric problem is:
   (a) 3 (b) 4 (c) 6

PROBLEMS

11.1 The X, Y, Z coordinates of the nodes of a tetrahedron element, in inches, are shown in Fig. 11.9.
  a. Derive the matrix $[B]$.
  b. Derive the stiffness matrix of the element assuming that $E = 30 \times 10^6$ psi and $\nu = 0.32$.

11.2 Find the nodal displacements and the stress distribution in the element shown in Fig. 11.9 by fixing the face 123. Assume the loads applied at node 4 as $P_X = 50$ lb, $P_Y = 100$ lb, and $P_Z = -150$ lb.

11.3 A uniform pressure of 100 psi is applied on the face 234 of the tetrahedron element shown in Fig. 11.9. Determine the corresponding load vector of the element.

11.4 If the temperature of the element shown in Fig. 11.9 is increased by 50 °F while all the nodes are constrained, determine the corresponding load vector. Assume the coefficient of thermal expansion as $\alpha = 6.5 \times 10^{-6}$ per °F.

11.5 The X, Y, Z coordinates of a hexahedron element are shown in Fig. 11.10. Derive the matrix $[J]$.

11.6 An axisymmetric ring element is shown in Fig. 11.11.
  a. Derive the matrix $[B]$.
  b. Derive the matrix $[D]$, for steel with $E = 30 \times 10^6$ psi and $\nu = 0.33$.
  c. Derive the element stiffness matrix, $[K]$.

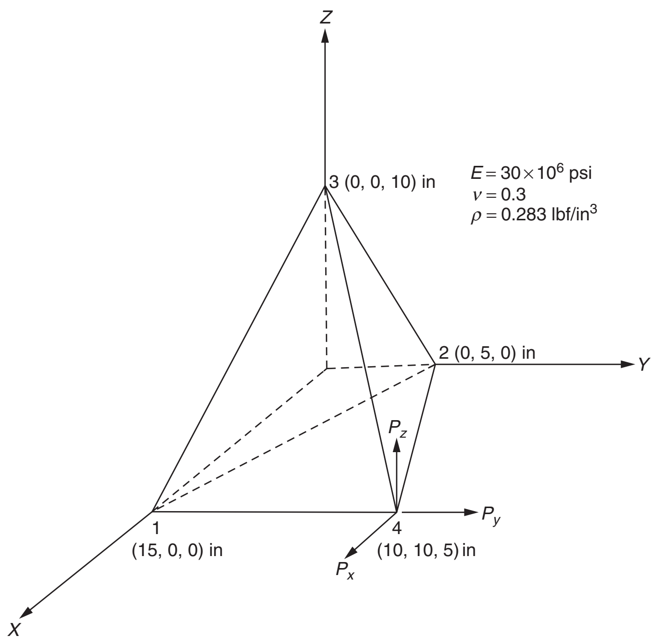

FIGURE 11.9 Tetrahedron element.

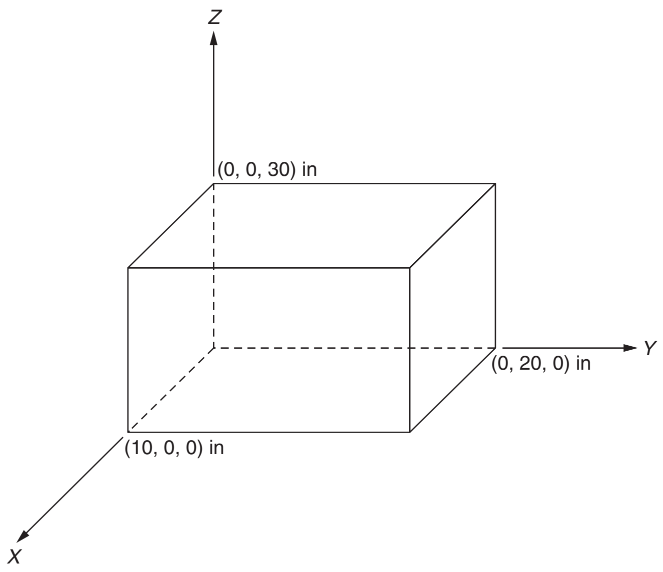

FIGURE 11.10 Hexahedron element.

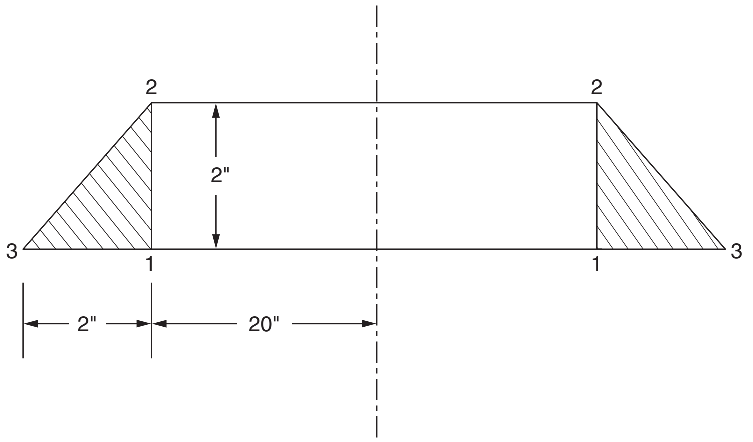

FIGURE 11.11 Axisymmetric ring element.

11.7 If the element shown in Fig. 11.11 is subjected to an initial strain due to an increase in temperature of 50 °F, determine the corresponding load vector. Assume a value of $\alpha = 6.5 \times 10^{-6}$ per °F.

11.8 If the face 23 of the element shown in Fig. 11.11 is subjected to a uniform pressure of 200 psi, determine the corresponding load vector.

11.9 A hexagonal plate with a circular hole is subjected to a uniform pressure on the inside surface as shown in Fig. 11.12A. Due to the symmetry of the geometry and the load, only a 30° segment of the plate can be considered for the finite element analysis [Fig. 11.12B]. Indicate a procedure for incorporating the boundary conditions along the X and s axes.

Hint: The symmetry conditions require that the nodes along the X and s axes should have zero displacement in a direction normal to the X and s axes, respectively. If the global degrees of freedom at node are denoted as Q_{2i-1} and Q_{2i}, then the boundary condition becomes a multipoint constraint that can be expressed as [11.8]

$$-Q_{2i-1} \sin\theta + Q_{2i} \cos\theta = 0$$

A method of incorporating this type of constraint was indicated in Section 6.5.

11.10 Write a program called SØLID for the analysis of three-dimensional solid bodies using tetrahedron elements. Find the tip deflection of the short cantilever beam discussed in Section 11.3.6 using this SØLID subroutine.

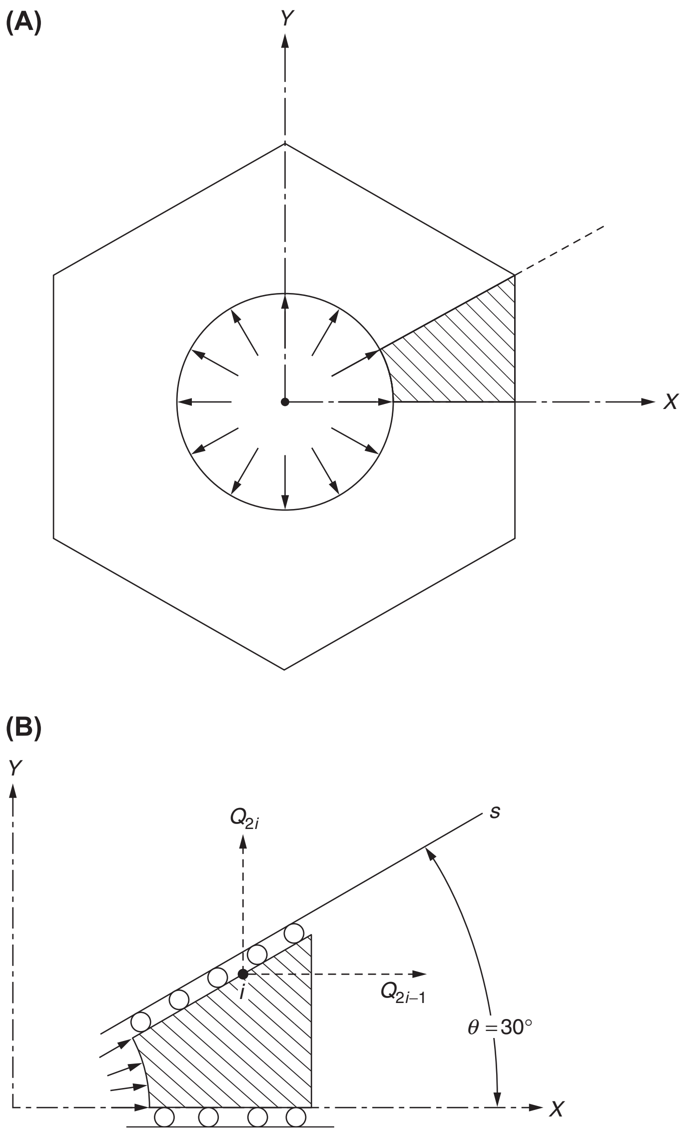

FIGURE 11.12 Hexagonal plate with circular hole.

11.11 For a tetrahedron element with nodes i, j, k, and l in the global coordinate system (shown in Fig. 11.13), magnetic forces of varying intensity

$$\vec{\vec{\phi}} = \begin{Bmatrix} \phi_x(x, y, z) \\ \phi_y(x, y, z) \\ \phi_z(x, y, z) \end{Bmatrix}$$

act throughout the element. Derive an expression for the corresponding nodal load vector of the element.

11.12 For a tetrahedron element with nodes i, j, k, and l in the global coordinate system (shown in Fig. 11.13), surface tractions

$$\vec{\vec{\Phi}} = \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix}$$

act on the face jkl of the element. Derive the expression for the corresponding nodal load vector of the element.

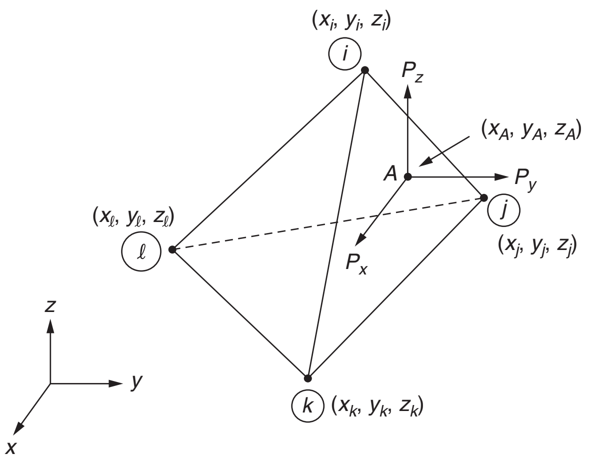

FIGURE 11.13 Tetrahedron element.

11.13 For a tetrahedron element with nodes i, j, k, and l in the global coordinate system (shown in Fig. 11.13), surface tractions

$$\vec{\vec{\Phi}} = \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix}$$

act on the face kli of the element. Derive the expression for the corresponding nodal load vector of the element.

11.14 For a tetrahedron element with nodes i, j, k, and l in the global coordinate system (shown in Fig. 11.13), surface tractions

$$\vec{\vec{\Phi}} = \begin{Bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{Bmatrix}$$

act on the face lij of the element. Derive the expression for the corresponding nodal load vector of the element.

11.15 In a three-dimensional problem modeled with tetrahedron elements, external concentrated forces of magnitude

$$\vec{P}_A = \begin{Bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{Bmatrix}$$

act at point A, which happens to lie somewhere on the face ijk of a tetrahedron element e with nodes i, j, k, and l in the global coordinate system (as shown in Fig. 11.13). Derive the corresponding nodal load vector of the element e.

11.16 In a three-dimensional problem modeled with tetrahedron elements, external concentrated forces of magnitude

$$\vec{P}_A = \begin{Bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{Bmatrix}$$

act at point A, which happens to lie somewhere on the face jkl of a tetrahedron element e with nodes i, j, k, and l in the global coordinate system. Derive the corresponding nodal load vector of the element e.

11.17 In a three-dimensional problem modeled with tetrahedron elements, external concentrated forces of magnitude

$$\vec{P}_A = \begin{Bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{Bmatrix}$$

act at point A, which happens to lie somewhere on the face kli of a tetrahedron element e with nodes i, j, k, and l in the global coordinate system. Derive the corresponding nodal load vector of the element e.

11.18 In a three-dimensional problem modeled with tetrahedron elements, external concentrated forces of magnitude

$$\vec{P}_A = \begin{Bmatrix} P_{Ax} \\ P_{Ay} \\ P_{Az} \end{Bmatrix}$$

act at point A, which happens to lie somewhere on the face lij of a tetrahedron element e with nodes i, j, k, and l in the global coordinate system. Derive the corresponding nodal load vector of the element e.

11.19 A uniformly distributed pressure of magnitude $\overline{\Phi} = 1$ MPa acts normal to the face ij as shown in Fig. 11.14 (all around the ring) of a triangular ring element. Determine the equivalent nodal load vector of the element.

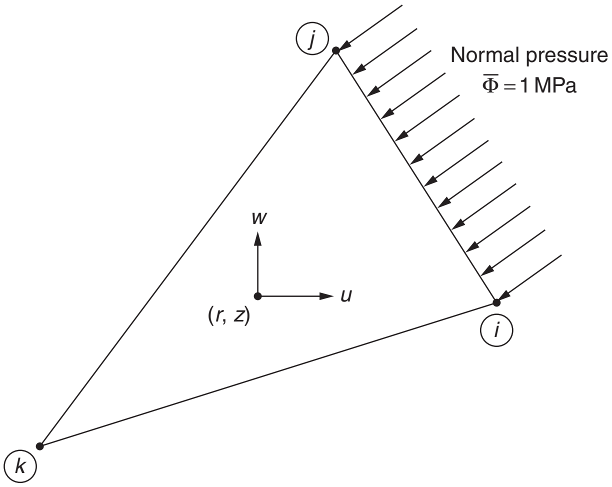

FIGURE 11.14 Triangular ring element under uniform pressure.

11.20 A distributed pressure varies linearly from 1 MPa at node i to 2 MPa at node j of the face ij as shown in Fig. 11.15 (all around the ring) of a triangular ring element. Determine the equivalent nodal load vector of the element. Hint: Find the r and z components of the normal pressures at points i and j as $(\overline{\Phi}_{ri}, \overline{\Phi}_{zi})$ and $(\overline{\Phi}_{rj}, \overline{\Phi}_{zj})$, respectively. Use $\overline{\Phi}_r = N_1\overline{\Phi}_{ri} + N_2\overline{\Phi}_{rj}$ and $\overline{\Phi}_z = N_1\overline{\Phi}_{zi} + N_2\overline{\Phi}_{zj}$ and the relations $\int_{s_{ij}} N_1^2\,\mathrm{d}s = \frac{1}{3}s_{ij}$, $\int_{s_{ij}} N_2^2\,\mathrm{d}s = \frac{1}{3}s_{ij}$, $\int_{s_{ij}} N_1N_2\,\mathrm{d}s = \frac{1}{6}s_{ij}$ with $s_{ij} = \sqrt{(r_j - r_i)^2 + (z_j - z_i)^2}$.

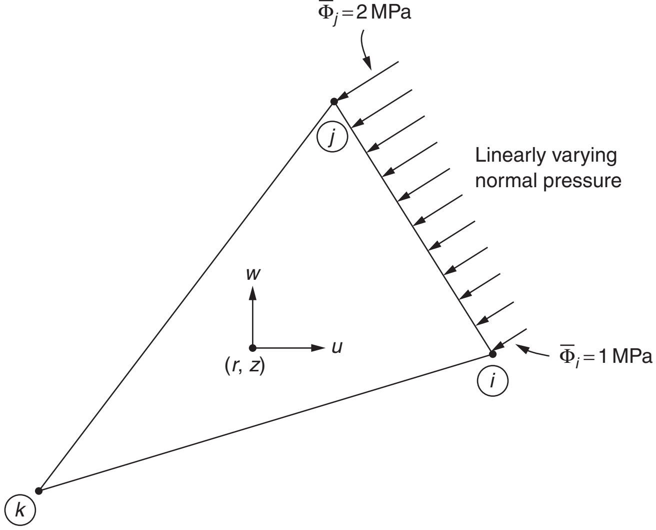

FIGURE 11.15 Triangular ring element under varying pressure.

11.21 A long thick-walled cylindrical pressure vessel of inner radius 0.75 m and outer radius 1.0 m is subjected to an internal pressure of 5 MPa. Considering an axial length of 0.2 m, determine the radial displacement of the inner surface of the pressure vessel. The material properties of the pressure vessel are given by $E = 207$ GPa and $v = 0.3$. Assume that the outer surface of the pressure vessel is constrained from any displacement. Use two triangular ring elements for idealization as shown in Fig. 11.16.

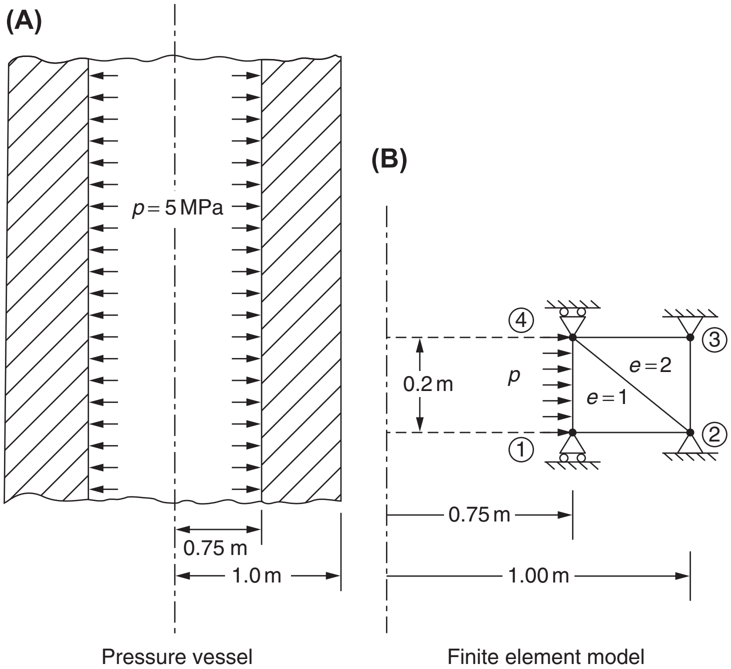

FIGURE 11.16 Thick-walled cylindrical pressure vessel.

11.22 Solve Example 11.9 by using the approximate values $L_i = \underline{L}_i, L_j = \underline{L}_j$, and $L_k = \underline{L}_k$, evaluated at the centroid $(\underline{r}, \underline{z})$ of the element cross section, in the integral of Eq. (11.45).
11.23 Find the nodal load vector of a triangular ring element if surface tractions $\overline{\Phi}_r$ and $\overline{\Phi}_z$ act on the surface jk.
11.24 Find the nodal load vector of a triangular ring element if surface tractions $\overline{\Phi}_r$ and $\overline{\Phi}_z$ act on the surface ki.

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Chapter 12

Dynamic Analysis

Chapter Outline

12.1 DYNAMIC EQUATIONS OF MOTION

In dynamic problems the displacements, velocities, strains, stresses, and loads are all time dependent. The procedure involved in deriving the finite element equations of a dynamic problem can be stated by the following steps:

Step 1: Idealize the body into E finite elements.
Step 2: Assume the displacement model of element e as

$$\vec{U}(x,y,z,t) = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} = [N(x,y,z)]\vec{Q}^{(e)}(t) \tag{12.1}$$

where $\vec{U}$ is the vector of displacements, $[N]$ is the matrix of shape functions, and $\vec{Q}^{(e)}$ is the vector of nodal displacements that is assumed to be a function of time t.

Step 3: Derive the element characteristic (stiffness and mass) matrices and characteristic (load) vector.

From Eq. (12.1), the strains can be expressed as

$$\vec{\varepsilon} = [B]\vec{Q}^{(e)} \tag{12.2}$$

and the stresses as

$$\vec{\sigma} = [D]\vec{\varepsilon} = [D][B]\vec{Q}^{(e)} \tag{12.3}$$

By differentiating Eq. (12.1) with respect to time, the velocity field can be obtained as

$$\dot{\vec{U}}(x,y,z,t) = [N(x,y,z)]\dot{\vec{Q}}^{(e)}(t) \tag{12.4}$$

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where $\overset{\cdot}{\vec{Q}}^{(e)}$ is the vector of nodal velocities. To derive the dynamic equations of motion of a structure, we can use either Lagrange equations [12.1] or Hamilton's principle stated in Section 8.3.2. The Lagrange equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial L}{\partial \dot{Q}}\right\} - \left\{\frac{\partial L}{\partial Q}\right\} + \left\{\frac{\partial R}{\partial \dot{Q}}\right\} = \{0\} \tag{12.5}$$

where

$$L = T - \pi_p \tag{12.6}$$

is called the Lagrangian function, T is the kinetic energy, π_p is the potential energy, R is the dissipation function, Q is the nodal displacement, and $\dot{Q}$ is the nodal velocity. The kinetic and potential energies of an element e can be expressed as

$$T^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \rho \dot{\vec{U}}^T \dot{\vec{U}}\,\mathrm{d}V \tag{12.7}$$

and

$$\pi_P^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \vec{\varepsilon}^T \vec{\sigma}\,\mathrm{d}V - \iint_{S_1^{(e)}} \vec{U}^T \vec{\overline{\Phi}}\,\mathrm{d}S_1 - \iiint_{V^{(e)}} \vec{U}^T \vec{\overline{\phi}}\,\mathrm{d}V \tag{12.8}$$

where $V^{(e)}$ is the volume, ρ is the density, and $\dot{\vec{U}}$ is the vector of velocities of element e. By assuming the existence of dissipative forces proportional to the relative velocities, the dissipation function of the element e can be expressed as

$$R^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \mu \dot{\vec{U}}^T \dot{\vec{U}}\,\mathrm{d}V \tag{12.9}$$

where μ can be called the damping coefficient. In Eqs. (12.7) to (12.9), the volume integral has to be taken over the volume of the element, and in Eq. (12.8) the surface integral has to be taken over that portion of the surface of the element on which distributed surface forces are prescribed.

By using Eqs. (12.1) to (12.3), the expressions for T, π_p, and R can be written as

$$T = \sum_{e=1}^{E} T^{(e)} = \frac{1}{2}\dot{\underset{\sim}{\vec{Q}}}^T\left[\sum_{e=1}^{E}\iiint_{V^{(e)}} \rho [N]^T[N]\mathrm{d}V\right]\dot{\underset{\sim}{\vec{Q}}} \tag{12.10}$$

$$\begin{aligned}\pi_p = \sum_{e=1}^{E}\pi_p^{(e)} = \frac{1}{2}\underset{\sim}{\vec{Q}}^T\left[\sum_{e=1}^{E}\iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V\right]\underset{\sim}{\vec{Q}} - \\ \underset{\sim}{\vec{Q}}^T\left(\sum_{e=1}^{E}\iint_{S_1^{(e)}}[N]^T\vec{\overline{\Phi}}(t)\mathrm{d}S_1 + \iiint_{V^{(e)}}[N]^T\vec{\overline{\phi}}(t)\mathrm{d}V\right) - \underset{\sim}{\vec{Q}}^T\underset{\sim}{\vec{P}}_c(t)\end{aligned} \tag{12.11}$$

$$R = \sum_{e=1}^{E} R^{(e)} = \frac{1}{2}\dot{\underset{\sim}{\vec{Q}}}^T\left[\sum_{e=1}^{E}\iiint_{V^{(e)}} \mu [N]^T[N]\mathrm{d}V\right]\dot{\underset{\sim}{\vec{Q}}} \tag{12.12}$$

where $\underset{\sim}{\vec{Q}}$ is the global nodal displacement vector, $\dot{\underset{\sim}{\vec{Q}}}$ is the global nodal velocity vector, and $\underset{\sim}{\vec{P}}_c$ is the vector of concentrated nodal forces of the structure or body. The matrices involving the integrals can be defined as follows:

$$\left[M^{(e)}\right] = \text{element mass matrix} = \iiint_{V^{(e)}} \rho [N]^T[N]\mathrm{d}V \tag{12.13}$$

$$\left[K^{(e)}\right] = \text{element stiffness matrix} = \iiint_{V^{(e)}} [B]^T[D][B]\mathrm{d}V \tag{12.14}$$

$$\left[C^{(e)}\right] = \text{element damping matrix} = \iiint_{V^{(e)}} \mu [N]^T[N]\mathrm{d}V \tag{12.15}$$

$$\vec{P}_s^{(e)} = \text{vector of element nodal forces produced by surface forces} = \iint_{S_1^{(e)}} [N]^T\vec{\overline{\Phi}}\cdot\mathrm{d}S_1 \tag{12.16}$$

$$\vec{P}_b^{(e)} = \text{vector of element nodal forces produced by body forces} = \iiint_{V^{(e)}} [N]^T \vec{\phi} \cdot dV \tag{12.17}$$

Step 4: Assemble the element matrices and vectors and derive the overall system equations of motion. Eqs. (12.10) to (12.12) can be written as

$$T = \frac{1}{2} \dot{\vec{\underset{\sim}{Q}}}^T \left[\underset{\sim}{M}\right] \dot{\vec{\underset{\sim}{Q}}} \tag{12.18}$$

$$\pi_P = \frac{1}{2} \vec{\underset{\sim}{Q}}^T \left[\underset{\sim}{K}\right] \vec{\underset{\sim}{Q}} - \vec{\underset{\sim}{Q}}^T \vec{P} \tag{12.19}$$

$$R = \frac{1}{2} \dot{\vec{\underset{\sim}{Q}}}^T \left[\underset{\sim}{C}\right] \dot{\vec{\underset{\sim}{Q}}} \tag{12.20}$$

where

$$\left[\underset{\sim}{M}\right] = \text{master mass matrix of the structure} = \sum_{e=1}^{E} \left[M^{(e)}\right]$$

$$\left[\underset{\sim}{K}\right] = \text{master stiffness matrix of the structure} = \sum_{e=1}^{E} \left[K^{(e)}\right]$$

$$\left[\underset{\sim}{C}\right] = \text{master damping matrix of the structure} = \sum_{e=1}^{E} \left[C^{(e)}\right]$$

$$\vec{\underset{\sim}{P}}(t) = \text{total load vector} = \sum_{e=1}^{E} \left(P_s^{(e)}(t) + P_b^{(e)}(t)\right) + \vec{\underset{\sim}{P}}_c(t)$$

By substituting Eqs. (12.18) to (12.20) into Eq. (12.5), we obtain the desired dynamic equations of motion of the structure or body as

$$\left[\underset{\sim}{M}\right] \ddot{\vec{\underset{\sim}{Q}}}(t) + \left[\underset{\sim}{C}\right] \dot{\vec{\underset{\sim}{Q}}}(t) + \left[\underset{\sim}{K}\right] \vec{\underset{\sim}{Q}}(t) = \vec{\underset{\sim}{P}}(t) \tag{12.21}$$

where $\ddot{\vec{\underset{\sim}{Q}}}$ is the vector of nodal accelerations in the global system. If damping is neglected, the equations of motion can be written as

$$\left[\underset{\sim}{M}\right] \ddot{\vec{\underset{\sim}{Q}}} + \left[\underset{\sim}{K}\right] \vec{\underset{\sim}{Q}} = \vec{\underset{\sim}{P}} \tag{12.22}$$

Step 5: Solve the equations of motion by applying the boundary and initial conditions. Eq. (12.21) or (12.22) can be solved by using any of the techniques discussed in Section 7.4 for propagation problems.
Step 6: Once the time history of nodal displacements, $\vec{\underset{\sim}{Q}}(t)$, is known, the time histories of stresses and strains in the elements can be found as in the case of static problems
Special space−time finite elements have also been developed for the solution of dynamic solid and structural mechanics problems [12.2,12.3].

12.2 CONSISTENT AND LUMPED MASS MATRICES

Eq. (12.13) for the mass matrix was first derived by Archer [12.4] and is called the consistent mass matrix of the element. It is called consistent because the same displacement model that is used for deriving the element stiffness matrix is used for the derivation of mass matrix. It is of interest to note that several dynamic problems have been and are being solved with simpler forms of mass matrices. The simplest form of mass matrix that can be used is that obtained by placing point (concentrated) masses m_i at node points i in the directions of the assumed displacement degrees of freedom. The concentrated masses refer to translational and rotational inertia of the element and are calculated by assuming that the material within the mean locations on either side of the particular displacement behaves like a rigid body while the remainder of the element does not participate in the motion. Thus, this assumption excludes the dynamic coupling that exists between the element displacements, and hence the resulting element mass matrix is purely diagonal, and is called the lumped mass matrix.

The lumped mass matrices will lead to nearly exact results if small but massive objects are placed at the nodes of a lightweight structure. The consistent mass matrices will be exact if the actual deformed shape (under dynamic conditions) is contained in the displacement shape functions $[N]$. Since the deformed shape under dynamic conditions is not known, frequently the static displacement distribution is used for $[N]$. Hence, the resulting mass distribution will only be approximate; however, the accuracy is generally adequate for most practical purposes. Since lumped element matrices are diagonal, the assembled or overall mass matrix of the structure requires less storage space than the consistent mass matrix. Moreover, the diagonal lumped mass matrices greatly facilitate the desired computations.

12.2.1 Consistent and Lumped Mass Matrices of a Bar Element

The displacement model (linear) of the bar element shown in Fig. 12.1A is given by

$$u(x) = [N]\vec{q}^{(e)} \tag{12.23}$$

where

$$[N] = \left[\left(1 - \frac{x}{l}\right)\left(\frac{x}{l}\right)\right] \tag{12.24}$$

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^{(e)} = \begin{Bmatrix} u(x=0) \\ u(x=l) \end{Bmatrix}^{(e)} \tag{12.25}$$

and u is the axial displacement parallel to the x axis. The consistent mass matrix of the element is given by

$$[m^{(e)}] = \iiint_{V^{(e)}} \rho [N]^T [N] \mathrm{d}V = \frac{\rho A l}{6} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array} \tag{12.26}$$

where A is the uniform cross-sectional area, and l is the length of the element. Thus, the consistent mass matrices, in general, are fully populated. On the other hand, the lumped mass matrix of the element can be obtained (by dividing the total mass of the element equally between the two nodes) as

$$[m^{(e)}]_l = \frac{\rho A l}{2} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array} \tag{12.27}$$

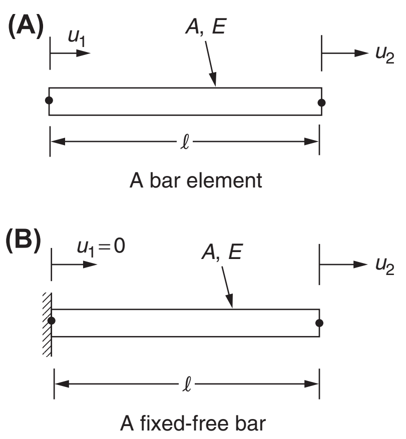

FIGURE 12.1 Bar element.

12.2.2 Consistent and Lumped Mass Matrices of a Space Truss Element

In a space truss element, a point located at a distance x from the left end (origin of local x axis) undergoes an axial displacement with components $u(x)$, $v(x)$, and $w(x)$ along the global X, Y, and Z directions, respectively. Since the displacement variation is linear, we can express $u(x)$, $v(x)$, and $w(x)$ (see Fig. 12.2) as

$$\underset{3\times1}{\vec{U}}(x) = \begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix} = \underset{3\times6}{[N]}\ \underset{6\times1}{\vec{Q}^{(e)}} \tag{12.28}$$

where

$$[N] = \begin{bmatrix} \left(1-\frac{x}{l}\right) & 0 & 0 & \frac{x}{l} & 0 & 0 \\ 0 & \left(1-\frac{x}{l}\right) & 0 & 0 & \frac{x}{l} & 0 \\ 0 & 0 & \left(1-\frac{x}{l}\right) & 0 & 0 & \frac{x}{l} \end{bmatrix}$$

and

$$\vec{Q}^{(e)} = \begin{Bmatrix} Q_{3i-2} \\ Q_{3i-1} \\ Q_{3i} \\ Q_{3j-2} \\ Q_{3j-1} \\ Q_{3j} \end{Bmatrix} \tag{12.29}$$

where Q_{3i-2}, Q_{3i-1}, and Q_{3i} are the components of displacement of node i (local node 1), and Q_{3j-2}, Q_{3j-1}, and Q_{3j} are the components of displacement of node j (local node 2) in the global XYZ system. If the density (ρ) and cross-sectional area (A) of the bar are constant, the consistent mass matrix of the element can be obtained as

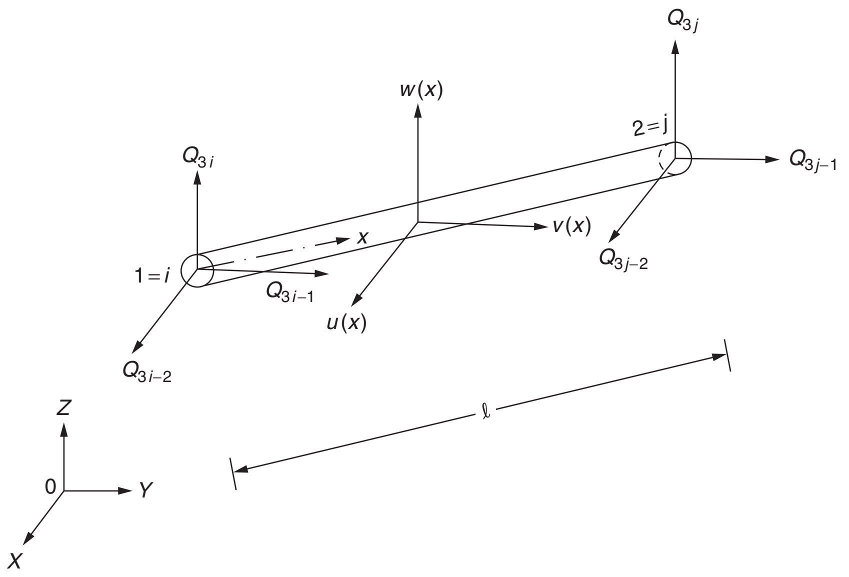

FIGURE 12.2 A truss element in space.

$$\left[m^{(e)}\right] = \left[M^{(e)}\right] = \iiint_{V^{(e)}} \rho[N]^T[N] \cdot \mathrm{dV}$$

$$= \frac{\rho A l}{6}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{12.30}$$

To find the lumped mass matrix of the element, as in the case of the bar element, the total element mass in each direction is distributed equally to the nodes of the element and the masses are associated with translational degrees of freedom in each of the X, Y, and Z directions. Thus, the lumped mass matrix of a space truss element is given by

$$\left[m^{(e)}\right]_l = \frac{\rho A l}{2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.31}$$

EXAMPLE 12.1

Derive the consistent and lumped mass matrices of a planar truss element shown in Fig. 12.3.

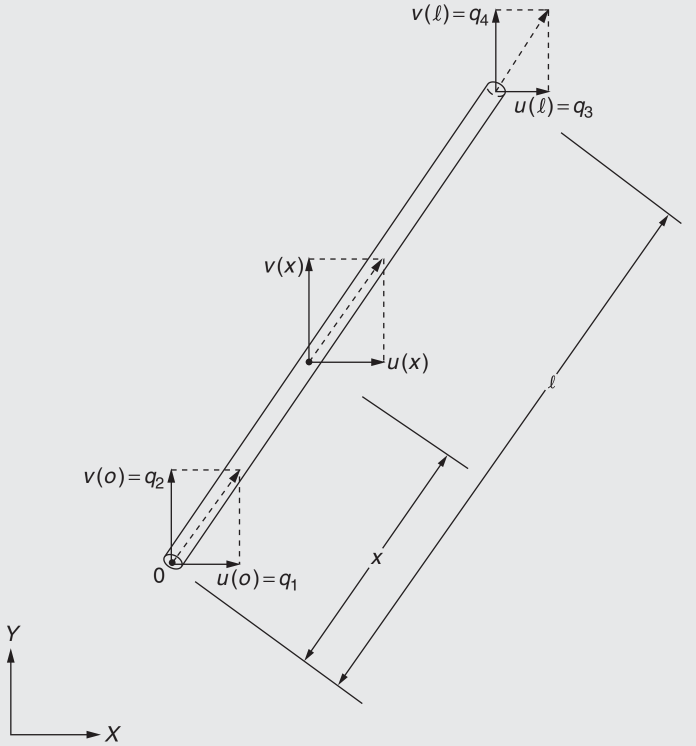

FIGURE 12.3 A planar truss element.

EXAMPLE 12.1 —cont'd

Solution

In a planar truss element, a point located at a distance x from the left end (origin of the local x axis) undergoes an axial displacement with components $u(x)$ and $v(x)$ along the global X and Y directions, respectively. Since the displacement variation is linear, we can express $u(x)$ and $v(x)$ as

$$\vec{U}(x) = [N]\vec{q}^{(e)} \tag{E.1}$$

where

$$\vec{U}(x) = \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix}, [N(x)] = \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \\ 0 & N_1(x) & 0 & N_2(x) \end{bmatrix}, \vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{E.2}$$

with

$$N_1(x) = 1 - \frac{x}{l} \quad \text{and} \quad N_2(x) = \frac{x}{l} \tag{E.3}$$

The consistent mass matrix of the element, $[m^{(e)}]$, can be evaluated as

$$\begin{aligned} \left[m^{(e)}\right] &= \iiint_{V^{(e)}} \rho [N]^T [N] \, \mathrm{d}V \\ &= \int_{x=0}^{l} \rho \begin{bmatrix} N_1(x) & 0 \\ 0 & N_1(x) \\ N_2(x) & 0 \\ 0 & N_2(x) \end{bmatrix} \begin{bmatrix} N_1(x) & 0 & N_2(x) & 0 \\ 0 & N_1(x) & 0 & N_2(x) \end{bmatrix} A \, \mathrm{d}x \end{aligned} \tag{E.4}$$

By carrying out the integrations in Eq. (E.4), we obtain the consistent mass matrix of the element as

$$\left[m^{(e)}\right] = \frac{\rho A l}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \tag{E.5}$$

To find the lumped mass matrix of the element, as in the case of the bar element, the total element mass in each direction is distributed equally to the nodes of the element and the masses are associated with translational degrees of freedom in both X and Y directions. Thus, the lumped mass matrix of a planar truss element is given by

$$\left[m^{(e)}\right]_l = \frac{\rho A l}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.6}$$

12.2.3 Consistent and Lumped Mass Matrices of a Uniform Beam Element

The transverse displacement $w(x)$ at a distance x from the origin can be expressed as (see Fig. 12.4)

$$w(x) = [N(x)]\vec{W}^{(e)} \tag{12.32}$$

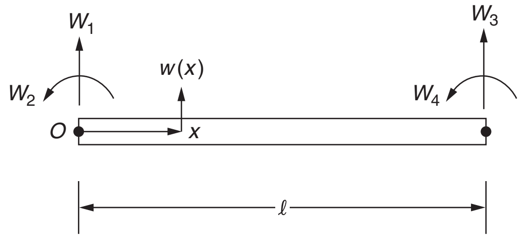

FIGURE 12.4 A beam element.

where $[N(x)]$ is given by Eq. (9.23) and

$$\vec{W}^{(e)} = \{ W_1 \quad W_2 \quad W_3 \quad W_4 \}^T \tag{12.33}$$

The consistent mass matrix of the element can be found as

$$\left[m^{(e)}\right] = \iiint_{V^{(e)}} \rho [N]^T [N] \mathrm{d}V = \rho \int_{x=0}^{l} [N]^T [N] \mathrm{d}x \iint_A \mathrm{d}A$$

$$= \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{12.34}$$

For the lumped mass matrix of a uniform beam element, the total mass of the element is distributed equally to the translational degrees of freedom at the two nodes. If rotatory inertia of the element is neglected, the rotational inertias (masses) associated with the rotational degrees of freedom will be zero. Thus, the lumped mass matrix of a beam element is given by

$$\left[m^{(e)}\right]_l = \frac{\rho A l}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12.35}$$

12.2.4 Consistent Mass Matrix of a Space Frame Element

A space frame element will have 12 degrees of freedom, 6 deflections, and 6 rotations, as shown in Fig. 9.9A. By taking the origin of the local coordinate system at node 1, the x axis along the length of the element, and the y and z axes along the principal axes of the element cross section, the displacement model can be expressed as

$$\vec{U}(x) = \begin{Bmatrix} u(x) \\ v(x) \\ w(x) \end{Bmatrix} = [N(x)]\vec{q}^{\,(e)} \tag{12.36}$$

where

$$[N(x)] = \left[\begin{array}{cccccccccccc} 1-\frac{x}{l} & 0 & 0 & 0 & 0 & 0 & \frac{x}{l} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{l^3}\left(2x^3-3lx^2+l^3\right) & 0 & 0 & 0 & \frac{1}{l^2}\left(x^3-2lx^2+l^2x\right) & 0 & -\frac{1}{l^3}\left(2x^3-3lx^2\right) & 0 & 0 & 0 & \frac{1}{l^2}\left(x^3-lx^2\right) \\ 0 & 0 & \frac{1}{l^3}\left(2x^3-3lx^2+l^3\right) & 0 & -\frac{1}{l^2}\left(x^3-2lx^2+l^2x\right) & 0 & 0 & 0 & -\frac{1}{l^3}\left(2x^3-3lx^2\right) & 0 & \frac{1}{l^2}\left(lx^2-x^3\right) & 0 \end{array}\right] \tag{12.37}$$

and

$$\vec{q}^{\,(e)} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{12} \end{Bmatrix}^{(e)} \tag{12.38}$$

The consistent mass matrix of the element in the local xyz system can be derived as

$$\left[m^{(e)}\right] = \iiint_{V^{(e)}} \rho [N]^T [N] \; \mathrm{d}V$$

$$= \rho A l \begin{bmatrix} \frac{1}{3} & & & & & & & & & & \text{Symmetric} & \\ 0 & \frac{13}{35} & & & & & & & & & & \\ 0 & 0 & \frac{13}{35} & & & & & & & & & \\ 0 & 0 & 0 & \frac{J}{3A} & & & & & & & & \\ 0 & 0 & -\frac{11}{210}l & 0 & \frac{l^2}{105} & & & & & & & \\ 0 & \frac{11}{210}l & 0 & 0 & 0 & \frac{l^2}{105} & & & & & & \\ \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & & & & & \\ 0 & \frac{9}{70} & 0 & 0 & 0 & \frac{13}{420}l & 0 & \frac{13}{35} & & & & \\ 0 & 0 & \frac{9}{70} & 0 & -\frac{13}{420}l & 0 & 0 & 0 & \frac{13}{35} & & & \\ 0 & 0 & 0 & \frac{J}{6A} & 0 & 0 & 0 & 0 & 0 & \frac{J}{3A} & & \\ 0 & 0 & \frac{13l}{420} & 0 & -\frac{l^2}{140} & 0 & 0 & 0 & \frac{11}{210}l & 0 & \frac{l^2}{105} & \\ 0 & -\frac{13}{420}l & 0 & 0 & 0 & -\frac{l^2}{140} & 0 & -\frac{11}{210}l & 0 & 0 & 0 & \frac{l^2}{105} \end{bmatrix} \tag{12.39}$$

where ρ is the density, A is the cross-sectional area, l is the length, and J is the polar moment of inertia of the element.

12.2.5 Consistent Mass Matrix of a Planar Frame Element

For the planar frame element shown in Figure 9.14, only axial and in-plane bending degrees of freedom will be there and the consistent mass matrix will be

$$\left[m^{(e)}\right] = \rho A l \begin{bmatrix} 1/3 & & & \text{Symmetric} & & \\ 0 & 13/35 & & & & \\ 0 & 11l/210 & l^2/105 & & & \\ 1/6 & 0 & 0 & 1/3 & & \\ 0 & 9/70 & 13l/420 & 0 & 13/35 & \\ 0 & -13l/420 & -l^2/140 & 0 & -11l/210 & l^2/105 \end{bmatrix} \tag{12.40}$$

Note

If the cross section of the frame (or beam) element is not small, the effects of rotatory inertia and shear deformation become important in the dynamic analysis. The derivation of stiffness and mass matrices of beam elements, including the effects of rotatory inertia and shear deformation, can be found in [12.5,12.6].

12.2.6 Consistent Mass Matrix of a Triangular Membrane Element

By considering all nine degrees of freedom of the element (shown in Figure 10.2), linear shape functions in terms of the local coordinates x and y can be used to express the displacement field as

$$\vec{U} = \begin{Bmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{Bmatrix} = [N]\vec{Q}^{(e)} \tag{12.41}$$

where

$$[N(x,y)] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 \end{bmatrix} \tag{12.42}$$

with $N_1(x, y)$, $N_2(x, y)$, and $N_3(x, y)$ given by Eq. (10.5), and

$$\vec{Q}^{(e)} = \{Q_{3i-2}\ Q_{3i-1}\ Q_{3i}\ Q_{3j-2}\ Q_{3j-1}\ Q_{3j}\ Q_{3k-2}\ Q_{3k-1}\ Q_{3k}\}^T \tag{12.43}$$

The consistent mass matrix of the element (applicable in any coordinate system) can be obtained as

$$\left[m^{(e)}\right] = \iiint_{V^{(e)}} \rho[N]^T[N]\,\mathrm{d}V \tag{12.44}$$

By carrying out the necessary integration (in the local xy coordinate system, for simplicity), the mass matrix can be derived as

$$\left[M^{(e)}\right] = \left[m^{(e)}\right] = \frac{\rho At}{12}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{12.45}$$

where t is the thickness of the element.

12.2.7 Consistent Mass Matrix of a Triangular Bending Element

For the triangular plate bending element shown in Figure 10.17, the stiffness matrix has been derived in Section 10.8 by assuming the displacement model

$$w(x,y) = [\eta]\vec{\alpha} \tag{12.46}$$

where $[\eta]$ and $\vec{\alpha}$ are given by Eqs. (10.77) and (10.78), respectively. By using Eqs. (12.46) and (10.80), the transverse displacement w can be expressed as

$$w(x,y) = \left([\eta]\left[\underset{\sim}{\eta}\right]^{-1}\right)\vec{q}^{(e)} \tag{12.47}$$

where $\left[\underset{\sim}{\eta}\right]$ is given by Eq. (10.81). Due to rotation of normals to the middle plane about the x and y axes, any point located at a distance of z from the middle plane will have in-plane displacement components given by

$$\left.\begin{aligned} u &= -z\cdot\frac{\partial w}{\partial x} \\ v &= -z\cdot\frac{\partial w}{\partial y} \end{aligned}\right\} \tag{12.48}$$

Thus, the three translational displacements can be expressed, using Eqs. (12.47) and (12.48), as

$$\underset{3\times 1}{\vec{U}}(x,y) = \begin{Bmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{Bmatrix} = \begin{bmatrix} -z\dfrac{\partial[\eta]}{\partial x} \\ -z\dfrac{\partial[\eta]}{\partial y} \\ [\eta] \end{bmatrix} \left[\underset{\sim}{\eta}\right]^{-1} \vec{q}^{(e)} = \underset{3\times 9}{[N_1]} \underset{9\times 9}{\left[\underset{\sim}{\eta}\right]^{-1}} \underset{9\times 1}{\vec{q}^{(e)}} \equiv [N]\vec{q}^{(e)} \tag{12.49}$$

where

$$[N_1] = \begin{bmatrix} 0 & -z & 0 & -2xz & -yz & 0 & -3x^2z & -z(y^2+2xy) & 0 \\ 0 & 0 & -z & 0 & -xz & -2yz & 0 & -z(2xy+x^2) & -3y^2z \\ 1 & x & y & x^2 & xy & y^2 & x^3 & (x^2y+xy^2) & y^3 \end{bmatrix} \tag{12.50}$$

and

$$[N] = [N_1]\left[\underset{\sim}{\eta}\right]^{-1} \tag{12.51}$$

The consistent mass matrix of the element can now be evaluated as

$$\begin{aligned} \left[m^{(e)}\right] &= \iiint_{V^{(e)}} \rho [N]^T[N]\,\mathrm{d}V \\ &= \iiint_{V^{(e)}} \rho \left(\left[\underset{\sim}{\eta}\right]^{-1}\right)^T [N_1]^T[N_1]\left[\underset{\sim}{\eta}\right]^{-1} \mathrm{d}V \end{aligned} \tag{12.52}$$

Eq. (12.52) denotes the mass matrix obtained by considering both translational (due to w) and rotatory (due to u and v) inertia of the element. If rotatory inertia is neglected, as is done in most practical computations, the consistent mass matrix can be obtained by setting simply $[N_1] \equiv [\eta]$ in Eq. (12.52). In this case we have

$$\begin{aligned} \left[m^{(e)}\right] &= \iiint_{V^{(e)}} \rho \left(\left[\underset{\sim}{\eta}\right]^{-1}\right)^T [\eta]^T[\eta]\left[\underset{\sim}{\eta}\right]^{-1} \mathrm{d}V \\ &= \rho t \left(\left[\underset{\sim}{\eta}\right]^{-1}\right)^T \left(\iint_A [\eta]^T[\eta]\mathrm{d}x\,\mathrm{d}y\right)\left[\underset{\sim}{\eta}\right]^{-1} \\ &= \rho t \left(\left[\underset{\sim}{\eta}\right]^{-1}\right)^T \int_{\text{area}} \\ &\times \begin{bmatrix} 1 & & & & & & & & \\ x & x^2 & & & & & & \text{Symmetric} & \\ y & xy & y^2 & & & & & & \\ x^2 & x^3 & x^2y & x^4 & & & & & \\ xy & x^2y & xy^2 & x^3y & x^2y^2 & & & & \\ y^2 & xy^2 & y^3 & x^2y^2 & xy^3 & y^4 & & & \\ x^3 & x^4 & x^3y & x^5 & x^4y & x^3y^2 & x^6 & & \\ (x^2y+xy^2) & (x^2y^2+x^3y) & (xy^3+x^2y^2) & (x^3y^2+x^4y) & (x^2y^3+x^3y^2) & (xy^4+x^2y^3) & (x^4y^2+x^5y) & (xy^2+x^2y)^2 & \\ y^3 & xy^3 & y^4 & x^2y^3 & xy^4 & y^5 & x^2y^3 & (xy^5+x^2y^4) & y^6 \end{bmatrix} \mathrm{d}x\,\mathrm{d}y \left[\underset{\sim}{\eta}\right]^{-1} \end{aligned} \tag{12.53}$$

Thus, the determination of the mass matrix $[m^{(e)}]$ involves the evaluation of integrals of the form

$$\iint_{\text{area}} x^i y^j \, dx \, dy, \quad i = 0 - 6 \quad \text{and} \quad j = 0 - 6 \tag{12.54}$$

Note that the highest powers of x and y appearing in the integrand of Eq. (12.54) are larger than the highest powers involved in the derivation of the stiffness matrix of the same element; see Eq. (10.87). This characteristic is true for all finite elements.

12.2.8 Consistent Mass Matrix of a Tetrahedron Element

For the solid tetrahedron element shown in Figure 11.1, the displacement field is given by Eq. (11.4). The element mass matrix in the global coordinate system can be found from the relation

$$\left[M^{(e)}\right] = \iiint_{V^{(e)}} \rho [N]^T [N] \, dV$$

After carrying out the lengthy volume integrations (using tetrahedral coordinates, for simplicity), the mass matrix can be obtained as

$$\left[M^{(e)}\right] = \frac{\rho V^{(e)}}{20} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{12.55}$$

12.3 CONSISTENT MASS MATRICES IN A GLOBAL COORDINATE SYSTEM

To reduce the computational effort, generally the consistent mass matrices of unassembled elements are derived in suitable local coordinate systems and then transformed into the global system selected for the assembled structure. If $\left[m^{(e)}\right]$, $\vec{q}^{(e)}$, and $\dot{\vec{q}}^{(e)}$ denote the mass matrix, nodal displacement vector, and nodal velocity vector in the local coordinate system, respectively, the kinetic energy associated with the motion of the element can be expressed as

$$T = \frac{1}{2} \dot{\vec{q}}^{(e)T} \left[m^{(e)}\right] \dot{\vec{q}}^{(e)} \tag{12.56}$$

If the element nodal displacements and nodal velocities are denoted as $\vec{Q}^{(e)}$ and $\dot{\vec{Q}}^{(e)}$ in the global system, we have the transformation relations

$$\vec{q}^{(e)} = [\lambda] \vec{Q}^{(e)}$$

and

$$\dot{\vec{q}}^{(e)} = [\lambda] \dot{\vec{Q}}^{(e)} \tag{12.57}$$

By substituting Eq. (12.57) into Eq. (12.56), we obtain

$$T = \frac{1}{2}\dot{\vec{Q}}^{(e)T}[\lambda]^T\left[m^{(e)}\right][\lambda]\dot{\vec{Q}}^{(e)} \tag{12.58}$$

By denoting the mass matrix of the element in the global coordinate system as $[M^{(e)}]$, the kinetic energy associated with the motion of the element can be expressed as

$$\underset{\sim}{T} = \frac{1}{2}\dot{\vec{Q}}^{(e)T}\left[M^{(e)}\right]\dot{\vec{Q}}^{(e)} \tag{12.59}$$

Since kinetic energy is a scalar quantity, it must be independent of the coordinate system. By equating Eqs. (12.58) and (12.59), we obtain the consistent mass matrix of the element in the global system as

$$\left[M^{(e)}\right] = [\lambda]^T\left[m^{(e)}\right][\lambda] \tag{12.60}$$

Note that this transformation relation is similar to the one used in the case of the element stiffness matrix.

Notes

1. In deriving the element mass matrix from the relation

$$\left[m^{(e)}\right] = \iiint_{V^{(e)}} \rho[N]^T[N]\cdot \mathrm{d}V$$

the matrix $[N]$ must refer to all nodal displacements even in the local coordinate system. Thus, for thin plates subjected to in-plane forces only (membrane elements), the transverse deflection must also be considered (in addition to the in-plane displacements considered in the derivation of element stiffness matrices) in formulating the matrix $[N]$.

2. For elements whose nodal degrees of freedom correspond to translational displacements only, the consistent mass matrix is invariant with respect to the orientation and position of the coordinate axes. Thus, the matrices $[m^{(e)}]$ and $[M^{(e)}]$ will be the same for pin-jointed bars, membrane elements, and three-dimensional elements such as solid tetrahedra having only translational degrees of freedom. On the other hand, for elements such as frame elements and plate bending elements, which have bending stiffness, the consistent mass matrices $[m^{(e)}]$ and $[M^{(e)}]$ will be different. For example, the transformation matrices needed for the derivation of element mass matrices in the global coordinate system from those given by Eqs. (12.39), (12.40), and Eq. (E.3) of Example 12.3 are given by Eqs. (9.41), (9.63), and (9.66), respectively.

12.4 FREE VIBRATION ANALYSIS

If we disturb any elastic structure in an appropriate manner initially at time $t = 0$ (i.e., by imposing properly selected initial displacements and then releasing these constraints), the structure can be made to oscillate harmonically. This oscillatory motion is a characteristic property of the structure and it depends on the distribution of mass and stiffness in the structure. If damping is present, the amplitudes of oscillations will decay progressively and if the magnitude of damping exceeds a certain critical value, the oscillatory character of the motion will cease altogether. On the other hand, if damping is absent, the oscillatory motion will continue indefinitely, with the amplitudes of oscillations depending on the initially imposed disturbance or displacement. The oscillatory motion occurs at certain frequencies known as natural frequencies or characteristic values, and it follows well-defined deformation patterns known as mode shapes or characteristic modes. The study of such free vibrations (free because the structure vibrates with no external forces after $t = 0$) is very important in finding the dynamic response of the elastic structure.

By assuming the external force vector $\vec{P}$ to be zero and the displacements to be harmonic as in

$$\vec{Q} = \underline{\vec{Q}}\cdot e^{i\omega t} \tag{12.61}$$

Eq. (12.22) gives the following free vibration equation:

$$\left[[K] - \omega^2[M]\right]\underline{\vec{Q}} = \vec{O} \tag{12.62}$$

where $\vec{\underset{\sim}{Q}}$ represents the amplitudes of the displacements $\vec{\underset{\sim}{Q}}$ (called the mode shape or eigenvector) and ω denotes the natural frequency of vibration. Eq. (12.62) is called a linear algebraic eigenvalue problem since neither $[K]$ nor $[M]$ is a function of the circular frequency ω, and it will have a nonzero solution for $\vec{\underset{\sim}{Q}}$ provided that the determinant of the coefficient matrix $([K] - \omega^2 [M])$ is zero—that is,

$$\left|[K] - \omega^2 [M]\right| = 0 \tag{12.63}$$

The various methods of finding the natural frequencies and mode shapes were discussed in Section 7.3. In general, all the eigenvalues of Eq. (12.63) will be different, and hence the structure will have n different natural frequencies. Only for these natural frequencies, a nonzero solution can be obtained for $\vec{\underset{\sim}{Q}}$ from Eq. (12.62). We designate the eigenvector (mode shape) corresponding to the j-th natural frequency (ω_j) as $\vec{\underset{\sim}{Q}}_j$.

It was assumed that the rigid body degrees of freedom were eliminated in deriving Eq. (12.62). If rigid body degrees of freedom are not eliminated in deriving the matrices $[K]$ and $[M]$, some of the natural frequencies ω would be zero. In such a case, for a general three-dimensional structure, there will be six rigid body degrees of freedom and hence six zero frequencies. It can be easily seen why $\omega = 0$ is a solution of Eq. (12.62). For $\omega = 0$, $\vec{Q} = \vec{\underset{\sim}{Q}} =$ constant vector in Eq. (12.61), and Eq. (12.62) gives

$$[K]\vec{\underset{\sim}{Q}}_{\text{rigid body}} = \vec{0} \tag{12.64}$$

which is obviously satisfied due to the fact that rigid body displacements alone do not produce any elastic restoring forces in the structure. The rigid body degrees of freedom in dynamic analysis can be eliminated by deleting the rows and columns corresponding to these degrees of freedom from the matrices $[K]$ and $[M]$ and by deleting the corresponding elements from displacement $\left(\vec{Q}\right)$ and load $\left(\vec{P}\right)$ vectors.

EXAMPLE 12.2

Find the natural frequency of vibration of a fixed-free bar in axial motion based on a one-element model using (a) consistent mass matrix and (b) lumped mass matrix.

Solution

a. The stiffness and consistent mass matrices of the bar, with a one-element model, are given by

$$[\underset{\sim}{K}] \equiv [K^{(e)}] = \frac{AE}{l}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ u_1 \\ u_2 \end{array}, [\underset{\sim}{M}] \equiv [M^{(e)}] = \frac{\rho A l}{6}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ u_1 \\ u_2 \end{array} \tag{E.1}$$

When the left end of the bar is fixed as shown in Fig. 12.1B, the system stiffness and consistent mass matrices take the form

$$[\underset{\sim}{K}] = \frac{AE}{l}\overset{u_2}{[1]}u_2, \quad [\underset{\sim}{M}] = \frac{\rho A l}{6}\overset{u_2}{[2]}u_2 \tag{E.2}$$

In this case, the eigenvalue problem is defined by the scalar equation:

$$\left(\frac{AE}{l} - \frac{\rho A l}{3}\omega^2\right)u_2 = 0 \tag{E.3}$$

From Eq. (E.3), the natural frequency of vibration of the bar can be found as

$$\left(\frac{AE}{l} - \frac{\rho A l}{3}\omega^2\right) = 0$$

or

$$\omega = \sqrt{3}\sqrt{\frac{E}{\rho l^2}} = 1.7320\sqrt{\frac{E}{\rho l^2}} \tag{E.4}$$

Continued

EXAMPLE 12.2 —cont'd

b. The lumped mass matrix of the bar, with a one-element model is given by

$$[\underset{\sim}{M}]_l \equiv [M^{(e)}]_l = \frac{\rho A l}{2} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array} \tag{E.5}$$

When the left end of the bar is fixed as shown in Fig. 12.1B, the system lumped mass matrix takes the form

$$[M]_l = \frac{\rho A l}{2} \begin{array}{c} u_2 \\ [1] \end{array} u_2 \tag{E.6}$$

The eigenvalue problem is given by the scalar equation:

$$\left(\frac{AE}{l} - \frac{\rho A l}{2}\omega^2\right) u_2 = 0 \tag{E.7}$$

From Eq. (E.7), the natural frequency of vibration of the bar can be found as

$$\left(\frac{AE}{l} - \frac{\rho A l}{2}\omega^2\right) = 0$$

or

$$\omega = \sqrt{2}\sqrt{\frac{E}{\rho l^2}} = 1.4142\sqrt{\frac{E}{\rho l^2}} \tag{E.8}$$

EXAMPLE 12.3

Find the natural frequencies and mode shapes of a uniform cantilever beam using one beam element and consistent mass matrix.

Solution

The natural frequencies and mode shapes of the cantilever beam, shown in Fig. 12.4, are given by the solution of the eigenvalue problem:

$$\left[[\underset{\sim}{K}] - \omega^2 [\underset{\sim}{M}]\right] \overrightarrow{\underset{\sim}{W}} = \vec{0} \tag{E.1}$$

where, for one element idealization, $\overrightarrow{\underset{\sim}{W}} = \{ W_1 \quad W_2 \quad W_3 \quad W_4 \}^T$,

$$[\underset{\sim}{K}] = [K^{(e)}] = \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \tag{E.2}$$

$$[\underset{\sim}{M}] = [M^{(e)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{E.3}$$

Applying the boundary conditions, $W_1 = W_2 = 0$, Eq. (E.1) reduces to

$$[[K] - \omega^2 [M]]\overrightarrow{W} = \vec{0} \tag{E.4}$$

with

$$\overrightarrow{W} = \{ W_3 \quad W_4 \}^T$$

EXAMPLE 12.3 —cont'd

$$[K] = \frac{2EI}{l^3}\begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} \tag{E.5}$$

and

$$[M] = \frac{\rho Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \tag{E.6}$$

By setting

$$\left|[K] - \omega^2[M]\right| = 0 \tag{E.7}$$

we obtain

$$\begin{vmatrix} \left(\frac{12EI}{l^3} - \frac{39\rho Al\omega^2}{105}\right) & \left(-\frac{6EI}{l^2} + \frac{11\rho Al^2\omega^2}{210}\right) \\ \left(-\frac{6EI}{l^2} + \frac{11\rho Al^2\omega^2}{210}\right) & \left(\frac{4EI}{l} - \frac{\rho Al^3\omega^2}{105}\right) \end{vmatrix} = 0$$

or

$$12\left(\frac{EI}{l^2}\right)^2 - \frac{34}{35}(\rho A\, E\, I)\omega^2 + \frac{35}{44,100}\left(\rho a\, l^2\right)^2\omega^4 = 0 \tag{E.8}$$

Using the notation

$$(\beta\, l)^4 = \frac{\rho A\, l^4\omega^2}{E\, I} \tag{E.9}$$

Eq. (E.3) can be rewritten as

$$\frac{35}{44,100}(\beta\, l)^8 - \frac{34}{35}(\beta\, l)^4 + 12 = 0 \tag{E.10}$$

which yields

$$(\beta\, l)^4 = 12.48,\ \ 1211.5$$

Thus, the two natural frequencies of the beam are given by

$$\omega_1 = 3.533\sqrt{\frac{E\, I}{\rho A\, l^4}},\quad \omega_2 = 34.81\sqrt{\frac{E\, I}{\rho A\, l^4}} \tag{E.11}$$

The mode shapes of the beam, $\overrightarrow{W} = \{W_3 \quad W_4\}^T$, can be determined by substituting ω_1 and ω_2 separately into Eq. (E.4) and solving for $\overrightarrow{W} = \{W_3 \quad W_4\}^T$. When ω_1, given by Eq. (E.11), is substituted, the first equation of Eq. (E.4) yields

$$\frac{EI}{l^3}\left[\left(12 - \frac{39}{105}(12.48)\right)W_3 + \left(-6l + \frac{11\, l}{210}(12.48)\right)W_4\right] = 0 \tag{E.12}$$

If the value of W_3 is assumed to be 1 (arbitrarily), Eq. (E.12) gives the value of W_4 as $1.3775/l$.
Thus, the mode shape corresponding to ω_1 is

$$\overrightarrow{W}\Big|_{\omega_1} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}\Bigg|_{\omega_1} = \begin{Bmatrix} 1 \\ 1.3775/l \end{Bmatrix} \tag{E.13}$$

Similarly, by substituting the value of ω_2, given by Eq. (E.11), into Eq. (E.4), we find

$$\frac{EI}{l^3}\left[\left(12 - \frac{39}{105}(1211.5)\right)W_3 + \left(-6l + \frac{11\, l}{210}(1211.5)\right)W_4\right] = 0 \tag{E.14}$$

If the value of W_3 is assumed to be 1 (arbitrarily), Eq. (E.14) gives the value of W_4 as $7.6225/l$.
Thus, the mode shape corresponding to ω_2 is

Continued

EXAMPLE 12.3 —cont'd

$$\vec{W}\Big|_{\omega_2} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}\Bigg|_{\omega_2} = \begin{Bmatrix} 1 \\ 7.6225/l \end{Bmatrix} \tag{E.15}$$

EXAMPLE 12.4

Find the natural frequencies and mode shapes of a uniform cantilever beam shown in Fig. 12.4 using one beam element and a lumped mass matrix.

Solution

In this case, the lumped mass matrix of the beam, without considering the rotatory inertia, is given by

$$[\underset{\sim}{M}]_l \equiv [M^{(e)}]_l = \frac{\rho A l}{2} \begin{array}{c} \begin{matrix} W_1 & W_2 & W_3 & W_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{matrix} \tag{E.1}$$

After applying the boundary conditions, Eq. (E.1) reduces to

$$[M]_l = \frac{\rho A l}{2} \begin{array}{c} \begin{matrix} W_3 & W_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{matrix} W_3 \\ W_4 \end{matrix} \tag{E.2}$$

The eigenvalue problem to determine the natural frequencies and mode shapes of the beam (see Eq. (E.4) of Example 12.3) takes the following form:

$$\left[[K] - \omega^2 [M]_l\right] \vec{W} = \vec{0} \tag{E.3}$$

with

$$\vec{W} = \{ W_3 \quad W_4 \}^T$$
$$[K] = \frac{2EI}{l^3} \begin{bmatrix} 6 & -3l \\ -3l & 2l^2 \end{bmatrix} \tag{E.4}$$

and $[M]_l$ is given by Eq. (E.2). By setting

$$\left| [K] - \omega^2 [M] \right| = 0 \tag{E.5}$$

we obtain

$$\begin{vmatrix} \left(\dfrac{12EI}{l^3} - \dfrac{\rho A l \omega^2}{2} \right) & -\dfrac{6EI}{l^2} \\ -\dfrac{6EI}{l^2} & \dfrac{4EI}{l} \end{vmatrix} = 0$$

or

$$\left(48 \frac{E^2 I^2}{l^4} - 2EI\rho A \omega^2 \right) - 36 \left(\frac{EI}{l^2} \right)^2 = 0 \tag{E.6}$$

which yields

$$\omega_1 = 2.4495 \sqrt{\frac{EI}{\rho A l^4}} \tag{E.7}$$

EXAMPLE 12.4 —cont'd

Note that with the lumped mass matrix, we obtain only one natural frequency because the rotatory inertia associated with the degrees of freedom W_2 and W_4 is not considered in $[M^{(e)}]_l$.

The mode shape corresponding to $\omega_1 = 2.4495\sqrt{\frac{EI}{\rho A l^4}}$ can be found from the first equation of (E.3) as

$$\frac{EI}{l^3}[(12-3)W_3 + (-6)W_4] = 0 \tag{E.8}$$

If the value of W_3 is assumed as 1 (arbitrarily), Eq. (E.8) gives the value of W_4 as 1.5/l. Thus, the mode shape corresponding to ω_1 is

$$\overrightarrow{W}\Big|_{\omega_2} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}\Bigg|_{\omega_2} = \begin{Bmatrix} 1 \\ 1.5/l \end{Bmatrix} \tag{E.9}$$

EXAMPLE 12.5

Find the natural frequencies of vibration of the two-bar truss shown in Fig. 12.5A using consistent mass matrices. Assume the Young's modulus (E) to be 2×10^6 N/cm^2 and density (ρ) as 0.0078 kg/cm^3.

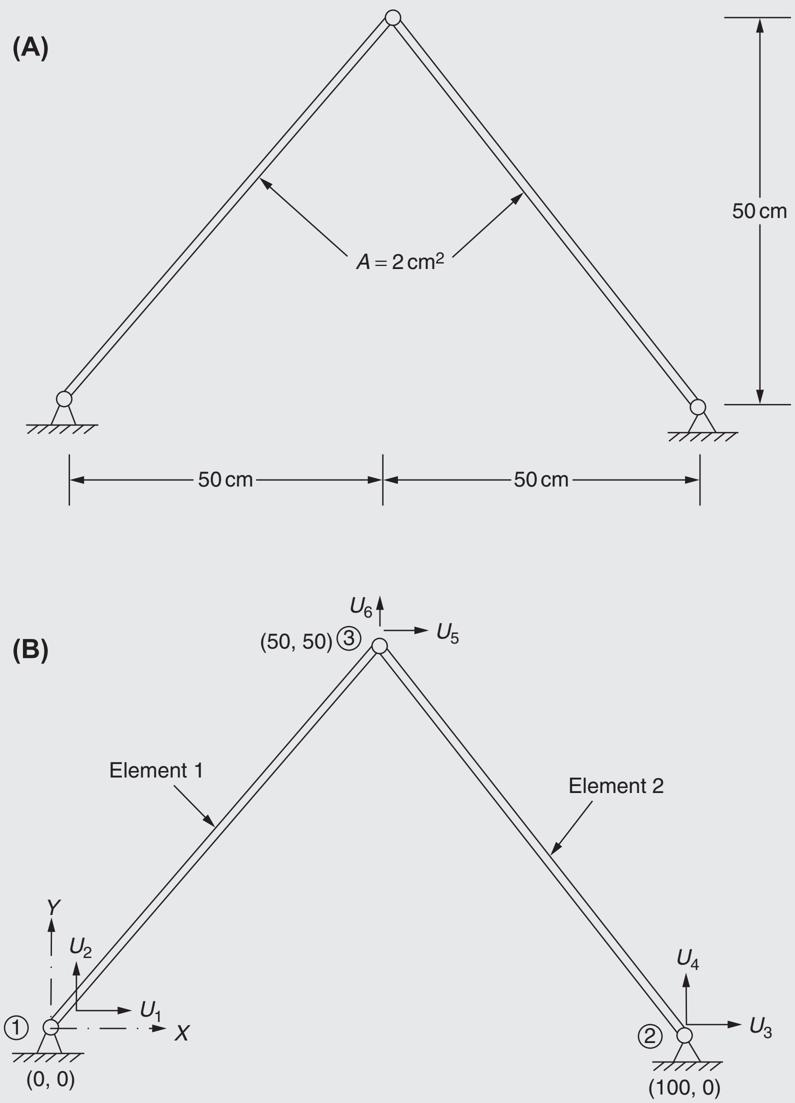

FIGURE 12.5 A two-bar truss.

Continued

EXAMPLE 12.5 —cont'd

Solution

The finite element model of the truss, along with the degrees of freedom, is shown in Fig. 12.5B. It can be seen that the degrees of freedom U_i, $i = 1, 2, 3, 4$ are zero (fixed) while U_5 and U_6 are free. The element stiffness matrices in the global XY-system (see Example 9.4) can be generated as follows:

$$[K^{(1)}] = 2.8284 \times 10^4 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_5 & U_6 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_5 \\ U_6 \end{array} \tag{E.1}$$

$$[K^{(2)}] = 2.8284 \times 10^4 \begin{array}{c} \begin{array}{cccc} U_5 & U_6 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \\ U_3 \\ U_4 \end{array} \tag{E.2}$$

The assembled stiffness matrix of the truss, after incorporating the boundary conditions, can be expressed as

$$[K] = 5.6568 \begin{array}{c} \begin{array}{cc} U_5 & U_6 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \end{array} \tag{E.3}$$

Noting that the length of elements is $l = 50\sqrt{2} = 70.7107$ cm and $\rho A l/6 = (0.0078)\ (2)\ (70.7107)/6 = 0.1838$ kg, the element mass matrices (consistent) (see Eq. (E.5) of Example 12.1) are given by:

$$[M^{(1)}] = 0.1838 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_5 & U_6 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_5 \\ U_6 \end{array} \tag{E.4}$$

$$[M^{(2)}] = 0.1838 \begin{array}{c} \begin{array}{cccc} U_5 & U_6 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \\ U_3 \\ U_4 \end{array} \tag{E.5}$$

The assembled mass matrix of the truss, after eliminating the fixed degrees of freedom, can be expressed as

$$[M] = 0.7352 \begin{array}{c} \begin{array}{cc} U_5 & U_6 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \end{array} \tag{E.6}$$

The eigenvalue problem is given by

$$\left[[K] - \omega^2 [M]\right] \vec{U} = \vec{0} \tag{E.7}$$

EXAMPLE 12.5 —cont'd

where ω is the natural frequency and $\vec{U} = \{U_5 \quad U_6\}^T$ is the mode shape. The natural frequencies of the truss are given by the positive roots of the equation:

$$\left|[K] - \omega^2[M]\right| = \left|5.6568\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 0.7352\omega^2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right| = 0$$

that is,

$$\begin{vmatrix}(5.6568 - 0.7352\,\omega^2) & 0\\ 0 & (5.6568 - 0.7352\,\omega^2)\end{vmatrix} = \left(5.6568 - 0.7352\,\omega^2\right)^2 = 0 \tag{E.8}$$

that is,

$$\omega = 2.7738 \text{ rad/s (repeated natural frequency)} \tag{E.9}$$

EXAMPLE 12.6

Find the natural frequencies of vibration of the two-bar truss shown in Fig. 12.5A using lumped mass matrices. Assume that Young's modulus (E) is 2×10^6 N/cm^2 and density (ρ) is 0.0078 kg/cm^3.

Solution

The finite element model of the truss, along with the degrees of freedom, is shown in Fig. 12.5B. It can be seen that the degrees of freedom U_i, $i = 1, 2, 3, 4$ are zero (fixed) while U_5 and U_6 are free. The assembled stiffness matrix of the truss, after incorporating the boundary conditions, is given by Eq. (E.3) of Example 12.5:

$$[K] = 5.6568 \begin{array}{c} \begin{array}{cc} U_5 & U_6 \end{array} \\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \end{array} \tag{E.1}$$

Noting that $\rho\, Al/2 = (0.0078)\,(2)\,(70.7107)/2 = 1.1031$ kg, the element mass matrices (lumped) (see Eq. (E.6) of Example 12.1) are given by

$$[M^{(1)}] = 0.1031 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_5 & U_6 \end{array} \\ \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_5 \\ U_6 \end{array} \tag{E.2}$$

$$[M^{(2)}] = 0.1031 \begin{array}{c} \begin{array}{cccc} U_5 & U_6 & U_3 & U_4 \end{array} \\ \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \\ U_3 \\ U_4 \end{array} \tag{E.3}$$

The assembled mass matrix of the truss, after eliminating the fixed degrees of freedom, can be expressed as

$$[M] = 2.2062 \begin{array}{c} \begin{array}{cc} U_5 & U_6 \end{array} \\ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \end{array} \begin{array}{c} \\ U_5 \\ U_6 \end{array} \tag{E.4}$$

Continued

EXAMPLE 12.6 —cont'd

The eigenvalue problem is given by

$$\left[[K] - \omega^2[M]\right]\vec{U} = \vec{0} \tag{E.5}$$

where ω is the natural frequency and $\vec{U} = \begin{Bmatrix} U_5 & U_6 \end{Bmatrix}^T$ is the mode shape. The natural frequencies of the truss are given by the positive roots of the equation:

$$\left|[K] - \omega^2[M]\right| = \left| 5.6568\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2.2062\,\omega^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \tag{E.6}$$

that is,

$$\begin{vmatrix} \left(5.6568 - 2.2062\,\omega^2\right) & 0 \\ 0 & \left(5.6568 - 2.2062\,\omega^2\right) \end{vmatrix} = \left(5.6568 - 2.2062\,\omega^2\right)^2 = 0 \tag{E.7}$$

that is,

$$\omega = 1.6013 \text{ rad/s (repeated natural frequency)} \tag{E.8}$$

EXAMPLE 12.7

Find the natural frequencies of longitudinal vibration of the unconstrained stepped bar shown in Fig. 12.6A.

Solution

We shall idealize the bar with two elements as shown in Fig. 12.6A. The stiffness and mass matrices of the two elements are given by

$$\left[K^{(1)}\right] = \frac{A^{(1)}E^{(1)}}{l^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{4AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[K^{(2)}\right] = \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[M^{(1)}\right] = \frac{\rho^{(1)}A^{(1)}l^{(1)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\left[M^{(2)}\right] = \frac{\rho^{(2)}A^{(2)}l^{(2)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The assembled stiffness and mass matrices are given by

$$\left[\underset{\sim}{K}\right] = \frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{E.1}$$

$$\left[\underset{\sim}{M}\right] = \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{E.2}$$

Since the bar is unconstrained (no degree of freedom is fixed), the frequency Eq. (12.63) becomes

$$\left| \frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2\frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0 \tag{E.3}$$

EXAMPLE 12.7 —cont'd

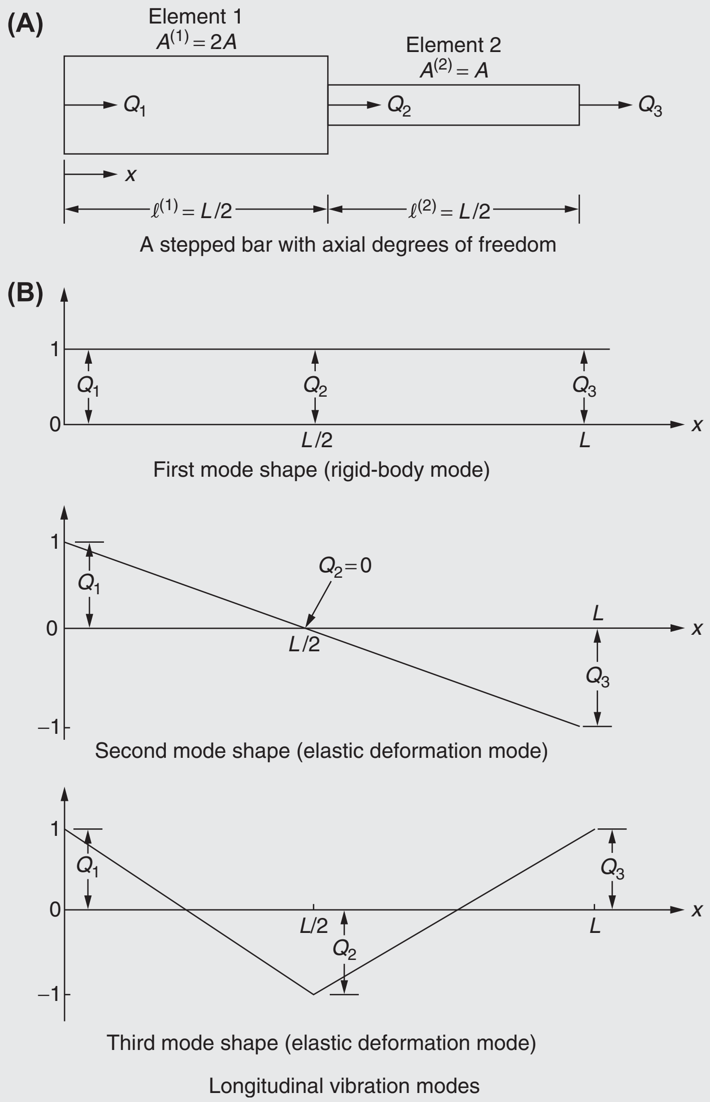

FIGURE 12.6 An unconstrained stepped bar and its mode shapes.

By defining

$$\beta^2 = \frac{\rho L^2 \omega^2}{24E} \tag{E.4}$$

Eq. (E.3) can be rewritten as

$$\begin{vmatrix} 2(1-2\beta^2) & -2(1+\beta^2) & 0 \\ -2(1+\beta^2) & 3(1-2\beta^2) & -(1+\beta^2) \\ 0 & -(1+\beta^2) & (1-2\beta^2) \end{vmatrix} = 0 \tag{E.5}$$

The expansion of this determinantal equation leads to

$$18\beta^2(1-2\beta^2)(\beta^2-2) = 0 \tag{E.6}$$

The roots of Eq. (E.6) give the natural frequencies of the bar as

Continued

EXAMPLE 12.7 —cont'd

$$\left.\begin{array}{llll}\text{When} & \beta^2 = 0: \omega_1^2 = 0 & \text{or} & \omega_1 = 0 \\ \text{When} & \beta^2 = \dfrac{1}{2}: \omega_2^2 = \dfrac{12E}{\rho L^2} & \text{or} & \omega_2 = 3.46\left[E/\left(\rho L^2\right)\right]^{1/2} \\ \text{When} & \beta^2 = 2: \omega_3^2 = \dfrac{48E}{\rho L^2} & \text{or} & \omega_3 = 6.92\left[E/\left(\rho L^2\right)\right]^{1/2}\end{array}\right\} \tag{E.7}$$

The first frequency, $\omega_1 = 0$, corresponds to the rigid-body mode, whereas the second and third frequencies correspond to elastic deformation modes. To find the mode shape $\underset{\sim}{\vec{Q}}_i$ corresponding to the natural frequency ω_i, we solve Eq. (12.62). Since Eq. (12.62) represents a system of homogeneous equations, we will be able to find only the relative magnitudes of the components of $\underset{\sim}{\vec{Q}}_i$.

For $\omega_1^2 = 0$, Eq. (12.62) gives $\underset{\sim}{\vec{Q}}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, whereas for $\omega_2^2 = 12E/(\rho L^2)$ and $\omega_3^2 = 48E/(\rho L^2)$, it gives $\underset{\sim}{\vec{Q}}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$ and $\underset{\sim}{\vec{Q}}_3 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$, respectively. These mode shapes are plotted in Fig. 12.6A, where the variation of displacement between the nodes has been assumed to be linear in accordance with the assumed displacement distribution of Eqs. (9.1) and (12.23).

[*M*]-Orthogonalization of Modes

Since only the relative magnitudes of the components of the mode shapes $\underset{\sim}{\vec{Q}}_i, i = 1, 2, 3,$ are known, the mode shapes can also be written as $a_i\underset{\sim}{\vec{Q}}_i$, where a_i is an arbitrary nonzero constant. In most of the dynamic response calculations, it is usual to choose the values of a_i so as to make the mode shapes orthogonal with respect to the mass matrix $[M]$ used in obtaining the modes $\underset{\sim}{\vec{Q}}_i$. This requires that

$$a_i\underset{\sim}{\vec{Q}}_i^T[M]a_j\underset{\sim}{\vec{Q}}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{12.65}$$

for all i and j. In the current example, the mass matrix is given by Eq. (E.2) of Example 12.7 and it can be verified that the condition $a_i\underset{\sim}{\vec{Q}}_i^T[M]a_j\underset{\sim}{\vec{Q}}_j = 0$ for $i \neq j$ is automatically satisfied for any a_i and a_j. To satisfy the condition $a_i\underset{\sim}{\vec{Q}}_i^T[M]a_j\underset{\sim}{\vec{Q}}_j = 1$ for $i = j$, we impose the conditions

$$a_i^2\underset{\sim}{\vec{Q}}_i^T[M]\underset{\sim}{\vec{Q}}_i = \frac{\rho A L a_i^2}{12}\underset{\sim}{\vec{Q}}_i^T\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\underset{\sim}{\vec{Q}}_i = 1$$

for $i = 1, 2, 3$, and obtain

$$a_i^2 = \frac{12}{\rho A L}\,\frac{1}{\underset{\sim}{\vec{Q}}_i^T\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\underset{\sim}{\vec{Q}}_i}, \quad i = 1, 2, 3 \tag{12.66}$$

Eq. (12.66) gives

$$\left.\begin{aligned} a_1 &= \left(\frac{2}{3\rho A L}\right)^{1/2} \\ a_2 &= \left(\frac{2}{\rho A L}\right)^{1/2} \\ a_3 &= \left(\frac{2}{\rho A L}\right)^{1/2} \end{aligned}\right\} \tag{12.67}$$

Thus, the $[M]$-orthogonal mode shapes of the stepped bar corresponding to the natural frequencies ω_1, ω_2, and ω_3 are given, respectively, by

$$\left(\frac{2}{3\rho AL}\right)^{1/2}\begin{Bmatrix}1\\1\\1\end{Bmatrix},\quad \left(\frac{2}{\rho AL}\right)^{1/2}\begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$

and

$$\left(\frac{2}{\rho AL}\right)^{1/2}\begin{Bmatrix}1\\-1\\1\end{Bmatrix}$$

EXAMPLE 12.8

Find the natural frequencies of longitudinal vibration of the constrained stepped bar shown in Fig. 12.7.

Solution

Since the left end of the bar is fixed, $Q_1 = 0$ and this degree of freedom has to be eliminated from the stiffness and mass matrices of Eqs. (E.1) and (E.2) of Example 12.7 to find the natural frequencies. This amounts to eliminating the rigid-body mode of the structure. For this, we delete the row and column corresponding to Q_1 from Eqs. (E.1) and (E.2) of Example 12.7 and write the frequency equation as

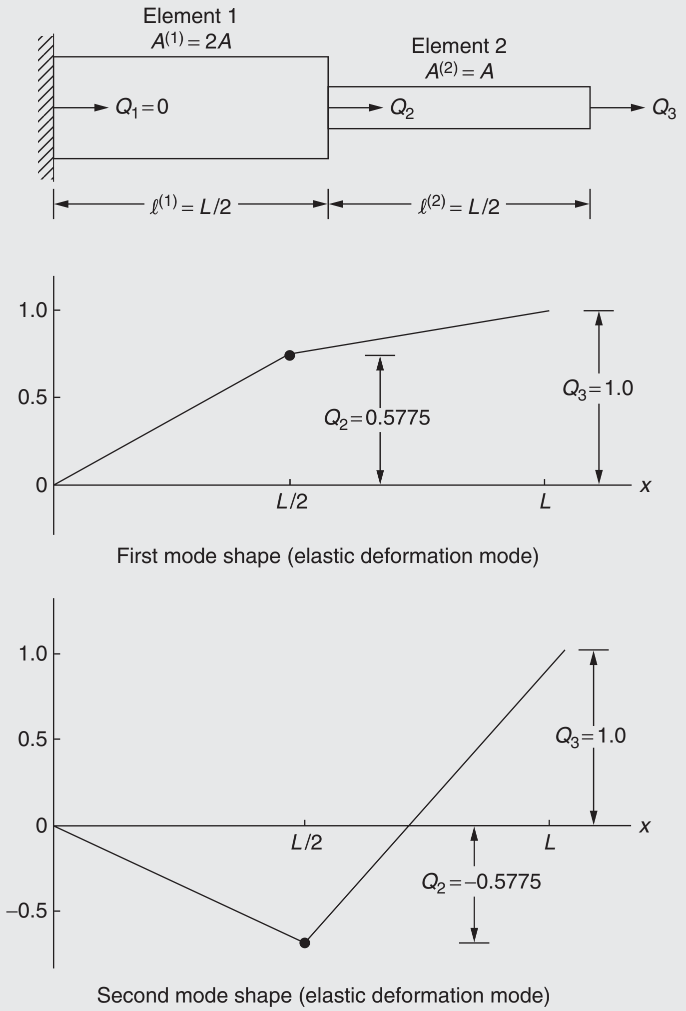

FIGURE 12.7 A constrained stepped bar and its mode shapes.

Continued

EXAMPLE 12.8 —cont'd

$$\left| \frac{2AE}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\rho A L \omega^2}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right| = 0 \tag{E.1}$$

Eq. (E.1) can be rewritten as

$$\begin{vmatrix} 3(1-2\beta^2) & -(1+\beta^2) \\ -(1+\beta^2) & (1-2\beta^2) \end{vmatrix} = 0 \tag{E.2}$$

The solution of Eq. (E.2) is given by

$$\beta_1^2 = \frac{7-3\sqrt{3}}{11} = 0.1640 \quad \text{and} \quad \beta_2^2 = \frac{7+3\sqrt{3}}{11} = 1.1087$$

or

$$\omega_1 = 1.985\sqrt{\frac{E}{\rho L^2}} \quad \text{and} \quad \omega_2 = 5.159\sqrt{\frac{E}{\rho L^2}} \tag{E.3}$$

The mode shapes corresponding to these natural frequencies can be found by solving the equation

$$\left[\frac{2AE}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\rho A L \omega_i^2}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \right] \underset{\sim}{\vec{Q}}_i = \vec{0}, \quad i = 1,2 \tag{E.4}$$

as

$$\underset{\sim}{\vec{Q}}_1 = \begin{Bmatrix} 0.5775 \\ 1.0 \end{Bmatrix} \quad \text{and} \quad \underset{\sim}{\vec{Q}}_2 = \begin{Bmatrix} -0.5775 \\ 1.0 \end{Bmatrix} \tag{E.5}$$

These mode shapes are plotted in Fig. 12.7. When orthogonalized with respect to the matrix $[M]$, these mode shapes give

$$a_1^2 \underset{\sim}{\vec{Q}}_1^T [M] \underset{\sim}{\vec{Q}}_1 = 1$$

or

$$a_1^2 = \frac{1}{(0.5775 \quad 1.0)\dfrac{\rho A L}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0.5775 \\ 1.0 \end{Bmatrix}}$$

or

$$a_1 = 1.526\big/(\rho A L)^{1/2} \tag{E.6}$$

$$a_2^2 \underset{\sim}{\vec{Q}}_2^T [M] \underset{\sim}{\vec{Q}}_2 = 1$$

or

$$a_2^2 = \frac{1}{(-0.5775 \quad 1.0)\dfrac{\rho A L}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} -0.5775 \\ 1.0 \end{Bmatrix}}$$

or

$$a_2 = 2.053\big/(\rho A L)^{1/2} \tag{E.7}$$

Thus, the $[M]$-orthogonal mode shapes of the stepped bar corresponding to ω_1 and ω_2 are given, respectively, by

$$\frac{1.526}{(\rho A L)^{1/2}}\begin{Bmatrix} 0.5775 \\ 1.0 \end{Bmatrix} = \frac{1}{(\rho A L)^{1/2}}\begin{Bmatrix} 0.8812 \\ 1.5260 \end{Bmatrix} \tag{E.8}$$

and

$$\frac{2.053}{(\rho A L)^{1/2}}\begin{Bmatrix} -0.5775 \\ 1.0 \end{Bmatrix} = \frac{1}{(\rho A L)^{1/2}}\begin{Bmatrix} -1.186 \\ 2.053 \end{Bmatrix} \tag{E.9}$$

12.5 DYNAMIC RESPONSE USING FINITE ELEMENT METHOD

When a structure is subjected to dynamic (time-dependent) loads, the displacements, strains, and stresses induced will also vary with time. The dynamic loads arise for a variety of reasons, such as gust loads due to atmospheric turbulence and impact forces due to landing on airplanes, wind and earthquake loads on buildings, and so on. The dynamic response calculations include the determination of displacements and stresses as functions of time at each point of the body or structure. The dynamic equations of motion for a damped elastic body have already been derived in Section 12.1 using the finite element procedure. These equations of motion can be solved by using any of the methods presented in Section 7.4 for solving propagation problems.

The direct integration approach considered in Section 7.4.2 involves the numerical integration of the equations of motion by marching in a series of time steps Δt evaluating accelerations, velocities, and displacements at each step. The basis of the mode superposition method discussed in Section 7.4.2 is that the modal matrix (i.e., the matrix formed by using the modes of the system) can be used to diagonalize the mass, damping, and stiffness matrices, and thus to uncouple the equations of motion. The solution of these independent equations, one corresponding to each degree of freedom, can be found by standard techniques and, finally, the solution of the original problem can be found by the superposition of the individual solutions. In this section, we consider the normal mode (or mode superposition or modal analysis) method of finding the dynamic response of an elastic body in some detail.

12.5.1 Uncoupling the Equations of Motion of an Undamped System

The equations of motion of an undamped elastic system (derived in Section 12.1) are given by

$$[M]\ddot{\vec{Q}} + [K]\dot{\vec{Q}} = \vec{P} \tag{12.68}$$

where $\vec{Q}$ and $\vec{P}$ are the time-dependent displacement and load vectors, respectively. Eq. (12.68) represents a system of n coupled second-order differential equations, where n is the number of degrees of freedom of the structure. We now present a method of uncoupling these equations.

Let the natural frequencies of the undamped eigenvalue problem

$$-\omega^2[M]\underline{\vec{Q}} + [K]\underline{\vec{Q}} = \vec{0} \tag{12.69}$$

be given by $\omega_1, \omega_2, \ldots, \omega_n$ with the corresponding eigenvectors given by $\underline{\vec{Q}}_1, \underline{\vec{Q}}_2, \ldots, \underline{\vec{Q}}_n$, respectively. By arranging the eigenvectors (normal modes) as columns, a matrix $\left[\underline{Q}\right]$, known as the modal matrix, can be defined as

$$\left[\underline{Q}\right] = \left[\underline{\vec{Q}}_1 \quad \underline{\vec{Q}}_2 \quad \cdots \quad \underline{\vec{Q}}_n\right] \tag{12.70}$$

Since the eigenvectors are $[M]$-orthogonal, we have

$$\underline{\vec{Q}}_i^T\left[M\right]\underline{\vec{Q}}_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} \tag{12.71}$$

Eqs. (12.70) and (12.71) lead to

$$\left[\underline{Q}\right]^T[M]\left[\underline{Q}\right] = [I] \tag{12.72}$$

where $[I]$ is the identity matrix of order n, and the eigenvalue problem, Eq. (12.69), can be restated as

$$[\grave{}\,\omega^2_{\grave{}}][M]\left[\underline{Q}\right] = [K]\left[\underline{Q}\right] \tag{12.73}$$

where

$$[\grave{}\,\omega^2_{\grave{}}] = \begin{bmatrix} \omega_1^2 & & & 0 \\ & \omega_2^2 & & \\ & & \ddots & \\ 0 & & & \omega_n^2 \end{bmatrix} \tag{12.74}$$

By premultiplying Eq. (12.73) by $\left[\underline{Q}\right]^T$, we obtain

$$\left[{}^{\backslash}\omega^2_{\backslash}\right]\left[\underline{Q}\right]^T[M]\left[\underline{Q}\right] = \left[\underline{Q}\right]^T[K]\left[\underline{Q}\right] \tag{12.75}$$

which, in view of Eq. (12.72), becomes

$$\left[{}^{\backslash}\omega^2_{\backslash}\right] = \left[\underline{Q}\right]^T[K]\left[\underline{Q}\right] \tag{12.76}$$

Since any n-dimensional vector can be expressed by superposing the eigenvectors,[1] we can express $\vec{Q}(t)$ as

$$\vec{Q}(t) = \left[\underline{Q}\right]\vec{\eta}(t) \tag{12.77}$$

where $\vec{\eta}(t)$ is a column vector consisting of a set of time-dependent generalized coordinates $\eta_1(t)$, $\eta_2(t)$, …, $\eta_n(t)$. By substituting Eq. (12.77) in Eq. (12.68), we obtain

$$[M]\left[\underline{Q}\right]\ddot{\vec{\eta}} + [K]\left[\underline{Q}\right]\vec{\eta} = \vec{P} \tag{12.78}$$

Premultiply both sides of Eq. (12.78) by $\left[\underline{Q}\right]^T$ and write

$$\left[\underline{Q}\right]^T[M]\left[\underline{Q}\right]\ddot{\vec{\eta}} + \left[\underline{Q}\right]^T[K]\left[\underline{Q}\right]\vec{\eta} = \left[\underline{Q}\right]^T\vec{P} \tag{12.79}$$

However, the normal modes satisfy Eqs. (12.72) and (12.76), and hence Eq. (12.79) reduces to

$$\ddot{\vec{\eta}} + \left[{}^{\backslash}\omega^2_{\backslash}\right]\vec{\eta} = \vec{N} \tag{12.80}$$

where

$$\vec{N} = \left[\underline{Q}\right]^T\vec{P}(t) \tag{12.81}$$

Eq. (12.80) represents a set of n uncoupled second-order differential equations of the type

$$\ddot{\eta}_i(t) + \omega_i^2\eta_i(t) = N_i(t), \quad i = 1, 2, \ldots, n \tag{12.82}$$

The reason for uncoupling the original equations of motion, Eq. (12.68), into the form of Eq. (12.82) is that the solution of n uncoupled differential equations is considerably easier than the solution of n coupled differential equations.

12.5.2 Uncoupling the Equations of Motion of a Damped System

The equations of motion of a damped elastic system are given by

$$[M]\ddot{\vec{Q}} + [C]\dot{\vec{Q}} + [K]\vec{Q} = \vec{P} \tag{12.83}$$

Generally little is known about the evaluation of the damping coefficients that are the elements of the damping matrix $[C]$. However, since the effect of damping is small compared to those of inertia and stiffness, the damping matrix $[C]$ is represented by simplified expressions. One simple way of expressing the damping matrix involves the representation of $[C]$ as a linear combination of mass and stiffness matrices as

$$[C] = a[M] + b[K] \tag{12.84}$$

where the constants a and b must be chosen to suit the problem at hand. In this case, the equations of motion, Eq. (12.83), will be uncoupled by the same transformation Eq. (12.77) as that for the undamped system. Thus, the use of Eqs. (12.77) and (12.84) in Eq. (12.83) leads to

$$\left[\underline{Q}\right]^T[M]\left[\underline{Q}\right]\ddot{\vec{\eta}} + \left(a\left[\underline{Q}\right]^T[M]\left[\underline{Q}\right] + b\left[\underline{Q}\right]^T[K]\left[\underline{Q}\right]\dot{\vec{\eta}} + \left[\underline{Q}\right]^T[K]\left[\underline{Q}\right]\right)\vec{\eta} = \left[\underline{Q}\right]^T\vec{P} \tag{12.85}$$

1. Because the eigenvectors are orthogonal, they will form an independent set of vectors, and hence they can be used as a basis for the decomposition of any arbitrary n-dimensional vector $\vec{Q}$. A proof of this statement, also known as the expansion theorem, can be found in [12.7].

In view of Eqs. (12.72) and (12.76), Eq. (12.85) can be expressed as

$$\ddot{\vec{\eta}} + \left(a[I] + b\left[{}^{\backprime}\omega^2_{\backprime}\right]\right)\dot{\vec{\eta}} + \left[{}^{\backprime}\omega^2_{\backprime}\right]\vec{\eta} = \vec{N} \tag{12.86}$$

where $\vec{N}$ is given by Eq. (12.81). Eq. (12.86) can be written in scalar form as

$$\ddot{\eta}_i(t) + \left(a + b\omega_i^2\right)\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = N_i(t), \quad i = 1, 2, \ldots, n \tag{12.87}$$

The quantity $\left(a + b\omega_i^2\right)$ is known as the modal damping constant in i-th normal mode, and it is common to define a quantity ζ_i known as modal damping ratio in i-th normal mode as

$$\zeta_i = \frac{a + b\omega_i^2}{2\omega_i} \tag{12.88}$$

so that the equations of motion in terms of generalized coordinates become

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = N_i(t), \quad i = 1, 2, \ldots, n \tag{12.89}$$

Thus, Eq. (12.89) denotes a set of n uncoupled second-order differential equations for the damped elastic system.

12.5.3 Solution of a General Second-Order Differential Equation

A general second-order differential equation (or one of the uncoupled equations of motion of a damped elastic system) can be expressed as Eq. (12.89). The solution of Eq. (12.89) consists of two parts: one called the homogeneous solution and the other known as the particular integral.

HOMOGENEOUS SOLUTION

The homogeneous solution can be obtained by solving the equation

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = 0 \tag{12.90}$$

By assuming a solution of the type

$$\eta_i(t) = A \cdot e^{\alpha t} \tag{12.91}$$

where A is a constant, Eq. (12.90) gives the following characteristic equation:

$$\alpha^2 + 2\zeta_i\omega_i\alpha + \omega_i^2 = 0 \tag{12.92}$$

The roots of Eq. (12.92) are given by

$$\alpha_{1,2} = -\zeta_i\omega_i \pm \omega_i\sqrt{\zeta_i^2 - 1} \tag{12.93}$$

Thus, the homogeneous solution of Eq. (12.89) can be expressed as

$$\eta_i(t) = A_1e^{\alpha_1 t} + A_2e^{\alpha_2 t} \tag{12.94}$$

where A_1 and A_2 are constants to be determined from the known initial displacement and velocity. Depending on the magnitude of ζ_i, the system is classified as underdamped, critically damped, and overdamped as follows:

1. **Underdamped case (when $\zeta_i < 1$):** If $\zeta_i < 1$, the solution given in Eq. (12.94) can be rewritten as

$$\begin{aligned}\eta_i(t) &= e^{-\zeta_i\omega_i t}\left(A_1e^{i\omega_{id}t} + A_2e^{-i\omega_{id}t}\right)\\ &= e^{-\zeta_i\omega_i t}(B_1 \cos \omega_{id}t + B_2 \sin \omega_{id}t)\\ &= C_1e^{-\zeta_i\omega_i t}\cos(\omega_{id}t - \phi)\end{aligned} \tag{12.95}$$

where $\omega_{id} = \omega_i\sqrt{1 - \zeta_i^2}$, and the constants B_1 and B_2 or C_1 and ϕ (ϕ is also known as phase angle) can be found from the initial conditions. Here, ω_{id} can be regarded as a natural frequency associated with the damped system.

2. **Critically damped case ($\zeta_i = 1$):** In this case, the roots α_1 and α_2 given by Eq. (12.90) will be equal:

$$\alpha_1 = \alpha_2 = -\omega_i \tag{12.96}$$

The solution of Eq. (12.90) is given by

$$\eta_i(t) = e^{-\omega_i t}(A_1 + A_2 t) \tag{12.97}$$

where A_1 and A_2 are constants of integration to be determined from the known initial conditions.

3. **Overdamped case ($\zeta_i > 1$):** When $\zeta_i > 1$, both the roots given by Eq. (12.93) will be negative and the solution given by Eq. (12.94) can be rewritten as

$$\begin{aligned}\eta_i(t) &= e^{-\zeta_i\omega_i t}\left(A_1 e^{\sqrt{\zeta_i^2-1}\omega_i t} + A_2 e^{-\sqrt{\zeta_i^2-1}\omega_i t}\right)\\ &= e^{-\zeta_i\omega_i t}\left(B_1 \cosh\sqrt{\zeta_i^2 - 1}\,\omega_i t + B_2 \sinh\sqrt{\zeta_i^2 - 1}\,\omega_i t\right)\end{aligned} \tag{12.98}$$

The solutions given by Eqs. (12.95), (12.97), and (12.98) are shown graphically in Fig. 12.8. Note that in the case of an underdamped system, the response oscillates within an envelope defined by $\eta_i(t) = \pm C_1 e^{\zeta_i\omega_i t}$ and the response dies out as time (t) increases. In the case of critical damping, the response is not periodic but dies out with time. Finally, in the case of overdamping, the response decreases monotonically with increasing time.

PARTICULAR INTEGRAL

By solving Eq. (12.89), the particular integral in the case of an underdamped system [12.7] can be obtained as

$$\eta_i(t) = \frac{1}{\omega_{id}}\int_0^t N_i(\tau)e^{-\zeta_i\omega_i(t-\tau)} \sin\omega_{id}(t-\tau)\mathrm{d}\tau \tag{12.99}$$

TOTAL SOLUTION

The total solution is given by the sum of homogeneous solution and the particular integral. If $\eta_i(0)$ and $\dot{\eta}_i(0)$ denote the initial conditions (i.e., values of $\eta_i(t)$ and $(\mathrm{d}\eta_i/\mathrm{d}t)\ (t)$ at $t = 0$), the total solution can be expressed as

$$\begin{aligned}\eta_i(t) = {}& \frac{1}{\omega_{id}}\int_0^t N_i(\tau)\cdot e^{-\zeta_i\omega_i(t-\tau)}\sin\omega_{id}(t-\tau)\mathrm{d}\tau + e^{-\zeta_i\omega_i t} \times \left[\omega_{id}t + \frac{\zeta_i}{(1-\zeta_i^2)^{1/2}}\sin\omega_{id}t\right]\eta_i(0)\\ &+ \left[\frac{1}{\omega_{id}}e^{-\zeta_i\omega_i t}\sin\omega_{id}t\right]\dot{\eta}_i(0)\end{aligned} \tag{12.100}$$

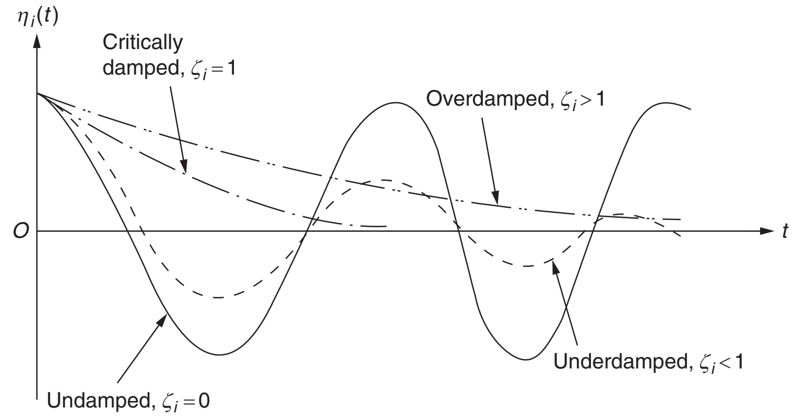

FIGURE 12.8 Response with different degrees of damping.

SOLUTION WHEN THE FORCING FUNCTION IS AN ARBITRARY FUNCTION OF TIME

The numerical solution of Eq. (12.99) when the forcing function $N_i(t)$ is an arbitrary function of time was disscussed in Section 7.4.2. Thus, by using the uncoupling procedure outlined in Sections 12.5.1 and 12.5.2, the response of any multi−degree of freedom system under any arbitrary loading conditions can be found.

EXAMPLE 12.9

Find the dynamic response of the stepped bar shown in Fig. 12.9A when an axial load of magnitude P_0 is applied at node 3 for a duration of time t_0 as shown in Fig. 12.9B.

Solution

The free vibration characteristics of this bar have already been determined in Example 12.8 as

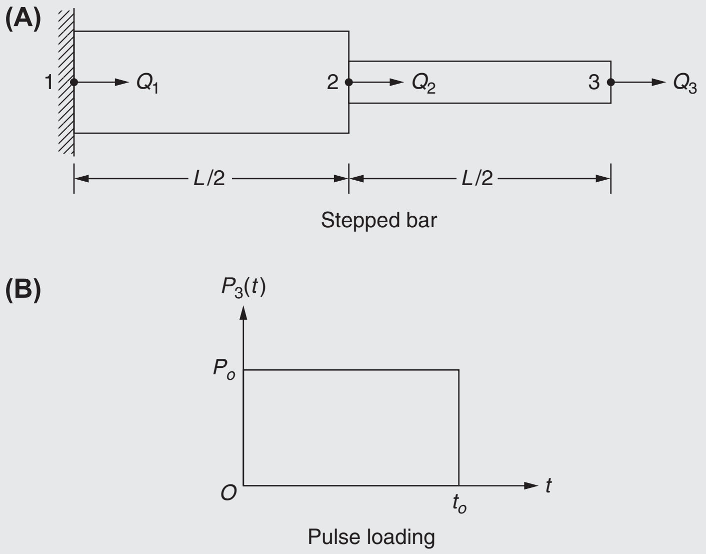

FIGURE 12.9 A stepped bar subjected to pulse loading.

$$\omega_1 = 1.985\left[E/\left(\rho L^2\right)\right]^{1/2} \tag{E.1}$$

$$\omega_2 = 5.159\left[E/\left(\rho L^2\right)\right]^{1/2} \tag{E.2}$$

$$\vec{\underset{\sim}{Q}}_1 = \frac{1}{(\rho AL)^{1/2}}\begin{Bmatrix} 0.8812 \\ 1.5260 \end{Bmatrix} \tag{E.3}$$

$$\vec{\underset{\sim}{Q}}_2 = \frac{1}{(\rho AL)^{1/2}}\begin{Bmatrix} -1.186 \\ 2.053 \end{Bmatrix} \tag{E.4}$$

$$\left[\underset{\sim}{Q}\right] = \frac{1}{(\rho AL)^{1/2}}\begin{bmatrix} 0.8812 & -1.1860 \\ 1.5260 & 2.0530 \end{bmatrix} \tag{E.5}$$

$$[M] = \frac{\rho AL}{12}\begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \tag{E.6}$$

$$[K] = \frac{2AE}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \tag{E.7}$$

Thus, it can be verified that

$$\left[\underset{\sim}{Q}\right]^T[M]\left[\underset{\sim}{Q}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{E.8}$$

Continued

EXAMPLE 12.9 —cont'd

and

$$\underline{[Q]}^T [K] \underline{[Q]} = \left(\frac{E}{\rho L^2}\right) \begin{bmatrix} (1.985)^2 & 0 \\ 0 & (5.159)^2 \end{bmatrix} \tag{E.9}$$

The generalized load vector is given by

$$\begin{aligned} \vec{N} &= \underline{[Q]}^T \vec{P}(t) = \frac{1}{(\rho AL)^{1/2}} \begin{bmatrix} 0.8812 & 1.5260 \\ -1.1860 & 2.0530 \end{bmatrix} \begin{Bmatrix} 0 \\ P_3 \end{Bmatrix} \\ &= \frac{1}{(\rho AL)^{1/2}} \begin{Bmatrix} 1.526 \\ 2.053 \end{Bmatrix} \cdot P_3 \end{aligned} \tag{E.10}$$

The undamped equations of motion, Eq. (12.80), are given by

$$\ddot{\vec{\eta}} + \frac{E}{\rho L^2} \begin{bmatrix} (1.985)^2 & 0 \\ 0 & (5.159)^2 \end{bmatrix} \vec{\eta} = \frac{1}{(\rho AL)^{1/2}} \begin{Bmatrix} 1.526 \\ 2.053 \end{Bmatrix} P_3 \tag{E.11}$$

which, in scalar form, represent

$$\ddot{\eta}_1 + \frac{3.941E}{\rho L^2}\eta_1 = \frac{1.526P_3}{(\rho AL)^{1/2}} \tag{E.12}$$

and

$$\ddot{\eta}_2 + \frac{26.62E}{\rho L^2}\eta_2 = \frac{2.063P_3}{(\rho AL)^{1/2}} \tag{E.13}$$

By assuming that all the initial displacements and velocities are zero, we obtain

$$\vec{Q}(t=0) = \vec{0} = \underline{[Q]}\,\vec{\eta}(0) \quad \text{and} \quad \dot{\vec{Q}}(t=0) = \vec{0} = \underline{[Q]}\,\dot{\vec{\eta}}(0) \tag{E.14}$$

so that

$$\vec{\eta}(0) = \vec{0} \quad \text{and} \quad \dot{\vec{\eta}}(0) = \vec{0} \tag{E.15}$$

Thus, the solutions of Eqs. (E.12) and (E.13) can be expressed (from Eq. 12.100) as follows:

$$\eta_1(t) = \frac{1}{\omega_1}\int_0^t N_1(\tau)\sin\ \omega_1(t-\tau)\,d\tau$$

that is,

$$\begin{aligned} \eta_1(t) &= \left(\frac{\rho L^2}{E}\right)^{1/2}\left(\frac{1}{1.985}\right)\int_0^t \left\{\frac{1.526P_3}{(\rho AL)^{1/2}}\right\}\sin\left\{\left(\frac{E}{\rho L^2}\right)^{1/2}(1.985)\ (t-\tau)\right\}d\tau \\ &= \left(\frac{L}{AE}\right)^{1/2}(0.7686)\int_0^t P_3(\tau)\sin\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-\tau)\right\}d\tau \end{aligned} \tag{E.16}$$

and

$$\eta_2(t) = \frac{1}{\omega_2}\int_0^t N_2(\tau)\sin\omega_2(t-\tau)d\tau$$

EXAMPLE 12.9 —cont'd

that is,

$$\begin{aligned}\eta_2(t) &= \left(\frac{\rho L^2}{E}\right)^{1/2}\left(\frac{1}{5.159}\right)\int_0^t \left\{\frac{2.053P_3}{(\rho AL)^{1/2}}\right\}\sin\left\{\left(\frac{E}{\rho L^2}\right)^{1/2}(5.159)\ (t-\tau)\right\}d\tau \\ &= \left(\frac{L}{AE}\right)^{1/2}(0.3979)\int_0^t P_3(\tau)\sin\left\{0.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-\tau)\right\}d\tau\end{aligned} \tag{E.17}$$

The solutions of Eqs. (E.16) and (E.17) for the given loading can be expressed as follows:
For $t < t_0$:

$$\eta_1(t) = 0.38720P_0\left(\frac{\rho L^3}{AE^2}\right)^{1/2}\left[1-\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.18}$$

and

$$\eta_2(t) = 0.07713P_0\left(\frac{\rho L^3}{AE^2}\right)^{1/2}\left[1-\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.19}$$

For $t > t_0$:

$$\eta_1(t) = 0.38720P_0\left(\frac{\rho L^3}{AE^2}\right)^{1/2}\left[\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\}-\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.20}$$

and

$$\eta_2(t) = 0.07713P_0\left(\frac{\rho L^3}{AE^2}\right)^{1/2}\left[\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\}-\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.21}$$

The physical displacements are given by

$$\begin{aligned}\vec{Q}(t) &= \begin{Bmatrix} Q_2(t) \\ Q_3(t)\end{Bmatrix} = \left[\underset{\sim}{Q}\right]\vec{\eta}(t) = \frac{1}{(\rho AL)^{1/2}}\begin{bmatrix} 0.8812 & -1.186 \\ 1.526 & 2.053\end{bmatrix}\begin{Bmatrix}\eta_1(t) \\ \eta_2(t)\end{Bmatrix} \\ &= \frac{1}{(\rho AL)^{1/2}}\begin{Bmatrix} 0.8812\ \eta_1(t) - 1.186\ \eta_1(t) \\ 1.526\ \ \eta_1(t) + 2.053\ \eta_2(t)\end{Bmatrix}\end{aligned} \tag{E.22}$$

Thus, for $t < t_0$:

$$Q_2(t) = \frac{P_0L}{AE}\left[0.24991 - 0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} + 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.23}$$

$$Q_3(t) = \frac{P_0L}{AE}\left[0.4324 - 0.5907\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} + 0.1583\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.24}$$

and for $t > t_0$:

$$\begin{aligned}Q_2(t) = \frac{P_0L}{AE}\Bigg[&0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} - 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} \\ &+ 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\Bigg]\end{aligned} \tag{E.25}$$

Continued

EXAMPLE 12.9 —cont'd

$$Q_3(t) = \frac{P_0 L}{AE}\left[0.5907\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.5907\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} + 0.1583\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.1583\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \tag{E.26}$$

To determine the average dynamic stresses in the two elements, we use

$$\begin{aligned}\sigma^{(1)}(t) &= 2E\left(\frac{Q_2-Q_1}{L}\right) = \frac{2P_0}{A}\left[0.24991 - 0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} + 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \quad \text{for } t<t_0 \\ &= \frac{2P_0}{A}\left[0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.34140\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} - 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} + 0.09149\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \quad \text{for } t>t_0\end{aligned} \tag{E.27}$$

and

$$\begin{aligned}\sigma^{(2)}(t) &= 2E\left(\frac{Q_3-Q_2}{L}\right) = \frac{2P_0}{A}\left[0.18249 - 0.24930\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\} + 0.06681\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \quad \text{for } t<t_0 \\ &= \frac{2P_0}{A}\left[0.24930\cos\left\{1.985\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.24930\cos\left\{1.985\left(\frac{E}{pL^2}\right)^{1/2}t\right\} + 0.24979\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}(t-t_0)\right\} - 0.24979\cos\left\{5.159\left(\frac{E}{\rho L^2}\right)^{1/2}t\right\}\right] \quad \text{for } t>t_0\end{aligned} \tag{E.28}$$

12.6 NONCONSERVATIVE STABILITY AND FLUTTER PROBLEMS

The stability of nonconservative systems was considered by the finite element method in Barsoum [12.8] and Mote and Matsumoto [12.9]. The problem of panel flutter was treated by Olson [12.10] and Kariappa and Somashekar [12.11]. The flutter analysis of three-dimensional structures (e.g., supersonic aircraft wing structures) that involve modeling by different types of finite elements was presented by Rao [12.12,12.13]. Flutter analysis involves the solution of a double eigenvalue problem that can be expressed as

$$\left[[K] - \omega^2[M] + [Q]\right]\vec{\xi} = \vec{0} \tag{12.101}$$

where $[K]$ and $[M]$ are the usual stiffness and mass matrices, respectively, ω is the flutter frequency, $[Q]$ is the aerodynamic matrix, and $\vec{\xi}$ is the vector of generalized coordinates. The matrix $[Q]$ is a function of flutter frequency ω and flutter velocity V, which are both unknown. For a nontrivial solution of $\vec{\xi}$, the determinant of the coefficient matrix of $\vec{\xi}$ must vanish. Thus, the flutter equation becomes

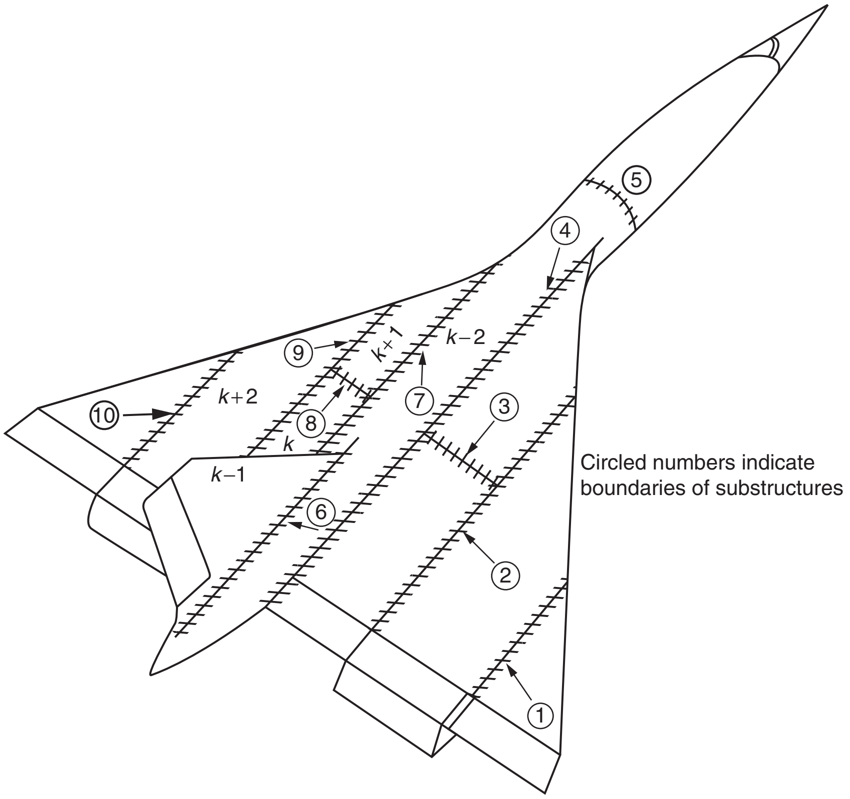

FIGURE 12.10 A large structure divided into substructures.

$$\left| [K] - \omega^2 [M] + [Q] \right| = 0 \tag{12.102}$$

Since two unknowns, ω and V, are in Eq. (12.102), the problem is called a double eigenvalue problem. The details of the generation of aerodynamic matrix $[Q]$ and the solution of Eq. (12.102) are given in Olson [12.10] and Rao [12.12].

12.7 SUBSTRUCTURES METHOD

In the finite element analysis of large systems, the number of equations to be solved for an accurate solution will be quite large. In such cases, the substructures method can be used to reduce the number of equations to manageable size. The system (or structure) is divided into a number of parts or segments, each called a substructure (see Fig. 12.10). Each substructure, in turn, is divided into several finite elements. The element matrix equations of each substructure are assembled to generate the substructure equations. By treating each substructure as a large element with many interior and exterior (boundary) nodes, and using a procedure known as static condensation [12.14], the equations of the substructure are reduced to a form involving only the exterior nodes of that particular substructure. The reduced substructure equations can then be assembled to obtain the overall system equations involving only the boundary unknowns of the various substructures. The number of these system equations is much less compared to the total number of unknowns. The solution of the system equations gives values of the boundary unknowns of each substructure. The known boundary nodal values can then be used as prescribed boundary conditions for each substructure to solve for the respective interior nodal unknowns. The concept of substructuring has been used for the analysis of static, dynamic, as well as nonlinear analyses [12.15,12.16].

REVIEW QUESTIONS

12.1 Give brief answers to the following questions.

1. What is the lumped mass matrix of a uniform bar element?
2. What is the consistent mass matrix of a uniform bar element?

3. What is the lumped mass matrix of a space truss element?
4. What is free vibration?
5. State the mathematical form of a linear algebraic eigenvalue problem.
6. How can we eliminate the possibility of rigid body motion(s) in the dynamic analysis of a structure?
7. What is meant by m-orthogonalization of natural modes?
8. How can we make a modal vector $\vec{Q}_i$ normal with respect to the mass matrix?
9. State two types of dynamic loads in real-life structures.

12.2 Fill in the blank with a suitable word.

1. In a dynamics problem, the displacements and stresses inside an element vary not only with position but also with ______.
2. The solution of the equations of motion of a structure requires both boundary conditions and ______ conditions.
3. The time histories are important in a ______ problem.
4. When the same displacement model is used in the derivation of stiffness and mass matrices, the resulting mass matrix is said to be ______.
5. The results given by the ______ mass matrices are expected to be more accurate.
6. The derivation of the mass matrix requires consideration of variations of all three ______ components.
7. For thick beam elements, the effects of rotary inertia and ______ deformation are to be considered.
8. The derivation of the consistent mass matrix of a triangular membrane element requires consideration of linear ______ of u and v within the local coordinates of the element.
9. The consistent mass matrix is ______ with respect to orientation and position of the coordinate axes for elements whose nodal degrees of freedom correspond to only translational displacements.
10. The vibratory motion at a ______ ______ follows a specific well-defined deformation pattern known as a mode shape.
11. Direct integration approach involves ______ integration of the equations of motion using small time steps.
12. Mode superposition approach involves representation of the dynamic response as the ______ of natural modes.
13. The coupled equations of motion $[M]\ddot{\vec{Q}} + [K]\vec{Q} = \vec{P}$ can be ______ by assuming $\vec{Q}(t) = [\underline{Q}]\vec{\eta}(t)$ where $\vec{\eta}(t)$ denotes the vector of n generalized coordinates.
14. A system with $\varsigma_i < 1$is said to be ______.
15. Substructures method is useful for ______ structures.

12.3 Indicate whether the following statement is true or false.

1. Lumped mass matrices do not consider the dynamic coupling between element displacements.
2. Lumped mass matrices are tridiagonal.
3. The mass matrix of a space truss element can be obtained from that of a bar element using coordinate transformation.
4. Some diagonal elements in the lumped mass matrix of a uniform beam element will be zero if the rotary inertia is not considered.
5. The consistent mass matrix of a planar frame element can be assembled from the consistent mass matrices of bar and beam elements.
6. The consistent mass matrix of a triangular plate bending element considers the variation of only the transverse displacement directly.
7. For membrane type of plate elements, the global and local consistent mass matrices will be identical.
8. Damping does not influence the nature of free vibration of a structure.
9. The study of free vibration involves finding the dynamic response of a system.
10. The natural frequency corresponding to a rigid body motion is zero.
11. In the mode superposition approach, the mass, damping, and stiffness matrices are diagonalized.

12. The solution of the undamped equations of motion $\ddot{\vec{\eta}} + diag[\omega^2]\vec{\eta} = [Q]^T \vec{P}\left(t\right) = \vec{N}\left(t\right)$ can be found by solving a second-order ordinary differential equation.

13. An overdamped system can have $\varsigma_i \geq 0$.

14. Flutter analysis involves the solution of a double eigenvalue problem.

12.4 Select the most appropriate answer among the multiple choices given.

1. The dynamic equations of motion of a structure cannot be derived using the following:
(a) Lagrange equations (b) Hamilton's principle (c) Rayleigh's method

2. The consistent mass matrix is also considered to be somewhat inaccurate because it uses the following in its derivation:
(a) Static displacement model
(b) Linear displacement model
(c) Linear stress–strain model

3. The order of the consistent mass matrix of a space frame element is:
(a) 6×6 (b) 12×12 (c) 9×9

4. In general, the consistent mass matrix of an element in the global coordinate system is:
(a) the same as the one in the local coordinate system
(b) $[\lambda]^T\ [m^{(e)}][\lambda]$
(c) $m_0\,[I]$where m_0 is the mass of the element

5. The motion of a structure during free vibration:
(a) diminishes with time (b) varies harmonically (c) increases exponentially

6. The natural frequencies of a structure are given by
(a) $([K] - \omega^2[M]) = 0$ (b) $[[K] - \omega^2[M]]\underline{\vec{Q}} = \vec{0}$ (c) $\left|[K] - \omega^2[M]\right| = 0$

7. The number of natural frequencies of a system found from the eigenvalue problem $[[K] - \omega^2[M]]\underline{\vec{Q}} = \vec{0}$ is:
(a) 6 (b) 1 (c) the same as the order of the matrices $[K]$ and $[M]$

8. How many frequencies can be zero for a rigid body in space (such as an airplane with no flexibility)?
(a) 1 (b) 6 (c) 3

9. For an undamped system $\left[\underline{Q}\right]$ satisfies the relation
(a) $\left[\underline{Q}\right]^T\left[M\right]\left[\underline{Q}\right] = \left[I\right]$ (b) $\left[\underline{Q}\right]^T\left[\underline{Q}\right] = \left[I\right]$ (c) $\left[\underline{Q}\right]^T\left[M\right]\left[\underline{Q}\right] = diag\left[\omega^2\right]$

10. The equations of motion of a damped system can be uncoupled:
(a) always
(b) only when damping matrix is a linear combination of mass and stiffness matrices
(c) only when $[c] = [0]$

11. The solution of the second order homogenous differential equation $\ddot{\eta}_i(t) + 2\,\varsigma\,\omega_i\,\dot{\eta}_i(t) + \omega_i^2\,\eta_i\,(t) = 0$ is given by $\eta_i(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$ where
(a) $\alpha_{1,2} = -\varsigma_i\,\omega_i$
(b) $\alpha_{1,2} = \pm\omega_i\sqrt{\varsigma_i^2 - 1}$
(c) $\alpha_{1,2} = -\varsigma_i\omega_i \pm \omega_i\sqrt{\varsigma_i^2 - 1}$

12. A critically damped system is one for which
(a) $\varsigma_i = 0$ (b) $\varsigma_i = \pm 1$ (c) $\varsigma_i \geq 0$

13. The damped response of a system is oscillatory only when:
(a) $\varsigma_i < 0$ (b) $\varsigma_i = 1$ (c) $\varsigma_i > 1$

PROBLEMS

12.1 Find the solution of Example 12.7 using the lumped mass matrix.

12.2 Find the solution of Example 12.8 using the lumped mass matrix.

12.3 Find the natural frequencies and modes of vibration for a two-element fixed–fixed beam.

12.4 Find the natural frequencies and modes of vibration for a one-element simply supported beam.

12.5 Find the natural frequencies and modes of vibration for a two-element simply supported beam by taking advantage of the symmetry about the midpoint.

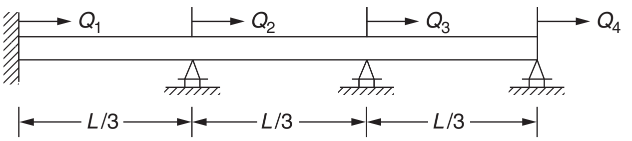

FIGURE 12.11 Rod with multiple supports.

12.6 Find the natural frequencies and mode shapes of the rod shown in Fig. 12.11 in axial vibration.

12.7 Sometimes it is desirable to suppress less important or unwanted degrees of freedom from the original system of equations

$$\underset{n\times n}{[K]}\ \underset{n\times 1}{\vec{X}} = \underset{n\times 1}{\vec{P}} \tag{P.1}$$

to reduce the size of the problem to be solved. This procedure, known as *static condensation* or *condensation of unwanted degrees of freedom*, consists of partitioning Eq. (P.1) as follows:

$$\begin{bmatrix} \underset{p\times p}{K_{11}} & \underset{p\times q}{K_{12}} \\ \underset{q\times p}{K_{21}} & \underset{q\times q}{K_{22}} \end{bmatrix} \begin{Bmatrix} \underset{p\times 1}{\vec{X}_1} \\ \underset{q\times 1}{\vec{X}_2} \end{Bmatrix} = \begin{Bmatrix} \underset{p\times 1}{\vec{P}_1} \\ \underset{q\times 1}{\vec{P}_2} \end{Bmatrix}, \quad p+q=n \tag{P.2}$$

where $\vec{X}_2$ is the vector of unwanted degrees of freedom. Eq. (P.2) gives

$$[K_{11}]\vec{X}_1 + [K_{12}]\vec{X}_2 = \vec{P}_1 \tag{P.3}$$

$$[K_{21}]\vec{X}_1 + [K_{22}]\vec{X}_2 = \vec{P}_2 \tag{P.4}$$

Solving Eq. (P.4) for $\vec{X}_2$ and substituting the result in Eq. (P.3) lead to the desired condensed set of equations

$$\underset{p\times p}{[\underline{K}]}\ \underset{p\times 1}{\vec{X}_1} = \vec{\underline{P}}_{p\times 1} \tag{P.5}$$

Derive the expressions of $\left[\underline{K}\right]$ and $\vec{\underline{P}}$.

12.8 Modify the program CST3D.m (see Section 23.5) to find the displacements and the first two natural frequencies of a box beam (similar to the one shown in Figure 10.9) with the following data:

Length = 100 in, width = 20 in, depth = 10 in, $t_c = 0.5$ in, $t_w = 1.0$ in, $E = 30\times 10^6$ psi, $v = 0.3$, $P_1 = P_2 = 1000$ lb

12.9 Find the natural frequencies of longitudinal vibration of the stepped bar shown in Fig. 12.12 using consistent mass matrices.

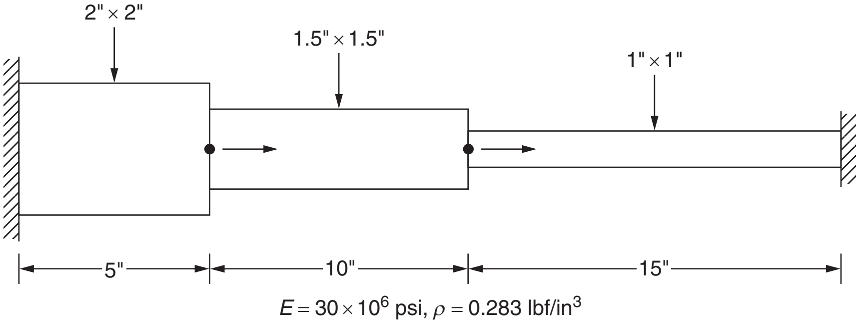

FIGURE 12.12 Three-stepped bar.

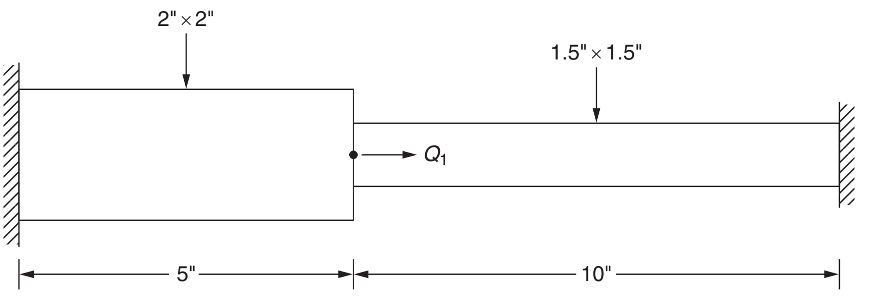

FIGURE 12.13 Two-stepped bar.

12.10 Solve Problem 12.9 using lumped mass matrices.

12.11 Find the natural frequencies of longitudinal vibration of the stepped bar shown in Fig. 12.13 using consistent mass matrices.

12.12 Solve Problem 12.11 using lumped mass matrices.

12.13 Find the mode shapes of the stepped bar shown in Fig. 12.12 corresponding to the natural frequencies found in Problem 12.9.

12.14 Find the mode shapes of the stepped bar shown in Fig. 12.12 corresponding to the natural frequencies found in Problem 12.10.

12.15 Orthogonalize the mode shapes found in Problem 12.13 with respect to the corresponding mass matrix.

12.16 Orthogonalize the mode shapes found in Problem 12.14 with respect to the corresponding mass matrix.

12.17 Find the consistent and lumped mass matrices of the bar element shown in Fig. 12.14 in the *XYZ* coordinate system.

12.18 a. Derive the stiffness and consistent mass matrices of the two-bar truss shown in Fig. 12.15.

b. Determine the natural frequencies of the truss (using the consistent mass matrix).

12.19 a. Derive the lumped mass matrix of the two-bar truss shown in Fig. 12.15.

b. Determine the natural frequencies of the truss (using the lumped mass matrix).

12.20 The properties of the two elements in the stepped beam shown in Fig. 12.16 are:

Element 1: $E = 30 \times 10^6$ psi, $\rho = 0.283$ lbf/in^3, cross-section = circular, 2-in diameter.
Element 2: $E = 11 \times 10^6$ psi, $\rho = 0.1$ lbf/in^3, cross-section = circular, 1-in diameter.
Find the natural frequencies of the stepped beam.

12.21 Find the mode shapes of the stepped beam considered in Problem 12.20.

12.22 Find the natural frequencies of the triangular plate shown in Fig. 12.17 using the consistent mass matrix. Use one triangular membrane element for modeling.

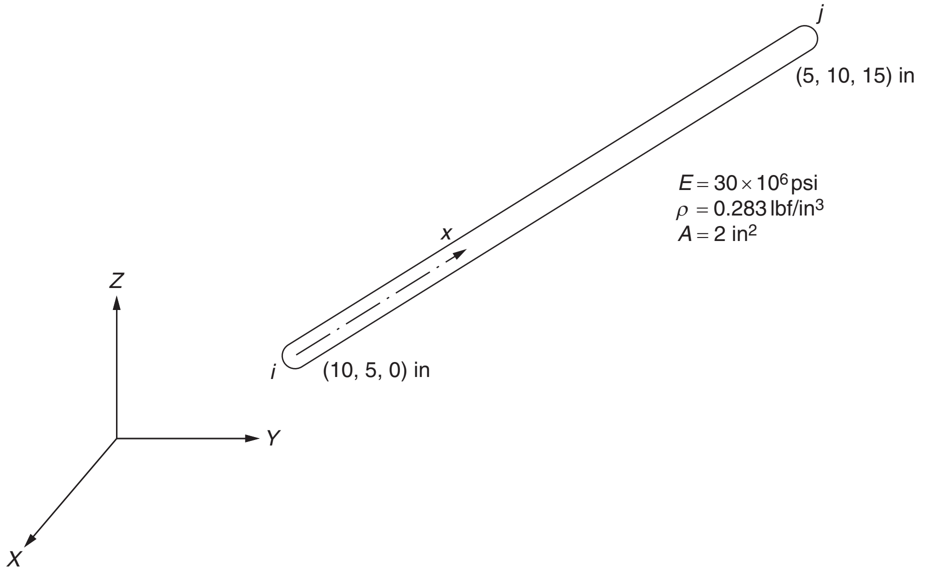

FIGURE 12.14 Bar element in three-dimensional space.

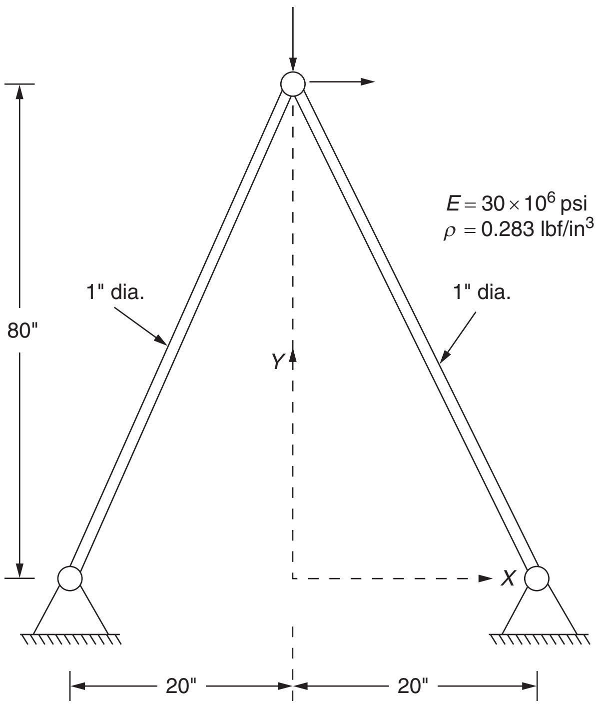

FIGURE 12.15 Two-bar truss.

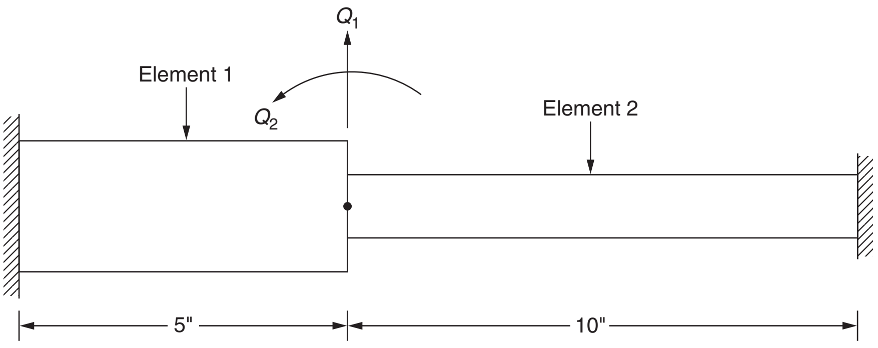

FIGURE 12.16 A stepped-bar.

12.23 Solve Problem 12.22 using the lumped mass matrix.

12.24 Consider the tetrahedron element shown in Fig. 12.18. Find the natural frequencies of the element by fixing the face 123.

12.25 Consider the stepped bar shown in Fig. 12.13. If the force shown in Fig. 12.19 is applied along Q_1, determine the dynamic response, $Q_1(t)$.

12.26 The cantilever beam shown in Fig. 12.20A is subjected to the force indicated in Fig. 12.20B along the direction of Q_1. Determine the responses $Q_1(t)$ and $Q_2(t)$.

12.27 a. Derive the consistent mass matrix of a bar element shown in Eq. (12.26) starting from the matrix of shape functions, $[N]$, given by Eq. (12.24).

b. Derive the consistent mass matrix of a space truss element shown in Eq. (12.30) starting from the matrix of shape functions, $[N]$, defined by Eq. (12.28).

12.28 Derive the consistent mass matrix of a uniform beam element shown in Fig. 12.4 by evaluating the integrals shown in Eq. (12.34).

12.29 Derive the consistent mass matrix of a planar frame element shown in Fig. 9.14.

12.30 Derive the consistent mass matrix of a uniform rod (or bar) element under torsion shown in Fig. 9.9C.

12.31 Derive the consistent mass matrix given by Eq. (12.39) of a space frame element shown in Fig. 9.9A.

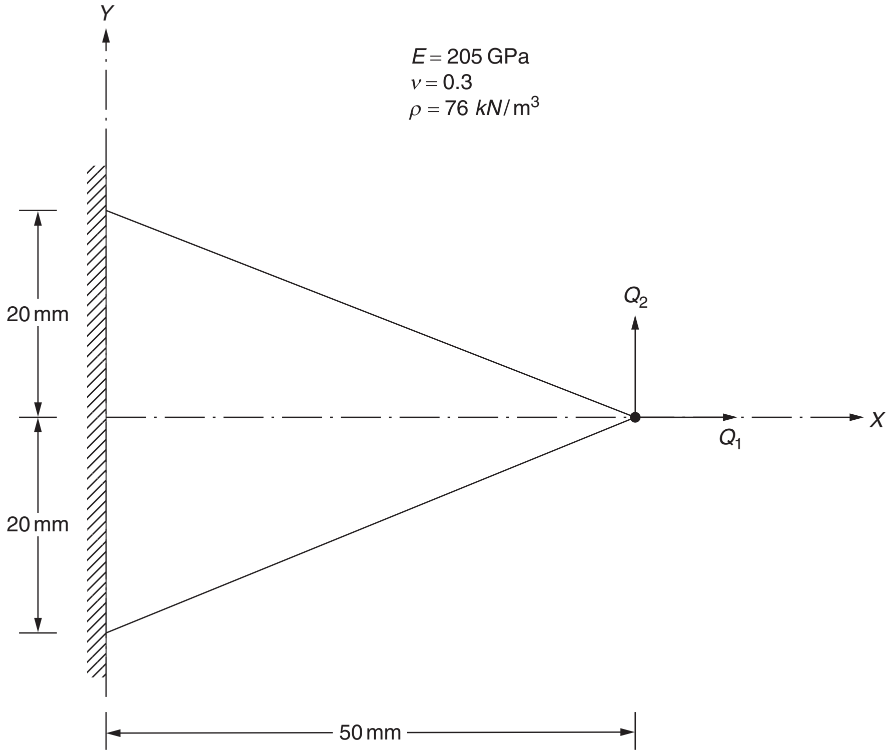

FIGURE 12.17 A triangular plate.

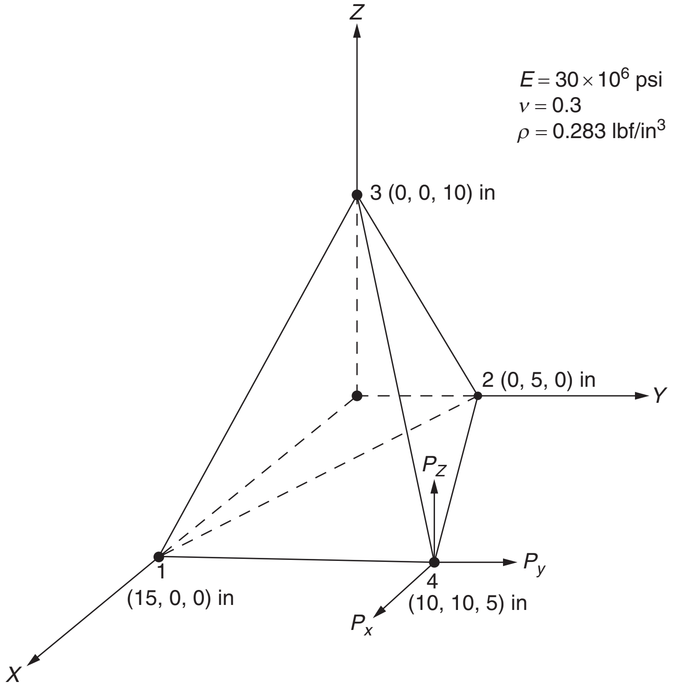

FIGURE 12.18 Tetrahedron element.

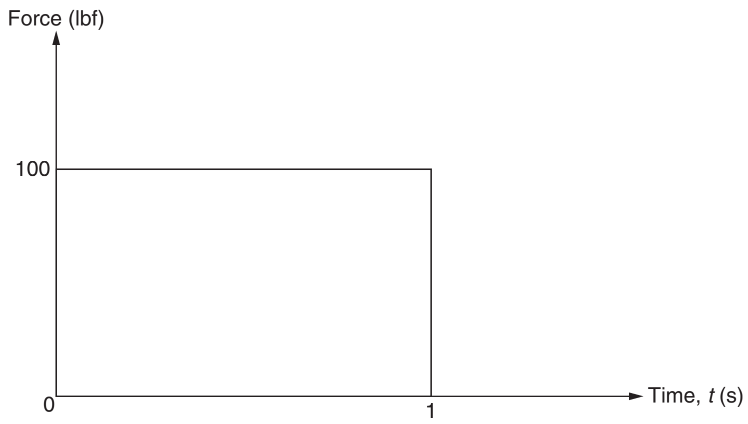

FIGURE 12.19 Force applied along Q_1.

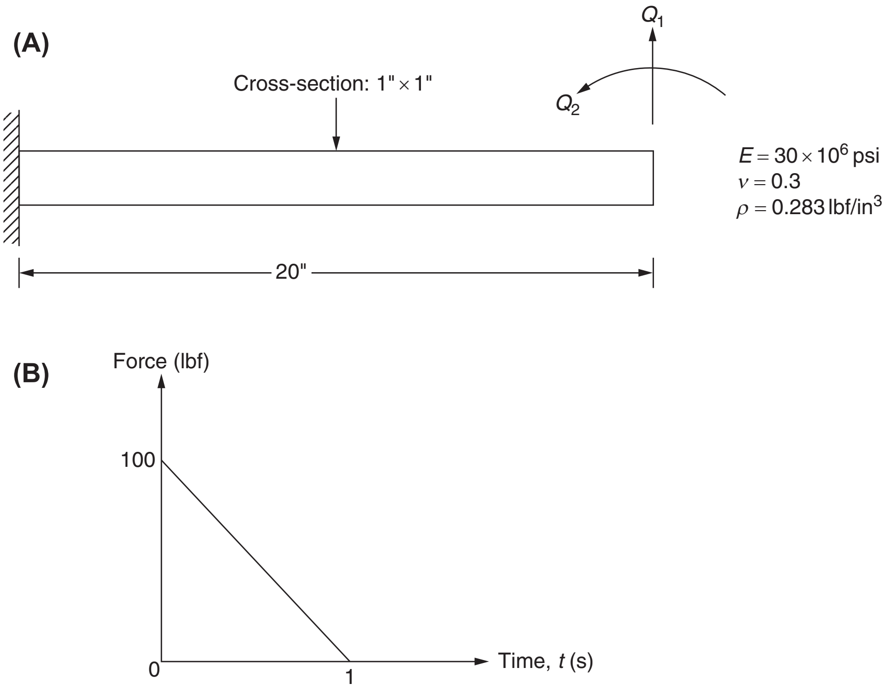

FIGURE 12.20 Cantilever beam with force applied along Q_1.

12.32 Derive the consistent mass matrix given by Eq. (12.45), for a triangular membrane element with nine degrees of freedom shown in Fig. 10.2.

12.33 Derive the consistent mass matrix given by Eq. (12.55) for a tetrahedron element.

12.34 Find the lumped mass matrix of a uniform beam element, shown in Fig. 12.4, by including its mass moment of inertia.

12.35 Find the lumped mass matrix of a uniform membrane element shown in Fig. 10.2.

12.36 Derive the consistent mass matrix of a two-node tapered bar element (with an axial degree of freedom at each node) with the area of cross section varying linearly along x as $A(x) = A_i\left(1-\frac{x}{l}\right)+A_j\left(\frac{x}{l}\right)$.

12.37 Find the lumped mass matrix of a two-node tapered bar element (with an axial degree of freedom at each node) with the area of cross-section varying linearly along x as $A(x) = A_i\left(1-\frac{x}{l}\right)+A_j\left(\frac{x}{l}\right)$.

12.38 Derive the consistent mass matrix of a two-node beam element with two degrees of freedom at each node (as shown in Fig. 12.4) when the area of cross section of the beam varies linearly as $A(x) = A_i\left(1-\frac{x}{l}\right)+A_j\left(\frac{x}{l}\right)$.

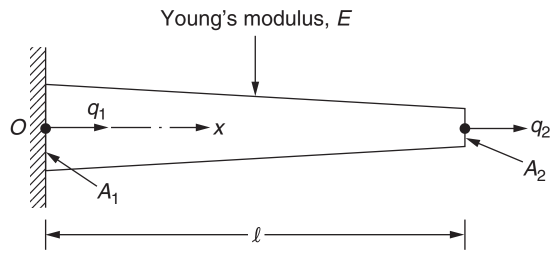

FIGURE 12.21 Constrained tapered bar element.

12.39 Find the lumped mass matrix of a two-node beam element with two degrees of freedom at each node (as shown in Fig. 12.4) when the area of cross section of the beam varies linearly as $A(x) = A_i\left(1-\frac{x}{l}\right) + A_j\left(\frac{x}{l}\right)$.

12.40 Find the natural frequencies of the free uniform bar shown in Fig. 12.1A using one finite element. Use the consistent mass matrix of the element.

12.41 Find the natural frequencies of the free uniform bar shown in Fig. 12.1A using one finite element. Use the lumped mass matrix of the element.

12.42 Find the natural frequencies of vibration of the constrained tapered bar element shown in Fig. 12.21 with the area of cross section varying as $A(x) = 2\ A\left(1-\frac{x}{2l}\right)$ using a one-bar element with the consistent mass matrix.

12.43 Find the natural frequencies of vibration of the constrained tapered bar element shown in Fig. 12.21 with the cross-section area varying as $A(x) = 2\ A\left(1-\frac{x}{2l}\right)$ using a one-bar element with the lumped mass matrix.

12.44 Find the natural frequencies of vibration of the free tapered bar element shown in Fig. 12.22 with the cross-section area varying as $A(x) = 2A\left(1-\frac{x}{2l}\right)$ using a one-bar element with the consistent mass matrix.

12.45 Find the natural frequencies of vibration of the free tapered bar element shown in Fig. 12.22 with the cross-section area varying as $A(x) = 2\ A\left(1-\frac{x}{2l}\right)$ using a one-bar element with the lumped mass matrix.

12.46 Fig. 12.23 shows a uniform beam fixed at $x = 0$ and simply supported at $x = l$. Find the natural frequency of vibration of the beam using a one-beam element with the consistent mass matrix.

12.47 Fig. 12.23 shows a uniform beam fixed at $x = 0$ and simply supported at $x = l$. Find the natural frequency of vibration of the beam using a one-beam element with the lumped mass matrix.

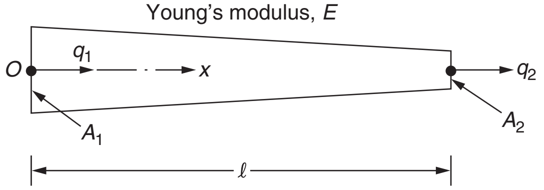

FIGURE 12.22 Free tapered bar element.

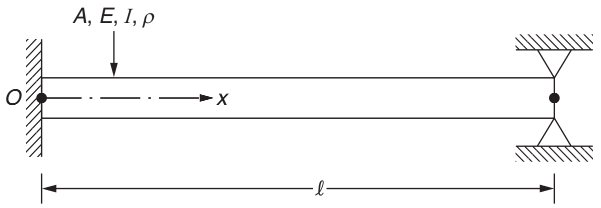

FIGURE 12.23 Fixed-simply supported beam.

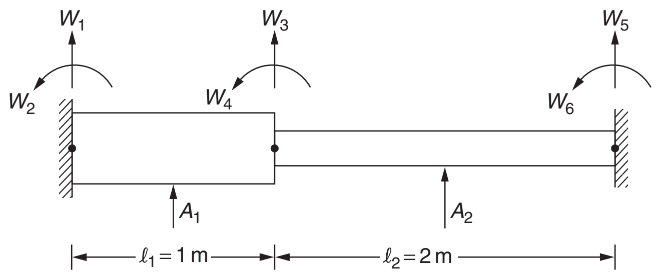

FIGURE 12.24 Stepped-bar with fixed ends.

12.48 Consider a stepped beam fixed at both the ends as shown in Fig. 12.24 with the following data: $\rho = 7850$ kg/m^3, $E = 207$ GPa, $A_1 =$ cross-sectional area of step $1 = 10$ cm $\times$ 10 cm, and $A_2 =$ cross-sectional area of step $2 = 5$ cm $\times$ 5 cm. Using a one-beam element for each step of the beam, determine the following:
a. The stiffness matrix and the mass matrix, using consistent mass matrices of elements, of the stepped beam.
b. The natural frequencies of vibration of the stepped beam.

12.49 Consider a stepped beam fixed at both the ends as shown in Fig. 12.24 with the following data: $\rho = 7850$ kg/m^3, $E = 207$ GPa, $A_1 =$ cross-sectional area of step $1 = 10$ cm $\times$ 10 cm, and $A_2 =$ cross-sectional area of step $2 = 5$ cm $\times$ 5 cm. Using one beam element for each step of the beam, determine the following:
a. The stiffness matrix and the mass matrix, using lumped mass matrices of the stepped beam elements.
b. The natural frequencies of vibration for the stepped beam.

12.50 Using a one-beam element idealization of the beam column shown in Fig. 12.25, find the natural frequencies of vibration. Use a lumped mass matrix of the beam column.

Data: $E = 207$ GPa, $\rho = 7850$ kg/m^3, $l = 1$ m, area of cross-section = round with 10-cm diameter.

12.51 Using a one-beam element idealization of the beam column shown in Fig. 12.25, find the natural frequencies of vibration. Use a consistent mass matrix of the beam column.

Data: $E = 207$ GPa, $\rho = 7850$ kg/m^3, $l = 1$ m, area of cross-section = round with 10-cm diameter.

12.52 Consider a rectangular plate in plane stress shown in Fig. 12.26. Using two triangular-plane membrane elements for idealization, derive the eigenvalue problem to find the natural frequencies of vibration of the plate. Use lumped mass matrices.

12.53 Consider a rectangular plate in plane stress shown in Fig. 12.26. Using two triangular plane membrane elements for idealization, derive the eigenvalue problem to find the natural frequencies of vibration of the plate. Use consistent mass matrices.

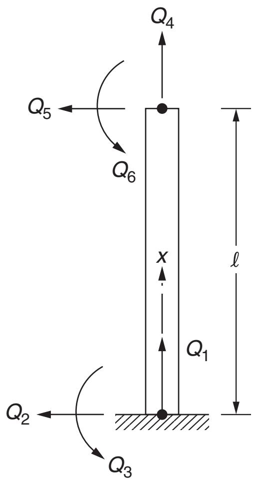

FIGURE 12.25 Beam-column.

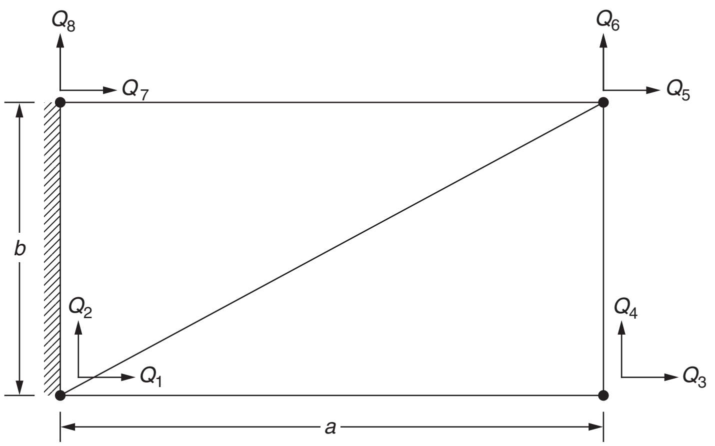

FIGURE 12.26 Rectangular element in plane stress.

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Part IV

Application to Heat Transfer Problems

Chapter 13

Formulation and Solution Procedure

Chapter Outline

13.1 INTRODUCTION

Knowledge of the temperature distribution within a body is important in many engineering problems. This information will be useful in computing the heat added to or removed from a body. Furthermore, if a heated body is not permitted to expand freely in all the directions, some stresses will be developed inside the body. The magnitude of these thermal stresses will influence the design of devices such as boilers, steam turbines, and jet engines. The first step in calculating the thermal stresses is to determine the temperature distribution within the body.

The objective of this chapter is to derive the finite element equations for the determination of temperature distribution within a conducting body. The basic unknown in heat transfer problems is temperature, similar to displacement in stress analysis problems. As indicated in Chapter 5, the finite element equations can be derived either by minimizing a suitable functional using a variational (Rayleigh−Ritz) approach or from the governing differential equation using a weighted residual (Galerkin) approach.

13.2 BASIC EQUATIONS OF HEAT TRANSFER

The basic equations of heat transfer, namely the energy balance and rate equations, are summarized in this section.

13.2.1 Energy Balance Equation

In the heat transfer analysis of any system, the following energy balance equation has to be satisfied because of conservation of energy:

$$\dot{E}_{in} + \dot{E}_g = \dot{E}_{out} + \dot{E}_{ie} \tag{13.1}$$

where the dot above a symbol signifies a time rate; $\dot{E}_{in}$ is the energy inflow into the system, $\dot{E}_g$ is the energy generated inside the system, $\dot{E}_{out}$ is the energy outflow from the system, and E_{ie} is the change in internal energy of the system.

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00013-4

13.2.2 Rate Equations

The rate equations, which describe the rates of energy flow, are given by the following equations.

FOR CONDUCTION

Definition: Conduction is the transfer of heat through materials without any net motion of the mass of the material. The rate of heat flow in x direction by conduction (q) is given by

$$q = kA\frac{\partial T}{\partial x} \tag{13.2}$$

where k is the thermal conductivity of the material, A is the area normal to x direction through which heat flows, T is the temperature, and x is the length parameter.

FOR CONVECTION

Definition: Convection is the process by which thermal energy is transferred between a solid and a fluid surrounding it. The rate of heat flow by convection (q) can be expressed as

$$q = hA(T - T_\infty) \tag{13.3}$$

where h is the heat transfer coefficient, A is the surface area of the body through which heat flows, T is the temperature of the surface of the body, and T_∞ is the temperature of the surrounding medium.

FOR RADIATION

Definition: Radiation heat transfer is the process by which the thermal energy is exchanged between two surfaces obeying the laws of electromagnetics.

The rate of heat flow by radiation (q) is governed by the relation

$$q = \sigma\ \varepsilon A\left(T^4 - T_\infty^4\right) \tag{13.4}$$

where σ is the Stefan–Boltzmann constant, ε is the emissivity of the surface, A is the surface area of the body through which heat flows, T is the absolute surface temperature of the body, and T_∞ is the absolute surrounding temperature.

ENERGY GENERATED IN A SOLID

Energy will be generated in a solid body whenever other forms of energy, such as chemical, nuclear, or electrical energy, are converted into thermal energy. The rate of heat generated $\left(\dot{E}_g\right)$ is governed by the equation

$$\dot{E}_g = \dot{q}V \tag{13.5}$$

where $\dot{q}$ is the strength of the heat source (rate of heat generated per unit volume per unit time) and V is the volume of the body.

ENERGY STORED IN A SOLID

Whenever the temperature of a solid body increases, thermal energy will be stored in it. The equation describing this phenomenon is given by

$$\dot{E}_S = \rho c V\frac{\partial T}{\partial t} \tag{13.6}$$

where $\dot{E}_S$ is the rate of energy storage in the body, ρ is the density of the material, c is the specific heat of the material, V is the volume of the body, T is the temperature of the body, and t is the time parameter.

13.3 GOVERNING EQUATION FOR THREE-DIMENSIONAL BODIES

Consider a small element of material in a solid body as shown in Fig. 13.1. The element is in the shape of a rectangular parallelepiped with sides dx, dy, and dz. The energy balance equation can be stated as follows [13.1]:

$$\begin{array}{l}\text{Heat inflow}\\ \text{during time } dt\end{array} + \begin{array}{l}\text{Heat generated}\\ \text{by internal}\\ \text{sources during } dt\end{array} = \begin{array}{l}\text{Heat outflow}\\ \text{during } dt\end{array} + \begin{array}{l}\text{Change in}\\ \text{internal}\\ \text{energy during } dt\end{array} \tag{13.7}$$

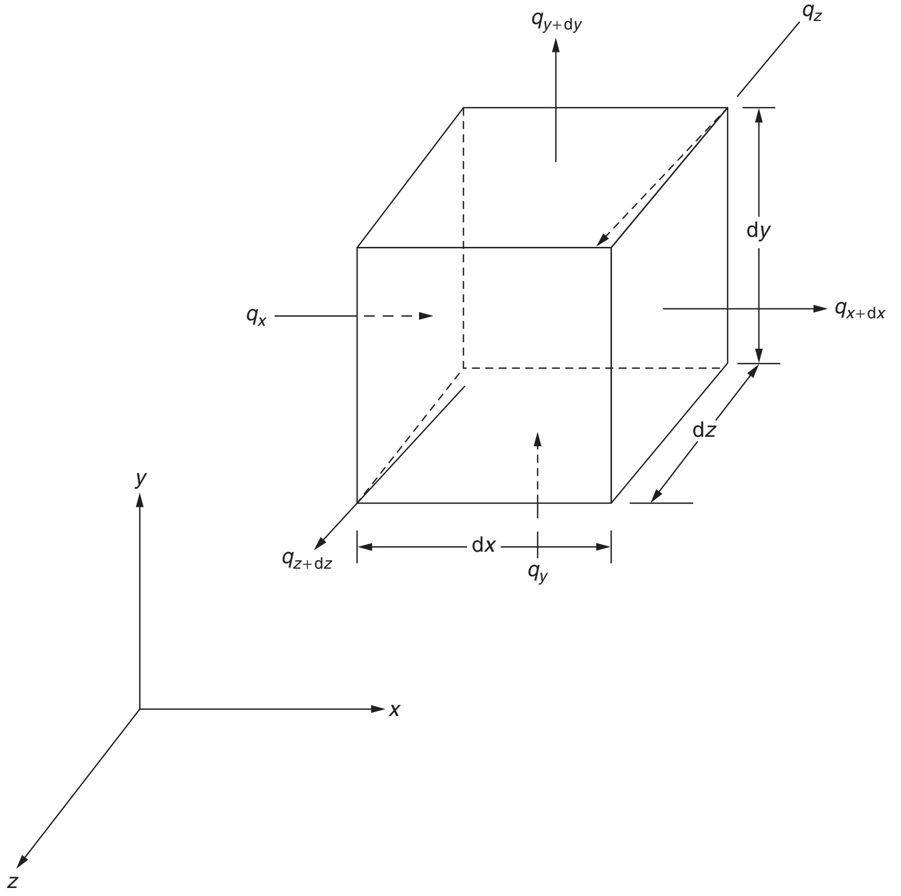

FIGURE 13.1 An element in Cartesian coordinates.

With the help of rate equations, Eq. (13.7) can be expressed as

$$(q_x + q_y + q_z)\mathrm{d}t + \dot{q}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ \mathrm{d}t = (q_{x+\mathrm{d}x} + q_{y+\mathrm{d}y} + q_{z+\mathrm{d}z})\mathrm{d}t + \rho c\ \mathrm{d}T\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \tag{13.8}$$

where

$$\begin{aligned} q_x &= \text{heat inflow rate into the face located at } x \\ &= -k_x A_x \frac{\partial T}{\partial x} = -k_x \frac{\partial T}{\partial x} \mathrm{d}y\ \mathrm{d}z \end{aligned} \tag{13.9}$$

$$\begin{aligned} q_{x+\mathrm{d}x} &= \text{heat outflow rate from the face located at } x + \mathrm{d}x \\ &= q|_{x+\mathrm{d}x} \approx q_x + \frac{\partial q_x}{\partial x}\mathrm{d}x \\ &= -k_x A_x \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k_x A_x \frac{\partial T}{\partial x}\right)\mathrm{d}x \\ &= -k_x \frac{\partial T}{\partial x}\mathrm{d}y\ \mathrm{d}z - \frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \end{aligned} \tag{13.10}$$

k_x is the thermal conductivity of the material in x direction, A_x is the area normal to the x direction through which heat flows $= \mathrm{d}y\ \mathrm{d}z$, T is the temperature, $\dot{q}$ is the rate of heat generated per unit volume (per unit time), ρ is the density of the material, and c is the specific heat of the material. By substituting Eqs. (13.9) and (13.10) and similar expressions for q_y, $q_{y+\mathrm{d}y}$, q_z, and $q_{z+\mathrm{d}z}$ into Eq. (13.8) and dividing each term by dx dy dz dt, we obtain

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \tag{13.11}$$

Eq. (13.11) is the differential equation governing the heat conduction in an orthotropic solid body. If the thermal conductivities in x, y, and z directions are assumed to be the same, $k_x = k_y = k_z = k =$ constant, Eq. (13.11) can be written as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{13.12}$$

where the constant $\alpha = (k/\rho c)$ is called the thermal diffusivity. Eq. (13.12) is the heat conduction equation that governs the temperature distribution and the conduction heat flow in a solid having uniform material properties (isotropic body). If heat sources are absent in the body, Eq. (13.12) reduces to the Fourier equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{13.13}$$

If the body is in a steady state (with heat sources present), Eq. (13.12) becomes the Poisson's equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=0 \tag{13.14}$$

If the body is in a steady state without any heat sources, Eq. (13.12) reduces to the Laplace equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \tag{13.15}$$

13.3.1 Governing Equation in Cylindrical Coordinate System

If a cylindrical coordinate system (with r, ϕ, z coordinates) is used instead of the Cartesian x, y, z system, Eq. (13.12) takes the form

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{13.16}$$

This equation can be derived by taking the element of the body as shown in Fig. 13.2.

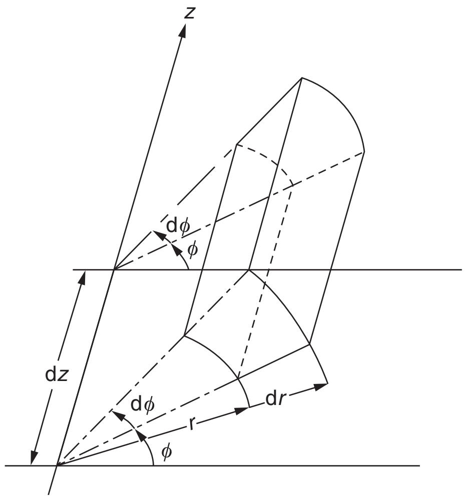

FIGURE 13.2 An element in cylindrical coordinates.

13.3.2 Governing Equation in Spherical Coordinate System

By considering an element of the body in a spherical r, ϕ, ψ coordinate system as indicated in Fig. 13.3, the general heat conduction Eq. (13.12) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\cdot\sin\phi}\cdot\frac{\partial}{\partial\phi}\left(\sin\phi\cdot\frac{\partial T}{\partial\phi}\right)+\frac{1}{r^2\cdot\sin^2\phi}\cdot\frac{\partial^2 T}{\partial\psi^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{13.17}$$

13.3.3 Boundary and Initial Conditions

Since the differential equation—Eq. (13.11) or (13.12)—is second order, two boundary conditions need to be specified. The possible boundary conditions are

$$T(x,y,z,t) = T_0 \quad \text{for} \quad t>0 \ \text{on} \ S_1 \tag{13.18}$$

$$k_x\cdot\frac{\partial T}{\partial x}\cdot l_x+k_y\cdot\frac{\partial T}{\partial y}\cdot l_y+k_z\cdot\frac{\partial T}{\partial z}\cdot l_z+q_0=0 \quad \text{for} \quad t>0 \ \text{on} \ S_2 \tag{13.19}$$

$$k_x\cdot\frac{\partial T}{\partial x}\cdot l_x+k_y\cdot\frac{\partial T}{\partial y}\cdot l_y+k_z\cdot\frac{\partial T}{\partial z}\cdot l_z+h(T-T_\infty)=0 \quad \text{for} \quad t>0 \ \text{on} \ S_3 \tag{13.20}$$

where q_0 is the boundary heat flux, h is the convection heat transfer coefficient, T_∞ is the surrounding temperature, and l_x, l_y, l_z are the direction cosines of the outward drawn normal to the boundary.

Eq. (13.18) indicates that the temperature is specified as T_0 (as $T_0(t)$ in an unsteady state problem) on the surface S_1. This boundary condition is applicable, for example, when the surface is in contact with a melting solid or a boiling liquid. There will be heat transfer in both these cases and the surface remains at the temperature of the phase change process. Eq. (13.19) represents the existence of a fixed or constant heat flux q_0 at the surface S_2. The equation basically states that the heat flux q_0 is related to the temperature gradient at the surface by Fourier's law. This boundary condition is realized when a thin film or patch electric heater is attached or bonded to the surface. A special case of this boundary condition corresponds to the perfectly insulated or adiabatic surface for which the temperature gradient is zero. Eq. (13.20) denotes the existence of convection heat transfer (heating or cooling of the body) at the surface S_3 of the body. Eq. (13.20) represents energy balance at the surface. This boundary condition is realized when cold (or hot) air flows around the hot (or cold) surface. The air may be blown by a fan to increase the convection heat transfer. The boundary condition stated in Eq. (13.18) is known as the Dirichlet condition and those stated in Eqs. (13.19) and (13.20) are called Neumann conditions.

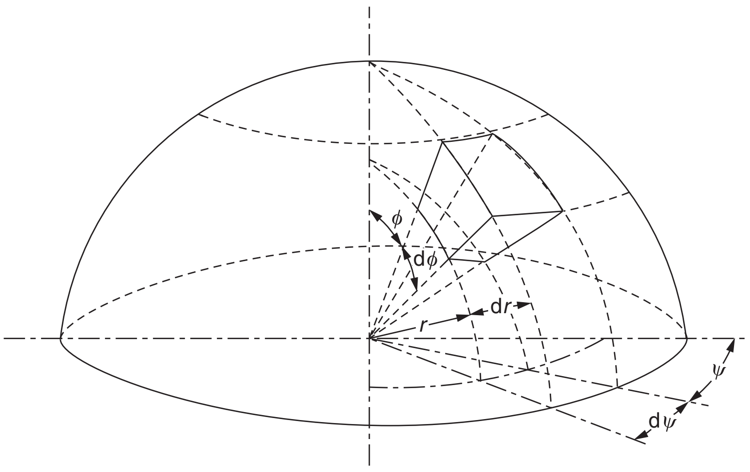

FIGURE 13.3 An element in spherical coordinates.

Furthermore, the differential equation, Eq. (13.11) or (13.12), is first-order in time t, and hence it requires one initial condition. The commonly used initial condition is

$$T(x, y, z, t = 0) = \overline{T}_0(x, y, z) \text{ in } V \tag{13.21}$$

where V is the domain (or volume) of the solid body and $\overline{T}_0$ is the specified temperature distribution at time zero.

EXAMPLE 13.1

A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of $k = 2$ W/m °C. During steady state operation, the furnace wall has a temperature of 800 °C on the inside and 300 °C on the outside. If one of the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall.

Solution

Assuming that the heat loss is due to conduction only, the rate of heat transfer (heat flowing through a unit surface area of the wall) is given by

$$q = k\frac{dT}{dx} \approx k\frac{\Delta T}{\Delta x} = 2.0\left(\frac{800 - 300}{0.2}\right) = 5000 \ \text{W/m}^2$$

The rate of heat loss can be determined as follows:

$$\begin{aligned}\text{Rate of heat loss} &= \text{Rate of heat transfer} \times \text{Surface area of the wall through which heat flows}\\ &= (5000)(2) = 10,000 \text{ W}\end{aligned}$$

EXAMPLE 13.2

A metal pipe of 10 cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20 °C while the outer surface of the pipe is at a temperature of 250 °C. If the heat transfer coefficient for free convection from the pipe to the air is $h = 20$ W/m^2 °C find the rate of heat loss from the pipe.

Solution

The convection heat loss from the pipe of unit length ($l = 1$ m) is given by

$$q = h\,P\,l\,(T - T_\infty) \tag{E.1}$$

where P is the perimeter of the pipe cross section, l is the length of the pipe, T is the temperature of the outer surface of the pipe, and T_∞ is the temperature of the air. Using the data given, Eq. (E.1) gives:

Heat loss per 1-m length of pipe (q) = 20 {π (0.1)} (1) (250—20) = 1445.1360 W.

EXAMPLE 13.3

A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20 °C while the outer surface of the pipe is at a temperature of 250 °C. If the emissivity of the outer surface of the pipe is 0.75, determine the rate of heat loss from the pipe to the surrounding air and walls of the room by radiation.

Solution

The radiation heat loss (q) from the pipe, of unit length ($l = 1$ m), is given by

$$q = \sigma\ \varepsilon\ A\left(T^4 - T_\infty^4\right) \tag{E.1}$$

where $\sigma = 5.7 \times 10{-}8$ W/m^2 °K^4, $\varepsilon = 0.75$, A = surface area from which radiation heat transfer occurs $= Pl = \pi\ Dl = \pi(0.1)(1) = 0.1\ \pi$ m^2, $T = 250$ °C = 523 °K, and $T_\infty = 20$ °C = 293 °K. Thus, Eq. (E.1) gives

$$\begin{aligned}\text{Heat loss per unit length of pipe } (q) &= 5.7 \times 10^{-8}(0.75)\,(0.1\pi)\left(523^4 - 293^4\right)\\ &= 905.8496 \text{ W}\end{aligned}$$

EXAMPLE 13.4

Compute the thermal diffusivity of the following building materials with given values of thermal conductivity (k), density (ρ), and specific heat (c_p):

Material	Thermal Conductivity, k W/m °C	Density, ρ Kg/m^3	Specific Heat, c_p J/kg °C
Plywood	0.12	540	1215
Brick	0.70	1920	835
Gypsum or plaster board	0.15	900	1080

Solution

The thermal diffusivity (α) is defined as

$$\alpha = \frac{k}{\rho \ c_p} \tag{E.1}$$

From the known data, the thermal diffusivities of the indicated materials can be determined from Eq. (E.1) as follows:

$$\text{Playwood: } \alpha = \frac{0.12}{(540)(1215)} = 183 \times 10^{-9} \ \text{m}^2/\text{s}$$

$$\text{Brick: } \alpha = \frac{0.70}{(1920)(835)} = 437 \times 10^{-9} \ \text{m}^2/\text{s}$$

$$\text{Gypsum or plaster board: } \alpha = \frac{0.15}{(900)(1080)} = 154 \times 10^{-9} \ \text{m}^2/\text{s}$$

EXAMPLE 13.5

In an automobile radiator, the coolant (assumed to be water) enters at 170 °F and leaves at 130 °F with a flow rate of 10 gallons per minute. If the surrounding air is at 75 °F, determine the surface area of the radiator. Assume the following data: Convection heat transfer coefficient = 12 BTU/(ft^2-hr °F), Specific heat of water at 140 °F = 1.0245 BTU/lb$_m$ °F, Density of water at 140 °F: 61.1794 lb$_m$/ft^3.

Solution

Assuming steady state, the heat lost through water must be equal to the heat gain by the surrounding air. Thus

$$m_w c_p (T_{in} - T_{out}) = h A_{rad} (T_w - T_\infty) \tag{E.1}$$

where m_w denotes the mass flow rate of water given by

$$m_w = \text{volume flow rate} \times \text{density of water} = \frac{10(0.1337)}{60}(61.1794) = 1.3631 \ \text{lbm/s}$$

Using the relation: 1 gallon = 231 in^3 = 0.1337 ft^3. Eq. (E.1) gives, for 1 h,

$$(1.3631 \times 3600)(1.0245)(170 - 130) = 12 A_{\text{rad}}(150 - 75) \tag{E.2}$$

where the average temperature of water is used for T_w as

$$T_w = \frac{T_{\text{IN}} + t_{\text{OUT}}}{2} = \frac{170 + 130}{2} = 150 \,^\circ F \tag{E.3}$$

The solution of Eq. (E.2) gives the surface area of the radiator as $A_{\text{rad}} = 223.4393$ ft^2.

Note

The large area required for the radiator surface area is accommodated within the available space of the radiator by using a very large number of fins, each fin designed to provide some of the heat exchanger surface area.

13.4 STATEMENT OF THE PROBLEM

13.4.1 In Differential Equation Form

The problem of finding the temperature distribution inside a solid body involves the solution of Eq. (13.11) or Eq. (13.12) subject to satisfaction of the boundary conditions of Eqs. (13.18) to (13.20) and the initial condition given by Eq. (13.21).

13.4.2 In Variational Form

The three-dimensional heat conduction problem can be stated in an equivalent variational form [13.2] as follows:

Find the temperature distribution $T(x, y, z, t)$ inside the solid body that minimizes the integral

$$I = \frac{1}{2}\iiint_V \left[k_x\left(\frac{\partial T}{\partial x}\right)^2 + k_y\left(\frac{\partial T}{\partial y}\right)^2 + k_z\left(\frac{\partial T}{\partial z}\right)^2 - 2\left(\dot{q} - \rho c \frac{\partial T}{\underset{\sim}{\partial t}}\right) T \right] \mathrm{d}V \tag{13.22}$$

and satisfies the boundary conditions of Eqs. (13.18) to (13.20) and the initial condition of Eq. (13.21). Here, the term $\left(\partial T \big/ \underset{\sim}{\partial t}\right)$ must be considered fixed while taking the variations. It can be verified that Eq. (13.11) is the Euler–Lagrange equation corresponding to the functional of Eq. (13.22). Generally it is not difficult to satisfy the boundary condition of Eq. (13.18), but Eqs. (13.19) and (13.20) present some difficulty. To overcome this difficulty, an integral pertaining to the boundary conditions of Eqs. (13.19) and (13.20) is added to the functional of Eq. (13.22) so that when the combined functional is minimized, the boundary conditions of Eqs. (13.19) and (13.20) would be automatically satisfied. The integral pertaining to Eqs. (13.19) and (13.20) is given by

$$\iint_{S_2} q_0 T \mathrm{d}S_2 + \iint_{S_3} \frac{1}{2} h (T - T_\infty)^2 \mathrm{d}S_3$$

Thus, the combined functional to be minimized will be

$$I = \frac{1}{2}\iiint_V \left[k_x\left(\frac{\partial T}{\partial x}\right)^2 + k_y\left(\frac{\partial T}{\partial y}\right)^2 + k_z\left(\frac{\partial T}{\partial z}\right)^2 - 2\left(\dot{q} - \rho c \frac{\partial T}{\underset{\sim}{\partial t}}\right) T \right] \mathrm{d}V + \iint_{S_2} q_0 T \mathrm{d}S_2 + \frac{1}{2}\iint_{S_3} h (T - T_\infty)^2 \mathrm{d}S_3 \tag{13.23}$$

13.5 DERIVATION OF FINITE ELEMENT EQUATIONS

The finite element equations for the heat conduction problem can be derived by using a variational approach or a Galerkin approach. We shall derive the equations using both approaches in this section.

13.5.1 Variational Approach

In this approach, we consider the minimization of the functional I given by Eq. (13.23) subject to the satisfaction of the boundary conditions of Eq. (13.18) and the initial conditions of Eq. (13.21). The step-by-step procedure involved in the derivation of finite element equations is as follows.

Step 1: Divide the domain V into E finite elements of p nodes each.

Step 2: Assume a suitable form of variation of T in each finite element and express $T^{(e)}$ (x, y, z, t) in element e as

$$T^{(e)}(x, y, z, t) = [N(x, y, z)]\overrightarrow{T}^{(e)} \tag{13.24}$$

where

$$[N(x, y, z)] = [N_1(x, y, z)\; N_2(x, y, z) \ldots N_p(x, y, z)]$$

$$\vec{T}^{(e)} = \begin{Bmatrix} T_1(t) \\ T_2(t) \\ \vdots \\ T_p(t) \end{Bmatrix}^{(e)}$$

$T_i(t)$ is the temperature of node i, and $N_i(x, y, z)$ is the interpolation function corresponding to node i of element e.
Step 3: Express the functional I as a sum of E elemental quantities $I^{(e)}$ as

$$I = \sum_{e=1}^{E} I^{(e)} \tag{13.25}$$

where

$$\begin{aligned} I^{(e)} &= \frac{1}{2}\iiint_{V(e)} \left[k_x \left(\frac{\partial T^{(e)}}{\partial x}\right)^2 + k_y \left(\frac{\partial T^{(e)}}{\partial y}\right)^2 + k_z \left(\frac{\partial T^{(e)}}{\partial z}\right)^2 - 2\left(\dot{q} - \rho c \frac{\partial T^{(e)}}{\underset{\sim}{\partial t}}\right) T^{(e)} \right] \mathrm{d}V \\ &+ \iint_{S_2^{(e)}} q_0 T^{(e)} \mathrm{d}S_2 + \frac{1}{2}\iint_{S_3^{(e)}} h\left(T^{(e)} - T_\infty\right)^2 \mathrm{d}S_3 \end{aligned} \tag{13.26}$$

For the minimization of the functional I, use the necessary conditions

$$\frac{\partial I}{\partial T_i} = \sum_{e=1}^{E} \frac{\partial I^{(e)}}{\partial T_i} = 0, \quad i = 1, 2, \ldots, M$$

where M is the total number of nodal temperature unknowns. From Eq. (13.26), we have

$$\begin{aligned} \frac{\partial I^{(e)}}{\partial T_i} &= \iiint_{V(e)} \left[k_x \frac{\partial T^{(e)}}{\partial x} \frac{\partial}{\partial T_i}\left(\frac{\partial T^{(e)}}{\partial x}\right) + k_y \frac{\partial T^{(e)}}{\partial y} \frac{\partial}{\partial T_i}\left(\frac{\partial T^{(e)}}{\partial y}\right) + k_z \frac{\partial T^{(e)}}{\partial z} \frac{\partial}{\partial T_i}\left(\frac{\partial T^{(e)}}{\partial z}\right) \right. \\ &\left. - \left(\dot{q} - \rho c \frac{\partial T^{(e)}}{\underset{\sim}{\partial t}}\right) \frac{\partial T^{(e)}}{\partial T_i} \right] \mathrm{d}V + \iint_{S_2^{(e)}} q_0 \frac{\partial T^{(e)}}{\partial T_i} \mathrm{d}S_2 + \iint_{S_3^{(e)}} h\left(T^{(e)} - T_\infty\right) \frac{\partial T^{(e)}}{\partial T_i} \cdot \mathrm{d}S_3 \end{aligned} \tag{13.27}$$

Note that the surface integrals do not appear in Eq. (13.27) if node i does not lie on S_2 and S_3. Eq. (13.24) gives

$$\left.\begin{aligned} \frac{\partial T^{(e)}}{\partial x} &= \left[\frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} \cdots \frac{\partial N_p}{\partial x}\right] \vec{T}^{(e)} \\ \frac{\partial}{\partial T_i}\left(\frac{\partial T^{(e)}}{\partial x}\right) &= \frac{\partial N_i}{\partial x} \\ \frac{\partial T^{(e)}}{\partial T_i} &= N_i \\ \frac{\partial T^{(e)}}{\underset{\sim}{\partial t}} &= [N] \dot{\vec{T}}^{(e)} \end{aligned}\right\} \tag{13.28}$$

where

$$\dot{\vec{T}}^{(e)} = \begin{Bmatrix} \partial T_1/\partial t \\ \vdots \\ \partial T_p/\partial t \end{Bmatrix}$$

Thus, Eq. (13.27) can be expressed as

$$\frac{\partial I^{(e)}}{\partial \vec{T}^{(e)}} = \left[K_1^{(e)}\right]\vec{T}^{(e)} - \vec{P}^{(e)} + \left[K_2^{(e)}\right]\vec{T}^{(e)} + \left[K_3^{(e)}\right]\dot{\vec{T}}^{(e)} \tag{13.29}$$

where the elements of $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, $\left[K_3^{(e)}\right]$, and $\vec{P}^{(e)}$ are given by

$$K_{1ij}^{(e)} = \iiint_{V^{(e)}} \left(k_x \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + k_y \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + k_z \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right) \cdot \mathrm{d}V \tag{13.30}$$

$$K_{2ij}^{(e)} = \iint_{S_3^{(e)}} h N_i N_j \cdot \mathrm{d}S_3 \tag{13.31}$$

$$K_{3ij}^{(e)} = \iiint_{V(e)} \rho c N_i N_j \cdot \mathrm{d}V \tag{13.32}$$

and

$$P_i^{(e)} = \iiint_{V^{(e)}} \dot{q} N_i \mathrm{d}v - \iint_{S_2^{(e)}} q_0 N_i \mathrm{d}S_2 + \iint_{S_3^{(e)}} h T_\infty N_i \mathrm{d}S_3 \tag{13.33}$$

Step 4: Rewrite Eq. (13.29) in matrix form as

$$\frac{\partial I}{\partial \underset{\sim}{\vec{T}}} = \sum_{e=1}^{E} \frac{\partial I^{(e)}}{\partial \vec{T}^{(e)}} = \sum_{e=1}^{E}\left(\left[\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right]\right]T^{(e)} - \vec{P}^{(e)} + \left[K_3^{(e)}\right]\dot{\vec{T}}^{(e)}\right) = \vec{0} \tag{13.34}$$

where $\underset{\sim}{\vec{T}}$ is the vector of nodal temperature unknowns of the system:

$$\underset{\sim}{\vec{T}} = \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \end{Bmatrix}$$

By using the familiar assembly process, Eq. (13.34) can be expressed as

$$[\underset{\sim}{K_3}]\dot{\underset{\sim}{\vec{T}}} + \left[\underset{\sim}{K}\right]\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{13.35}$$

where

$$[\underset{\sim}{K_3}] = \sum_{e=1}^{E}\left[K_3^{(e)}\right] \tag{13.36}$$

$$[\underset{\sim}{K}] = \sum_{e=1}^{E}\left[\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right]\right] \tag{13.37}$$

and

$$\underset{\sim}{\vec{P}} = \sum_{e=1}^{E}\vec{P}^{(e)} \tag{13.38}$$

Step 5: Eq. (13.35) are the desired equations that have to be solved after incorporating the boundary conditions specified over S_1 (Eq. (13.18) and the initial conditions stated in Eq. (13.21)).

13.5.2 Galerkin Approach

The finite element procedure using the Galerkin method can be described by the following steps.

Step 1: Divide the domain V into E finite elements of p nodes each.

Step 2: Assume a suitable form of variation of T in each finite element and express $T^{(e)}$ (x, y, z, t) in element e as

$$T^{(e)}(x,y,z,t) = [N(x,y,z)]\vec{T}^{(e)} \tag{13.39}$$

Step 3: In the Galerkin method, the integral of the weighted residue over the domain of the element is set equal to zero by taking the weights same as the interpolation functions N_i. Since the solution in Eq. (13.39) is not exact, substitution of Eq. (13.39) into the differential Eq. (13.11) gives a nonzero value instead of zero. This nonzero value will be the residue. Hence, the criterion to be satisfied at any instant of time is

$$\iiint_{V^{(e)}}\left[N_i\left\{\frac{\partial}{\partial x}\left(k_x\frac{\partial T^{(e)}}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T^{(e)}}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T^{(e)}}{\partial z}\right)+\dot{q}-\rho c\frac{\partial T^{(e)}}{\partial t}\right\}\right]\mathrm{d}V = 0, \quad i = 1,2,\ldots,p \tag{13.40}$$

By noting that the first integral term of Eq. (13.40) can be written as

$$\iiint_{V^{(e)}} N_i\frac{\partial}{\partial x}\left(k_x\frac{\partial T^{(e)}}{\partial x}\right)\mathrm{d}V = -\iiint_{V^{(e)}}\frac{\partial N_i}{\partial x}k_x\frac{\partial T^{(e)}}{\partial x}\mathrm{d}V + \iint_{S^{(e)}} N_i k_x\frac{\partial T^{(e)}}{\partial x}l_x\ \mathrm{d}S \tag{13.41}$$

where l_x is the x-direction cosine of the outward drawn normal, and with similar expressions for the second and third integral terms, Eq. (13.40) can be stated as

$$\begin{aligned}
&-\iiint_{V^{(e)}}\left[k_x\frac{\partial N_i}{\partial x}\frac{\partial T^{(e)}}{\partial x}+k_y\frac{\partial N_i}{\partial y}\frac{\partial T^{(e)}}{\partial y}+k_z\frac{\partial N_i}{\partial z}\frac{\partial T^{(e)}}{\partial z}\right]\mathrm{d}V\\
&+\iint_{S^{(e)}} N_i\left[k_x\frac{\partial T^{(e)}}{\partial x}l_x+k_y\frac{\partial T^{(e)}}{\partial y}l_y+k_z\frac{\partial T^{(e)}}{\partial z}l_z\right]\mathrm{d}S\\
&+\iiint_{V^{(e)}} N_i\left(\dot{q}-\rho c\frac{\partial T^{(e)}}{\partial t}\right)\mathrm{d}V = 0, \quad i = 1,2,\ldots,p
\end{aligned} \tag{13.42}$$

Since the boundary of the element $S^{(e)}$ is composed of $S_1^{(e)}, S_2^{(e)}$, and $S_3^{(e)}$, the surface integral of Eq. (13.42) over $S_1^{(e)}$ would be zero (since $T^{(e)}$ is prescribed to be a constant T_0 on $S_1^{(e)}$, the derivatives of $T^{(e)}$ with respect to x, y, and z would be zero). On the surfaces $S_2^{(e)}$ and $S_3^{(e)}$, the boundary conditions given by Eqs. (13.19) and (13.20) are to be satisfied. For this, the surface integral in Eq. (13.42) over $S_2^{(e)}$ and $S_3^{(e)}$ is written in equivalent form as

$$\iint_{S_2^{(e)}+S_3^{(e)}} N_i\left[k_x\frac{\partial T^{(e)}}{\partial x}l_x+k_y\frac{\partial T^{(e)}}{\partial y}l_y+k_z\frac{\partial T^{(e)}}{\partial z}l_z\right]\mathrm{d}S = -\iint_{S_2^{(e)}} N_i q_0\ \mathrm{d}S_2 - \iint_{S_3^{(e)}} h\left(T^{(e)}-T_\infty\right)dS_3 \tag{13.43}$$

By using Eqs. (13.39) and (13.43), Eq. (13.42) can be expressed in matrix form as

$$\left[K_1^{(e)}\right]\vec{T}^{(e)}+\left[K_2^{(e)}\right]\vec{T}^{(e)}+\left[K_3^{(e)}\right]\dot{\vec{T}}^{(e)}-\vec{P}^{(e)} = \vec{0} \tag{13.44}$$

where the elements of the matrices $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, $\left[K_3^{(e)}\right]$, and $\vec{P}^{(e)}$ are the same as those given in Eqs. (13.30) to (13.33).

Step 4: The element Eq. (13.44) can be assembled in the usual manner to obtain the overall equations as

$$\left[\underset{\sim}{K}_3\right]\dot{\underset{\sim}{\vec{T}}}+\left[\underset{\sim}{K}\right]\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{13.45}$$

where $\left[\underset{\sim}{K}_3\right]$, $\left[\underset{\sim}{K}\right]$, and $\underset{\sim}{\vec{P}}$ are the same as those defined by Eqs. (13.36) to (13.38). It can be seen that the same final equations, Eq. (13.35), are obtained in both the approaches.

Step 5: Eq. (13.35) have to be solved after incorporating the boundary conditions specified over S_1 and the initial conditions.

Notes

1. The expressions for $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, $\left[K_3^{(e)}\right]$, and $\vec{P}^{(e)}$ can be stated using matrix notation as

$$\left[K_1^{(e)}\right] = \iiint_{V^{(e)}} [B]^T[D][B]\,\mathrm{d}V \tag{13.46}$$

$$\left[K_2^{(e)}\right] = \iint_{S_3^{(e)}} h[N]^T[N]\mathrm{d}S_3 \tag{13.47}$$

$$\left[K_3^{(e)}\right] = \iiint_{V^{(e)}} \rho c[N]^T[N]\,\mathrm{d}V \tag{13.48}$$

$$\vec{P}^{(e)} = \vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)} \tag{13.49}$$

where

$$\vec{P}_1^{(e)} = \iiint_{V^{(e)}} \dot{q}[N]^T\mathrm{d}V \tag{13.50}$$

$$\vec{P}_2^{(e)} = \iint_{S_2^{(e)}} q_0[N]^T\mathrm{d}S_2 \tag{13.51}$$

$$\vec{P}_3^{(e)} = \iint_{S_3^{(e)}} hT_\infty[N]^T\mathrm{d}S_3 \tag{13.52}$$

$$[D] = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \tag{13.53}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_p}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_p}{\partial y} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots & \frac{\partial N_p}{\partial z} \end{bmatrix} \tag{13.54}$$

2. When all three modes of heat transfer are considered, the governing differential equation becomes nonlinear (due to the inclusion of radiation term). An iterative procedure is presented in Section 14.7 for the solution of heat transfer problems involving radiation.

REVIEW QUESTIONS

13.1 Give brief answers to the following questions.

1. Give two practical systems for which thermal stress analysis is very important.
2. What is the basic idea used in energy balance equations?
3. What are rate equations?
4. How is thermal diffusivity (α) defined?

13.2 Fill in the blank with a suitable word.

1. The basic unknown parameter in most thermal problems is ____________________.
2. The rate of heat flow by conduction (q) is ____________________.
3. The rate of heat flow by convection (q) is ____________________.
4. The rate of heat flow by radiation (q) is ____________________.

5. When the temperature is specified on part of the surface for an unsteady state problem, the boundary condition is known as ______________________ condition.
6. When energy balance is enforced on a surface due to a flow of cold (or hot) air on a hot (or cold) surface, the condition is called ______________________ condition.
7. The reason for using initial condition in the solution of an unsteady state problem is that the differential equation has ________________ derivative of order one.
8. A heat transfer problem can be stated either in ______________________________________ equation form or in ____________________________________ form.
9. In the finite element equations of a general heat transfer problem, there will be three square matrices corresponding to conduction, convection, and ________________ ____________________________.
10. In the finite element equations of a thermal problem, the right-hand side vector (similar to the load vector in structural problems) is composed of three vectors, associated with heat ____________________________, convection, and boundary ____________________ ____________________________.
11. Radiation heat transfer obeys the laws of ____.
12. The consideration of radiation heat transfer introduces nonlinearity to the system due to the presence of the __________________ power of temperature.
13. Thermal ____________________________ is defined as $\alpha = k/(\rho c)$.

13.3 Indicate whether the following statement is true or false.

1. In the variational approach, the functional to be minimized includes contributions corresponding to the Dirichlet conditions.
2. The Galerkin method also minimizes the functional considered in the variational approach.

13.4 Select the most appropriate answer from the multiple choices given.

1. The energy generated in a solid body is:
   (a) $k\ V$ (b) $\dot{q}\ V$ (c) $h\ V$
2. The energy absorbed in a solid body is given by
   (a) $k\ A\ \frac{\partial T}{\partial x}$ (b) $h\ A\ \frac{\partial T}{\partial t}$ (c) $\rho\ c\ V\ \frac{\partial T}{\partial t}$

13.5 Match the nature of following equations for an isotropic body.

1. Fourier equation	(a) Steady state (with no heat source)
2. Poisson equation	(b) Unsteady state (no heat generation)
3. Laplace equation	(c) Steady state

13.6 Match the nature of the following types of heat transfer.

1. Conduction	(a) Heat flows from one surface to another even with no medium
2. Convection	(b) Heat transfers through the material
3. Radiation	(c) Heat transfers from solid to fluid

PROBLEMS

13.1 Derive the heat conduction equation in cylindrical coordinates, Eq. (13.16), from Eq. (13.12).
Hint: Use the relations $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$ in Eq. (13.12).

13.2 Derive the heat conduction equation in spherical coordinates, Eq. (13.17), from Eq. (13.12).
Hint: Use the relations $x = r \sin\phi \cos \psi$, $y = r \sin\phi \sin \psi$, and $z = r \cos\phi$ in Eq. (13.12).

13.3 The steady-state one-dimensional heat conduction equation is given by:

In Cartesian coordinate system: $\frac{d}{dx}\left[k\frac{dT}{dx}\right] = 0$

In cylindrical coordinate system: $\frac{d}{dr}\left[rk\frac{dT}{dr}\right] = 0$

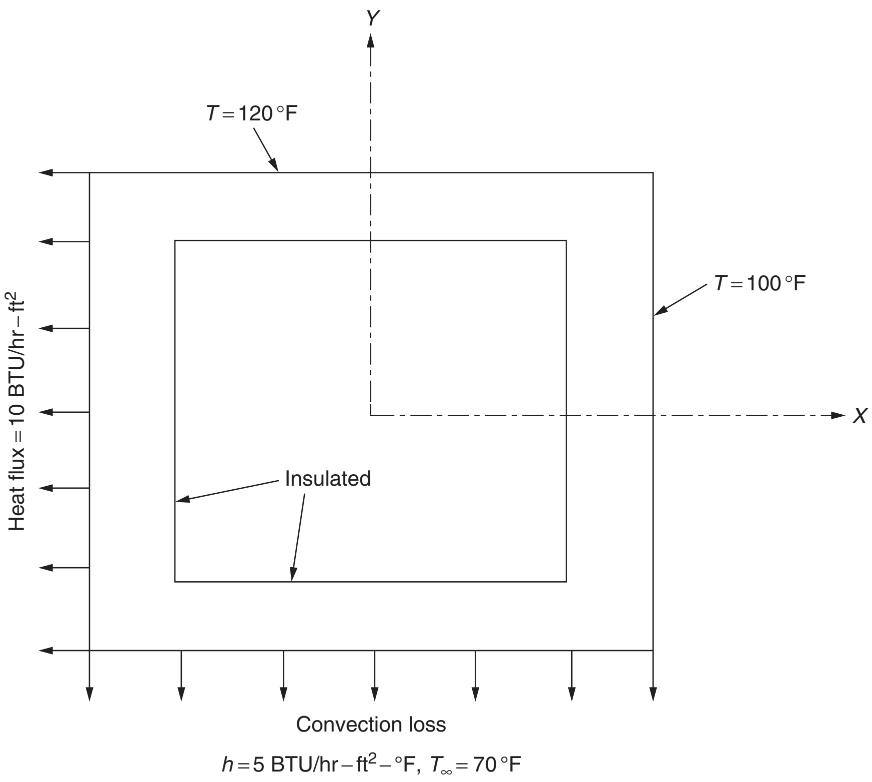

FIGURE 13.4 Heat transfer in a hollow rectangular section.

In spherical coordinate system: $\frac{\mathrm{d}}{\mathrm{d}r}\left[kr^2\frac{\mathrm{d}T}{\mathrm{d}r}\right] = 0$

Suggest a suitable temperature distribution model in each case for use in the finite element analysis.

13.4 Express the boundary conditions of Fig. 13.4 in the form of equations.

13.5 The thermal equilibrium equation for a one-dimensional problem, including conduction, convection, and radiation, can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[kA\frac{\mathrm{d}T}{\mathrm{d}x}\right] - hP(T - T_\infty) - \varepsilon\sigma P\left(T^4 - T_\infty^4\right) + \dot{q}A = 0; \quad 0 \le x \le L \tag{P.1}$$

where k is the conductivity coefficient, h is the convection coefficient, ε is the emissivity of the surface, σ is the Stefan–Boltzman constant, $\dot{q}$ is the heat generated per unit volume, A is the cross-sectional area, P is the perimeter, $T(x)$ is the temperature at location x, T_∞ is the ambient temperature, and L is the length of the body. Show that the variational functional I corresponding to Eq. (P.1) is given by

$$I = \int_{x=0}^{L}\left\{\dot{q}AT - \frac{1}{2}hPT^2 + hPT_\infty T - \frac{1}{5}\varepsilon\sigma PT^5 + \varepsilon\sigma PT_\infty^4 T - \frac{1}{2}kA\left[\frac{\mathrm{d}T}{\mathrm{d}x}\right]^2\right\}\mathrm{d}x \tag{P.2}$$

13.6 Derive the finite element equations corresponding to Eq. (P.1) of Problem 13.5 using the Galerkin approach.

13.7 Derive the finite element equations corresponding to Eqs. (P.1) and (P.2) of Problem 13.5 using a variational approach.

13.8 Heat transfer takes place by convection and radiation from the inner and outer surfaces of a hollow sphere. If the radii are R_i and R_o, the fluid (ambient) temperatures are T_i and T_o, convection heat transfer coefficients are h_i and h_o, and emissivities are ε_i and ε_o at the inner (i) and outer (o) surfaces of the hollow sphere, state the governing differential equation and the boundary conditions to be satisfied in finding the temperature distribution in the sphere, $T(r)$.

13.9 Compute the thermal diffusivity of the following metals with given values of thermal conductivity (k), density (ρ), and specific heat (c_p):

Metal	Thermal Conductivity, k W/m °C	Density, ρ kg/m^3	Specific Heat, c_p J/kg °C
Aluminum	235	2700	900
Copper	400	8930	385
Carbon steel	60.2	7855	432

13.10 Heat flows by conduction at a rate of 5 kW through the insulated board of a wall. The insulating board has a thickness of 2 cm, surface area of 5 m^2, and thermal conductivity of 0.4 W/m °C. If the temperature of the inner (hot) surface of the board is 350 °C, find the temperature of the outer surface of the board.

13.11 The glass window of a room has an area of 4 m^2, a thickness of 0.75 cm, and a thermal conductivity of 1.8 W/m °C. If the inside and outside surfaces of the window are 20 °C and −2 °C, respectively, determine the heat loss through the window.

13.12 A cast iron casting of dimensions 10 × 15 × 25 cm, shown in Fig. 13.5, is at a temperature of 500 °C. Cold air at a temperature of 20 °C is blown around and on the top of the casting. The convection coefficient between the casting and air is $h = 120$ W/m^2 °C. Determine the rate of heat transfer from the casting to the air.

13.13 A rectangular chip of surface area 0.3 m^2 is maintained at 200 °C. It is placed in an evacuated chamber whose walls are maintained at 20 °C. If the emissivity of the surface of the chip is 0.9, find the rate at which heat is transferred by radiation between the chip and the chamber walls.

13.14 A pipe with an outer diameter of 12 cm and length 5 m runs through a room. The pipe carries superheated steam that causes the temperature of the outer surface of the pipe to be 180 °C. The temperature of the air and the walls of the room is 20 °C. If the emissivity of the pipe surface is 0.85 and the convection heat transfer coefficient between the pipe and the surrounding air is $h = 15$ W/m^2 °C, determine the following:

a. Rate of heat loss from the pipe due to convection.

b. Rate of heat loss from the pipe due to radiation.

13.15 A uniform fin of length L and cross-sectional area A (shown in Fig. 13.6) has a thermal conductivity k. The temperature at the root of the fin, $T(x = 0)$, is maintained at a high temperature T_0 and the fin is surrounded by cold air at temperature T_∞. The convection heat transfer between the fin and air is h. The temperature distribution,

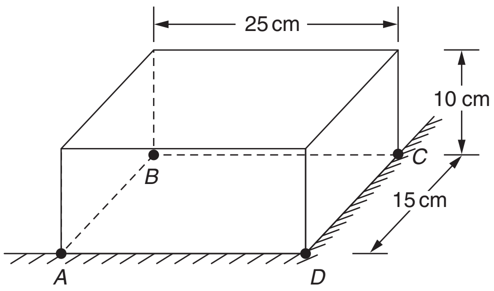

FIGURE 13.5 A cast iron casting.

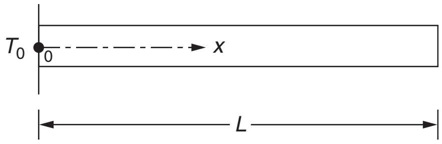

FIGURE 13.6 Uniform fin.

$T = T(x)$, along the fin, with convection loss occurring along the peripheral surface of the fin from $x = 0$ to $x = L$ and the right edge of the fin (at $x = L$) insulated, is given by [13.3]:

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh\ m\,(L - x)}{\cosh\ mL} \tag{P.3}$$

where

$$m = \sqrt{\frac{h\,P}{k\,A}} \tag{P.4}$$

and P is the perimeter of the cross section of the fin. Plot Eq. (P.3) for an aluminum fin with $k = 235$ W/m °C, $h = 20$ W/m^2 °C, $A = 2$ cm^2, $P = 6$ cm, $T_0 = 200$ °C, $T_\infty = 20$ °C, and L = 10 cm.

13.16 A uniform fin of length L and cross-sectional area A (shown in Fig. 13.6) has a thermal conductivity k. The temperature at the root of the fin, $T(x = 0)$, is maintained at a high temperature T_0 and the fin is surrounded by cold air at a temperature T_∞. The convection heat transfer between the fin and air is h. The temperature distribution, $T = T(x)$, along the fin, with convection loss occurring along the peripheral surface of the fin from $x = 0$ to $x = L$ as well as at the right edge of the fin (at $x = L$), is given by

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\cosh m\,(L - x) + \dfrac{h}{m\ k}\sinh m\,(L - x)}{\cosh mL + \dfrac{h}{m\ k}\sinh mL} \tag{P.5}$$

where m is given by Eq. (P.4) of Problem 13.15, and P is the perimeter of the cross section of the fin. Plot Eq. (P.5) for an aluminum fin with $k = 235$ W/m °C, $h = 20$ W/m^2 °C, $A = 2$ cm^2, $P = 6$ cm, $T_0 = 200$ °C, $T_\infty = 20$ °C, and L = 10 cm.

13.17 A uniform fin of length L and cross-sectional area A (shown in Fig. 13.6) has a thermal conductivity k. The temperature at the root of the fin, $T(x = 0)$, is maintained at a high temperature T_0 and the fin is surrounded by cold air at a temperature T_∞. The convection heat transfer between the fin and air is h. The temperature distribution, $T = T(x)$, along the fin, with convection loss occurring along the peripheral surface of the fin from $x = 0$ to $x = L$ and the right edge of the fin (at $x = L$) maintained at a temperature T_L, is given by

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\left(\dfrac{T_L - T_\infty}{T_0 - T_\infty}\right)\sinh mx + \sinh m\,(L - x)}{\sinh mL} \tag{P.6}$$

where m is given by Eq. (P.4) of Problem 13.15, and P is the perimeter of the cross section of the fin. Plot Eq. (P.6) for an aluminum fin with $k = 235$ W/m °C, $h = 20$ W/m^2 °C, $A = 2$ cm^2, $P = 6$ cm, $T_0 = 200$ °C, $T_\infty = 20$ °C, $T_L = 40$ °C, and L = 10 cm.

13.18 A uniform fin of length L and cross-sectional area A (shown in Fig. 13.6) has a thermal conductivity k. The temperature at the root of the fin, $T(x = 0)$, is maintained at a high temperature T_0 and the fin is surrounded by cold air at a temperature T_∞. If the fin is very long ($L \to \infty$), the temperature distribution, $T = T(x)$, along the fin with convection loss occurring along the peripheral surface of the fin is given by

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx} \tag{P.7}$$

where m is given by Eq. (P.4) of Problem 13.15, P is the perimeter of the cross section of the fin, and h is the convection heat transfer coefficient between the fin and air. Plot Eq. (P.7) for an aluminum fin with $k = 235$ W/m °C, $h = 20$ W/m^2 °C, $A = 2$ cm^2, $P = 6$ cm, $T_0 = 200$ °C, and $T_\infty = 20$ °C.

13.19 Fig. 13.7 shows a hollow sphere of inner radius R_i and outer radius R_o. The inner and outer surfaces of the sphere are maintained at the steady temperatures of T_i and T_o, respectively. Consider the thermal equilibrium of a small hollow spherical element of inner radius r and outer radius $r + \mathrm{d}r$ with no heat generation. Derive an expression for the rate of heat transfer through the hollow sphere by conduction using the Fourier law.

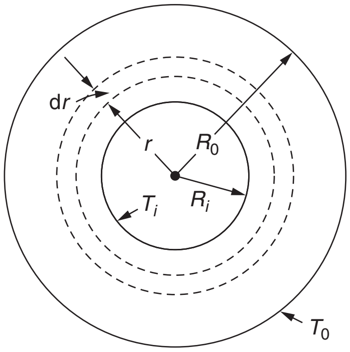

FIGURE 13.7 A hollow sphere.

13.20 Consider a hollow cylinder of inner radius R_i and outer radius R_o as shown in Fig. 13.8. The temperatures of the inside and outside surfaces of the cylinder are maintained at the steady values T_i and T_o, respectively. If heat transfer occurs only by conduction with no heat generation, the temperature distribution in the cylinder in the radial direction is given by

$$T(r) = \frac{T_i - T_o}{\ln\left(\frac{R_i}{R_o}\right)} \ln\left(\frac{r}{R_o}\right) + T_o \tag{P.8}$$

and the rate of heat transfer in a cylinder of length L is given by

$$q = \frac{2\pi\, L\, k\, (T_i - T_o)}{\ln\left(\frac{R_o}{R_i}\right)} \tag{P.9}$$

Plot Eqs. (P.8) and (P.9) for the following data: $R_i = 0.1$ m, $R_o = 0.2$ m, $L = 1$ m, $k = 401$ W/m °C, $T_i = 120$ °C and $T_o = 20$ °C.

13.21 Fig. 13.9 shows a slab of thickness 5 cm in the x direction is maintained at 500 °C on the surface at $x = 0$ while the surface at $x = 5$ cm is exposed to a large wall maintained at 20 °C. Assuming a vacuum between the slab and the wall, find the temperature of the slab on the surface PQRS located at $x = 5$ cm. Assume the wall as a black body, the emissivity of the surface of the wall (PQRS) as $\varepsilon = 0$, and the wall to be in a steady state.

13.22 In Problem 13.21, assume that a stream of air flows between the slab and the wall (instead of vacuum). Assume that the temperature of the air is 25 °C that provides a heat transfer coefficient of 80 W/m^2 K for the right-hand surface of the slab and the thermal conductivity of the slab is $k = 5$ Watts/(m-K). Determine (a) the surface temperature, and (b) the heat transfer rate.

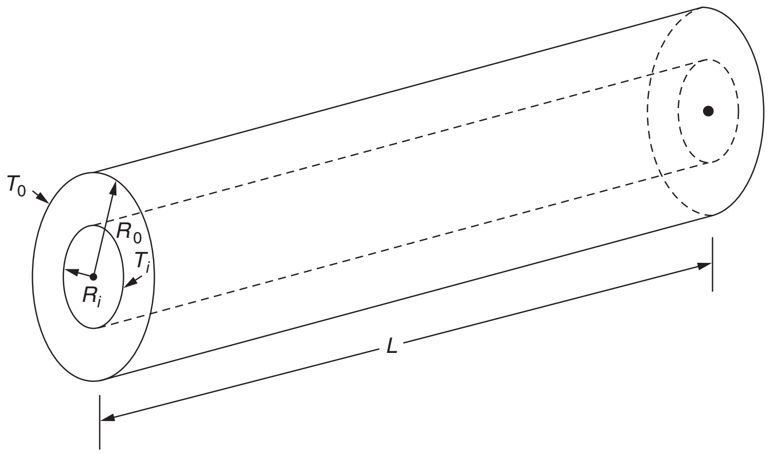

FIGURE 13.8 A hollow cylinder.

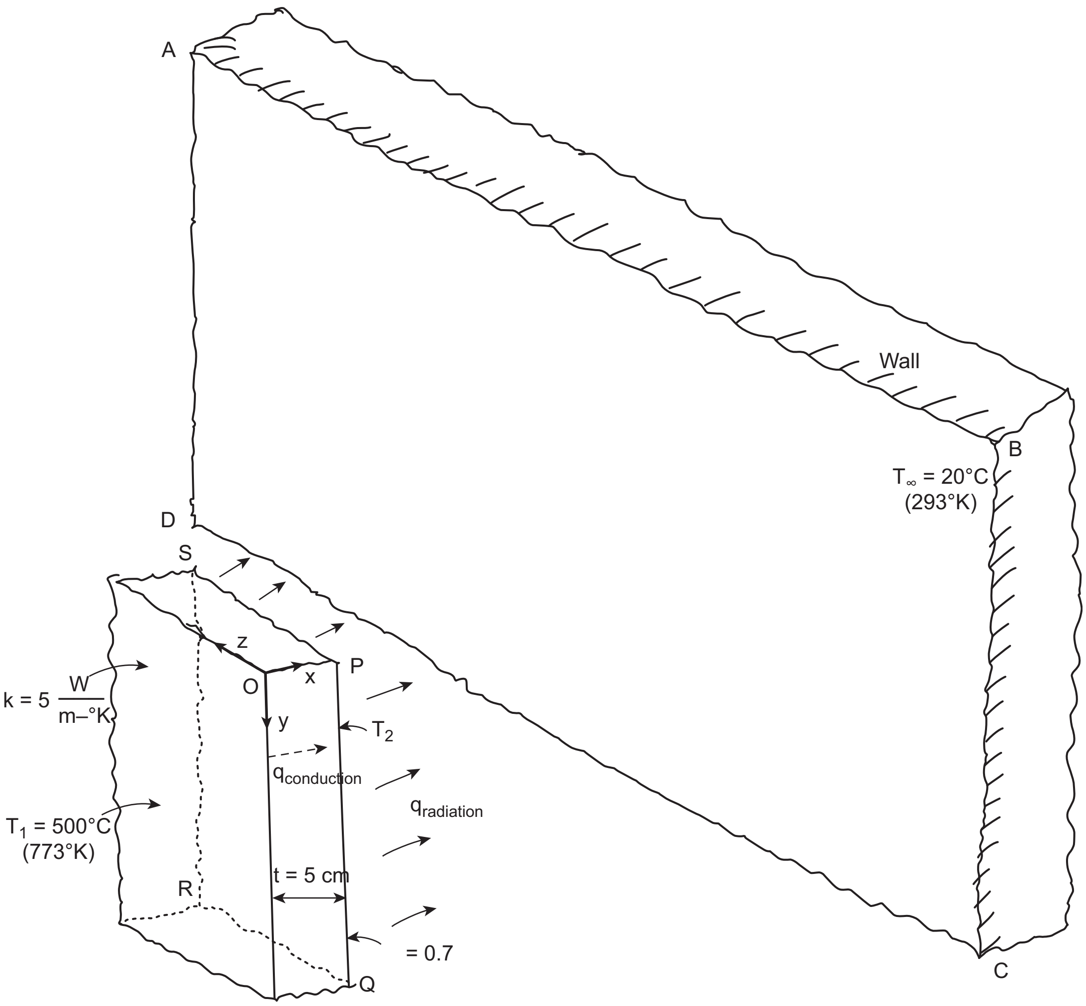

FIGURE 13.9 Radiation heat transfer between a slab and a wall.

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Chapter 14

One-Dimensional Problems

Chapter Outline

14.1 INTRODUCTION

For a one-dimensional heat transfer problem, the governing differential equation is given by

$$k\frac{d^2T}{dx^2} + \dot{q} = 0 \tag{14.1}$$

The boundary conditions are

$$T(x = 0) = T_0(\text{temperature specified}) \tag{14.2}$$

$$k\frac{dT}{dx}l_x + h(T - T_\infty) + q_0 = 0 \ \text{ on the surface} \\ (\text{combined heat flux and convection specified}) \tag{14.3}$$

A fin is a common example of a one-dimensional heat transfer problem. One end of the fin is connected to a heat source (whose temperature is known) and heat will be lost to the surroundings through the perimeter surface and the end. We now consider the analysis of uniform and tapered fins.

14.2 STRAIGHT UNIFORM FIN ANALYSIS

Step 1: Idealize the rod into several finite elements as shown in Fig. 14.1B.
Step 2: Assume a linear temperature variation inside any element e as

$$T^{(e)}(x) = a_1 + a_2x = [N(x)]\overrightarrow{q}^{(e)} \tag{14.4}$$

where

$$[N(x)] = [N_i(x)N_j(x)] \tag{14.5}$$

$$N_i(x) \equiv N_1(x) = 1 - \frac{x}{l^{(e)}} \tag{14.6}$$

$$N_j(x) \equiv N_2(x) = \frac{x}{l^{(e)}} \tag{14.7}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00014-6

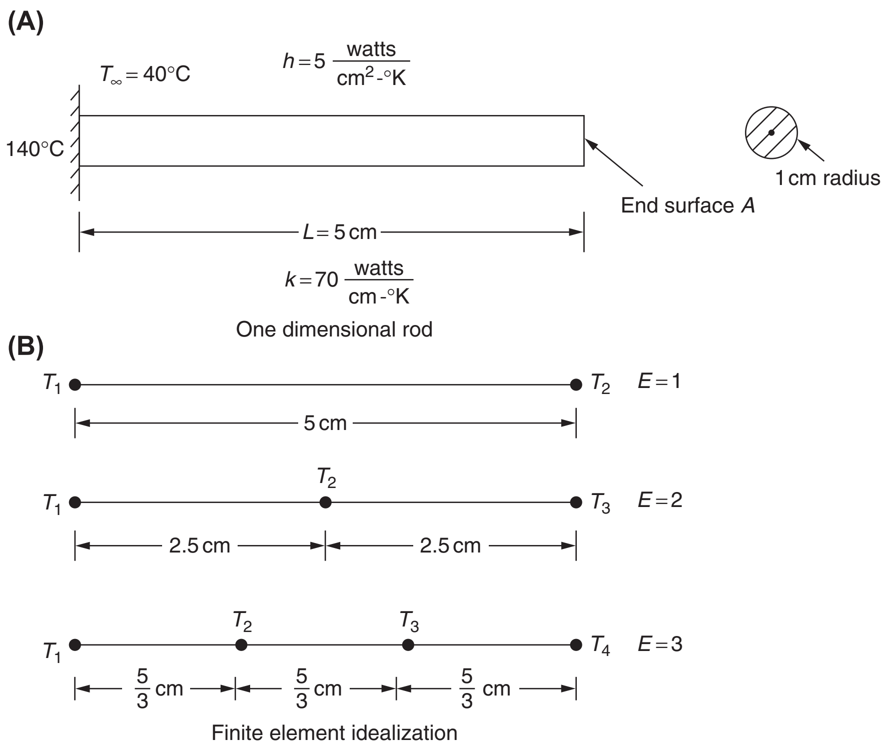

FIGURE 14.1 A one-dimensional fin.

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \equiv \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} \tag{14.8}$$

i and j indicate the global node numbers corresponding to the left- and right-hand side nodes, and $l^{(e)}$ is the length of element e.

Step 3: Derivation of element matrices:

Since this is a one-dimensional problem, Eqs. (13.53) and (13.54) reduce to

$$[D] = [k] \text{and} [B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{l^{(e)}} & \frac{1}{l^{(e)}} \end{bmatrix} \tag{14.9}$$

Eqs. (13.46) to (13.49) become

$$\begin{aligned} \left[K_1^{(e)}\right] &= \iiint_{V^{(e)}} [B]^T [D][B] \mathrm{d}V = \int_{x=0}^{l^{(e)}} \begin{Bmatrix} -\frac{1}{l^{(e)}} \\ \frac{1}{l^{(e)}} \end{Bmatrix} [k] \left\{ -\frac{1}{l^{(e)}} \; \frac{1}{l^{(e)}} \right\} A \, \mathrm{d}x \\ &= \frac{Ak}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{14.10}$$

$$\begin{aligned} \left[K_2^{(e)}\right] &= \int_{S_3^{(e)}} h[N]^T \left[N\right] \mathrm{d}S_3^{(e)} = \int_{x=0}^{l^{(e)}} h \begin{Bmatrix} 1 - x/l^{(e)} \\ x/l^{(e)} \end{Bmatrix} \left\{ \left(1 - \frac{x}{l^{(e)}}\right) \frac{x}{l^{(e)}} \right\} P \, \mathrm{d}x \\ &= \frac{hPl^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned} \tag{14.11}$$

since $dS_3 = P\,dx$, where P is the perimeter.

$$\left[K_3^{(e)}\right] = [0] \tag{14.12}$$

since this is a steady-state problem.

$$\begin{aligned}\vec{P}^{(e)} &= \int_{x=0}^{l^{(e)}} \dot{q} \begin{Bmatrix} 1-\dfrac{x}{l^{(e)}} \\ \dfrac{x}{l^{(e)}} \end{Bmatrix} A\,dx - \int_{x=0}^{l^{(e)}} q_0 \begin{Bmatrix} 1-\dfrac{x}{l^{(e)}} \\ \dfrac{x}{l^{(e)}} \end{Bmatrix} P\,dx + \int_{x=0}^{l^{(e)}} hT_\infty \begin{Bmatrix} 1-\dfrac{x}{l^{(e)}} \\ \dfrac{x}{l^{(e)}} \end{Bmatrix} P\,dx \\ &= \frac{\dot{q}Al^{(e)}}{2}\begin{Bmatrix}1\\1\end{Bmatrix} - \frac{q_0Pl^{(e)}}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{hT_\infty Pl^{(e)}}{2}\begin{Bmatrix}1\\1\end{Bmatrix}\end{aligned} \tag{14.13}$$

Step 4: Assembled equations: The element matrices can be assembled to obtain the overall equations as (Eq. (13.45))

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{14.14}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E}\left(\frac{Ak}{l^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{hPl^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right) \tag{14.15}$$

and

$$\underset{\sim}{\vec{P}} = \sum_{e=1}^{E}\vec{P}^{(e)} = \sum_{e=1}^{E}\frac{1}{2}\left(\dot{q}Al^{(e)} - q_0Pl^{(e)} + hT_\infty Pl^{(e)}\right)\begin{Bmatrix}1\\1\end{Bmatrix} \tag{14.16}$$

Step 5: The assembled equations (14.14) are to be solved, after incorporating the boundary conditions stated in Eq. (14.2), to find the nodal temperatures.

EXAMPLE 14.1

Find the temperature distribution in the one-dimensional fin shown in Fig. 14.1A using one finite element.

Solution

Here, $\dot{q} = q_0 = 0$, and hence, for $E = 1$, Eq. (14.14) gives

$$\begin{bmatrix}\left(\dfrac{Ak}{L}+\dfrac{2hPL}{6}\right) & \left(-\dfrac{Ak}{L}+\dfrac{hPL}{6}\right)\\ \left(-\dfrac{Ak}{L}+\dfrac{hPL}{6}\right) & \left(\dfrac{Ak}{L}+\dfrac{2hPL}{6}\right)\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix} = \frac{hPT_\infty L}{2}\begin{Bmatrix}1\\1\end{Bmatrix} \tag{E.1}$$

By dividing throughout by (Ak/L), Eq. (E.1) can be written as

$$\begin{bmatrix}\left(1+\dfrac{2hPL^2}{6kA}\right) & \left(-1+\dfrac{hPL^2}{6kA}\right)\\ \left(-1+\dfrac{hPL^2}{6kA}\right) & \left(1+\dfrac{2hPL^2}{6kA}\right)\end{bmatrix}\begin{Bmatrix}T_1\\T_2\end{Bmatrix} = \frac{hPT_\infty L^2}{2kA}\begin{Bmatrix}1\\1\end{Bmatrix} \tag{E.2}$$

For the data given, $P = 2\pi$ cm and $A = \pi$ cm^2, and hence

$$\frac{hPL^2}{kA} = \frac{(5)(2\pi)(5^2)}{(70)(\pi)} = \frac{25}{7} \text{ and } \frac{hPT_\infty L^2}{2kA} = \frac{(5)(2\pi)(40)(5^2)}{2(70)(\pi)} = \frac{500}{7}$$

Continued

EXAMPLE 14.1 —cont'd

Thus, Eq. (E.1) becomes

$$\begin{bmatrix} 92 & -17 \\ -17 & 92 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} 3000 \\ 3000 \end{bmatrix} \tag{E.3}$$

In order to incorporate the boundary condition $T_1 = 140°C$, we replace the first equation of (E.3) by $T_1 = 140$ and rewrite the second equation of (E.3) as

$$92T_2 = 3000 + 17T_1 = 3000 + 17(140) = 5380$$

from which the unknown temperature T_2 can be found as $T_2 = 58.48°C$.

While solving Eq. (E.3) on a computer, the boundary condition $T_1 = 140$ can be incorporated by modifying Eq. (E.3) as

$$\begin{bmatrix} 1 & 0 \\ 0 & 92 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 3000 + 17 \times 140 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 5380 \end{Bmatrix} \tag{E.4}$$

Note

Once the nodal temperatures are known, the temperature distribution in the fin (finite elements) can be found from the assumed interpolation model, Eq. (14.4).

EXAMPLE 14.2

Find the temperature distribution in the one-dimensional fin shown in Fig. 14.1A using two finite elements.

Solution

In this case, Eq. (14.14) represents the assembly of two element equations and leads to

$$\begin{bmatrix} a_1 & a_2 & 0 \\ a_2 & 2a_1 & a_2 \\ 0 & a_2 & a_1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} b \\ 2b \\ b \end{Bmatrix} \tag{E.1}$$

where

$$a_1 = 1 + \frac{2hPl^2}{24kA} = 1 + \frac{2}{24}\left(\frac{25}{7}\right) = \frac{109}{84}$$

$$a_2 = -1 + \frac{hPl^2}{24kA} = -1 + \frac{1}{24}\left(\frac{25}{7}\right) = -\frac{143}{168}$$

$$b = \frac{hPT_\infty l^2}{8kA} = \frac{500}{28}$$

As before, we modify Eq. (E.1) to incorporate the boundary condition $T_1 = 140$ as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2a_1 & a_2 \\ 0 & a_2 & a_1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 2b - a_2 \times T_1 \\ b - 0 \times T_1 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 2b - 140a_2 \\ b \end{Bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{218}{84} & -\dfrac{143}{168} \\ 0 & -\dfrac{143}{168} & \dfrac{109}{84} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 140 \\ \dfrac{13,010}{84} \\ \dfrac{1500}{84} \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives

$$T_1 = 140°C, \quad T_2 = 81.77°C, \quad \text{and} \quad T_3 = 67.39°C \tag{E.3}$$

14.3 CONVECTION LOSS FROM END SURFACE OF FIN

In Examples 14.1 and 14.2, it is assumed that the convection heat loss occurs only from the perimeter surface and not from the end surface A (Fig. 14.1A). If the convection loss from the end surface is also to be considered, the following method can be adopted. Let the convection heat loss occur from the surface at the right-side node of element e. Then the surface integral of Eq. (13.47) or Eq. (14.11) should extend over this surface also. Thus, in Eq. (14.11), the following term should also be included:

$$\iint_{S_3^{(e)}} h[N]^T[N]\mathrm{d}S_3 = \iint_A h\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}\{N_1 \quad N_2\}\mathrm{d}S_3 \tag{14.17}$$

corresponding to the surface at the right-side node 2.

Since we are interested in the surface at node 2 (right-side node), we substitute $N_1(x = l^{(e)}) = 0$ and $N_2(x = l^{(e)}) = 1$ in Eq. (14.17) to obtain

$$\iint_A h\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\{0 \quad 1\}\mathrm{d}S_3 = \iint_A h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathrm{d}S_3 = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{14.18}$$

Similarly, the surface integral over S_3 in Eq. (13.52) or Eq. (14.13) should extend over the free end surface also. Thus, the additional term to be included in the vector $\overrightarrow{P}^{(e)}$ is given by

$$\iint_{S_3^{(e)}} hT_\infty[N]^T\mathrm{d}S_3 = hT_\infty\iint_A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\mathrm{d}S_3 = hT_\infty A\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{14.19}$$

EXAMPLE 14.3

Find the temperature distribution in the fin shown in Fig. 14.1A by including the effect of convection from the end surface A using one finite element.

Solution

In this case, Eq. (E.1) of Example 14.1 will be modified as

$$\begin{bmatrix} \left(\frac{Ak}{L}+\frac{2hPL}{6}+0\right) & \left(-\frac{Ak}{L}+\frac{hPL}{6}+0\right) \\ \left(-\frac{Ak}{L}+\frac{hPL}{6}+0\right) & \left(\frac{Ak}{L}+\frac{hPL}{6}+hA\right) \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} \frac{hPT_\infty L}{2}+0 \\ \frac{hPT_\infty L}{2}+hAT_\infty \end{Bmatrix} \tag{E.1}$$

For the given data, Eq. (E.1) reduces to (after multiplying throughout by (L/Ak))

$$\begin{bmatrix} \left(1+\frac{25}{21}\right) & \left(-1+\frac{25}{42}\right) \\ \left(-1+\frac{25}{42}\right) & \left(1+\frac{25}{21}+\frac{5}{14}\right) \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} \frac{500}{7} \\ \frac{500}{7}+\frac{100}{7} \end{Bmatrix}$$

or

$$\begin{bmatrix} 92 & -17 \\ -17 & 107 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ 3600 \end{Bmatrix} \tag{E.2}$$

After incorporating the boundary condition, $T_1 = 140$, Eq. (E.2) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 107 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ 3600+17T_1 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 5980 \end{Bmatrix} \tag{E.3}$$

from which the solution can be obtained as

$$T_1 = 140°\mathrm{C} \text{ and } T_2 = 55.89°\mathrm{C}$$

This result indicates that the convection loss from the end surface causes the temperature at the end to reduce from 58.48°C to 55.89°C.

EXAMPLE 14.4

Find the temperature distribution in the fin shown in Fig. 14.1A by including the effect of convection from the end surface A using two finite elements.

Solution

In this case, the element matrices and vectors are given by

$$\left[K_1^{(1)}\right] = \frac{(\pi)(70)}{(2.5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 28\pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[K_2^{(1)}\right] = \frac{(5)(2\pi)(2.5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 4.1667\pi\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{1}{2}(5)(40)(2\pi)(2.5)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 500\pi\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\left[K_1^{(2)}\right] = \frac{(\pi)(70)}{2.5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 28\pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[K_2^{(2)}\right] = \frac{(5)(2\pi)(2.5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (5)(\pi)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8.3334\pi & 4.1667\pi \\ 4.1667\pi & 13.3334\pi \end{bmatrix}$$

$$\vec{P}^{(2)} = \frac{1}{2}(5)(40)(2\pi)(2.5)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + (5)(40)\pi\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 500\pi \\ 700\pi \end{Bmatrix}$$

The assembled equations can be written as

$$\begin{bmatrix} 36.3334\pi & -23.8333\pi & 0 \\ -23.8333\pi & 72.6668\pi & -23.8333\pi \\ 0 & -23.8333\pi & 41.3334\pi \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 500\pi \\ 1000\pi \\ 700\pi \end{Bmatrix} \tag{E.1}$$

After incorporating the boundary condition $T_1 = 140$, Eq. (E.1) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 72.6668 & -23.8333 \\ 0 & -23.8333 & 41.3334 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ 1000 + 23.8333\,T_1 \\ 700 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 4336 \\ 700 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives

$$T_1 = 140°\text{C}, \quad T_2 = 80.44°\text{C}, \quad \text{and} \quad T_3 = 63.36°\text{C} \tag{E.3}$$

The temperatures at nodes 2 and 3 can be seen to be lower compared to the corresponding values given in Eq. (E.3) of Example 14.2 due to convection loss from the end surface.

EXAMPLE 14.5

For the fin shown in Fig. 14.1A, a colder body (an ice pack) is in contact at the right end (at $x = L$) so that heat flux (loss) occurs at the rate of $q_0 = 300$ W/cm^2. Find the temperature distribution in the fin by considering convection loss along the lateral (cylindrical) surface of the fin and the stated heat flux condition at $x = L$. Use one finite element for modeling the fin.

Notes

1. The vector $\vec{P}^{(e)}$ due to heat flux q_0 is given by

$$\vec{P}^{(e)} = \iint_{S_2^{(e)}} q_0 [N]^T \mathrm{d}S_2 = \iint_{S_2^{(e)}} q_0 \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \mathrm{d}S_2 \tag{E.1}$$

As in the case of convection loss from the right end of the fin, where $N_1 = 0$ and $N_2 = 1$, Eq. (E.1) becomes

$$\vec{P}^{(e)} = q_0 A \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{E.2}$$

2. q_0 is considered positive (negative) when heat gain (loss) occurs to the body.
3. If no heat transfer takes place from any surface, that particular surface is said to be insulated.

Solution
Here $\dot{q} = 0$ and $q_0 \neq 0$; hence for $E = 1$, Eq. (14.14) gives

$$\begin{bmatrix} \left(\frac{Ak}{L} + \frac{2hPL}{6}\right) & \left(-\frac{Ak}{L} + \frac{hPL}{6}\right) \\ \left(-\frac{Ak}{L} + \frac{hPL}{6}\right) & \left(\frac{Ak}{L} + \frac{2hPL}{6}\right) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} \frac{hPT_\infty L}{2} + 0 \\ \frac{hPT_\infty L}{2} + q_0 A \end{Bmatrix} \tag{E.3}$$

By dividing throughout by (Ak/L), Eq. (E.3) can be written as

$$\begin{bmatrix} \left(1 + \frac{2hPL^2}{6kA}\right) & \left(-1 + \frac{hPL^2}{6kA}\right) \\ \left(-1 + \frac{hPL^2}{6kA}\right) & \left(1 + \frac{2hPL^2}{6kA}\right) \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} \frac{hPT_\infty L^2}{2kA} + 0 \\ \frac{hPT_\infty L^2}{2kA} + \frac{q_0 L}{k} \end{Bmatrix} \tag{E.4}$$

For the data given, $P = 2\pi$ cm, $A = \pi$ cm^2, and $q_0 = -300$ W/cm^2, and hence

$$\frac{hPL^2}{kA} = \frac{(5)(2\pi)(5^2)}{(70)(\pi)} = \frac{25}{7}, \frac{hPT_\infty L^2}{2kA} = \frac{(5)(2\pi)(40)(5^2)}{2(70)(\pi)} = \frac{500}{7}$$

and

$$\frac{q_0 L}{k} = \frac{(-300)(5)}{70} = -\frac{150}{7}$$

Thus, Eq. (E.3) becomes

$$\begin{bmatrix} 92 & -17 \\ -17 & 92 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ 2100 \end{Bmatrix} \tag{E.5}$$

In order to incorporate the boundary condition $T_1 = 140°$C, we replace the first equation of (E.5) by $T_1 = 140$ and rewrite the second equation of (E.5) as

$$92T_2 = 2100 + 17T_1 = 2100 + 17(140) = 4480$$

from which the unknown temperature T_2 can be found as $T_2 = 48.69°$C.
While solving Eq. (E.5) on a computer, the boundary condition $T_1 = 140$ can be incorporated by modifying Eq. (E.5) as

$$\begin{bmatrix} 1 & 0 \\ 0 & 92 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 2100 + 17 \times 140 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 4480 \end{Bmatrix} \tag{E.6}$$

Note
Once the nodal temperatures are known, the temperature distribution in the fin (finite elements) can be found from the assumed interpolation model, Eq. (14.4).

EXAMPLE 14.6

Find the temperature distribution in the one-dimensional fin described in Example 14.5 using two finite elements for modeling the fin.

Solution

In this case, the element matrices and vectors are given by

$$\left[K_1^{(1)}\right] = \frac{(\pi)(70)}{(2.5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 28\pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[K_2^{(1)}\right] = \frac{(5)(2\pi)(2.5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 4.1667\pi\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{1}{2}(5)(40)(2\pi)(2.5)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 500\pi\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\left[K_1^{(2)}\right] = \frac{(\pi)(70)}{(2.5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 28\pi\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left[K_2^{(2)}\right] = \frac{(5)(2\pi)(2.5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8.3334\pi & 4.1667\pi \\ 4.1667\pi & 8.3334\pi \end{bmatrix}$$

$$\vec{P}^{(2)} = \frac{1}{2}(5)(40)(2\pi)(2.5)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + (-300)\pi\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 500\pi \\ 200\pi \end{Bmatrix}$$

The assembled equations can be written as

$$\begin{bmatrix} 36.3334\pi & -23.8333\pi & 0 \\ -23.8333\pi & 72.6668\pi & -23.8333\pi \\ 0 & -23.8333\pi & 36.3334\pi \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 500\pi \\ 1000\pi \\ 200\pi \end{Bmatrix} \tag{E.1}$$

After incorporating the boundary condition $T_1 = 140$, Eq. (E.1) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 72.6668 & -23.8333 \\ 0 & -23.8333 & 36.3334 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ 1000 + 23.8333\ T_1 \\ 200 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 4336 \\ 200 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives

$$T_1 = 140°\text{C}, \quad T_2 = 78.33°\text{C}, \quad \text{and} \quad T_3 = 56.88°\text{C} \tag{E.3}$$

The temperatures at nodes 2 and 3 are lower compared to the corresponding values given in Eq. (E.3) of Example 14.2 due to heat flux loss from the end surface.

14.4 TAPERED FIN ANALYSIS

In a tapered fin, the area of cross section A varies with x. By assuming a linear variation of area from node i (local node 1) to node j (local node 2) of element e, the area of cross section at a distance x from node i can be expressed as

$$A(x) = A_i + \frac{(A_j - A_i)x}{l^{(e)}} = A_iN_i(x) + A_jN_j(x) \tag{14.20}$$

where N_i and N_j are the linear shape functions defined in Eq. (14.5), and A_i and A_j are the cross-sectional areas of element e at nodes i and j, respectively.

The matrices $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, $\left[K_3^{(e)}\right]$, and $\vec{P}^{(e)}$ can be obtained as (from Eqs. (13.46) to (13.49))

$$\left[K_1^{(e)}\right] = \iiint_{V^{(e)}} [B]^T [D] [B] \, \mathrm{d}V = \int_{x=0}^{l^{(e)}} \begin{Bmatrix} \left(-\dfrac{1}{l^{(e)}}\right) \\ \left(\dfrac{1}{l^{(e)}}\right) \end{Bmatrix} [k] \left\{ \left(-\frac{1}{l^{(e)}}\right) \left(\frac{1}{l^{(e)}}\right) \right\} A(x)\mathrm{d}x \tag{14.21}$$

$$= \frac{k}{l^{(e)}} \left(\frac{A_i + A_j}{2}\right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{k\overline{A}^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where $\overline{A}^{(e)}$ is the average area of the element e. Since the evaluation of the integral in $\left[K_2^{(e)}\right]$ involves the perimeter P, we can use a similar procedure by writing P as

$$P(x) = P_i N_i(x) + P_j N_j(x) \tag{14.22}$$

where P_i and P_j are the perimeters of element e at nodes i and j, respectively, we obtain

$$\left[K_2^{(e)}\right] = h \iint_{S_3^{(e)}} [N]^T [N] \mathrm{d}S_3 = h \int_{x=0}^{l^{(e)}} \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} P(x)\mathrm{d}x \tag{14.23}$$

The integrals of Eq. (14.23) can be evaluated as

$$\int_{x=0}^{l^{(e)}} N_i^2(x) P(x) \mathrm{d}x = \frac{l^{(e)}}{12}(3P_i + P_j) \tag{14.24}$$

$$\int_{x=0}^{l^{(e)}} N_i(x) N_j(x) P(x) \mathrm{d}x = \frac{l^{(e)}}{12}(P_i + P_j) \tag{14.25}$$

$$\int_{x=0}^{l^{(e)}} N_j^2(x) P(x) \mathrm{d}x = \frac{l^{(e)}}{12}(P_i + 3P_j) \tag{14.26}$$

$$\left[K_2^{(e)}\right] = \frac{hl^{(e)}}{12} \begin{bmatrix} (3P_i + P_j) & (P_i + P_j) \\ (P_i + P_j) & (P_i + 3P_j) \end{bmatrix}^{(e)} \tag{14.27}$$

Since this is a steady-state problem with $\dot{q} = q_0 = 0$, we have

$$\left[K_3^{(e)}\right] = [0] \tag{14.28}$$

$$\vec{P}^{(e)} = \iint_{S_3^{(e)}} hT_\infty [N]^T \mathrm{d}S_3 = \frac{hT_\infty l^{(e)}}{6} \begin{Bmatrix} 2P_i + P_j \\ P_i + 2P_j \end{Bmatrix}^{(e)} \tag{14.29}$$

Once the element matrices are available, the overall equations can be obtained using Eq. (13.35).

EXAMPLE 14.7

Find the temperature distribution in the tapered fin shown in Fig. 14.2 using one finite element.

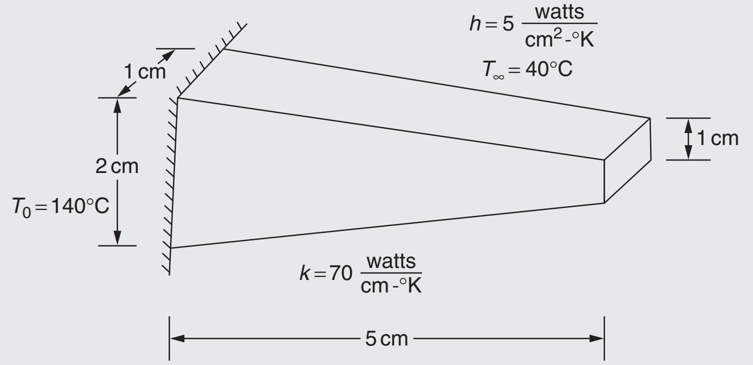

FIGURE 14.2 A tapered fin.

Continued

EXAMPLE 14.7 —cont'd

Solution

For $E = 1$, $l^{(1)} = 5$ cm, $A_i = 2$ cm^2, $A_j = 1$ cm^2, $\overline{A} = 1.5$ cm^2, $P_i = 6$ cm, and $P_j = 4$ cm. Thus, Eqs. (14.21), (14.27), and (14.29) give

$$\left[K_1^{(e)}\right] = \frac{(70)(1.5)}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 21 & -21 \\ -21 & 21 \end{bmatrix}$$

$$\left[K_2^{(e)}\right] = \frac{(5)(5)}{12}\begin{bmatrix} (3\times 6+4) & (6+4) \\ (6+4) & (6+3\times 4) \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 275 & 125 \\ 125 & 225 \end{bmatrix}$$

$$\vec{P}^{(e)} = \frac{(5)(40)(5)}{6}\begin{Bmatrix} 2\times 6+4 \\ 6+2\times 4 \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 8000 \\ 7000 \end{Bmatrix}$$

Hence, Eq. (13.35) gives

$$\begin{bmatrix} 401 & -1 \\ -1 & 351 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 16,000 \\ 14,000 \end{Bmatrix} \tag{E.1}$$

When the boundary condition, $T_1 = 140°$C, is incorporated, Eq. (E.1) gets modified to

$$\begin{bmatrix} 1 & 0 \\ 0 & 351 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ 14,000 + T_1 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 14,140 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives

$$T_1 = 140°\text{C} \quad \text{and} \quad T_2 = 40.28°\text{C} \tag{E.3}$$

EXAMPLE 14.8

Find the temperature distribution in the tapered fin shown in Fig. 14.2 using two finite elements.

Solution

For the first element ($e = 1$), we have $l^{(1)} = 2.5$ cm, $A_i = 2$ cm^2, $A_j = 1.5$ cm^2, $\overline{A} = 1.75$ cm^2, $P_i = 6$ cm, and $P_j = 5$ cm.

$$\left[K_1^{(1)}\right] = \frac{(70)(1.75)}{(2.5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 49 & -49 \\ -49 & 49 \end{bmatrix}$$

$$\left[K_2^{(1)}\right] = \frac{(5)(2.5)}{(12)}\begin{bmatrix} (3\times 6+5) & (6+5) \\ (6+5) & (6+3\times 5) \end{bmatrix} = \begin{bmatrix} 23.95 & 11.45 \\ 11.45 & 21.90 \end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{(5)(40)(2.5)}{6}\begin{Bmatrix} 2\times 6+5 \\ 6+2\times 5 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 8500 \\ 8000 \end{Bmatrix}$$

For the second element ($e = 2$), we have $l^{(2)} = 2.5$ cm, $A_i = 1.5$ cm^2, $A_j = 1$ cm^2, $\overline{A} = 1.25$ cm^2, $P_i = 5$ cm, and $P_j = 4$ cm.

$$\left[K_1^{(2)}\right] = \frac{(70)(1.25)}{(2.5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 35 & -35 \\ -35 & 35 \end{bmatrix}$$

$$\left[K_2^{(2)}\right] = \frac{(5)(2.5)}{12}\begin{bmatrix} (3\times 5+4) & (5+4) \\ (5+4) & (5+3\times 4) \end{bmatrix} = \begin{bmatrix} 19.8 & 9.4 \\ 9.4 & 17.7 \end{bmatrix}$$

$$\vec{P}^{(2)} = \frac{(5)(40)(2.5)}{6}\begin{Bmatrix} 2\times 5+4 \\ 5+2\times 4 \end{Bmatrix} = \frac{1}{6}\begin{Bmatrix} 7000 \\ 6500 \end{Bmatrix}$$

EXAMPLE 14.8 —cont'd

The overall or assembled system equations (13.35) will be

$$\begin{bmatrix} 72.95 & -37.55 & 0 \\ -37.55 & 125.70 & -25.60 \\ 0 & -25.60 & 52.70 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} \frac{8500}{6} \\ \frac{15,000}{6} \\ \frac{6500}{6} \end{Bmatrix} \tag{E.1}$$

Eq. (E.1), when modified to incorporate the boundary condition $T_1 = 140°\text{C}$, appears as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 125.70 & -25.60 \\ 0 & -25.60 & 52.70 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ \frac{15,000}{6} + 37.55\,T_1 \\ \frac{6500}{6} \end{Bmatrix} = \begin{Bmatrix} 140.0 \\ 7757.0 \\ 1083.3 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives the nodal temperatures

$$T_1 = 140°\text{C}, \quad T_2 = 73.13°\text{C}, \quad \text{and} \quad T_3 = 56.07°\text{C} \tag{E.3}$$

14.5 ANALYSIS OF UNIFORM FINS USING QUADRATIC ELEMENTS

The solution of uniform one-dimensional heat transfer problems is considered using a quadratic model for the variation of temperature in the element. The step-by-step procedure is as follows.

Step 1: Idealize the fin into E finite elements as shown in Fig. 14.3.

Step 2: Assume a quadratic variation of temperature inside any element e as

$$T^{(e)}(x) = a_1 + a_2 x + a_3 x^2 = [N(x)]\vec{q}^{(e)} \tag{14.30}$$

where

$$[N(x)] = [N_i(x) \;\; N_j(x) \;\; N_k(x)] \tag{14.31}$$

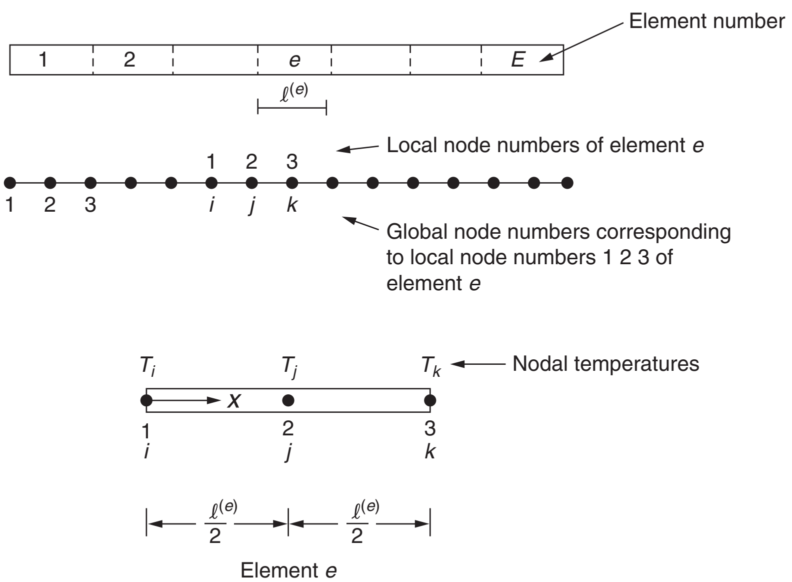

FIGURE 14.3 A quadratic element.

$$N_i(x) = \left(1 - \frac{2x}{l^{(e)}}\right)\left(1 - \frac{x}{l^{(e)}}\right) \tag{14.32}$$

$$N_j(x) = \frac{4x}{l^{(e)}}\left(1 - \frac{x}{l^{(e)}}\right) \tag{14.33}$$

$$N_k(x) = -\frac{x}{l^{(e)}}\left(1 - \frac{2x}{l^{(e)}}\right) \tag{14.34}$$

and

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

where i, j, and k denote the global node numbers corresponding to local nodes 1 (left end), 2 (middle), and 3 (right end), respectively.

Step 3: Derivation of element matrices:

For the quadratic element, we have

$$[D] = [k] \tag{14.35}$$

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{4x}{l^{(e)2}} - \frac{3}{l^{(e)}} & \frac{4}{l^{(e)}} - \frac{8x}{l^{(e)2}} & \frac{4x}{l^{(e)2}} - \frac{1}{l^{(e)}} \end{bmatrix} \tag{14.36}$$

The definitions of $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, $\left[K_3^{(e)}\right]$, and $\vec{P}^{(e)}$ remain the same as those given in Eqs. (13.46) to (13.49), but the integrals have to be reevaluated using the quadratic displacement model of Eq. (14.30). This gives

$$\left[K_1^{(e)}\right] = kA \int_{x=0}^{l^{(e)}} \mathrm{d}x \begin{bmatrix} \left(\frac{4x}{l^{(e)^2}} - \frac{3}{l^{(e)}}\right)^2 & \left(\frac{4x}{l^{(e)^2}} - \frac{3}{l^{(e)}}\right)\left(\frac{4}{l^{(e)}} - \frac{8x}{l^{(e)^2}}\right) & \left(\frac{4x}{l^{(e)^2}} - \frac{3}{l^{(e)}}\right)\left(\frac{4x}{l^{(e)^2}} - \frac{1}{l^{(e)}}\right) \\ & \left(\frac{4}{l^{(e)}} - \frac{8x}{l^{(e)^2}}\right)^2 & \left(\frac{4}{l^{(e)}} - \frac{8x}{l^{(e)^2}}\right)\left(\frac{4x}{l^{(e)^2}} - \frac{1}{l^{(e)}}\right) \\ \text{Symmetric} & & \left(\frac{4x}{l^{(e)^2}} - \frac{1}{l^{(e)}}\right)^2 \end{bmatrix}$$

$$= \frac{kA}{3l^{(e)}} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{14.37}$$

where A is the cross-sectional area of the element, and

$$\left[K_2^{(e)}\right] = h \cdot P \int_{x=0}^{l(e)} \begin{bmatrix} N_i^2(x) & N_i(x)\cdot N_j(x) & N_i(x)\cdot N_k(x) \\ N_i(x)\cdot N_j(x) & N_j^2(x) & N_j(x)\cdot N_k(x) \\ N_i(x)\cdot N_k(x) & N_j(x)\cdot N_k(x) & N_k^2(x) \end{bmatrix} \mathrm{d}x$$

$$= \frac{hPl^{(e)}}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{14.38}$$

where P is the perimeter of the element. For steady-state problems,

$$\left[K_3^{(e)}\right] = [0] \tag{14.39}$$

$$\vec{P}^{(e)} = \dot{q}A\int_{x=0}^{l^{(e)}}\begin{Bmatrix} N_i(x) \\ N_j(x) \\ N_k(x)\end{Bmatrix}\mathrm{d}x - q_0P\int_{x=0}^{l^{(e)}}\begin{Bmatrix} N_i(x) \\ N_j(x) \\ N_k(x)\end{Bmatrix}\mathrm{d}x + hT_\infty P\int_{x=0}^{l^{(e)}}\begin{Bmatrix} N_i(x) \\ N_j(x) \\ N_k(x)\end{Bmatrix}\mathrm{d}x \tag{14.40}$$

where $\mathrm{d}V^{(e)}$, $\mathrm{d}S_2^{(e)}$, and $\mathrm{d}S_3^{(e)}$ were replaced by $A\,\mathrm{d}x$, $P\,\mathrm{d}x$, and $P\,\mathrm{d}x$, respectively. With the help of Eqs. (14.32) to (14.34), Eq. (14.40) can be expressed as

$$\vec{P}^{(e)} = \frac{\dot{q}Al^{(e)}}{6}\begin{Bmatrix} 1 \\ 4 \\ 1\end{Bmatrix} - \frac{q_0Pl^{(e)}}{6}\begin{Bmatrix} 1 \\ 4 \\ 1\end{Bmatrix} + \frac{kT_\infty Pl^{(e)}}{6}\begin{Bmatrix} 1 \\ 4 \\ 1\end{Bmatrix} \tag{14.41}$$

If convection occurs from the free end of the element, such as node i, then $N_i(x=0)=1$, $N_j(x=0)=N_k(x=0)=0$, and hence the additional surface integral term to be added to the matrix $\left[K_2^{(e)}\right]$ will be

$$\begin{aligned}\iint_{S_3^{(e)}} h[N]^T[N]\,\mathrm{d}S_3 &= h\iint_{S_3^{(e)}}\begin{bmatrix} N_i^2(x=0) & N_i(x=0)\cdot N_j(x=0) & N_i(x=0)\cdot N_k(x=0) \\ N_i(x=0)\cdot N_j(x=0) & N_j^2(x=0) & N_j(x=0)\cdot N_k(x=0) \\ N_i(x=0)\cdot N_k(x=0) & N_j(x=0)\cdot N_k(x=0) & N_k^2(x=0)\end{bmatrix}\mathrm{d}S_3 \\ &= hA_i\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\end{aligned} \tag{14.42}$$

where A_i is the cross-sectional area of the rod at node i. Similarly, the additional surface integral term to be added to $\vec{P}^{(e)}$ due to convection from the free end (e.g., node i) will be

$$\iint_{S_3^{(e)}} hT_\infty[N]^T\mathrm{d}S_3 = hT_\infty\iint_{S_3^{(e)}}\begin{Bmatrix} N_i(x=0) \\ N_j(x=0) \\ N_k(x=0)\end{Bmatrix}\mathrm{d}S_3 = hT_\infty A_i\begin{Bmatrix} 1 \\ 0 \\ 0\end{Bmatrix} \tag{14.43}$$

EXAMPLE 14.9

Find the temperature distribution in the fin shown in Fig. 14.1A using one quadratic element.

Solution

For simplicity, we neglect convection from the free end. Then, Eqs. (14.37), (14.38), and (14.41) become

$$\left[K_1^{(1)}\right] = \frac{(70)(\pi)}{3(5)}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7\end{bmatrix} = \frac{\pi}{3}\begin{bmatrix} 98 & -112 & 14 \\ -112 & 224 & -112 \\ 14 & -112 & 98\end{bmatrix}$$

$$\left[K_2^{(1)}\right] = \frac{(5)(2\pi)(5)}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4\end{bmatrix} = \frac{\pi}{3}\begin{bmatrix} 20 & 10 & -5 \\ 10 & 80 & 10 \\ -5 & 10 & 20\end{bmatrix}$$

$$\vec{P}^{(1)} = \frac{(5)(40)(2\pi)(5)}{6}\begin{Bmatrix} 1 \\ 4 \\ 1\end{Bmatrix} = \frac{\pi}{3}\begin{Bmatrix} 1000 \\ 4000 \\ 1000\end{Bmatrix}$$

Continued

EXAMPLE 14.9 —cont'd

The system equations can be expressed as

$$\begin{bmatrix} 118 & -102 & 9 \\ -102 & 304 & -102 \\ 9 & -102 & 118 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 4000 \\ 1000 \end{Bmatrix} \tag{E.1}$$

where $T_1 = T(x=0)$, $T_2 = T(x=2.5)$, and $T_3 = T(x=5.0)$. By incorporating the boundary condition $T_1 = 140$, Eq. (E.1) can be modified as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 304 & -102 \\ 0 & -102 & 118 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 4000 + 102T_1 \\ 1000 - 9T_1 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 18,280 \\ -260 \end{Bmatrix} \tag{E.2}$$

The solution of Eq. (E.2) gives

$$T_1 = 140.0°\text{C}, \quad T_2 = 83.64°\text{C}, \quad \text{and} \quad T_3 = 70.09°\text{C} \tag{E.3}$$

This shows that the quadratic element yields a temperature of 70.09°C at the end node, while the linear finite element yielded a temperature of 58.48°C (for $E = 1$, in Example 14.1) and 67.39°C (for $E = 2$ in Example 14.2).

14.6 UNSTEADY STATE PROBLEMS

Time-dependent or unsteady state problems are very common in heat transfer. For some of these time-dependent problems, the transient period occurs between the starting of the physical process and the reaching of the steady-state condition. For some problems, the steady-state condition is never obtained, and in these cases the transient period includes the entire life of the process. Several finite element procedures have been suggested for the solution of transient heat transfer problems [14.1,14.2]. We consider the finite element solution of time-dependent heat transfer problems briefly in this section. The governing differential equation for an unsteady state heat transfer problem is given by Eq. (13.11) and the associated boundary and initial conditions are given by Eqs. (13.18) to (13.21). In general all the parameters k_x, k_y, k_z, $\dot{q}$, and ρc will be time dependent. The finite element solution of this problem leads to a set of first-order linear differential equations, Eq. (13.35). It can be seen that the term $[\underset{\sim}{K}_3]\,\dot{\underset{\sim}{\vec{T}}}$ is the additional term that appears because of the unsteady state. The associated element matrix is defined as (see Eqs. (13.32) and (13.48)):

$$\left[\underset{\sim}{K}_3^{(e)}\right] = \iiint_{V(e)} \rho c [N]^T [N]\,\mathrm{d}V \tag{14.44}$$

which is also known as the element capacitance matrix.

14.6.1 Derivation of Element Capacitance Matrices

For the straight uniform one-dimensional element considered in Section 14.2, the shape function matrix is given by Eq. (14.5). By writing the element volume as $\mathrm{d}V = A^{(e)}\mathrm{d}x$, where $A^{(e)}$ is the cross-sectional area of the element e, Eq. (14.44) can be expressed as

$$\begin{aligned} \left[K_3^{(e)}\right] &= (\rho c)^{(e)} \int_{x=0}^{l^{(e)}} \begin{Bmatrix} \left(1 - \dfrac{x}{l^{(e)}}\right) \\ \dfrac{x}{l^{(e)}} \end{Bmatrix} \left\{\left(1 - \dfrac{x}{l^{(e)}}\right) \; \dfrac{x}{l^{(e)}}\right\} A^{(e)}\mathrm{d}x \\ &= \frac{(\rho c)^{(e)} A^{(e)} l^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned} \tag{14.45}$$

where $(\rho c)^{(e)}$ $A^{(e)}$ is assumed to be a constant for the element e. For the linearly tapered fin element considered in Section 14.4, $dV = A(x)dx = \left[A_i + \left((A_j - A_i)/l^{(e)}\right)x\right]dx$, and hence Eq. (14.44), becomes

$$\left[K_3^{(e)}\right] = (\rho c)^{(e)} \int_{x=0}^{l^{(e)}} \begin{Bmatrix} \left(-\dfrac{1}{l^{(e)}}\right) \\ \left(\dfrac{1}{l^{(e)}}\right) \end{Bmatrix} \left\{ -\frac{1}{l^{(e)}} \; \frac{1}{l^{(e)}} \right\} \cdot \left[A_i + \left(\frac{A_j - A_i}{l^{(e)}}\right)x\right] dx$$
$$= \frac{(\rho c)^{(e)} \overline{A}^{(e)}}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{14.46}$$

where $\overline{A}^{(e)}$ is the average cross-sectional area of the element e. For the straight uniform fin considered in Section 14.5, using a quadratic model (defined by Eq. 14.30), Eq. (14.44) becomes

$$\left[K_3^{(e)}\right] = (\rho\, c)^{(e)} A^{(e)} \int_{x=0}^{l^{(e)}} \begin{bmatrix} N_i^2 & N_i N_j & N_i N_k \\ N_i N_j & N_j^2 & N_j N_k \\ N_i N_k & N_j N_k & N_k^2 \end{bmatrix} dx$$
$$= \frac{(\rho\, c)^{(e)} A^{(e)} l^{(e)}}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{14.47}$$

where $A^{(e)}$ is the cross-sectional area of the element.

EXAMPLE 14.10

Find the time-dependent temperature distribution in a plane wall that is insulated on one face and is subjected to a step change in surface temperature on the other face as shown in Fig. 14.4A using one finite element.

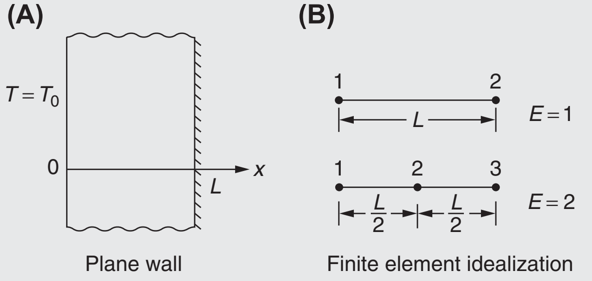

FIGURE 14.4 Plane wall subjected to a step-change in temperature.

Solution

The finite element equations for this one-dimensional transient problem are given by (Eq. 13.35)

$$[\underset{\sim}{K_3}]\, \dot{\vec{\underset{\sim}{T}}} + \left[\underset{\sim}{K}\right] \vec{\underset{\sim}{T}} = \vec{\underset{\sim}{P}} \tag{E.1}$$

where the element matrices, with the assumption of linear temperature variation, are given by

$$\left[\underset{\sim}{K_1^{(e)}}\right] = \frac{A^{(e)} k^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{E.2}$$

Continued

EXAMPLE 14.10 —cont'd

$$\left[K_2^{(e)}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ since no convection condition is specified} \tag{E.3}$$

$$\left[K_3^{(e)}\right] = \frac{(\rho c)^{(e)} A^{(e)} l^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{E.4}$$

$$\vec{P}^{(e)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ since no } \dot{q}, q_0, \text{ and } h \text{ are specified in the problem} \tag{E.5}$$

For $E = 1$, T_1 and T_2 denote the temperatures of nodes 1 and 2, and Eq. (E.1) becomes

$$\frac{\rho c A L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dfrac{dT_1}{dt} \\ \dfrac{dT_2}{dt} \end{Bmatrix} + \frac{Ak}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.6}$$

Eq. (E.6) is to be modified to satisfy the boundary condition at $x = 0$. Since T_2 is the only unknown in the problem, we can delete the first equation of (E.6) and set $T_1 = T_0$ and $(dT_1/dt) = 0$ in Eq. (E.6) to obtain

$$\frac{dT_2}{dt} = -\frac{3k}{\rho c L^2}(T_2 - T_0) \tag{E.7}$$

By defining $\theta = T_2 - T_0$, Eq. (E.7) can be written as

$$\frac{d\theta}{dt} + \alpha\theta = 0 \tag{E.8}$$

where $\alpha = (3k/\rho c L^2)$. The solution of Eq. (E.8) is given by

$$\theta(t) = a_1 e^{-\alpha t} \tag{E.9}$$

where a_1 is a constant whose value can be determined from the known initial condition, $T_2(t = 0) = \overline{T}_0$:

$$\theta(t = 0) = \overline{T}_0 - T_0 = a_1 \cdot e^{-\alpha(0)} = a_1 \tag{E.10}$$

Thus, the solution of Eq. (E.7) is

$$T_2(t) = T_0 + \left(\overline{T}_0 - T_0\right)\cdot e^{-\left(3k/\rho c L^2\right)t} \tag{E.11}$$

EXAMPLE 14.11

Find the time-dependent temperature distribution in a plane wall that is insulated on one face and is subjected to a step change in surface temperature on the other face as shown in Fig. 14.4A using two finite elements.

Solution

For $E = 2$, T_1, T_2, and T_3 indicate the temperatures of nodes 1, 2, and 3 and Eq. (E.1) of Example 14.8 can be derived as (see Problem 14.33)

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} dT_1/dt \\ dT_2/dt \\ dT_3/dt \end{Bmatrix} + \frac{24k}{\rho c L^2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

As before, we delete the first equation from Eq. (E.1) and substitute the boundary condition $T_1 = T_0$ (and hence $(dT_1/dt) = 0$) in the remaining two equations to obtain

$$\left.\begin{aligned} 4\frac{dT_2}{dt} + \frac{dT_3}{dt} + \frac{24k}{\rho c L^2}(-T_0 + 2T_2 - T_3) &= 0 \\ \frac{dT_2}{dt} + 2\frac{dT_3}{dt} + \frac{24k}{\rho c L^2}(-T_2 + T_3) &= 0 \end{aligned}\right\} \tag{E.2}$$

EXAMPLE 14.11 —cont'd

By defining $\theta_2 = T_2 - T_0$ and $\theta_3 = T_3 - T_0$, Eq. (E.2) can be expressed as

$$\left.\begin{aligned} 4\frac{d\theta_2}{dt} + \frac{d\theta_3}{dt} + \frac{24k}{\rho c L^2}(2\theta_2 - \theta_3) &= 0 \\ \frac{d\theta_2}{dt} + 2\frac{d\theta_3}{dt} + \frac{24k}{\rho c L^2}(-\theta_2 + \theta_3) &= 0 \end{aligned}\right\} \tag{E.3}$$

These equations can be solved using the initial conditions on $\theta_2(t)$ and $\theta_3(t)$ (from the known or specified initial conditions on T_2 and T_3).

14.6.2 Finite Difference Solution in Time Domain

The solution of the unsteady state equations, namely Eq. (13.35), based on the fourth-order Runge—Kutta integration procedure was given in Chapter 7. We now present an alternative approach using the finite difference scheme for solving these equations. This scheme is based on approximating the first-time derivative of T as

$$\left.\frac{dT}{dt}\right|_t = \frac{T_1 - T_0}{\Delta t} \tag{14.48}$$

where $T_1 = T(t + (\Delta t/2))$, $T_0 = T(t - (\Delta t/2))$, and Δt is a small time step. Thus, $\dot{\vec{\underset{\sim}{T}}} = \left(d\vec{\underset{\sim}{T}}\Big/dt\right)$ can be replaced by

$$\left.\frac{d\vec{\underset{\sim}{T}}}{dt}\right|_t = \frac{1}{\Delta t}\left(\vec{\underset{\sim}{T}}_1 - \vec{\underset{\sim}{T}}_0\right) \tag{14.49}$$

Since $\dot{\vec{\underset{\sim}{T}}}$ is evaluated at the middle point of the time interval Δt, the quantities $\vec{\underset{\sim}{T}}$ and $\vec{\underset{\sim}{P}}$ involved in Eq. (13.35) are also to be evaluated at this point. These quantities can be approximated as

$$\left.\vec{\underset{\sim}{T}}\right|_t = \frac{1}{2}\left(\vec{\underset{\sim}{T}}_1 - \vec{\underset{\sim}{T}}_0\right) \tag{14.50}$$

and

$$\left.\vec{\underset{\sim}{P}}\right|_t = \frac{1}{2}\left(\vec{\underset{\sim}{P}}_1 - \vec{\underset{\sim}{P}}_0\right) \tag{14.51}$$

where

$$\vec{\underset{\sim}{P}}_1 = \vec{\underset{\sim}{P}}\left(t + \frac{\Delta t}{2}\right) \quad \text{and} \quad \vec{\underset{\sim}{P}}_0 = \vec{\underset{\sim}{P}}\left(t - \frac{\Delta t}{2}\right) \tag{14.52}$$

By substituting Eqs. (14.49) to (14.51) into Eq. (13.35), we obtain

$$\frac{1}{\Delta t}\left[\underset{\sim}{K}_3\right]\left(\vec{\underset{\sim}{T}}_1 - \vec{\underset{\sim}{T}}_0\right) + \frac{1}{2}\left[\underset{\sim}{K}\right]\left(\vec{\underset{\sim}{T}}_1 + \vec{\underset{\sim}{T}}_0\right) = \left.\vec{\underset{\sim}{P}}\right|_t$$

or

$$\left(\left[\underset{\sim}{K}\right] + \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right]\right)\vec{\underset{\sim}{T}}_1 = \left(-\left[\underset{\sim}{K}\right] + \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right]\right)\vec{\underset{\sim}{T}}_0 + \left(\vec{\underset{\sim}{P}}_1 + \vec{\underset{\sim}{P}}_0\right) \tag{14.53}$$

This equation shows that the nodal temperatures $\vec{\underset{\sim}{T}}$ at time $t + \Delta t$ can be computed once the nodal temperatures at time t are known since $\vec{\underset{\sim}{P}}_1$ can be computed before solving Eq. (14.53). Thus, the known initial conditions (on nodal temperatures) can be used to find the solution at subsequent time steps.

Note

Eq. (14.53) has been derived by evaluating the derivative at the middle point of the time interval. The nodal values (i.e., at time t) of $\vec{\underset{\sim}{T}}$ can be computed after solving Eq. (14.53) using Eq. (14.50). In fact, by using Eq. (14.50), Eq. (14.53) can be rewritten as

$$\left(\left[\underset{\sim}{K}\right] + \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right]\right)\vec{\underset{\sim}{T}}\Big|_t = \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right]\vec{\underset{\sim}{T}}_0 - \vec{\underset{\sim}{P}}\Big|_t \tag{14.54}$$

where the nodal temperatures $\vec{\underset{\sim}{T}}\Big|_t$ can be directly obtained.

EXAMPLE 14.12

Derive the recursive relations, Eq. (14.53), for the one-dimensional fin shown in Fig. 14.5 with the following data:

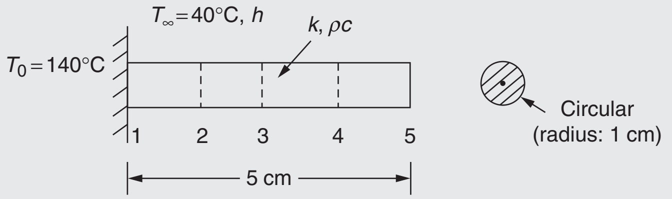

FIGURE 14.5 Transient heat transfer in a one-dimensional fin.

$$k = 70\frac{\text{watts}}{\text{cm} - ^\circ\text{K}},\quad h = 10\frac{\text{watts}}{\text{cm}^2 - ^\circ\text{K}},\quad T_\infty = 40^\circ\text{C},\quad T_0 = 140^\circ\text{C},$$

$$\rho c = 20\frac{\text{Joules}}{\text{cm} - ^\circ\text{K}},\quad \Delta t = 2\text{ minutes}$$

Solution

We divide the fin into four finite elements ($E = 4$) so that the element matrices and vectors become

$$\left[K_1^{(e)}\right] = \frac{(\pi)(70)}{(1.25)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 175.93 & -175.93 \\ -175.93 & 175.93 \end{bmatrix}$$

$$\left[K_2^{(e)}\right] = \frac{(10)(2\pi)(1.25)}{(6)}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 39.26 & 19.63 \\ 19.63 & 39.26 \end{bmatrix}$$

$$\left[K_3^{(e)}\right] = \frac{(20)(\pi)(1.25)}{(6)}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 39.26 & 19.63 \\ 19.63 & 39.26 \end{bmatrix}$$

$$\vec{P}^{(e)} = \vec{P}_3^{(e)} - \vec{P}_3^{(e)} + \vec{P}_3^{(e)} = \vec{P}_3^{(e)} = \frac{(10)(40)(2\pi)(1.25)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1570.80 \\ 1570.80 \end{Bmatrix}$$

The assembled matrices and vectors are

$$\left[\underset{\sim}{K}_1\right] = \begin{bmatrix} 175.93 & -175.93 & 0 & 0 & 0 \\ -175.93 & 351.86 & -175.93 & 0 & 0 \\ 0 & -175.93 & 351.86 & -175.93 & 0 \\ 0 & 0 & -175.93 & 351.86 & -175.93 \\ 0 & 0 & 0 & -175.93 & 175.93 \end{bmatrix} \tag{E.1}$$

EXAMPLE 14.12 —cont'd

$$\left[\underset{\sim}{K}_3\right] = \left[\underset{\sim}{K}_2\right] = \begin{bmatrix} 39.26 & 19.93 & 0 & 0 & 0 \\ 19.63 & 78.52 & 19.63 & 0 & 0 \\ 0 & 19.63 & 78.52 & 19.63 & 0 \\ 0 & 0 & 19.63 & 78.52 & 19.63 \\ 0 & 0 & 0 & 19.63 & 39.26 \end{bmatrix} \tag{E.2}$$

$$\overrightarrow{\underset{\sim}{P}} = \begin{Bmatrix} 1570.80 \\ 3141.60 \\ 3141.60 \\ 3141.60 \\ 1570.80 \end{Bmatrix} \tag{E.3}$$

Since $\Delta t = 1/30$ h, we have

$$[A] = \left[\underset{\sim}{K}\right] + \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right] = \left[\underset{\sim}{K}_1\right] + \left[\underset{\sim}{K}_2\right] + 60\left[\underset{\sim}{K}_3\right] = \begin{bmatrix} 2570.79 & 1021.50 & 0 & 0 & 0 \\ 1021.50 & 5141.58 & 1021.50 & 0 & 0 \\ 0 & 1021.50 & 5141.58 & 1021.50 & 0 \\ 0 & 0 & 1021.50 & 5141.58 & 1021.50 \\ 0 & 0 & 0 & 1021.50 & 2570.79 \end{bmatrix} \tag{E.4}$$

and

$$[B] = -\left[\underset{\sim}{K}\right] + \frac{2}{\Delta t}\left[\underset{\sim}{K}_3\right] = -\left[\underset{\sim}{K}_1\right] - \left[\underset{\sim}{K}_2\right] + 60\left[\underset{\sim}{K}_3\right] = \begin{bmatrix} 2140.41 & 1334.10 & 0 & 0 & 0 \\ 1334.10 & 4280.82 & 1334.10 & 0 & 0 \\ 0 & 1334.10 & 4280.82 & 1334.10 & 0 \\ 0 & 0 & 1334.10 & 4280.82 & 1334.10 \\ 0 & 0 & 0 & 1334.10 & 2140.41 \end{bmatrix} \tag{E.5}$$

Hence, the desired recursive relation is

$$[A]\overrightarrow{\underset{\sim}{T}}_1 = [B]\overrightarrow{\underset{\sim}{T}}_0 + \overrightarrow{\underset{\sim}{P}} \tag{E.6}$$

where $[A]$, $[B]$, and $\overrightarrow{\underset{\sim}{P}}$ are given by Eqs. (E.4), (E.5), and (E.3), respectively.

14.7 HEAT TRANSFER PROBLEMS WITH RADIATION

The rate of heat flow by radiation (q) is governed by the relation

$$q = \sigma \varepsilon A\left(T^4 - T_\infty^4\right) \tag{14.55}$$

where σ is the Stefan–Boltzmann constant, ε is the emissivity of the surface, A is the surface area of the body through which heat flows, T is the absolute surface temperature of the body, and T_∞ is the absolute surrounding temperature. Thus, the inclusion of the radiation boundary condition makes a heat transfer problem nonlinear due to the nonlinear relation of Eq. (14.55). Hence, an iterative procedure is to be adopted to find the finite element solution of the problem. For example, for a one-dimensional problem, the governing differential equation is

$$k\frac{\partial^2 T}{\partial x^2} + \dot{q} = \rho c\frac{\partial T}{\partial t} \tag{14.56}$$

If heat flux is specified on the surface of the rod and if both convection and radiation losses take place from the surface, the boundary conditions of the problem can be expressed as

$$T(x = 0, t) = T_0 \tag{14.57}$$

and

$$\left.\begin{array}{c} k\dfrac{\partial T}{\partial x} l_x + h(T - T_\infty) + q_o + \sigma\varepsilon\left(T^4 - T_\infty^4\right) = 0 \\ \text{on the surface} \end{array}\right\} \tag{14.58}$$

The initial conditions can be specified as

$$T(x, t = 0) = \overline{T}_0 \tag{14.59}$$

For convenience, we define a radiation heat transfer coefficient (h_r) as

$$h_r = \sigma\varepsilon\left(T^2 + T_\infty^2\right)(T + T_\infty) \tag{14.60}$$

so that Eq. (14.58) can be expressed as

$$\left.\begin{array}{c} k\dfrac{\partial T}{\partial x} l_x + h(T - T_\infty) + q_o + h_r(T - T_\infty) = 0 \\ \text{on the surface} \end{array}\right\} \tag{14.61}$$

The inclusion of the convection term $h(T - T_\infty)$ in the finite element analysis resulted in the matrix (Eq. 13.31)

$$\left[K_2^{(e)}\right] = \iint_{S_3^{(e)}} h[N]^T[N]\mathrm{d}S_3 \tag{14.62}$$

and the vector (Eq. 13.33)

$$\vec{P}_3^{(e)} = \iint_{S_3^{(e)}} hT_\infty[N]^T\mathrm{d}S_3 \tag{14.63}$$

Assuming, for the time being, that h_r is independent of the temperature T, and proceeding as in the case of the term $h(T - T_\infty)$, we obtain the additional matrix

$$\left[K_4^{(e)}\right] = \iint_{S_4^{(e)}} h_r[N]^T[N]\mathrm{d}S_4 \tag{14.64}$$

and the additional vector

$$\vec{P}_4^{(e)} = \iint_{S_4^{(e)}} h_rT_\infty[N]^T\mathrm{d}S_4 \tag{14.65}$$

to be assembled in generating the matrix $\left[\underset{\sim}{K}\right]$ and the vector $\underset{\sim}{\vec{P}}$, respectively. In Eqs. (14.64) and (14.65), $S_4^{(e)}$ denotes the surface of the element e from which radiation loss takes place. Since h_r was assumed to be a constant in deriving Eqs. (14.64) and (14.65), its value needs to be changed subsequently. Since the correct solution $\left(\underset{\sim}{\vec{T}}\right)$ cannot be found unless the correct value of h_r is used in Eqs. (14.64) and (14.65), the following iterative procedure can be adopted:

1. Set the iteration number as $n = 1$ and assume $h_r^{(e)} = 0$.
2. Generate $\left[K_4^{(e)}\right]$ and $\vec{P}_4^{(e)}$ using Eqs. (14.64) and (14.65) using the latest values of $h_r^{(e)}$.
3. Assemble the element matrices and vectors to obtain the overall Eq. (13.35) with $\left[\underset{\sim}{K}\right] = \sum_e = 1^E\left[\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right] + \left[K_4^{(e)}\right]\right]$ and $\underset{\sim}{\vec{P}} = \sum_{e=1}^{E}\left[\vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)} + \vec{P}_4^{(e)}\right]$.

4. Solve Eq. (13.35) and find $\underset{\sim}{\vec{T}}$.
5. From the known nodal temperatures $\underset{\sim}{\vec{T}}$, find the new value of $h_r^{(e)}$ using Eq. (14.60) (the average of the two nodal temperatures of the element, $(T_i + T_j)/2$, can be used as $T_{av}^{(e)}$ in place of T):

$$h_r^{(e)} = \sigma\varepsilon\left(T_{av}^{(e)^2} + T_\infty^2\right)\left(T_{av}^{(e)} + T_\infty\right) \tag{14.66}$$

If $n > 1$, test for the convergence of the method. If

$$\left|\frac{\left[h_r^{(e)}\right]_n - \left[h_r^{(e)}\right]_{n-1}}{\left[h_r^{(e)}\right]_{n-1}}\right| \le \delta_1 \tag{14.67}$$

and

$$\left\|\left(\underset{\sim}{\vec{T}}^{(\text{total})}\right)_n - \left(\underset{\sim}{\vec{T}}^{(\text{total})}\right)_{n-1}\right\| \le \delta_2 \tag{14.68}$$

where δ_1 and δ_2 are specified small numbers, the method is assumed to have converged. Hence, stop the method by taking $\underset{\sim}{\vec{T}}\Big|_{\text{correct}} = \left(\underset{\sim}{\vec{T}}\right)_n$. On the other hand, if either of the inequalities of Eqs. (14.67) and (14.68) is not satisfied, set the new iteration number as $n = n + 1$, and go to step 2.

EXAMPLE 14.13

Find the steady-state temperature distribution in the one-dimensional fin shown in Fig. 14.1A by considering both convection and radiation losses from its perimeter surface using a one-element idealization. Assume $\varepsilon = 0.1$ and $\sigma = 5.7 \times 10^{-8}\text{W/cm}^2\text{–K}^4$.

Solution

For linear temperature variation inside the element, the matrix $\left[K_4^{(e)}\right]$ and the vector $\vec{P}_4^{(e)}$ can be obtained as

$$\left[K_4^{(e)}\right] = \frac{h_r P l^{(e)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{P}_4^{(e)} = \frac{h_r T_\infty P l^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

For a one-element idealization ($E = 1$), the matrices $\left[K_1^{(e)}\right]$, $\left[K_2^{(e)}\right]$, and $\vec{P}_3^{(e)}$ can be derived as

$$\left[K_1^{(1)}\right] = \frac{(\pi)(70)}{(5)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \pi\begin{bmatrix} 14 & -14 \\ -14 & 14 \end{bmatrix}$$

$$\left[K_2^{(1)}\right] = \frac{(5)(2\pi)(5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \pi\begin{bmatrix} 16.67 & 8.33 \\ 8.33 & 16.67 \end{bmatrix}$$

$$\vec{P}_3^{(1)} = \frac{(5)(40)(2\pi)(5)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \pi\begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix}$$

ITERATION 1

By using $h_r^{(1)} = 0$, the matrices $\left[K_4^{(1)}\right]$ and $\vec{P}_4^{(1)}$ can be obtained as $\left[K_4^{(1)}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{P}_4^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. The overall Eq. (13.35) becomes

$$\begin{bmatrix} 30.67 & -5.67 \\ -5.67 & 30.67 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \end{Bmatrix} \tag{E.1}$$

Continued

ITERATION 1 —cont'd

After incorporating the boundary condition $T_1 = 140$, Eq. (E.1) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 30.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_1 \\ 1000 + 5.67\,T_1 \end{Bmatrix} = \begin{Bmatrix} 140.0 \\ 1793.8 \end{Bmatrix} \tag{E.2}$$

from which the solution can be obtained as

$$\vec{\underset{\sim}{T}} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140.00 \\ 58.48 \end{Bmatrix} \tag{E.3}$$

The average temperature of the nodes of the element can be computed as

$$T_{av}^{(1)} = \frac{T_1 + T_2}{2} = 99.24°\text{C}$$

Thus, the values of $T_{av}^{(1)}$ and T_∞ to be used in the computation of $h_r^{(1)}$ are 372.24 °K and 313 °K, respectively. The solution of Eq. (14.66) gives the value of

$$h_r^{(1)} = \left(5.7 \times 10^{-8}\right)(0.1)\left(372.24^2 + 313^2\right)(372.24 + 313) = 0.9234$$

ITERATION 2

By using the current value of $h_r^{(1)}$, we can derive

$$\left[K_4^{(1)}\right] = \frac{(0.9234)(2\pi)(5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \pi\begin{bmatrix} 3.078 & 1.539 \\ 1.539 & 3.078 \end{bmatrix}$$

$$\vec{P}_4^{(1)} = \frac{(0.9234)(40)(2\pi)(5)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \pi\begin{Bmatrix} 184.68 \\ 184.68 \end{Bmatrix}$$

Thus, the overall Eq. (13.35) can be written as

$$\begin{bmatrix} 33.745 & -4.128 \\ -4.128 & 33.745 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 1184.68 \\ 1184.68 \end{Bmatrix} \tag{E.4}$$

The application of the boundary condition ($T_1 = 140$) leads to

$$\begin{bmatrix} 1 & 0 \\ 0 & 33.745 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140.00 \\ 1762.00 \end{Bmatrix} \tag{E.5}$$

Eq. (E.5) gives the solution

$$\vec{\underset{\sim}{T}} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140.00 \\ 52.25 \end{Bmatrix} \tag{E.6}$$

Thus, $T_{av}^{(1)} = 96.125°\text{C} = 369.125°\text{K}$ and the new value of $h_r^{(1)}$ can be obtained as

$$h_r^{(1)} = \left(5.7 \times 10^{-8}\right)(0.1)\left(369.125^2 + 313^2\right)(369.125 + 313) = 0.9103$$

ITERATION 3

With the present value $h_r^{(1)}$, we can obtain

$$\left[K_4^{(1)}\right] = \frac{(0.9103)(2\pi)(5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \pi\begin{bmatrix} 3.034 & 1.517 \\ 1.517 & 3.034 \end{bmatrix}$$

ITERATION 3 —cont'd

and

$$\vec{P}_4^{(e)} = \frac{(0.9103)(40)(2\pi)(5)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \pi\begin{Bmatrix} 182.06 \\ 182.06 \end{Bmatrix}$$

Thus, the overall Eq. (13.35) can be expressed as

$$\begin{bmatrix} 33.701 & -4.150 \\ -4.150 & 33.701 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 1182.06 \\ 1182.06 \end{Bmatrix} \tag{E.7}$$

After incorporating the known condition, $T_1 = 140$, Eq. (E.7) gives

$$\begin{bmatrix} 1 & 0 \\ 0 & 33.701 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140.00 \\ 1763.06 \end{Bmatrix} \tag{E.8}$$

from which the solution can be obtained as

$$\vec{\underset{\sim}{T}} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 140.00 \\ 52.31 \end{Bmatrix} \tag{E.9}$$

This solution gives $T_{av}^{(1)} = (T_1 + T_2)/2 = 96.155°\text{C} = 369.155°\text{K}$ and $h_r^{(1)} = (5.7 \times 10^{-8})$ (0.1) $(369.155^2 + 313^2)$ $(369.155 + 313) = 0.9104$.

Since the difference between this value of $h_r^{(1)}$ and the previous value is very small, we assume convergence, and hence the solution of Eq. (E.9) can be taken as the correct solution of the problem.

REVIEW QUESTIONS

14.1 Give brief answers to the following questions.

1. State the governing differential equation of a one-dimensional heat transfer problem.
2. Give a practical example of a one-dimensional heat transfer.
3. What is the finite difference expression for the first derivative $\frac{\partial T}{\partial t}$ at $t = t_0$?
4. What are element capacitance matrices?
5. State the temperature variation for a quadratic element.
6. State a simple method of modeling a one-dimensional fin whose cross section varies exponentially.

14.2 Fill in the blank with a suitable word.

1. For a one-dimensional fin of length l and circular cross section with area A and perimeter P, the convection heat transfer occurs over a total area of ________________.
2. The radiation term introduces nonlinearity because of the presence of the term ________________.
3. A transient heat transfer problem involves boundary conditions as well as ________________ condition.
4. In general, the boundary conditions of a one-dimensional heat transfer problem involves ________________ and combined heat ________________ and specification of ________________ loss.

14.3 Indicate whether the following statement is true or false.

1. The solution of a radiation heat transfer problem requires an iterative process.
2. It is more convenient to solve a transient heat transfer problem using only finite differences.
3. The convection term in heat transfer analysis gives rise to an element matrix and an element vector.
4. The conduction coefficient appears on the right-hand side vector in a heat transfer problem.

14.4 Select the most appropriate answer from the multiple choices given.

1. The following term is defined as the radiation heat transfer coefficient (h_r):
   (a) $\sigma\,\varepsilon\left(T^4 - T_\infty^4\right)$ (b) $\sigma\,\varepsilon\left(T^4 + T_\infty^4\right)$ (c) $\sigma\,\varepsilon\left(T^2 + T_\infty^2\right)(T + T_\infty)$

PROBLEMS

14.1 A composite wall, made up of two materials, is shown in Fig. 14.6. The temperature on the left side of the wall is specified as 80 °F while convection takes place on the right side of the wall. Find the temperature distribution in the wall using two linear elements.

14.2 A fin, of size 1 × 10 × 50 in, extends from a wall as shown in Fig. 14.7. If the wall temperature is maintained at 500 °F and the ambient temperature is 70 °F, determine the temperature distribution in the fin using three one-dimensional elements in the x direction. Assume $k = 40$ BTU/hr-ft °F and $h = 120$ BTU/hr-ft^2 °F.

14.3 Determine the amount of heat transferred from the fin considered in Problem 14.2.

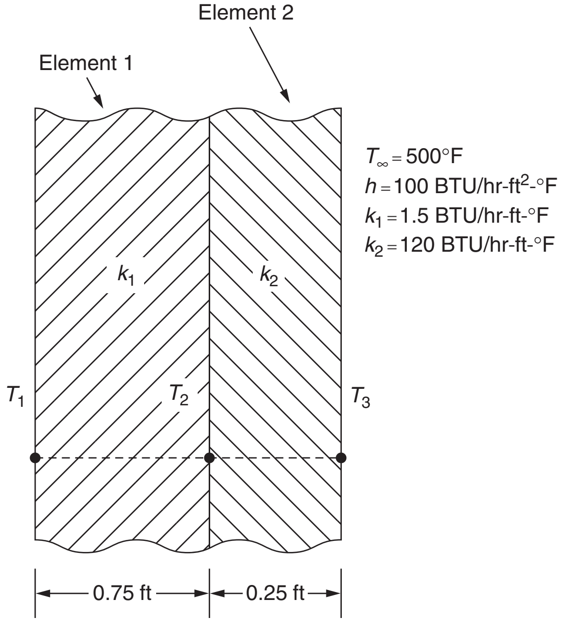

FIGURE 14.6 A composite wall.

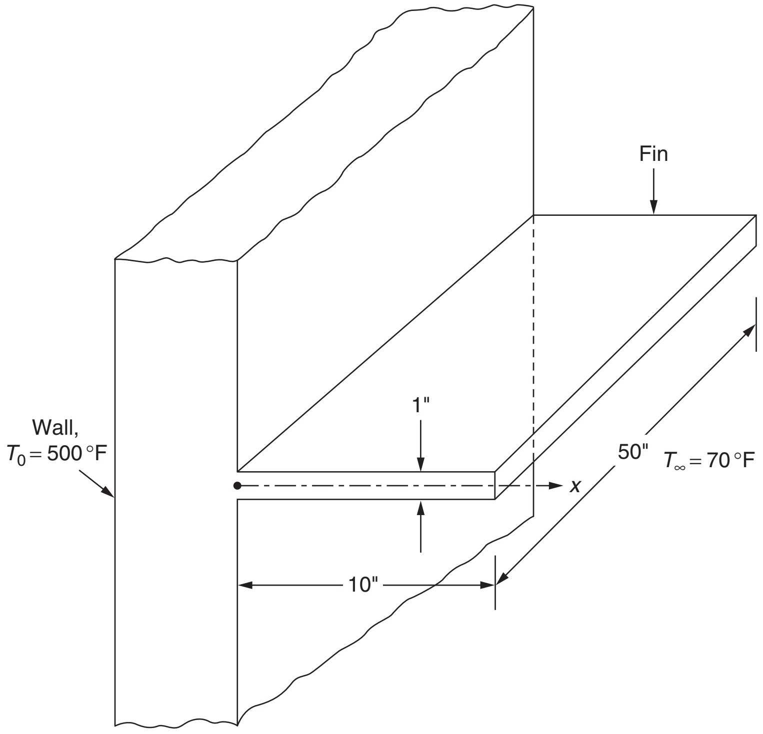

FIGURE 14.7 Fin extending from a wall.

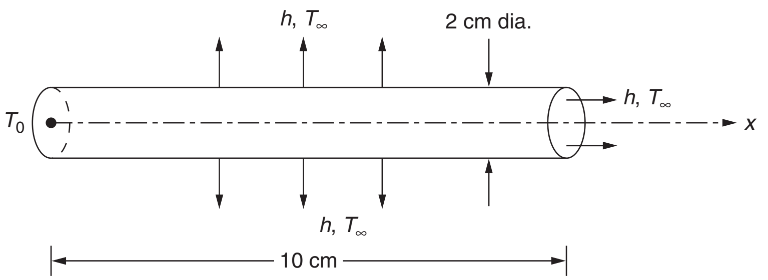

FIGURE 14.8 Uniform aluminum fin.

14.4 One side of a brick wall, of width 5 m, height 4 m, and thickness 0.5 m, is exposed to a temperature of −30°C, while the other side is maintained at 30°C. If the thermal conductivity (k) is 0.75 W/m°C and the heat transfer coefficient on the colder side of the wall (h) is 5 W/m^2°C, determine the following:
a. Temperature distribution in the wall using two one-dimensional elements in the thickness.
b. Heat loss from the wall.

14.5 Fig. 14.8 shows a uniform aluminum fin of diameter 2 cm. The root (left end) of the fin is maintained at a temperature of $T_0 = 100$°C while convection takes place from the lateral (circular) surface and the right (flat) edge of the fin. Assuming $k = 200$ W/m°C, $h = 1000$ W/m^2°C, and $T_\infty = 20$°C, determine the temperature distribution in the fin using a two-element idealization.

14.6 Solve Problem 14.5 by neglecting heat convection from the right-hand edge of the fin.

14.7 Solve Problem 14.5 by assuming the fin diameter to be varying linearly from 4 cm at the root to 1 cm at the right end.

14.8 A uniform steel fin of length 10 in, with a rectangular cross section 2 × 1 in, is shown in Fig. 14.9. If heat transfer takes place by convection from all the surfaces while the left side (root) of the fin is maintained at $T_0 = 500$ °F, determine the temperature distribution in the fin. Assume that $k = 9$ BTU/hr-ft °F, $h = 2500$ BTU/hr-ft^2 °F, and $T_\infty = 50$ °F. Use two finite elements.

14.9 Solve Problem 14.8 using three finite elements.

14.10 Derive the finite element equations corresponding to Eqs. (14.56) to (14.59) without assuming the radiation heat transfer coefficient (h_r) to be a constant.

14.11 A wall consists of 4-cm-thick wood, 10-cm-thick fiber glass insulation, and 1-cm-thick plaster. If the temperatures on the wood and plaster faces are 20 and −20°C, respectively, determine the temperature distribution in the wall. Assume thermal conductivities of wood, glass fiber, and plaster as 0.17, 0.035, and 0.5 W/m°C, respectively, and the heat transfer coefficient on the colder side of the wall as 25 W/m^2°C.

14.12 The radial temperature distribution in an annular fin (Fig. 14.10) is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[ktr\frac{\mathrm{d}T}{\mathrm{d}r}\right] - 2hr(T - T_\infty) = 0$$

with boundary conditions

$$T(r_i) = T_0(\text{temperature specified})$$

$$\frac{\mathrm{d}T}{\mathrm{d}r}(r_0) = 0 \text{ (insulated)}$$

Derive the finite element equations corresponding to this problem.

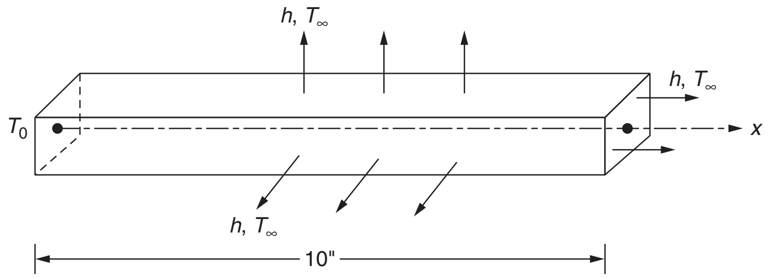

FIGURE 14.9 Uniform fin with a rectangular section.

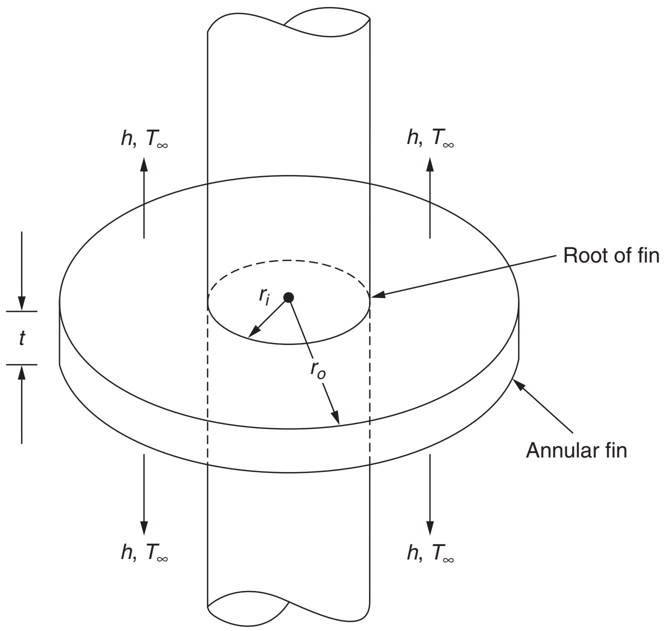

FIGURE 14.10 An annular fin.

14.13 Derive the element matrix $[K^{(e)}]$ and the vector $\overrightarrow{P}^{(e)}$ for a one-dimensional element for which the thermal conductivity k varies linearly between the two nodes.

14.14 Using the finite element method, find the tip temperature and the heat loss from the tapered fin shown in Fig. 14.11. Assume that (a) the temperature is uniform in the y direction, (b) the heat transfer from the fin edges (one is shown hatched) is negligible, and (c) there is no temperature variation in the z direction.

14.15 A plane wall of thickness 15 cm has an initial temperature distribution given by $T(x, t = 0) = 500 \sin(\pi x/L)$, where $x = 0$ and $x = L$ denote the two faces of the wall. The temperature of each face is kept at zero and the wall is allowed to approach thermal equilibrium as time increases. Find the time variation of temperature distribution in the wall for $\alpha = (k/\rho c) = 10$ cm^2/hr using the finite element method.

14.16 Derive the matrix $\left[K_4^{(e)}\right]$ corresponding to radiation heat transfer for a tapered one-dimensional element.

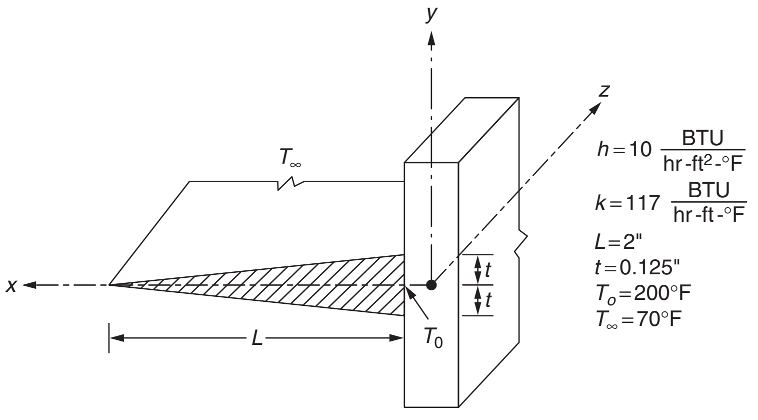

FIGURE 14.11 A tapered fin.

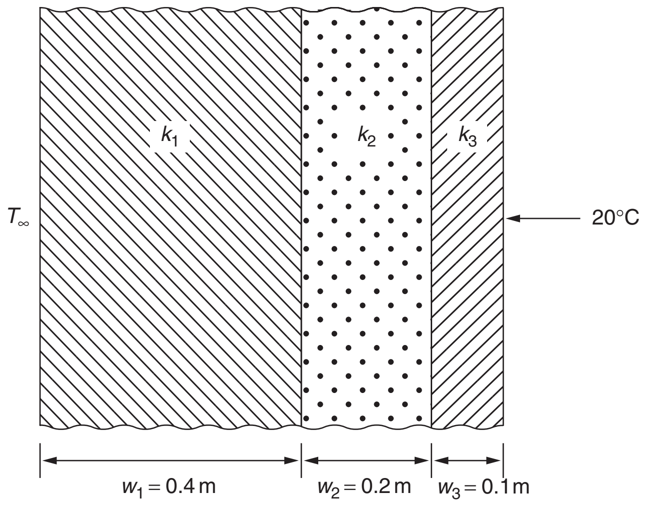

FIGURE 14.12 A composite wall consisting of three materials.

14.17 Derive the matrix $\left[K_4^{(e)}\right]$ corresponding to radiation heat transfer for a one-dimensional element using quadratic temperature variation within the element.

14.18 Find the steady-state temperature distribution in the tapered fin shown in Fig. 14.2 by considering both convection and radiation from its perimeter surface. Take $\varepsilon = 0.1$ and $\sigma = 5.7 \times 10^{-8}$ W/m^2 °K^4.

14.19 A composite wall, made up of three different materials as shown in Fig. 14.12, is used in the enclosure of a furnace. The inner (furnace side) surface of the wall is exposed to the furnace temperature of $T_\infty = 600$°C and convection heat transfer occurs on the inner surface of the wall. The outer surface of the wall is maintained at the atmospheric temperature of 20°C. Determine the temperature distribution in the wall for the following data: $w_1 = 0.4$ m, $w_2 = 0.2$ m, $w_3 = 0.1$ m, $k_1 = 10$ W/m°C, $k_2 = 25$ W/m°C, $k_3 = 60$ W/m°C, $h = 50$ W/m^{2}°C.

14.20 A thick metal plate, with a thermal conductivity coefficient of $k = 1500$ W/m°C, has a thickness of 0.25 m in the x direction and infinite (very large) dimensions in the y and z directions as shown in Fig. 14.13. Because the dimensions of the plate in the y and z directions are very large, it can be considered an infinite slab and can be modeled as a one-dimensional heat transfer problem (in the x direction). One side of a plate (the left face, at $x = 0$) is

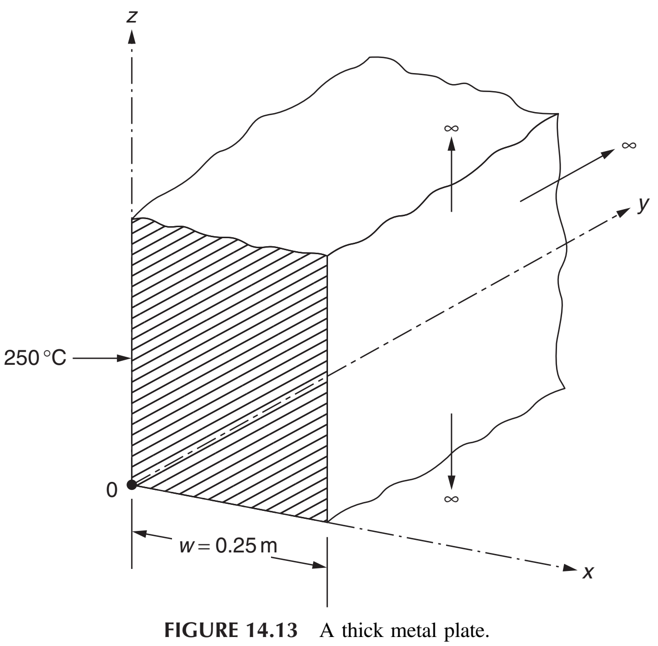

FIGURE 14.13 A thick metal plate.

maintained at 250°C and the other side (the right face, at $x = w = 0.25$ m) is exposed to the room temperature of 20°C. Determine the temperature distribution in the thickness of the plate using two finite elements.

14.21 For the plate described in Problem 14.20, one side of a plate (the left face, at $x = 0$) is maintained at 250°C and the other side (the right face, at $x = w = 0.25$ m) is insulated. Determine the temperature distribution along the thickness or x direction of the plate using two finite elements.

14.22 For the plate described in Problem 14.20, one side of a plate (the left face, at $x = 0$) is maintained at 250°C and the other side (the right face, at $x = w = 0.25$ m) is subject to a heat flux boundary condition with $q_0 = 20$ W. Determine the temperature distribution along the thickness or x direction of the plate using two one-dimensional elements.

14.23 An aluminum plate fin of length $L = 20$ cm, width $W = 30$ cm, and thickness $h = 0.2$ cm is shown in Fig. 14.14. Because the thickness of the fin is very small compared to its length and width, it can be modeled as a one-dimensional fin with heat flowing only in the length (x) direction. Heat is transferred from the top and bottom surfaces of the fin to the surrounding air by convection. Find the temperature distribution along the fin and the heat transferred from the fin to the surrounding air by convection for the following data:

$$k = 250\ \text{W/m} - {}^\circ\text{C},\quad h = 400\ \text{W/m}^2 - {}^\circ\text{C},\quad T_0 = T(x = 0) = 300^\circ\text{C},\quad \text{and}\quad T_\infty = 30^\circ\text{C}$$

Assume that the temperature of the end surface (S) is maintained at 30°C.

14.24 For the plate fin described in Problem 14.23 and shown in Fig. 14.14, find the temperature distribution in the fin and the heat transferred from the fin by convection if convection heat transfer takes place from the end surface (S) also (in addition to the top and bottom surfaces of the fin).

14.25 For the plate fin described in Problem 14.23 and shown in Fig. 14.14, find the temperature distribution in the fin and the heat transferred from the fin by convection if the end surface (S) is insulated.

14.26 In a one-dimensional heat transfer problem, the system equations, before applying the boundary conditions, are given by

$$\begin{bmatrix} 0.15 & -0.15 & 0 \\ -0.15 & 0.30 & -0.15 \\ 0 & -0.15 & 0.15 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

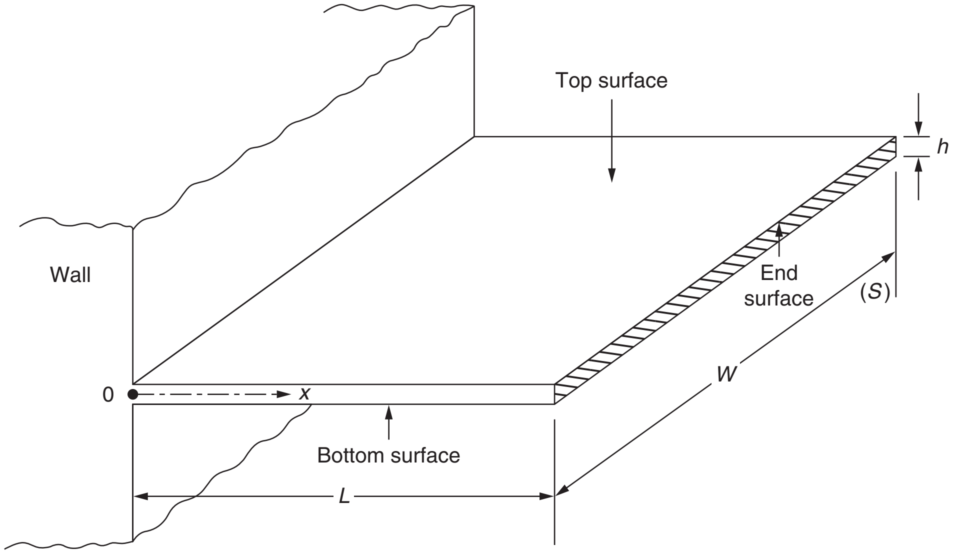

FIGURE 14.14 A plate fin.

where T_1, T_2, and T_3 are the temperatures at nodes 1, 2, and 3, respectively. If the boundary conditions of the problem are specified as $T_1 = 250°C$ and $T_3 = 100°C$, derive the final system equations after applying the boundary conditions. Find the solution of the resulting equations.

14.27 Find the temperature distribution in the aluminum fin described in Problem 14.5 under the following conditions:
a. The convection heat transfer from the right (flat) edge is negligible.
b. Heat flux of $q_0 = 200$ W/cm^2 (due to an electric heating pad in contact) at the right (flat) edge.
Use one finite element for idealization.

14.28 Find the temperature distribution in the aluminum fin described in Problem 14.5 under the following conditions:
a. The convection heat transfer from the right (flat) edge is negligible.
b. Heat flux of $q_0 = 200$ W/cm^2 (due to an electric heating pad in contact) at the right (flat) edge.
Use two finite elements for idealization.

14.29 Find the temperature distribution in the steel rectangular fin described in Problem 14.8 under the following conditions:
a. The convection heat transfer from the right (flat) edge is negligible.
b. Heat flux of $q_0 = 100$ W/cm^2 (due to an electric heating pad in contact) at the right (flat) edge.
Use one finite element for idealization.

14.30 Find the temperature distribution in the steel rectangular fin described in Problem 14.8 under the following conditions:
a. The convection heat transfer from the right (flat) edge is negligible.
b. Heat flux of $q_0 = 100$ W/cm^2 (due to an electric heating pad in contact) at the right (flat) edge.
Use two finite elements for idealization.

14.31 Carry out the integrations required and derive Eqs. (14.37), (14.38), and (14.41) for a quadratic element.

14.32 Carry out the integrations required and derive the element capacitance matrices given by Eqs. (14.45) to (14.47).

14.33 Derive Eq. (E.1) of Example 14.11.

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Chapter 15

Two-Dimensional Problems

Chapter Outline

15.1 INTRODUCTION

For a two-dimensional steady-state problem, the governing differential equation is (Fig. 15.1A)

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + \dot{q} = 0 \tag{15.1}$$

and the boundary conditions are

$$T = T_o(x, y) \text{ on } S_1 \tag{15.2}$$

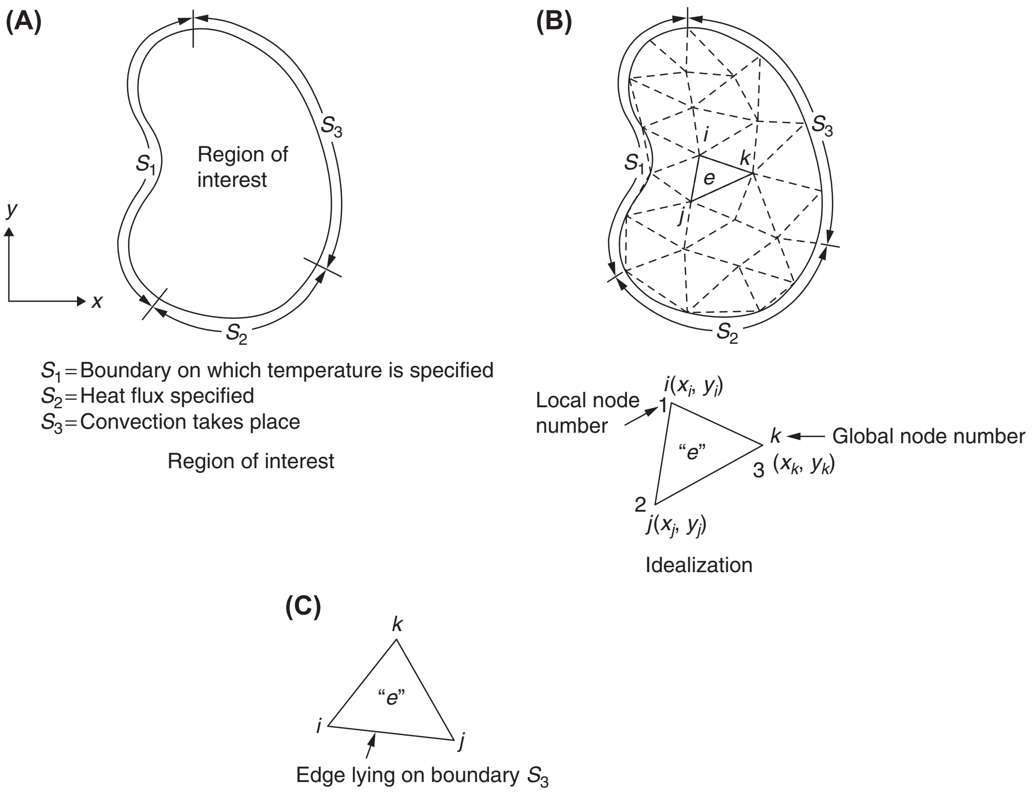


FIGURE 15.1 Two-dimensional problem.

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00015-8

$$k_x \frac{\partial T}{\partial x} l_x + k_y \frac{\partial T}{\partial y} l_y + q = 0 \text{ on } S_2 \tag{15.3}$$

$$k_x \frac{\partial T}{\partial x} l_x + k_y \frac{\partial T}{\partial y} l_y + h(T - T_\infty) = 0 \text{ on } S_3 \tag{15.4}$$

where k_x and k_y are thermal conductivities in the principal (x and y) directions, $\dot{q}$ is the strength of heat source, q is the magnitude of boundary heat flux, $h(T - T_\infty)$ is the surface heat flow due to convection, and l_x and l_y are the direction cosines of the outward normal to the surface.

15.2 SOLUTION

The finite element solution of this problem can be obtained as

Step 1: Idealize the solution region with triangular elements as shown in Fig. 15.1B.

Step 2: Assuming a linear variation of temperature $T^{(e)}$ inside the finite element e,

$$T^{(e)}(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y = [N(x,y)]\vec{q}^{(e)} \tag{15.5}$$

where

$$[N(x,y)] = \begin{Bmatrix} N_i(x,y) \\ N_j(x,y) \\ N_k(x,y) \end{Bmatrix}^T = \begin{Bmatrix} (a_i + xb_i + yc_i)/2A^{(e)} \\ (a_j + xb_j + yc_j)/2A^{(e)} \\ (a_k + xb_k + yc_k)/2A^{(e)} \end{Bmatrix}^T \tag{15.6}$$

$$\vec{q}^{(e)} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \equiv \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \tag{15.7}$$

and $A^{(e)}$ is the area and T_i, T_j, and T_k are the nodal temperatures of element e. The expressions for a_i, b_i, c_i, and $A^{(e)}$ are given by Eqs. (3.32) and (3.31), respectively.

Step 3: Derivation of element matrices. Once the matrix $[N(x, y)]$ is defined, Eq. (15.6), the matrix $[B]$ of Eq. (13.54) can be computed as

$$[B] = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{bmatrix} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \tag{15.8}$$

Using

$$[D] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{15.9}$$

Eq. (13.46) gives

$$\left[K_1^{(e)}\right] = \frac{1}{4A^{(e)^2}} \iiint_{V^{(e)}} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \cdot \mathrm{d}V \tag{15.10}$$

Assuming a unit thickness, the elemental volume can be expressed as $\mathrm{d}V = \mathrm{d}A$. Thus, Eq. (15.10) becomes

$$\left[K_1^{(e)}\right] = \frac{k_x}{4A^{(e)}} \begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{bmatrix} + \frac{k_y}{4A^{(e)}} \begin{bmatrix} c_i^2 & c_i c_j & c_i c_k \\ c_i c_j & c_j^2 & c_j c_k \\ c_i c_k & c_j c_k & c_k^2 \end{bmatrix} \tag{15.11}$$

For an isotropic material with $k_x = k_y = k$, Eq. (15.11) reduces to

$$\left[K_1^{(e)}\right] = \frac{k}{4A^{(e)}}\begin{bmatrix} \left(b_i^2 + c_i^2\right) & (b_ib_j + c_ic_j) & (b_ib_k + c_ic_k) \\ & \left(b_j^2 + c_j^2\right) & (b_jb_k + c_jc_k) \\ \text{Symmetric} & & \left(b_k^2 + c_k^2\right) \end{bmatrix} \tag{15.12}$$

To determine the matrix $\left[K_2^{(e)}\right]$, integration over the surface $\left[S_3^{(e)}\right]$ is to be performed:

$$\left[K_2^{(e)}\right] = h\iint_{S_3^{(e)}} \begin{bmatrix} N_i^2 & N_iN_j & N_iN_k \\ N_iN_j & N_j^2 & N_jN_k \\ N_iN_k & N_jN_k & N_k^2 \end{bmatrix} \mathrm{d}S_3 \tag{15.13}$$

Thus, the surface $S_3^{(e)}$ that experiences the convection phenomenon must be known for evaluating the integrals of Eq. (15.13). Let the edge ij of element e lie on the boundary S_3 as shown in Fig. 15.1C so that $N_k = 0$ along this edge. Then Eq. (15.13) becomes

$$\left[K_2^{(e)}\right] = h\iint_{S_3^{(e)}} \begin{bmatrix} N_i^2 & N_iN_j & 0 \\ N_iN_j & N_j^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \mathrm{d}S_3 \tag{15.14}$$

Note that if the edge ik (or jk) is subjected to convection instead of the edge ij, $N_j = 0$ (or $N_i = 0$) in Eq. (15.13). To evaluate the integrals of Eq. (15.14) conveniently, we can use the triangular or area coordinates introduced in Section 3.9.2. Because the temperature is assumed to vary linearly inside the element, we have $N_i = L_1$, $N_j = L_2$, $N_k = L_3$. Along the edge ij, $N_k = L_3 = 0$, and hence Eq. (15.14) becomes

$$\left[K_2^{(e)}\right] = h\int_{s=s_i}^{s_j} \begin{bmatrix} L_1^2 & L_1L_2 & 0 \\ L_1L_2 & L_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{d}s \tag{15.15}$$

where s denotes the direction along the edge ij, and $\mathrm{d}S_3$ was replaced by $t \cdot \mathrm{d}s = \mathrm{d}s$ since a unit thickness has been assumed for the element e. The integrals of Eq. (15.15) can be evaluated using Eq. (3.77) to find

$$\left[K_2^{(e)}\right] = \frac{hs_{ji}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15.16}$$

The integrals involved in Eq. (13.49) can be evaluated using triangular coordinates as follows:

$$\vec{P}_1^{(e)} = \iiint_{V^{(e)}} \dot{q}[N]^T \mathrm{d}V = \dot{q}_o \iint_{A^{(e)}} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \mathrm{d}A = \frac{\dot{q}_o A^{(e)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{15.17}$$

The integral in

$$\vec{P}_2^{(e)} = \iint_{S_2^{(e)}} q[N]^T \mathrm{d}S_2 \tag{15.18}$$

depends on the edge that lies on the heat flux boundary S_2. If the edge ij lies on S_2, $N_k = L_3 = 0$ and $\mathrm{d}S_2 = t\mathrm{d}s = \mathrm{d}s$ as in Eq. (15.15) and hence

$$\vec{P}_2^{(e)} = q\int_{s=s_i}^{s_j} \begin{Bmatrix} L_1 \\ L_2 \\ 0 \end{Bmatrix} \mathrm{d}s = \frac{qs_{ji}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{15.19}$$

Similarly, the vector $\vec{P}_3^{(e)}$ can be obtained as

$$\vec{P}_3^{(e)} = \iint_{S_3^{(e)}} hT_\infty [N]^T \mathrm{d}S_3 = \frac{hT_\infty s_{ji}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \text{if the edge } ij \text{ lies on } S_3 \tag{15.20}$$

Note that if the heat flux (q) or the convection heat transfer (h) occurs from two sides of the element e, then the surface integral becomes a sum of the integral for each side.

Step 4: The assembled equations (13.35) can be expressed as

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{15.21}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E}\left(\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right]\right) \tag{15.22}$$

and

$$\underset{\sim}{\vec{P}} = \sum_{e=1}^{E}\left(\vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)}\right) \tag{15.23}$$

Step 5: The overall equations (15.21) are to be solved, after incorporating the boundary conditions, to obtain the values of nodal temperatures.

EXAMPLE 15.1

Compute the element matrices and vectors for the element shown in Fig. 15.2 when the edges jk and ki experience convection heat loss.

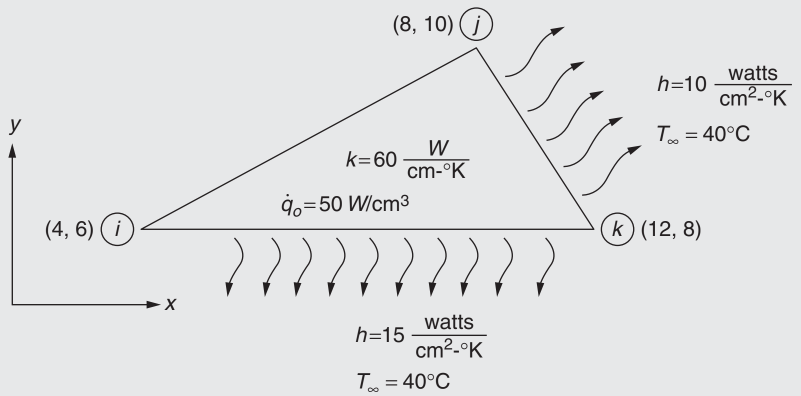


FIGURE 15.2 A triangular element.

Solution

From the data given in Fig. 15.2, we can compute the required quantities as

$$b_i = (y_j - y_k) = (10-8) = 2$$

$$b_j = (y_k - y_i) = (8-6) = 2$$

$$b_k = (y_i - y_j) = (6-10) = -4$$

$$c_i = x_k - x_j = 12 - 8 = 4$$

$$c_j = x_i - x_k = 4 - 12 = -8$$

$$c_k = x_j - x_i = 8 - 4 = 4$$

$$A^{(e)} = \tfrac{1}{2}|[(-4)(2) - (-4)(-4)]| = \frac{1}{2}|(-8-16)| = 12$$

$$s_{kj} = s_k - s_j = \text{length of edge } jk = \left[(x_k - x_j)^2 + (y_k - y_j)^2\right]^{1/2} = 4.47$$

$$s_{ik} = s_i - s_k = \text{length of edge } ki = \left[(x_i - x_k)^2 + (y_i - y_k)^2\right]^{1/2} = 8.25$$

EXAMPLE 15.1 —cont'd

Substitution of these values in Eqs. (15.12) and (15.16) to (15.20) gives

$$\left[K_1^{(e)}\right] = \frac{60}{4 \times 12}\begin{bmatrix} (4+16) & (4-32) & (-8+16) \\ & (4+64) & (-8-32) \\ \text{Symmetric} & & (16+16) \end{bmatrix} = \begin{bmatrix} 25 & -35 & 10 \\ -35 & 85 & -50 \\ 10 & -50 & 40 \end{bmatrix}$$

$$\left[K_2^{(e)}\right] = \frac{h_{ik}s_{ik}}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \frac{h_{kj}s_{kj}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \frac{(15)(8.25)}{6}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \frac{(10)(4.47)}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 41.250 & 0 & 20.625 \\ 0 & 14.900 & 7.450 \\ 20.625 & 7.450 & 56.150 \end{bmatrix}$$

$$\vec{P}_1^{(e)} = \frac{\dot{q}_o A^{(e)}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{(50)(12)}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 200 \\ 200 \end{Bmatrix}$$

$$\vec{P}_2^{(e)} = \vec{0} \text{ since no boundary heat flux is specified}$$

$$P_3^{(e)} = \frac{(hT_\infty)_{kj}s_{kj}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \frac{(hT_\infty)_{ik}s_{ik}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$= \frac{(10)(40)(4.47)}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \frac{(15)(40)(8.25)}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2475 \\ 894 \\ 3369 \end{Bmatrix}$$

EXAMPLE 15.2

Find the temperature distribution in a square region with uniform energy generation as shown in Fig. 15.3A. Assume that there is no temperature variation in the z direction. Take $k = 30$ W/cm °K, $L = 10$ cm, $T_\infty = 50$ °C, and $\dot{q} = \dot{q}_0 = 100$ W/cm^3.

Solution

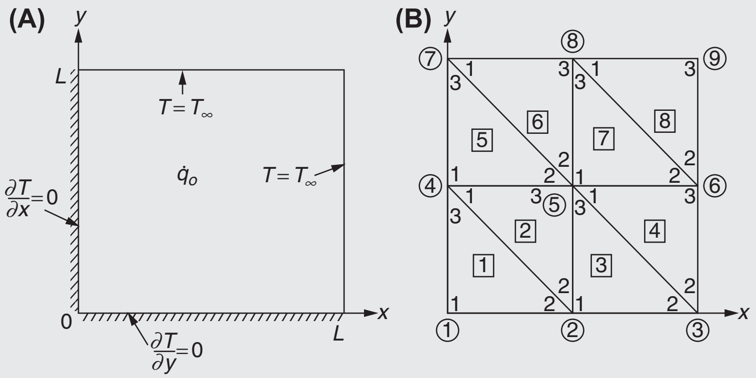
Step 1: Divide the solution region into eight triangular elements as shown in Fig. 15.3B. Label the local corner numbers of the elements counterclockwise starting from the lower left corner (only for convenience). The information needed for subsequent calculations is given here:

Node Number (i)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Coordinates of node i (x_i, y_i)	(0,0)	$\left(\frac{L}{2},0\right)$	(L, 0)	$\left(0,\frac{L}{2}\right)$	$\left(\frac{L}{2},\frac{L}{2}\right)$	$\left(L,\frac{L}{2}\right)$	(0, L)	$\left(\frac{L}{2},L\right)$	(L, L)

Element Number (e)		**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Global node numbers i, j, and k corresponding to local nodes 1, 2, and 3	i	1	4	2	5	4	7	5	8
	j	2	2	3	3	5	5	6	6
	k	4	5	5	6	7	8	8	9

Continued

EXAMPLE 15.2 —cont'd



A square region with uniform energy generation

Finite element idealization

FIGURE 15.3 Square region with uniform heat generation.

Step 2: Computation of $[N(x, y)]$ of Eq. (15.6) for various elements:

The information needed for the computation of $[N(x, y)]$ is given next (a_i, a_j, and a_k are not computed because they are not needed in the computations).

Step 3: Derivation of element matrices and vectors:

a. $\left[K_1^{(e)}\right]$ matrices (Eq. 15.12):

$$[K_1^{(1)}] = \frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+\frac{L^2}{4}\right) & \left(-\frac{L^2}{4}+0\right) & \left(0-\frac{L^2}{4}\right) \\ & \left(\frac{L^2}{4}+0\right) & (0+0) \\ \text{Symmetric} & & \left(0+\frac{L^2}{4}\right) \end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{array}{ccc} T_1 & T_2 & T_4 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ T_1 \\ T_2 \\ T_4 \end{array}$$

$$[K_1^{(2)}] = \frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+\frac{L^2}{4}\right) & \left(-\frac{L^2}{4}+0\right) & \left(0-\frac{L^2}{4}\right) \\ & \left(\frac{L^2}{4}+0\right) & (0+0) \\ \text{Symmetric} & & \left(0+\frac{L^2}{4}\right) \end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{array}{ccc} T_4 & T_2 & T_5 \end{array} \\ \begin{bmatrix} -2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ T_4 \\ T_2 \\ T_5 \end{array}$$

$$[K_1^{(3)}] = \frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+0\right) & (0+0) & \left(-\frac{L^2}{4}+0\right) \\ & \left(0+\frac{L^2}{4}\right) & \left(0-\frac{L^2}{4}\right) \\ \text{Symmetric} & & \left(\frac{L^2}{4}+\frac{L^2}{4}\right) \end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{array}{ccc} T_2 & T_3 & T_5 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \end{array}\begin{array}{c} \\ T_2 \\ T_3 \\ T_5 \end{array}$$

Element Number e	x_i	x_j	x_k	y_i	y_j	y_k	$c_k = x_j - x_i$	$c_i = x_k - x_j$	$c_j = x_i - x_k$	$b_k = y_i - y_j$	$b_i = y_j - y_k$	$b_j = y_k - y_j$	$A^{(e)} = \frac{1}{2}\left\|(x_{ij}y_{jk}\text{-}x_{jk}y_{ij})\right\|$	
1	0	$\frac{L}{2}$	0	0	0	$\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	0	$-\frac{L}{2}$	$\frac{L}{2}$	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
2	0	$\frac{L}{2}$	$\frac{L}{2}$	$\frac{L}{2}$	0	$\frac{L}{2}$	$\frac{L}{2}$	0	$-\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
3	$\frac{L}{2}$	L	$\frac{L}{2}$	0	0	$\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	0	$-\frac{L}{2}$	$\frac{L}{2}$	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
4	$\frac{L}{2}$	L	L	$\frac{L}{2}$	0	$\frac{L}{2}$	$\frac{L}{2}$	0	$-\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
5	0	$\frac{L}{2}$	0	$\frac{L}{2}$	$\frac{L}{2}$	L	$\frac{L}{2}$	$-\frac{L}{2}$	0	0	$-\frac{L}{2}$	$\frac{L}{2}$	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
6	0	$\frac{L}{2}$	$\frac{L}{2}$	L	$\frac{L}{2}$	L	$\frac{L}{2}$	0	$-\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
7	$\frac{L}{2}$	L	$\frac{L}{2}$	$\frac{L}{2}$	$\frac{L}{2}$	L	$\frac{L}{2}$	$-\frac{L}{2}$	0	0	$-\frac{L}{2}$	$\frac{L}{2}$	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$
8	$\frac{L}{2}$	L	L	L	$\frac{L}{2}$	L	$\frac{L}{2}$	0	$-\frac{L}{2}$	$\frac{L}{2}$	$-\frac{L}{2}$	0	$\frac{1}{2}\left\|\frac{L^2}{4} - 0\right\|$	$= \frac{L^2}{8}$

EXAMPLE 15.2 —cont'd

$$\left[K_1^{(4)}\right]=\frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+0\right) & (0+0) & \left(-\frac{L^2}{4}+0\right)\\ & \left(0+\frac{L^2}{4}\right) & \left(0-\frac{L^2}{4}\right)\\ \text{Symmetric} & & \left(\frac{L^2}{4}+\frac{L^2}{4}\right)\end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{matrix} T_5 & T_3 & T_6\end{matrix}\\ \begin{bmatrix}1&0&-1\\0&1&-1\\-1&-1&2\end{bmatrix}\end{array}\begin{matrix}T_5\\T_3\\T_6\end{matrix}$$

$$\left[K_1^{(5)}\right]=\frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+\frac{L^2}{4}\right) & \left(-\frac{L^2}{4}+0\right) & \left(0-\frac{L^2}{4}\right)\\ & \left(\frac{L^2}{4}+0\right) & (0+0)\\ \text{Symmetric} & & \left(0+\frac{L^2}{4}\right)\end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{matrix} T_4 & T_5 & T_7\end{matrix}\\ \begin{bmatrix}2&-1&-1\\-1&1&0\\-1&0&1\end{bmatrix}\end{array}\begin{matrix}T_4\\T_5\\T_7\end{matrix}$$

$$\left[K_1^{(6)}\right]=\frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+0\right) & (0+0) & \left(-\frac{L^2}{4}+0\right)\\ & \left(0+\frac{L^2}{4}\right) & \left(0-\frac{L^2}{4}\right)\\ \text{Symmetric} & & \left(\frac{L^2}{4}+\frac{L^2}{4}\right)\end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{matrix} T_7 & T_5 & T_8\end{matrix}\\ \begin{bmatrix}1&0&-1\\0&1&-1\\-1&-1&2\end{bmatrix}\end{array}\begin{matrix}T_7\\T_5\\T_8\end{matrix}$$

$$\left[K_1^{(7)}\right]=\frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+\frac{L^2}{4}\right) & \left(-\frac{L^2}{4}+0\right) & \left(0-\frac{L^2}{4}\right)\\ & \left(\frac{L^2}{4}+0\right) & (0+0)\\ \text{Symmetric} & & \left(0+\frac{L^2}{4}\right)\end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{matrix} T_5 & T_6 & T_8\end{matrix}\\ \begin{bmatrix}2&-1&-1\\-1&1&0\\-1&0&1\end{bmatrix}\end{array}\begin{matrix}T_5\\T_6\\T_8\end{matrix}$$

$$\left[K_1^{8}\right]=\frac{2k}{L^2}\begin{bmatrix} \left(\frac{L^2}{4}+0\right) & (0+0) & \left(-\frac{L^2}{4}+0\right)\\ & \left(0+\frac{L^2}{4}\right) & \left(0-\frac{L^2}{4}\right)\\ \text{Symmetric} & & \left(\frac{L^2}{4}+\frac{L^2}{4}\right)\end{bmatrix} = \frac{k}{2}\begin{array}{c} \begin{matrix} T_8 & T_6 & T_9\end{matrix}\\ \begin{bmatrix}1&0&-1\\0&1&-1\\-1&-1&2\end{bmatrix}\end{array}\begin{matrix}T_8\\T_6\\T_9\end{matrix}$$

b. $\left[K_2^{(e)}\right]$ matrices (Eq. 15.16) and $\vec{P}_3^{(e)}$ vectors (Eq. 15.20):
Because no convective boundary condition is specified in the problem, we have

$$\left[K_2^{(e)}\right] = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \quad \vec{P}_3^{(e)} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad \text{for} \quad e = 1, 2, \ldots, 8$$

EXAMPLE 15.2 —cont'd

c. $\overrightarrow{P}_1^{(e)}$ vectors (Eq. 15.17):

Since $A^{(e)}$ is the same for $e = 1-8$, we obtain

$$\overrightarrow{P}_1^{(e)} = \frac{\dot{q}_0 L^2}{24}\begin{Bmatrix}1\\1\\1\end{Bmatrix}, \quad e = 1,2,\ldots,8$$

d. $\overrightarrow{P}_2^{(e)}$ vectors (Eq. 15.19):

Since no boundary heat flux is specified in the problem, we have

$$\overrightarrow{P}_2^{(e)} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}, \quad e = 1,2,\ldots,8$$

Step 4: The element matrices and vectors derived in step 3 are assembled to obtain the overall system matrices and vectors as follows:

$$\left[\underset{\sim}{K}\right] = \left[\underset{\sim}{K}_1\right] = \sum_{e=1}^{8}\left[K_1^{(e)}\right] = \frac{k}{2}$$

$$\times\begin{array}{c}\begin{array}{ccccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 \end{array}\\ \left[\begin{array}{ccccccccc}
2 & -1 & & -1 & & & & & \\
-1 & 1+1+2 & -1 & 0+0 & -1-1 & & & & \\
 & -1 & 1+1 & & 0+0 & -1 & & & \\
-1 & 0+0 & & 1+1+2 & -1-1 & & -1 & & \\
 & -1-1 & 0+0 & -1-1 & 2+1+1+1+1+2 & -1-1 & 0+0 & -1-1 & \\
 & & -1 & & -1-1 & 2+1+1 & & 0+0 & -1 \\
 & & & -1 & 0+0 & & 1+1 & -1 & \\
 & & & & -1-1 & 0+0 & -1 & 2+1+1 & -1 \\
 & & & & & -1 & & -1 & 2
\end{array}\right]\end{array}\begin{array}{c} \\ T_1\\T_2\\T_3\\T_4\\T_5\\T_6\\T_7\\T_8\\T_9\end{array}$$

$$= \frac{k}{2}\begin{bmatrix}
2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
-1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0\\
-1 & 0 & 0 & 4 & -2 & 0 & -1 & 0 & 0\\
0 & -2 & 0 & -2 & 8 & -2 & 0 & -2 & 0\\
0 & 0 & -1 & 0 & -2 & 4 & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0\\
0 & 0 & 0 & 0 & -2 & 0 & -1 & 4 & -1\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2
\end{bmatrix} \tag{E.1}$$

Continued

EXAMPLE 15.2 —cont'd

$$\vec{\underset{\sim}{P}} = \vec{\underset{\sim}{P}}_1 = \sum_{e=1}^{8} \vec{P}_1^{(e)} = \frac{\dot{q}_o L^2}{24} \begin{Bmatrix} 1 \\ 1+1+1 \\ 1+1 \\ 1+1+1 \\ 1+1+1+1+1+1 \\ 1+1+1 \\ 1+1 \\ 1+1+1 \\ 1 \end{Bmatrix} \begin{matrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{matrix} = \frac{\dot{q}_o L^2}{24} \begin{Bmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 6 \\ 3 \\ 2 \\ 3 \\ 1 \end{Bmatrix} \tag{E.2}$$

Thus, the overall system equations are given by Eq. (15.21), where $[\underset{\sim}{K}]$ and $\vec{\underset{\sim}{P}}$ are given by Eqs. (E.1) and (E.2), and

$$\vec{\underset{\sim}{T}} = \{T_1 \;\; T_2 \ldots \;\; T_9\}^T \tag{E.3}$$

Step 5: The boundary conditions to be incorporated are $T_3 = T_6 = T_7 = T_8 = T_9 = T_\infty$. The following procedure can be adopted to incorporate these boundary conditions in Eq. (15.21) without destroying the symmetry of the matrix. To incorporate the condition $T_3 = T_\infty$, for example, transfer all the off-diagonal elements of the third column (that get multiplied by T_3) to the right-hand side of the equation. These elements are then set equal to zero on the left-hand side. Then, in the third row of $[\underset{\sim}{K}]$, the off-diagonal elements are set equal to 0 and the diagonal element is set equal to 1. Replace the third component of the new right-hand side by T_∞(value of T_3). Thus, after the incorporation of the boundary condition $T_3 = T_\infty$, Eq. (15.21) will appear as follows:

$$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -2 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 8 & -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \frac{\dot{q}_o L^2}{12k} \begin{Bmatrix} 1 \\ 3 \\ 0 \\ 3 \\ 6 \\ 3 \\ 2 \\ 3 \\ 1 \end{Bmatrix} - T_\infty \begin{Bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.4}$$

It can be observed that the third equation of (E.4) is now decoupled from the remaining equations and has the desired solution $T_3 = T_\infty$ as specified by the boundary condition. After incorporating the remaining boundary conditions, namely $T_6 = T_7 = T_8 = T_9 = T_\infty$, the final equations will appear as follows:

$$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \frac{\dot{q}_o L^2}{12k} \begin{Bmatrix} 1 \\ 3 \\ 0 \\ 3 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + T_\infty \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 4 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.5}$$

The solution of Eq. (E.5) gives the following result:

$$T_1 = 133.3°\text{C}, \quad T_2 = 119.4°\text{C}, \quad T_3 = 50.0°\text{C}, \quad T_4 = 119.4°\text{C}, \quad T_5 = 105.6°\text{C},$$
$$T_6 = 50.0°\text{C}, \quad T_7 = 50.0°\text{C}, \quad T_8 = 50.0°\text{C}, \quad T_9 = 50.0°\text{C}$$

15.3 UNSTEADY STATE PROBLEMS

The finite element equations governing the unsteady state problem are given by Eq. (13.35). It can be seen that the term $\left[\underset{\sim}{K}_3\right] \underset{\sim}{\dot{\vec{T}}}$ represents the unsteady state part. The element matrix $\left[K_3^{(e)}\right]$ can be evaluated using the definition given in Eq. (13.48). Since the shape function matrix used for the triangular element (in terms of natural coordinates) is

$$[N(x,y)] = [L_1 \quad L_2 \quad L_3] \tag{15.24}$$

for unit thickness of the element, we obtain [15.1–15.3]:

$$\begin{aligned}\left[K_3^{(e)}\right] &= (\rho c)^{(e)} \iint_{A^{(e)}} \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_1L_2 & L_2^2 & L_2L_3 \\ L_1L_3 & L_2L_3 & L_3^2 \end{bmatrix} \cdot \mathrm{d}A \\ &= \frac{(\rho c)^{(e)} A^{(e)}}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\end{aligned} \tag{15.25}$$

REVIEW QUESTIONS

15.1 Give brief answers to the following questions.

1. State the governing differential equation for a two-dimensional steady-state heat transfer problem for an isotropic conducting material.
2. Indicate the nature of boundary conditions for a two-dimensional steady-state heat transfer problem.

15.2 Fill in the blank with a suitable word.

1. The temperature variation is assumed to be ____________________ in a triangular element with three nodes.

15.3 Indicate whether the following statement is true or false.

1. The unsteady state contributes the matrices $\left[K_3^{(e)}\right]$ in a two-dimensional heat transfer problem.
2. The triangular coordinates and area coordinates are different for a triangular element.

15.4 Select the most appropriate answer from the multiple choices given.

1. For a triangular element with nodes i, j, and k, the following area coordinate is zero along the edge ij:
   (a) L_1 (b) L_2 (c) L_3

PROBLEMS

15.1 Find the temperature distribution in the square plate shown in Fig. 15.4.

15.2 If convection takes place from the triangular faces rather than the edges for the element ijk shown in Fig. 15.5, evaluate the surface integrals that contribute to the matrix $[K^{(e)}]$ and the vector $\vec{P}^{(e)}$.

15.3 The temperature distribution in an isotropic plate of thickness t is given by the equation

$$\frac{\partial}{\partial x}\left[kt\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[kt\frac{\partial T}{\partial y}\right] + \dot{q} = 0 \tag{P.1}$$

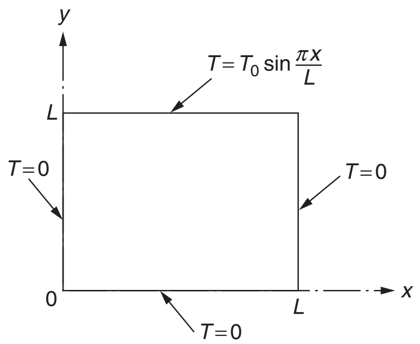


FIGURE 15.4 A square plate.

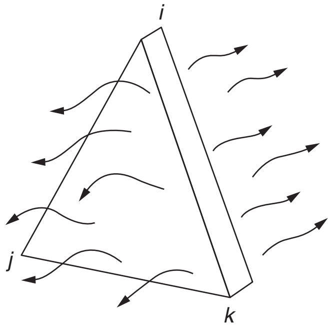


FIGURE 15.5 A triangular plate.

with the following boundary conditions (including radiation heat transfer):

$$T = T_0(x, y) \text{ on } S_1 \tag{P.2}$$

$$k\frac{\partial T}{\partial x}l_x + k\frac{\partial T}{\partial y}l_y + q = 0 \text{ on } S_2 \tag{P.3}$$

$$k\frac{\partial T}{\partial x}l_x + k\frac{\partial T}{\partial y}l_y + h(T - T_\infty) = 0 \text{ on } S_3 \tag{P.4}$$

$$k\frac{\partial T}{\partial x}l_x + k\frac{\partial T}{\partial y}l_y + \varepsilon\ \sigma\left(T^4 - T_\infty^4\right) = 0 \text{ on } S_4 \tag{P.5}$$

where S_4 denotes the surface from which radiation heat transfer takes place. Derive the variational functional I corresponding to Eqs. (P.1) to (P.5).

15.4 Derive the finite element equations corresponding to Eqs. (P.1) to (P.5) of Problem 15.3 using the Galerkin method.

15.5 Evaluate the integrals in Eq. (15.13) and derive the matrix $\left[K_2^{(e)}\right]$ assuming that convection takes place along the edge jk of element e.

15.6 Evaluate the integrals in Eq. (15.13) and derive the matrix $\left[K_2^{(e)}\right]$ assuming that convection takes place along the edge ki of element e.

15.7 If heat flux and convection heat transfer take place from the edge jk of element e, derive the corresponding vectors $\overrightarrow{P}_2^{(e)}$ and $\overrightarrow{P}_3^{(e)}$.

15.8 If heat flux and convection heat transfer take place from the edge ki of element e, derive the corresponding vectors $\overrightarrow{P}_2^{(e)}$ and $\overrightarrow{P}_3^{(e)}$.

15.9 Explain why the element matrices resulting from conduction and boundary convection, $\left[K_1^{(e)}\right]$ and $\left[K_2^{(e)}\right]$, are always symmetric.

15.10 Evaluate the conduction matrix, $\left[K_1^{(e)}\right]$, for an isotropic rectangular element with four nodes. Use linear temperature variation in x and y directions.

15.11 A three-noded triangular plate element from a finite element grid is shown in Fig. 15.6. The element has a thickness of 0.2 in and is made of aluminum with $k = 115$ BTU/hr-ft °F. Convection heat transfer takes place from all three edges and the two triangular faces of the element to an ambient temperature of 70 °F with a convection coefficient of 100 BTU/hr-ft^2 °F. Determine the characteristic matrices $\left[K_1^{(e)}\right]$ and $\left[K_2^{(e)}\right]$ of the element.

15.12 If an internal heat source of $\dot{q} = 1000$ BTU/hr-ft^3 is present at the centroid and a heat flux of 50 BTU/hr-ft^2 is imposed on each of the three faces of the triangular element considered in Problem 15.11, determine the characteristic vectors $\vec{P}_1^{(e)}$, $\vec{P}_2^{(e)}$, and $\vec{P}_3^{(e)}$ of the element.

15.13 Consider the trapezoidal plate discretized into four elements and five nodes as shown in Fig. 15.7. If $\left[K_{ij}^{(e)}\right] \equiv \left[K_1^{(e)}\right]$ denotes the characteristic (conduction) matrix of element e ($e = 1, 2, 3, 4$), express the global (assembled) characteristic matrix. Can the bandwidth of the global matrix be reduced by renumbering the nodes? If so, give the details.

15.14 Consider a rectangular element of sides a and b and thickness t idealized as two triangular elements and one rectangular element as shown in Figs. 15.8A and 15.8B, respectively.

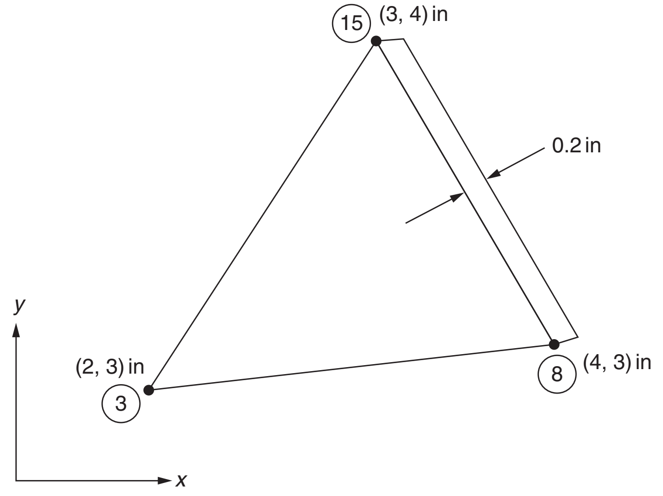


FIGURE 15.6 Triangular element with convection from all edges.

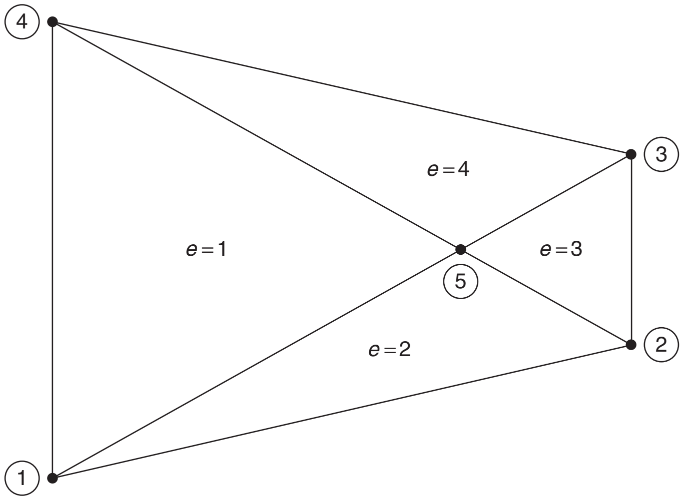


FIGURE 15.7 Trapezoidal plate.

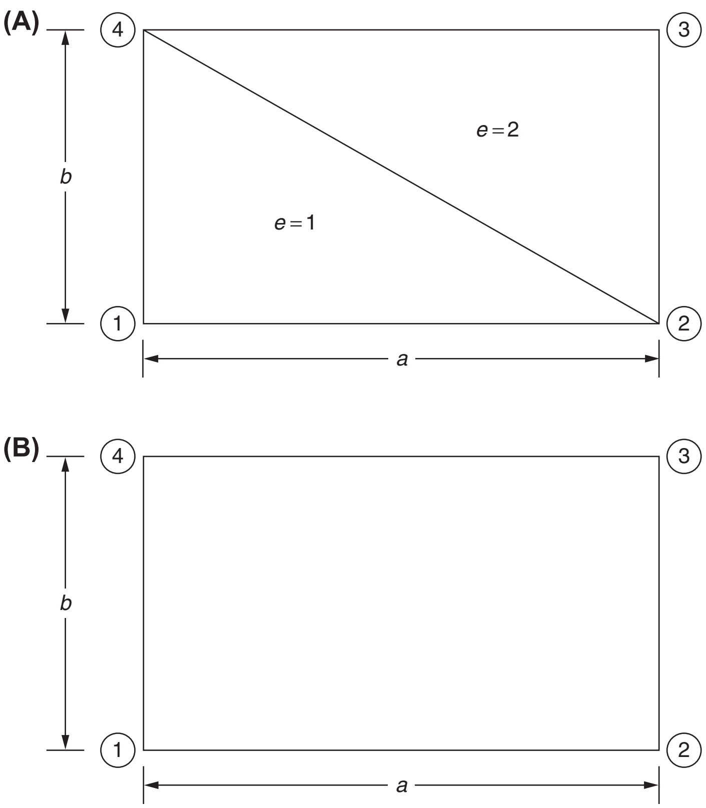


FIGURE 15.8 Rectangular element with two different idealizations.

a. Derive the assembled characteristic (conduction) matrix, $[K_1]$, for the rectangle.
b. Compare the result of (a) with the characteristic (conduction) matrix of a rectangular element given by

$$[K_1]_{rect} = \frac{ktb}{4a}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

15.15 The (X, Y) coordinates of the nodes of a triangular element of thickness 0.2 cm are shown in Fig. 15.9. Convection takes place from all three edges of the element. If $\dot{q} = 200$ W/cm^3, $k = 100$ W/m °C, $h = 150$ W/cm^2 °C, and $T_\infty = 30$ °C, determine the following:

a. Element matrices $\left[K_1^{(e)}\right]$ and $\left[K_2^{(e)}\right]$.

b. Element vectors $\overrightarrow{P}_1^{(e)}$ and $\overrightarrow{P}_3^{(e)}$.

15.16 Derive the element matrices and vectors for the element shown in Fig. 15.10 for the following data: $k_x = 40$ W/°C-cm, $k_y = 60$ W/°C-cm, $h = 5$ W/cm^2 °C, $\dot{q} = 5$ W/cm^3, and $T_\infty = 10$ °C. Assume $t = 1$ cm.

15.17 Derive the element matrices and vectors for the element shown in Fig. 15.11 for the following data: $k_x = 40$ W/°C-cm, $k_y = 60$ W/°C-cm, $h = 5$ W/cm^2 °C, $\dot{q} = 5$ W/cm^3, and $T_\infty = 10$ °C. Assume $t = 1$ cm.

15.18 Derive the element matrices and vectors for the element shown in Fig. 15.12 for the following data: $k_x = 40$ W/°C-cm, $k_y = 60$ W/°C-cm, $h = 5$ W/cm^2 °C, $\dot{q} = 5$ W/cm^3, and $T_\infty = 10$ °C. Assume $t = 1$ cm.

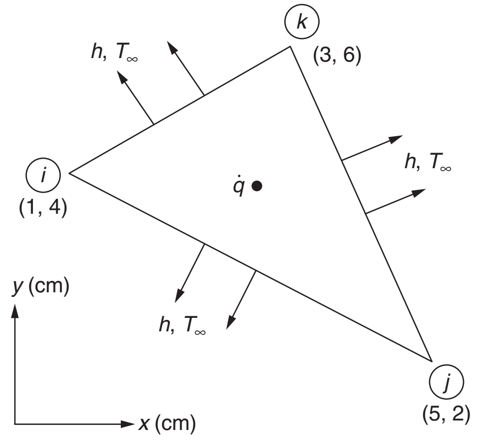


FIGURE 15.9 Triangular element with heat generation and convection.

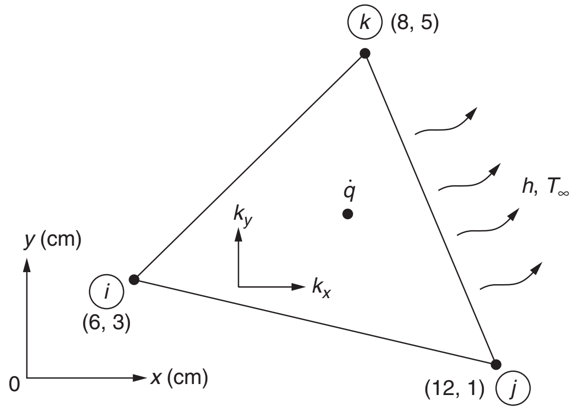


FIGURE 15.10 Triangular element with convection along edge *jk*.

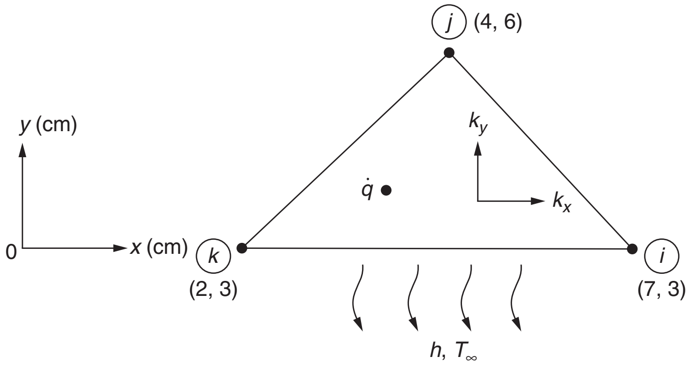


FIGURE 15.11 Triangular element with convection along edge *ki*.

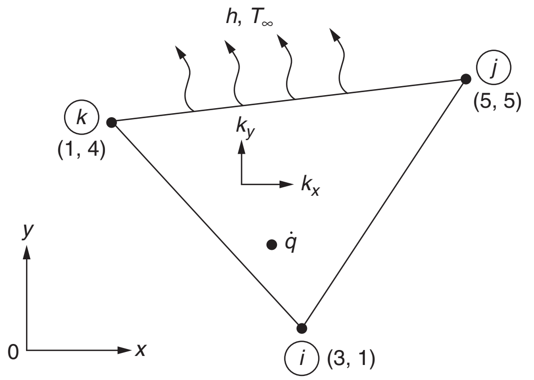


FIGURE 15.12 Triangular element with convection along edge *jk*.

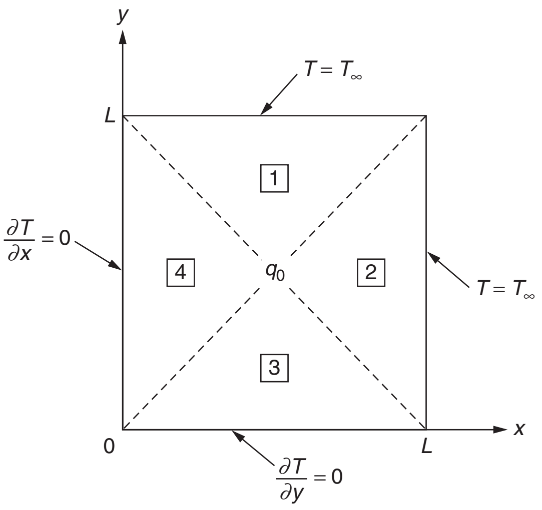


FIGURE 15.13 Idealization of a square region.

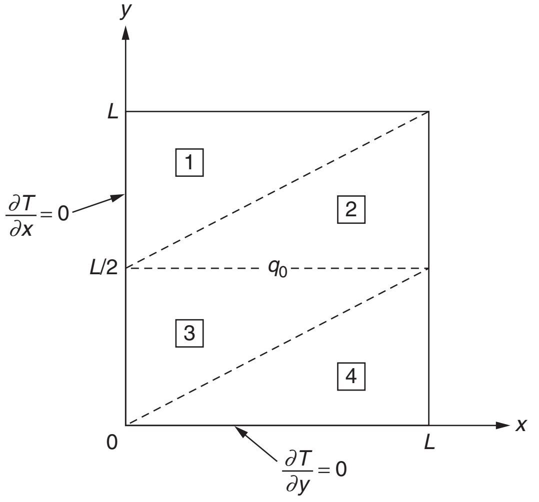


FIGURE 15.14 Square region using a different idealization.

15.19 Find the temperature distribution in a square region with uniform energy generation as shown in Fig. 15.13. Assume that there is no temperature variation in the z direction. Take $k = 30$ W/cm °C, $L = 10$ cm, $T_\infty = 50$ °C, and $\dot{q} = \dot{q}_0 = 5\ \mathrm{W/cm^3}$. Use four triangular elements as indicated by the dotted lines in Fig. 15.13. Compare the nodal temperatures with those found in Example 15.2.

15.20 Solve Problem 15.19 using the four element idealization indicated by the dotted lines in Fig. 15.14.

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Chapter 16

Three-Dimensional Problems

Chapter Outline

16.1 INTRODUCTION

Equations governing heat transfer in three-dimensional bodies were given in Section 13.3. Certain types of three-dimensional problems, namely axisymmetric ones, can be modeled and solved using ring elements. The solution of axisymmetric problems using triangular ring elements and three-dimensional problems using tetrahedron elements is considered in this chapter. For simplicity, linear interpolation functions, in terms of natural coordinates, are used in the analysis.

16.2 AXISYMMETRIC PROBLEMS

The differential equation of heat conduction for an axisymmetric case, in cylindrical coordinates, is given by (see Eq. 13.16):

$$\frac{\partial}{\partial r}\left(rk_r\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(rk_z\frac{\partial T}{\partial z}\right)+r\dot{q}=0 \tag{16.1}$$

The boundary conditions associated with the problem are

1. $T = T_0(r,z)$ on S_1 (temperature specified on surface S_1) (16.2)

2. $\dfrac{\partial T}{\partial n} =$ on S_2 (insulated boundary condition on surface S_2) (16.3)

3. $k_r r\dfrac{\partial T}{\partial r}l_r + k_z r\dfrac{\partial T}{\partial z}l_z + rh(T-T_\infty) = 0$ on S_3 (convective boundary condition on surface S_3) (16.4)

4. $k_r r\dfrac{\partial T}{\partial r}l_r + k_z r\dfrac{\partial T}{\partial z}l_z + rq = 0$ on S_4 (heat flux input specified on surface S_4) (16.5)

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00016-X

Here, k_r and k_z indicate the thermal conductivities of the solid in r and z directions, n represents the normal direction to the surface, l_r and l_z denote the direction cosines of the outward drawn normal (n), and $h\,(T - T_\infty)$ is the surface heat flow owing to convection.

The problem defined by Eqs. (16.1) − (16.5) can be stated in variational form as follows: Find the temperature distribution $T(r, z)$ that minimizes the functional:

$$I = \frac{1}{2}\iiint_V \left[k_r r\left(\frac{\partial T}{\partial r}\right)^2 + k_z r\left(\frac{\partial T}{\partial z}\right)^2 - 2\dot{q}rT\right]\mathrm{d}V + \frac{1}{2}\iint_{S_3} hr\left(T^2 - 2T_\infty T\right)\mathrm{d}S_3 + \iint_{S_4} rqT\,\mathrm{d}S_4 \tag{16.6}$$

and satisfies the boundary conditions specified by Eqs. (16.2) and (16.3). The finite element solution of the problem is given by the following steps.

Step 1: Replace the solid body of revolution by an assembly of triangular ring elements as shown in Fig. 16.1.
Step 2: We use a natural coordinate system and assume linear variation of temperature inside an element e so that the temperature $T^{(e)}$ can be expressed as:

$$T^{(e)} = [N]\overrightarrow{q}^{(e)} \tag{16.7}$$

where

$$[N] = [N_i \quad N_j \quad N_k] \equiv [L_1 \quad L_2 \quad L_3] \tag{16.8}$$

and

$$\overrightarrow{q}^{(e)} = \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} \tag{16.9}$$

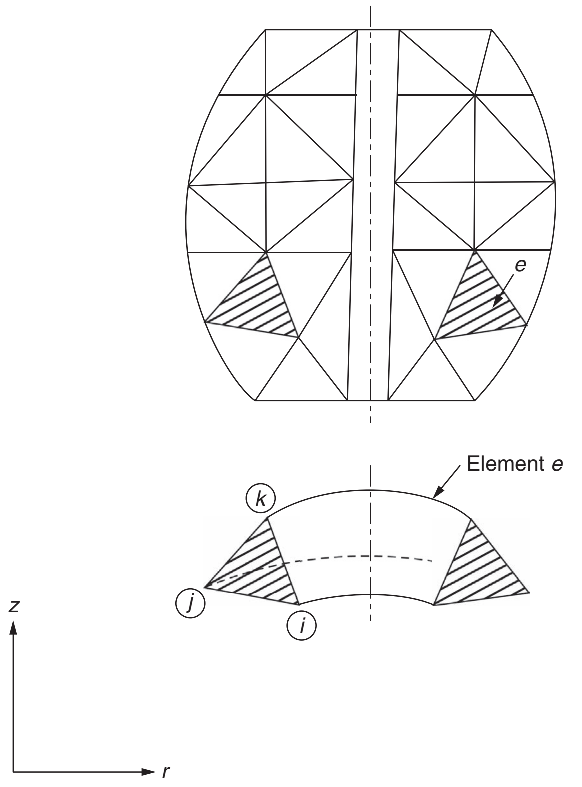

FIGURE 16.1 Idealization of an axisymmetric body with triangular ring elements.

The natural coordinates L_1, L_2, and L_3 are related to the global cylindrical coordinates (r, z) of nodes i, j, and k as:

$$\begin{Bmatrix} 1 \\ r \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ r_i & r_j & r_k \\ z_i & z_j & z_k \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \tag{16.10}$$

or, equivalently,

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A^{(e)}} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} 1 \\ r \\ z \end{Bmatrix} \tag{16.11}$$

where

$$\begin{aligned} a_1 &= r_j z_k - r_k z_j \\ a_2 &= r_k z_i - r_i z_k \\ a_3 &= r_i z_j - r_j z_i \\ b_1 &= z_j - z_k \\ b_2 &= z_k - z_i \\ b_3 &= z_i - z_j \\ c_1 &= r_k - r_j \\ c_2 &= r_i - r_k \\ c_3 &= r_j - r_i \end{aligned} \tag{16.12}$$

and $A^{(e)}$ is the area of triangle ijk given by:

$$A^{(e)} = \frac{1}{2}\left[r_i(z_j - z_k) + r_j(z_k - z_i) + r_k(z_i - z_j)\right] \tag{16.13}$$

Step 3: The element matrices and vectors can be derived using Eqs. (13.46) − (13.49) as follows: Noting that:

$$[D] = \begin{bmatrix} rk_r & 0 \\ 0 & rk_z \end{bmatrix} \tag{16.14}$$

and

$$[B] = \begin{bmatrix} \dfrac{\partial N_i}{\partial r} & \dfrac{\partial N_j}{\partial r} & \dfrac{\partial N_k}{\partial r} \\ \dfrac{\partial N_i}{\partial z} & \dfrac{\partial N_j}{\partial z} & \dfrac{\partial N_k}{\partial z} \end{bmatrix} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{16.15}$$

and by writing $\mathrm{d}V^{(e)}$ as $2\pi r \cdot \mathrm{d}A$, where $\mathrm{d}A$ is the differential area of the triangle ijk, Eq. (13.46) gives:

$$\begin{aligned} \left[K_1^{(e)}\right] &= 2\pi \iint_{A^{(e)}} r[B]^T[D][B]\mathrm{d}A \\ &= \frac{2\pi k_r}{4A^{(e)2}} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} \iint_{A^{(e)}} r^2 \mathrm{d}A + \frac{2\pi k_z}{4A^{(e)2}} \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^2 & c_2 c_3 \\ c_1 c_3 & c_2 c_3 & c_3^2 \end{bmatrix} \iint_{A^{(e)}} r^2 \mathrm{d}A \end{aligned} \tag{16.16}$$

The radial distance r can be expressed in terms of the natural coordinates L_1, L_2, and L_3 as:

$$r = r_i L_1 + r_j L_2 + r_k L_3 \tag{16.17}$$

Thus, the integral term in Eq. (16.16) can be expressed as:

$$\overline{R}^2 \equiv \iint_{A^{(e)}} r^2 \mathrm{d}A = \iint_{A^{(e)}} (r_i \; r_j \; r_k) \begin{bmatrix} L_1^2 & L_1 L_2 & L_1 L_3 \\ L_1 L_2 & L_2^2 & L_2 L_3 \\ L_1 L_3 & L_2 L_3 & L_3^2 \end{bmatrix} \begin{Bmatrix} r_i \\ r_j \\ r_k \end{Bmatrix} \mathrm{d}A \tag{16.18}$$

By using the integration formula for natural coordinates, Eq. (3.78), Eq. (16.18) can be written as:

$$\overline{R}^2 = \iint_{A^{(e)}} r^2 \mathrm{d}A = \frac{1}{12} (r_i \; r_j \; r_k) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} r_i \\ r_j \\ r_k \end{Bmatrix} \tag{16.19}$$

and hence:

$$\left[K_1^{(e)}\right] = \frac{\pi k_r \overline{R}^2}{2A^{(e)}} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} + \frac{\pi k_z \overline{R}^2}{2A^{(e)}} \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_1 c_2 & c_2^2 & c_2 c_3 \\ c_1 c_3 & c_2 c_3 & c_3^2 \end{bmatrix} \tag{16.20}$$

For isotropic materials with $k_r = k_z = k$, Eq. (16.20) becomes:

$$\left[K_1^{(e)}\right] = \frac{\pi k \overline{R}^2}{2A^{(e)}} \begin{bmatrix} (b_1^2 + c_1^2) & (b_1 b_2 + c_1 c_2) & (b_1 b_3 + c_1 c_3) \\ (b_1 b_2 + c_1 c_2) & (b_2^2 + c_2^2) & (b_2 b_3 + c_2 c_3) \\ (b_1 b_3 + c_1 c_3) & (b_2 b_3 + c_2 c_3) & (b_3^2 + c_3^2) \end{bmatrix} \tag{16.21}$$

To evaluate the surface integral of Eq. (13.47), we assume that the edge ij lies on the surface S_3 from which heat convection takes place. Along this edge, $L_3 = 0$ and $\mathrm{d}S_3 = 2\pi r \, \mathrm{d}s$, so that Eq. (13.47) gives:

$$\left[K_2^{(e)}\right] = 2\pi h \int_{s=s_i}^{s_j} \begin{Bmatrix} L_1 \\ L_2 \\ 0 \end{Bmatrix} \{ L_1 \quad L_2 \quad 0 \} r \mathrm{d}s = 2\pi h \int_{s=s_i}^{s_j} \begin{bmatrix} r L_1^2 & r L_1 L_2 & 0 \\ r L_1 L_2 & r L_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{d}s \tag{16.22}$$

By substituting Eq. (16.17) for r and by using the relation:

$$\int_{s=s_i}^{s_j} L_1^p L_2^q \mathrm{d}s = s_{ji} \frac{p!q!}{(p+q+1)!} \tag{16.23}$$

where $s_{ji} = s_j - s_i =$ length of the edge ij, Eq. (16.22) gives:

$$\left[K_2^{(e)}\right] = \frac{\pi h s_{ji}}{6} \begin{bmatrix} (3r_i + r_j) & (r_i + r_j) & 0 \\ (r_i + r_j) & (r_i + 3r_j) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{16.24}$$

To evaluate the volume integral for $\vec{P}_1^{(e)}$ as:

$$\vec{P}_1^{(e)} = \iiint_{V^{(e)}} r\dot{q}[N]^T \mathrm{d}V \tag{16.25}$$

we use the approximation $r\dot{q} = r_c\dot{q} =$ constant, where $r_c = (r_i + r_j + r_k)/3$, and the relation $\mathrm{d}V = 2\pi r \cdot \mathrm{d}A$ to obtain:

$$\vec{P}_1^{(e)} = 2\pi r_c \dot{q} \iint_{A^{(e)}} \begin{Bmatrix} rL_1 \\ rL_2 \\ rL_3 \end{Bmatrix} \mathrm{d}A \tag{16.26}$$

With the help of Eq. (16.17), Eq. (16.26) can be evaluated to obtain:

$$\vec{P}_1^{(e)} = \frac{\pi r_c \dot{q} A^{(e)}}{6} \begin{Bmatrix} (2r_i + r_j + r_k) \\ (r_i + 2r_j + r_k) \\ (r_i + r_j + 2r_k) \end{Bmatrix} \tag{16.27}$$

The surface integral involved in the definition of $\vec{P}_3^{(e)}$ can be evaluated as in the case of Eq. (16.24). Thus, if the edge ij lies on the surface S_3,

$$\vec{P}_3^{(e)} = \iint_{S_3^{(e)}} hT_\infty [N]^T \mathrm{d}S_3 = 2\pi h T_\infty \int_{s=s_i}^{s_j} \begin{Bmatrix} rL_1 \\ rL_2 \\ 0 \end{Bmatrix} \mathrm{d}s = \frac{\pi h T_\infty s_{ji}}{3} \begin{Bmatrix} (2r_i + r_j) \\ (r_i + 2r_j) \\ 0 \end{Bmatrix} \tag{16.28}$$

Similarly, expressions for $\vec{P}_2^{(e)}$ can be obtained as:

$$\vec{P}_2^{(e)} = \iint_{S_2^{(e)}} q[N]^T \mathrm{d}S_2 = \frac{\pi q s_{ji}}{3} \begin{Bmatrix} (2r_i + r_j) \\ (r_i + 2r_j) \\ 0 \end{Bmatrix} \quad \text{if the edge } ij \text{ lies on } S_2 \tag{16.29}$$

Step 4: Once the element matrices and vectors are available, the overall or system equations can be derived as:

$$[\underset{\sim}{K}] \underset{\sim}{\vec{T}} = \underset{\sim}{\vec{P}} \tag{16.30}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E} \left[\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right] \right] \tag{16.31}$$

and

$$\underset{\sim}{\vec{P}} = \sum_{e=1}^{E} \left(\vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)} \right) \tag{16.32}$$

Step 5: The solution of the problem can be obtained by solving Eq. (16.30) after incorporating the known boundary conditions.

EXAMPLE 16.1
Derive the element matrices and vectors for the element shown in Fig. 16.2.

Solution
From the data shown in Fig. 16.2, the required element properties can be computed as:

$$\begin{aligned} b_1 &= z_j - z_k = 2 - 6 = -4 \\ b_2 &= z_k - z_i = 6 - 2 = 4 \\ b_3 &= z_i - z_j = 2 - 2 = 0 \\ c_1 &= r_k - r_j = 7 - 7 = 0 \\ c_2 &= r_i - r_k = 4 - 7 = -3 \\ c_3 &= r_j - r_i = 7 - 4 = 3 \end{aligned}$$

$$A^{(e)} = \frac{1}{2}[4(2-6) + 7(6-2) + 7(2-2)] = 6$$

$$\bar{R}^2 = \frac{1}{12}(4 \quad 7 \quad 7)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{Bmatrix} 4 \\ 7 \\ 7 \end{Bmatrix} = \frac{438}{12} = 36.5$$

$$r_c = (r_i + r_j + r_k)/3 = (4 + 7 + 7)/3 = 6$$

$$s_{kj} = \left[(r_k - r_j)^2 + (z_k - z_j)^2\right]^{1/2} = \left[(7-7)^2 + (6-2)^2\right]^{1/2} = 4$$

$$s_{ji} = \left[(r_j - r_i)^2 + (z_j - z_i)^2\right]^{1/2} = \left[(7-4)^2 + (2-2)^2\right]^{1/2} = 3$$

From Eq. (16.21), $\left[K_1^{(e)}\right]$ can be obtained as:

$$\left[K_1^{(e)}\right] = \begin{bmatrix} 9175 & -9175 & 0 \\ -9175 & 14330 & -5160 \\ 0 & -5160 & 5160 \end{bmatrix}$$

Because convection occurs along the two edges *ij* and *jk*, the $\left[K_2^{(e)}\right]$ matrix can be written as:

$$\left[K_2^{(e)}\right] = \frac{\pi(h)_{ij}s_{ji}}{6}\begin{bmatrix} (3r_i + r_j) & (r_i + r_j) & 0 \\ (r_i + r_j) & (r_i + 3r_j) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\pi(h)_{jk}s_{kj}}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & (3r_j + r_k) & (r_j + r_k) \\ 0 & (r_j + r_k) & (r_j + 3r_k) \end{bmatrix}$$

$$= \frac{\pi(15)(3)}{6}\begin{bmatrix} (12+7) & (4+7) & 0 \\ (4+7) & (4+21) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\pi(10)(4)}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & (21+7) & (7+7) \\ 0 & (7+7) & (7+21) \end{bmatrix}$$

$$= \begin{bmatrix} 447.7 & 259.2 & 0 \\ 259.2 & 1176.0 & 293.2 \\ 0 & 293.2 & 586.4 \end{bmatrix}$$

Eq. (16.27) gives:

$$\vec{P}_1^{(e)} = \frac{\pi(60)(50)(6)}{6}\begin{Bmatrix} (8+7+7) \\ (4+14+7) \\ (4+7+14) \end{Bmatrix} = \begin{Bmatrix} 20734.5 \\ 23561.9 \\ 23561.9 \end{Bmatrix}$$

EXAMPLE 16.1 —cont'd

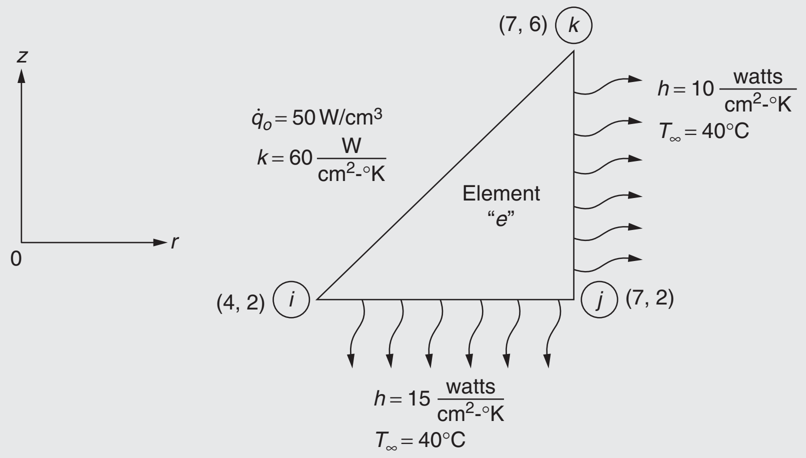

FIGURE 16.2 Axisymmetric Triangular Element.

Because no boundary heat flux is specified, $\vec{P}_2^{(e)} = \vec{0}$. From Eq. (16.28) and a similar equation for the edge jk, we obtain:

$$\vec{P}_3^{(e)} = \frac{\pi (hT_\infty)_{ij} s_{ji}}{3} \begin{Bmatrix} (2r_i + r_j) \\ (r_i + 2r_j) \\ 0 \end{Bmatrix} + \frac{\pi (hT_\infty)_{jk} s_{kj}}{3} \begin{Bmatrix} 0 \\ (2r_j + r_k) \\ (r_j + 2r_k) \end{Bmatrix}$$

$$= \frac{\pi(15)(40)(3)}{3} \begin{Bmatrix} (8+7) \\ (4+14) \\ 0 \end{Bmatrix} + \frac{\pi(10)(40)(4)}{3} \begin{Bmatrix} 0 \\ (14+7) \\ (7+14) \end{Bmatrix}$$

$$= \begin{Bmatrix} 28,274.3 \\ 69,115.0 \\ 35,185.8 \end{Bmatrix}$$

Thus,

$$\left[K^{(e)}\right] = \left[K_1^{(e)}\right] + \left[K_2^{(e)}\right] = \begin{bmatrix} 9622.7 & -8915.8 & 0 \\ -8915.8 & 15,506.0 & -4866.8 \\ 0 & -4866.8 & 5746.4 \end{bmatrix}$$

and

$$\vec{P}^{(e)} = \vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)} = \begin{Bmatrix} 49,008.8 \\ 92,676.9 \\ 58,747.7 \end{Bmatrix}$$

16.3 THREE-DIMENSIONAL HEAT TRANSFER PROBLEMS

The governing differential equation for steady-state heat conduction in a solid body is given by Eq. (13.11) with the right hand−side term as zero and the boundary conditions as in Eqs. (13.18)−(13.20). The finite element solution of these equations can be obtained by using the following procedure.

Step 1: Divide the solid body into E tetrahedron elements.

Step 2: We use a natural coordinate system and assume linear variation of temperature inside an element e so that temperature $T^{(e)}$ can be expressed as:

$$T^{(e)} = [N]\vec{q}^{(e)} \tag{16.33}$$

where

$$[N] = [N_i \quad N_j \quad N_k \quad N_l] \equiv [L_1 \quad L_2 \quad L_3 \quad L_4] \tag{16.34}$$

and

$$\vec{q}^{(e)} = \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_l \end{Bmatrix} \tag{16.35}$$

The natural coordinates L_1, L_2, L_3, and L_4 are related to the global Cartesian coordinates of nodes i, j, k, and l by Eq. (3.84).

Step 3: The element matrices and vectors can be derived using Eqs. (13.46)−(13.49) as:

$$[D] = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \tag{16.36}$$

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_l}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_l}{\partial y} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_l}{\partial z} \end{bmatrix} = \frac{1}{6V^{(e)}} \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} \tag{16.37}$$

$$\begin{aligned} \left[K_1^{(e)}\right] &= \iiint_{V^{(e)}} [B]^T [D][B]\mathrm{d}V = \frac{k_x}{36V^{(e)}} \begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 & b_1b_4 \\ b_1b_2 & b_2^2 & b_2b_3 & b_2b_4 \\ b_1b_3 & b_2b_3 & b_3^2 & b_3b_4 \\ b_1b_4 & b_2b_4 & b_3b_4 & b_4^2 \end{bmatrix} \\ &+ \frac{k_y}{36V^{(e)}} \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 & c_1c_4 \\ c_1c_2 & c_2^2 & c_2c_3 & c_2c_4 \\ c_1c_3 & c_2c_3 & c_3^2 & c_3c_4 \\ c_1c_4 & c_2c_4 & c_3c_4 & c_4^2 \end{bmatrix} + \frac{k_z}{36V^{(e)}} \begin{bmatrix} d_1^2 & d_1d_2 & d_1d_3 & d_1d_4 \\ d_1d_2 & d_2^2 & d_2d_3 & d_2d_4 \\ d_1d_3 & d_2d_3 & d_3^2 & d_3d_4 \\ d_1d_4 & d_2d_4 & d_3d_4 & d_4^2 \end{bmatrix} \end{aligned} \tag{16.38}$$

For an isotropic material with $k_x = k_y = k_z = k$, Eq. (16.38) becomes:

$$\left[K_1^{(e)}\right] = \frac{k}{36V^{(e)}} \times \begin{bmatrix} \left(b_1^2+c_1^2+d_1^2\right) & (b_1b_2+c_1c_2+d_1d_2) & (b_1b_3+c_1c_3+d_1d_3) & (b_1b_4+c_1c_4+d_1d_4) \\ & \left(b_2^2+c_2^2+d_2^2\right) & (b_2b_3+c_2c_3+d_2d_3) & (b_2b_4+c_2c_4+d_2d_4) \\ & & \left(b_3^2+c_3^2+d_3^2\right) & (b_3b_4+c_3c_4+d_3d_4) \\ \text{Symmetric} & & & \left(b_4^2+c_4^2+d_4^2\right) \end{bmatrix} \tag{16.39}$$

The matrix $\left[K_2^{(e)}\right]$ is given by:

$$\left[K_2^{(e)}\right] = \iint_{S_3^{(e)}} h \begin{bmatrix} N_i^2 & N_iN_j & N_iN_k & N_iN_l \\ & N_j^2 & N_jN_k & N_jN_l \\ & & N_k^2 & N_kN_l \\ \text{Symmetric} & & & N_l^2 \end{bmatrix} dS_3 \tag{16.40}$$

If the face ijk of the element experiences convection, $N_l = 0$ along this face, and hence Eq. (16.40) gives:

$$\left[K_2^{(e)}\right] = \frac{hA_{ijk}}{12}\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16.41}$$

where A_{ijk} is the surface area of face ijk. There are three other forms of Eq. (16.41): one for each of the other faces, jkl, kli, and lij. In each case, the value of the diagonal terms will be 2 and the values of the nonzero off-diagonal terms will be 1. The coefficients in the row and the column associated with the node not lying on the surface will be zero.

$$\vec{P}_1^{(e)} = \iiint_{V^{(e)}} \dot{q} \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_l \end{Bmatrix} dV = \frac{\dot{q}_0 V^{(e)}}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{16.42}$$

If the face ijk lies on the surface S_2 on which heat flux is specified,

$$\vec{P}_2^{(e)} = \iint_{S_2^{(e)}} q \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_l \end{Bmatrix} dS_2 = q\iint_{S_2^{(e)}} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ 0 \end{Bmatrix} dS_2 = \frac{qA_{ijk}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{16.43}$$

and similarly, if convection loss occurs from the face ijk,

$$\vec{P}_3^{(e)} = \iint_{S_2^{(e)}} hT_\infty \begin{Bmatrix} N_i \\ N_j \\ N_k \\ N_l \end{Bmatrix} dS_3 = hT_\infty \iint_{S_2^{(e)}} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ 0 \end{Bmatrix} dS_3 = \frac{hT_\infty A_{ijk}}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{16.44}$$

There are three other forms of Eqs. (16.43) and (16.44). In these equations, the zero coefficient will be located in the row corresponding to the node not lying on the face.

EXAMPLE 16.2

Derive the element equations for the element shown in Fig. 16.3.

Solution

From the given data, the required element properties can be computed as:

$$V^{(e)} = \frac{1}{6}\begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & 4 & 3 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 2 & 4 \end{vmatrix} = \frac{5}{6}$$

$$b_1 = -\begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{vmatrix} = 10, \quad c_1 = -\begin{vmatrix} 4 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = -17$$

$$d_1 = -\begin{vmatrix} 4 & 3 & 1 \\ -1 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 1$$

$$b_2 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 0, \quad c_2 = -\begin{vmatrix} -1 & 1 & 0 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 2$$

$$d_2 = -\begin{vmatrix} -1 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$b_3 = -\begin{vmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{vmatrix} = -5, \quad c_3 = -\begin{vmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 4 & 1 & 1 \end{vmatrix} = 10$$

$$d_3 = -\begin{vmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 4 & 3 & 1 \end{vmatrix} = 0,$$

$$b_4 = -\begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 5, \quad c_4 = -\begin{vmatrix} 0 & 1 & 2 \\ 4 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -11,$$

$$d_4 = -\begin{vmatrix} 0 & 1 & 1 \\ 4 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2$$

To compute area A_{jkl}, we use the formula:

$$A_{jkl} = [s(s-\alpha)(s-\beta)(s-\gamma)]^{1/2}$$

EXAMPLE 16.2 —cont'd

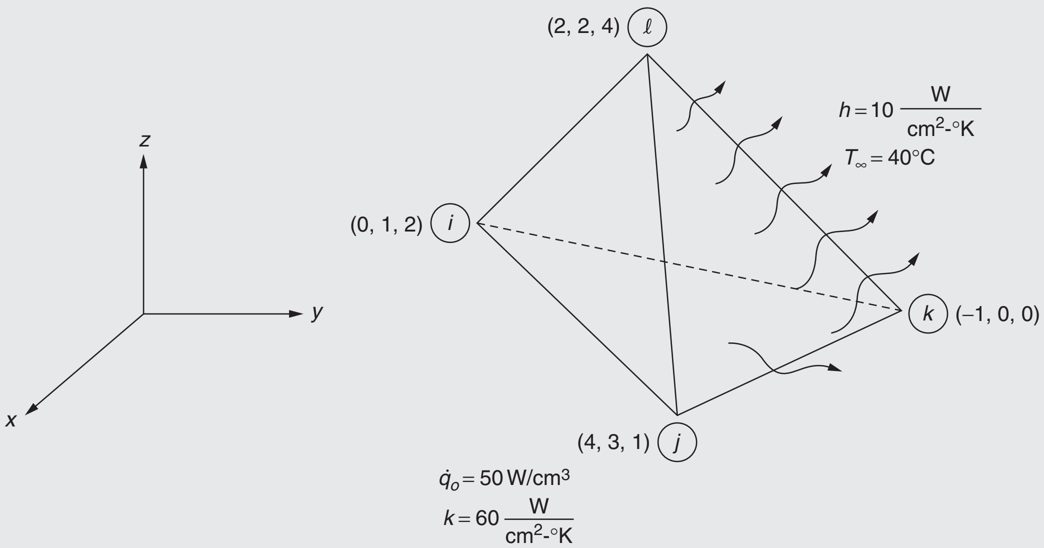

FIGURE 16.3 Tetrahedron element.

where α, β, and γ are the lengths of the sides of the triangle:

$$\alpha = \text{length } jk = \left[(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2\right]^{1/2}$$
$$= (25 + 9 + 1)^{(1/2)} = 5.916$$

$$\beta = \text{length } kl = \left[(x_l - x_k)^2 + (y_l - y_k)^2 + (z_l - z_k)^2\right]^{1/2}$$
$$= (9 + 4 + 16)^{(1/2)} = 5.385$$

$$\gamma = \text{length } lj = \left[(x_j - x_l)^2 + (y_j - y_l)^2 + (z_j - z_l)^2\right]^{1/2}$$
$$= (4 + 1 + 9)^{(1/2)} = 3.742$$

$$s = \frac{1}{2}(\alpha + \beta + \gamma) = \frac{1}{2}(5.916 + 5.385 + 3.742) = 7.522$$

$$A_{jkl} = [7.522(7.522 - 5.916)(7.522 - 5.385)(7.522 - 3.742)]^{(1/2)} = 9.877$$

Eq. (16.39) gives:

$$\left[K_1^{(e)}\right] = \frac{60 \times 6}{36 \times 5}\begin{bmatrix} (100+289+1) & (0-34-1) & (-50-170+0) & (50+187+2) \\ & (0+4+1) & (0+20+0) & (0-22-2) \\ & & (25+100+0) & (-25-110+0) \\ \text{Symmetric} & & & (25+121+4) \end{bmatrix}$$

$$= \begin{bmatrix} 780 & -70 & -440 & 478 \\ & 10 & 40 & -48 \\ & & 250 & -270 \\ \text{Symmetric} & & & 300 \end{bmatrix}$$

Continued

EXAMPLE 16.2 —cont'd

Matrix $\left[K_2^{(e)}\right]$ will be a modification of Eq. (16.41):

$$\left[K_2^{(e)}\right] = \frac{h \cdot A_{jkl}}{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} = \frac{(10)(9.877)}{12} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 16.462 & 8.231 & 8.231 \\ 0 & 8.231 & 16.462 & 8.231 \\ 0 & 8.231 & 8.231 & 16.462 \end{bmatrix}$$

$$\vec{P}_1^{(e)} = \frac{50 \times 5}{6 \times 4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 10.42 \\ 10.42 \\ 10.42 \\ 10.42 \end{Bmatrix}$$

$$\vec{P}_2^{(e)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{since no boundary heat flux is specified}$$

$$\vec{P}_3^{(e)} = \frac{10 \times 40 \times 9.877}{3} \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1316.92 \\ 1316.92 \\ 1316.92 \end{Bmatrix}$$

$$\left[K^{(e)}\right] = \left[K_1^{(e)}\right] + \left[K_2^{(e)}\right] = \begin{bmatrix} 780.000 & -70.000 & -440.000 & 478.000 \\ & 26.462 & 48.231 & -39.769 \\ & & 266.462 & -261.769 \\ \text{Symmetric} & & & 316.462 \end{bmatrix} \tag{E.1}$$

$$\vec{P}^{(e)} = \vec{P}_1^{(e)} - \vec{P}_2^{(e)} + \vec{P}_3^{(e)} = \begin{Bmatrix} 10.42 \\ 1327.34 \\ 1327.34 \\ 1327.34 \end{Bmatrix} \tag{E.2}$$

EXAMPLE 16.2 —cont'd

The element equations are:

$$[K^{(e)}]\vec{q}^{(e)} = \vec{P}^{(e)}$$

where $[K^{(e)}]$ and $\vec{P}^{(e)}$ are given by Eqs. (E.1) and (E.2), respectively, and

$$\vec{q}^{(e)} = \begin{Bmatrix} T_i \\ T_j \\ T_k \\ T_l \end{Bmatrix}$$

16.4 UNSTEADY-STATE PROBLEMS

The finite element equations governing unsteady-state problems are given by Eq. (13.35). It can be seen that the term $[\underset{\sim}{K}_3]\dot{\underset{\sim}{\vec{T}}}$ represents the unsteady-state part [16.1−16.3]. Element matrix $\left[K_3^{(e)}\right]$ can be evaluated using the definition given in Eq. (13.48).

16.4.1 Axisymmetric Problems

For a triangular ring element, in terms of natural coordinates, matrix $[N(r, z)]$ is given by:

$$[N] = [L_1 \quad L_2 \quad L_3] \tag{16.45}$$

By expressing $dV = 2\pi r\, dA$, Eq. (13.48) can be written as:

$$\left[K_3^{(e)}\right] = \iiint_{V^{(e)}} \rho c[N]^T[N]dV$$

$$= (\rho c)^{(e)} 2\pi \iint_{A^{(e)}} \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_1L_2 & L_2^2 & L_2L_3 \\ L_1L_3 & L_2L_3 & L_3^2 \end{bmatrix} (r_iL_1 + r_jL_2 + r_kL_3)dA \tag{16.46}$$

where Eq. (16.17) has been substituted for r. By carrying out the area integrations indicated in Eq. (16.46), we obtain:

$$\left[K_3^{(e)}\right] = \frac{\pi(\rho c)^{(e)}A^{(e)}}{30}\begin{bmatrix} (6r_i + 2r_j + 2r_k) & (2r_i + 2r_j + r_k) & (2r_i + r_j + 2r_k) \\ & (2r_i + 6r_j + 2r_k) & (r_i + 2r_j + 2r_k) \\ \text{Symmetric} & & (2r_i + 2r_j + 6r_k) \end{bmatrix} \tag{16.47}$$

16.4.2 Three-Dimensional Problems

The shape function matrix for the tetrahedron element is given by:

$$[N(x, y, z)] = [L_1 \quad L_2 \quad L_3 \quad L_4] \tag{16.48}$$

With this, the $\left[K_3^{(e)}\right]$ matrix can be derived as:

$$\left[K_3^{(e)}\right] = (\rho c)^{(e)} \iiint_{V^{(e)}} \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_1L_2 & L_2^2 & L_2L_3 & L_2L_4 \\ L_1L_3 & L_2L_3 & L_3^2 & L_3L_4 \\ L_1L_4 & L_2L_4 & L_3L_4 & L_4^2 \end{bmatrix} \mathrm{d}V \tag{16.49}$$

$$= \frac{(\rho c)^{(e)} \cdot V^{(e)}}{20} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

REVIEW QUESTIONS

16.1 Give brief answers to the following questions.

1. State the differential equation of heat conduction for an axisymmetric problem in cylindrical coordinates.
2. What type of boundary condition needs to be enforced for an insulated surface?
3. State the governing differential equation of steady-state heat transfer for a three-dimensional problem (assuming an isotropic material).

16.2 Fill in the blank with a suitable word.

1. For the analysis of heat transfer using linear temperature variation for three-dimensional problem, the simplest element type that can be used is ________.
2. Consideration of unsteady-state heat transfer requires the generation of ________ matrices involving the term $\rho\, c$.
3. The size of element matrices in an axisymmetric heat transfer problem is ________ whereas that of a three-dimensional heat transfer problem is 4×4 when linear temperature variation is assumed.

16.3 Select the most appropriate answer from the multiple choices given.

1. For an axisymmetric heat transfer problem stated in cylindrical coordinates (r, θ, z), the temperature does not vary along the coordinate:
   (a) r (b) θ (c) z
2. For an axisymmetric problem, the following type of finite element can be used:
   (a) Triangular ring type of element about the z-axis
   (b) Triangular ring type of element about the r-axis
   (c) Triangular ring type of element about the θ axis
3. The steady-state heat transfer problem with no heat generation is given by
   (a) Poisson equation (b) Fourier equation (c) Laplace equation

PROBLEMS

16.1 If radiation takes place on surface S_5 of an axisymmetric problem, state the boundary condition and indicate a method for deriving the corresponding finite element equations.

16.2 Derive the finite element equations corresponding to Eqs. (16.1)–(16.5) using the Galerkin approach.

16.3 If convection heat transfer takes place from the face corresponding to edge jk of a triangular ring element, derive matrix $\left[K_2^{(e)}\right]$ and vector $\vec{P}_3^{(e)}$.

16.4 If convection heat transfer takes place from the face corresponding to edge ki of a triangular ring element, derive matrix $\left[K_2^{(e)}\right]$ and vector $\vec{P}_3^{(e)}$.

16.5 Evaluate the conduction matrix, $\left[K_1^{(e)}\right]$, for an isotropic, axisymmetric ring element of a rectangular cross-section with four nodes. Use linear temperature variation in the r and z directions.

16.6 A three-node axisymmetric aluminum triangular ring element from a finite element grid is shown in Fig. 16.4. Convection heat transfer takes place from all of the faces (edges) of the triangle with a convection coefficient of 100 BTU/hr-ft^2-°F. If $k = 115$ BTU/hr-ft-°F, determine the characteristic matrices $\left[K_1^{(e)}\right]$ and $\left[K_2^{(e)}\right]$ of the element.

16.7 If convection takes place from face ijl of a tetrahedron element in a solid body, derive matrix $\left[K_2^{(e)}\right]$ and vector $\vec{P}_3^{(e)}$.

16.8 If convection takes place from face jkl of a tetrahedron element in a solid body, derive matrix $\left[K_2^{(e)}\right]$ and vector $\vec{P}_3^{(e)}$.

16.9 If convection takes place from face ikl of a tetrahedron element in a solid body, derive matrix $\left[K_2^{(e)}\right]$ and vector $\vec{P}_3^{(e)}$.

16.10 Derive the element equations for the tetrahedron element of a three-dimensional body shown in Fig. 16.5. Assume $k = 100$ BTU/hr-ft-°F, $h = 150$ BTU/hr-ft^2-°F from face ijk, and $\dot{q} = 500$ BTU/hr-ft^3.

16.11 Evaluate matrix $\left[K_3^{(e)}\right]$ for the triangular ring element shown in Fig. 16.2. Assume that $\rho c = 20$ J/cm °C.

16.12 Consider the axisymmetric triangular element shown in Fig. 16.6. The material of the element is copper with a thermal conductivity of $k = 401$ W/m °C. Determine the following:

a. The element characteristic matrix corresponding to conduction in the r and z directions.

b. The element characteristic matrix corresponding to convection if the side (face) ij of the element undergoes convection to air at 30°C with $h = 50$ W/m^2 °C.

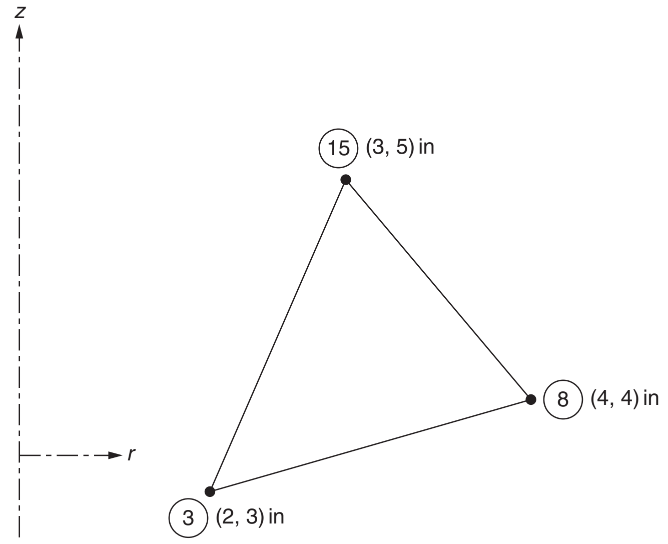

FIGURE 16.4 Triangular ring element.

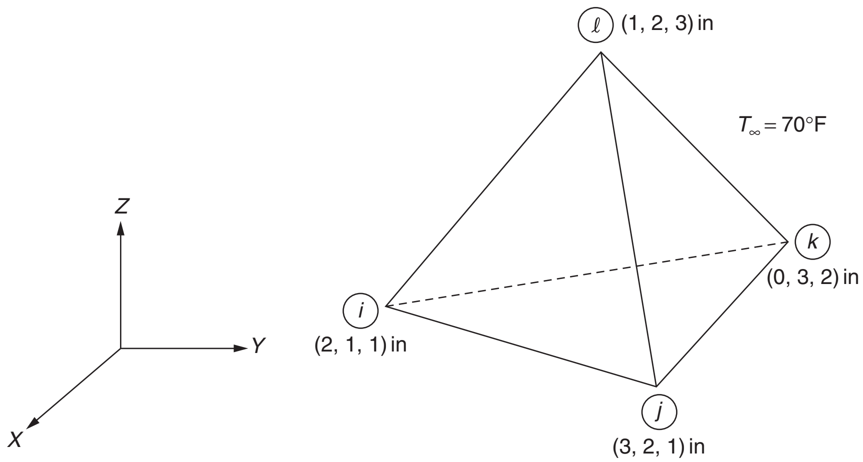

FIGURE 16.5 Tetrahedron element.

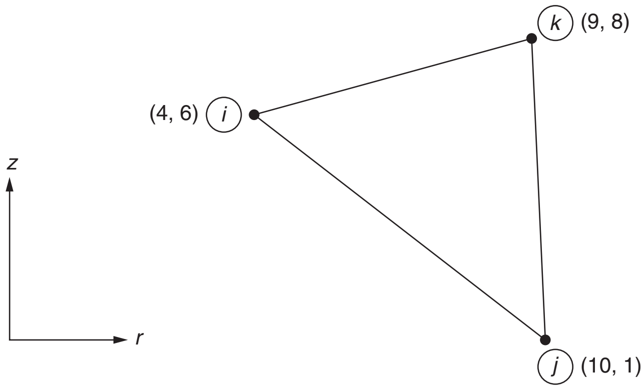

FIGURE 16.6 Axisymmetric triangular element (copper).

16.13 Consider the axisymmetric triangular element described in Problem 16.12 and Fig. 16.6. Determine the characteristic vector of the element as a result of the following:

a. An internal source of 75 W/cm^3

b. Convection along boundary (face) *ij*

c. Boundary heat flux of 50 W/cm^2 occurring on face *ij*

16.14 Consider the axisymmetric triangular element shown in Fig. 16.7. The material of the element is aluminum with a thermal conductivity of $k = 124$ BTU/hr-ft-°F. Determine the following:

a. The element characteristic matrix corresponding to conduction in the r and z directions.

b. The element characteristic matrix corresponding to convection if the side (face) *ki* of the element undergoes convection to air at 65°F with $h = 70$ Btu/hr-ft^2-°F.

16.15 Consider the axisymmetric triangular element described in Problem 16.14 and Fig. 16.7. Determine the characteristic vector of the element resulting from:

a. An internal source of 50 BTU/hr-ft^3

b. Convection along the boundary (face) *ki*

c. Boundary heat flux of 100 BTU/hr-ft^2 occurring on the face *ki*

16.16 The nodal temperatures of the axisymmetric triangular element described in Problem 16.12 are $T_i = 200°C$, $T_j = 170°C$, and $T_k = 220°C$. Determine the average heat flow rates (heat fluxes) in the element by conduction in the r and z directions. To which point in the element are these heat flow rates usually associated?

16.17 The nodal temperatures of the axisymmetric triangular element described in Problem 16.14 are $T_i = 350°F$, $T_j = 390°F$, and $T_k = 460°F$. Determine the average heat flow rates (heat fluxes) in the element by conduction in the r and z directions. To which point in the element are these heat flow rates usually associated?

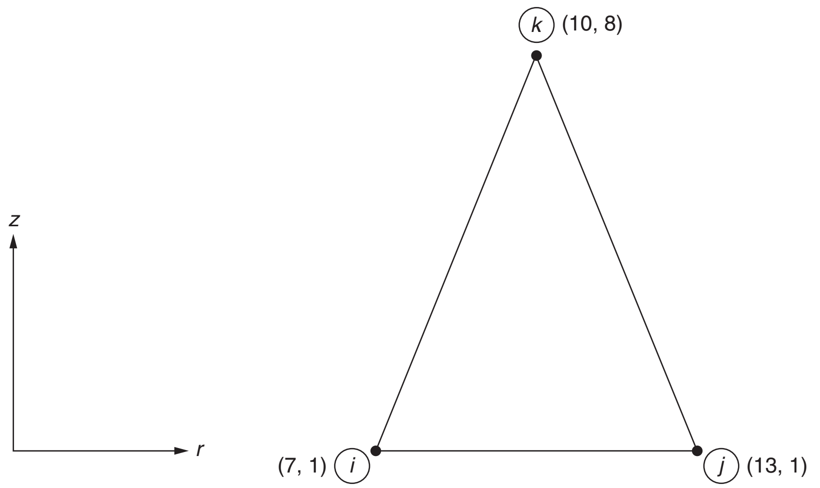

FIGURE 16.7 Axisymmetric triangular element (aluminum).

16.18 Derive the differential equation and the boundary conditions, if any, corresponding to the functional given by Eq. (16.6) using the Euler−Lagrange equation.

16.19 Derive the finite element equations for an axisymmetric triangular element if radiation heat transfer takes place on the edge (face) *ij*.

16.20 The nodal coordinates of a tetrahedron element (in centimeters) are given by:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i = \begin{Bmatrix} 15 \\ 0 \\ 0 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_j = \begin{Bmatrix} 0 \\ 5 \\ 0 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_k = \begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_l = \begin{Bmatrix} 10 \\ 10 \\ 5 \end{Bmatrix}$$

The material of the element is aluminum with a thermal conductivity of 250 W/m-°C. Find the matrices corresponding to heat conduction along the x, y, and z directions and the conductivity matrix of the element, $\left[K_1^{(e)}\right]$.

16.21 If the material of the tetrahedron element considered in Problem 16.20 is copper with a thermal conductivity of 401 W/m °C, find the matrices corresponding to heat conduction along the x, y, and z directions and the conductivity matrix of the element, $\left[K_1^{(e)}\right]$.

16.22 The nodal coordinates of a tetrahedron element (in centimeters) are given by:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_i = \begin{Bmatrix} 5 \\ 4 \\ 0 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_j = \begin{Bmatrix} 5 \\ 6 \\ 0 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_k = \begin{Bmatrix} 5 \\ 5 \\ 4 \end{Bmatrix}, \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_l = \begin{Bmatrix} 8 \\ 4 \\ 2 \end{Bmatrix}$$

The material of the element is carbon steel with a thermal conductivity of 54 W/m-°C. Find the matrices corresponding to heat conduction along the x, y, and z directions and the conductivity matrix of the element, $\left[K_1^{(e)}\right]$.

16.23 If face *ijk* of the tetrahedron element described in Problem 16.20 is part of the boundary of the thermal system that is exposed to heat convection with $h = 100$ W/m^2 °C and $T_\infty = 20$°C, find the element characteristic matrix and the characteristic vector corresponding to heat convection.

16.24 If face *kli* of the tetrahedron element described in Problem 16.22 is part of the boundary of the thermal system that is exposed to heat convection with $h = 50$ W/m^2 °C and $T_\infty = 30$°C, find the element characteristic matrix and the characteristic vector corresponding to heat convection.

16.25 The tetrahedron element described in Problem 16.20 has a 50-W/cm^{3}n internal heat source. Determine the corresponding element characteristic vector.

16.26 If the tetrahedron element described in Problem 16.22 has a 100-W/cm^{3}n internal heat source, find the corresponding element characteristic vector.

16.27 If face *ijk* of the tetrahedron element described in Problem 16.20 is part of the boundary of the thermal system that is subjected to a 100-W/m^2 heat flux, determine the element characteristic vector corresponding to the specified heat flux.

16.28 If face *kli* of the tetrahedron element described in Problem 16.22 is part of the boundary of the thermal system that is subjected to a 50-W/m^2 heat flux, determine the element characteristic vector corresponding to the specified heat flux.

16.29 For the tetrahedron element described in Problem 16.20, determine the element characteristic vector owing from a point source of 100 W located at $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_0 = \begin{Bmatrix} 5 \\ 5 \\ 2 \end{Bmatrix}$.

16.30 For the tetrahedron element described in Problem 16.22, determine the element characteristic vector owing to a point source of 50 W located at $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_0 = \begin{Bmatrix} 6 \\ 4 \\ 2 \end{Bmatrix}$.

16.31 Derive the finite element equations for a tetrahedron element if radiation heat transfer takes place on face *ijk*.

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Part V

Application to Fluid Mechanics Problems

Chapter 17

Basic Equations of Fluid Mechanics

Chapter Outline

17.1 INTRODUCTION

Although the finite element method was extensively developed for structural and solid mechanics problems, it was not considered a powerful tool for the solution of fluid mechanics problems until recently. One of the reasons is the success achieved with the more traditional finite difference procedures in solving fluid flow problems. In recent years, significant contributions have been made in the solution of different types of fluid flow problems using the finite element method. This chapter presents a summary of the basic concepts and equations of fluid mechanics.

17.2 BASIC CHARACTERISTICS OF FLUIDS

A fluid is a substance (gas or liquid) that will deform continuously under the action of applied surface (shearing) stresses. The magnitude of the stress depends on the *rate of angular deformation*. On the other hand, a solid can be defined as a substance that will deform by an amount proportional to the stress applied after which static equilibrium will result. Here, the magnitude of the shear stress depends on the *magnitude of angular deformation*.

Different fluids show different relations between stress and the rate of deformation. Depending on the nature of relation followed between stress and rate of deformation, fluids can be classified as Newtonian and non-Newtonian fluids. A Newtonian fluid is one in which the shear stress is directly proportional to the rate of deformation starting with zero stress and zero deformation. The constant of proportionality is defined as μ, the absolute or dynamic viscosity. Common examples of Newtonian fluids are air and water. A non-Newtonian fluid is one that has a variable proportionality between stress and rate of deformation. Common examples of non-Newtonian fluids are some plastics, colloidal suspensions, and emulsions. Fluids can also be classified as compressible and incompressible. Usually, liquids are treated as incompressible, whereas gases and vapors are assumed to be compressible.

A flow field is described in terms of the velocities and accelerations of fluid particles at different times and at different points throughout the fluid-filled space. For the graphical representation of fluid motion, it is convenient to introduce the

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concepts of streamlines and path lines. A *streamline* is an imaginary line that connects a series of points in space at a given instant in such a manner that all particles falling on the line at that instant have velocities whose vectors are tangent to the line. Thus, the streamlines represent the direction of motion at each point along the line at the given instant. A *path line* is the locus of points through which a fluid particle of fixed identity passes as it moves in space. For a steady flow the streamlines and path lines are identical, whereas they are, in general, different for an unsteady flow.

A flow may be termed as inviscid or viscous depending on the importance of consideration of viscosity of the fluid in the analysis. An *inviscid flow* is a frictionless flow characterized by zero viscosity. A *viscous flow* is one in which the fluid is assumed to have nonzero viscosity. Although no real fluid is inviscid, there are several flow situations in which the effect of viscosity of the fluid can be neglected. For example, in the analysis of a flow over a body surface, the viscosity effects are considered in a thin region close to the flow boundary (known as boundary layer), whereas the viscosity effect is neglected in the rest of the flow.

Depending on the dynamic macroscopic behavior of the fluid flow, we have laminar, transition, and turbulent motion. A *laminar flow* is an orderly state of flow in which macroscopic fluid particles move in layers. A *turbulent flow* is one in which the fluid particles have irregular, fluctuating motions and erratic paths. In this case, macroscopic mixing occurs both lateral to and in the direction of the main flow. A *transition flow* occurs whenever a laminar flow becomes unstable and approaches a turbulent flow.

17.3 METHODS OF DESCRIBING THE MOTION OF A FLUID

The motion of a group of particles in a fluid can be described by either the Lagrangian method or the Eulerian method. In the Lagrangian method, the coordinates of the moving particles are represented as functions of time. This means that at some arbitrary time t_0, the coordinates of a particle (x_0, y_0, z_0) are identified and that thereafter we follow that particle through the fluid flow. Thus, the position of the particle at any other instant is given by a set of equations of the form

$$x = f_1(x_0, y_0, z_0, t), \quad y = f_2(x_0, y_0, z_0, t), \quad z = f_3(x_0, y_0, z_0, t)$$

The Lagrangian approach is not generally used in fluid mechanics because it leads to more cumbersome equations. In the Eulerian method, we observe the flow characteristics in the vicinity of a fixed point as the particles pass by. Thus, in this approach the velocities at various points are expressed as functions of time as

$$u = f_1(x, y, z, t), \quad v = f_2(x, y, z, t), \quad w = f_3(x, y, z, t)$$

where u, v, and w are the components of velocity in x, y, and z directions, respectively.

The velocity change in the vicinity of a point in the x direction is given by

$$\mathrm{d}u = \frac{\partial u}{\partial t}\mathrm{d}t + \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial z}\mathrm{d}z \tag{17.1}$$

(total derivative expressed in terms of partial derivatives).

The small distances moved by a particle in time dt can be expressed as

$$\mathrm{d}x = u\,\mathrm{d}t, \quad \mathrm{d}y = v\,\mathrm{d}t, \quad \mathrm{d}z = w\,\mathrm{d}t \tag{17.2}$$

Thus, dividing Eq. (17.1) by dt and using Eq. (17.2) leads to the *total* or *substantial derivative* of the velocity u (x component of acceleration) as

$$a_x = \frac{\mathrm{d}u}{\mathrm{d}t} \equiv \frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{17.3a}$$

The other components of acceleration can be expressed in a similar manner as

$$a_y = \frac{\mathrm{d}v}{\mathrm{d}t} \equiv \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{17.3b}$$

$$a_z = \frac{\mathrm{d}w}{\mathrm{d}t} \equiv \frac{\mathrm{D}w}{\mathrm{D}t} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{17.3c}$$

17.4 CONTINUITY EQUATION

To derive the continuity equation, consider a differential control volume of size dx dy dz as shown in Fig. 17.1. Assuming that the density and the velocity are functions of space and time, we obtain the flux of mass per second for the three directions x, y, and z, respectively, as $-\frac{\partial}{\partial x}(\rho u)\cdot$ dy dz, $-\frac{\partial}{\partial y}(\rho v)\cdot$ dx dz, and $-\frac{\partial}{\partial z}(\rho w)\cdot$ dx dy. From the principle of conservation of matter, the sum of these must be equal to the time rate of change of mass, $\frac{\partial}{\partial t}(\rho \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z)$.

Since the control volume is independent of time, we can cancel dx dy dz from all the terms and obtain

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \tag{17.4a}$$

where ρ is the mass density; u, v, and w are the x, y, and z components of velocity, respectively; and t is time. By using the vector notation

$$\vec{V} = u\vec{i}+v\vec{j}+w\vec{k} = \text{velocity vector}$$

and

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k} = \text{gradient vector}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the unit vectors in x, y, and z directions, respectively. Eq. (17.4a) can be expressed as

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot\rho\vec{\nabla} = 0 \tag{17.4b}$$

This equation can also be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\mathrm{D}(\rho)}{\mathrm{D}t} \tag{17.4c}$$

where (D/Dt) represents the total or substantial derivative with respect to time. Eq. (17.4) represents the general three-dimensional continuity equation for a fluid in unsteady flow. If the fluid is incompressible, the time rate of

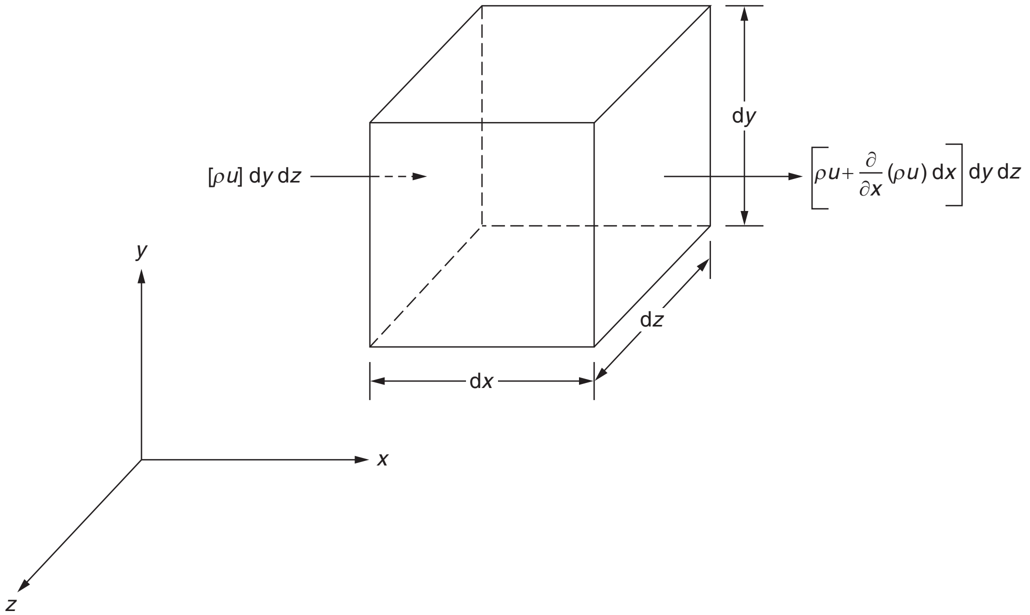

FIGURE 17.1 Differential control volume for conservation of mass.

volume expansion of a fluid element will be zero and hence the continuity equation, for both steady and unsteady flows, becomes

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \tag{17.5}$$

17.5 EQUATIONS OF MOTION OR MOMENTUM EQUATIONS

17.5.1 State of Stress in a Fluid

The state of stress in a fluid is characterized, as in the case of a solid, by the six stress components σ_{xx}, σ_{yy}, σ_{zz}, σ_{xy}, σ_{yz}, and σ_{zx}. However, if the fluid is at rest, the shear stress components σ_{xy}, σ_{yz}, and σ_{zx} will be zero and all the normal stress components will be the same and equal to the negative of the hydrostatic pressure, p — that is, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$.

17.5.2 Relation Between Stress and Rate of Strain for Newtonian Fluids

STRESS—STRAIN RELATIONS FOR SOLIDS

In solids the stresses σ_{ij} are related to the strains ε_{ij} according to Hooke's law, Eq. (8.7):

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - v(\sigma_{yy}+\sigma_{zz})], \ldots \\ \varepsilon_{xy} &= \frac{\sigma_{xy}}{G}, \ldots \end{aligned}\right\} \tag{17.6}$$

If an element of solid having sides dx, dy, and dz is deformed into an element having sides $(1+\varepsilon_{xx})\mathrm{d}x$, $(1+\varepsilon_{yy})\mathrm{d}y$, and $(1+\varepsilon_{zz})\mathrm{d}z$, the volume dilation of the element (e) is defined as

$$\begin{aligned} e &= \frac{\text{change in volume of the element}}{\text{original volume of the element}} \\ &= \frac{(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z} \\ &= \varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} \end{aligned} \tag{17.7}$$

Using Eq. (17.6), Eq. (17.7) can be expressed as

$$e = \frac{1-2v}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) = \frac{1-2v}{e}3\overline{\sigma} \tag{17.8}$$

where $\overline{\sigma}$ is the arithmetic mean of the three normal stresses defined as

$$\overline{\sigma} = (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})/3 \tag{17.9}$$

Young's modulus and Poisson's ratio are related as

$$G = \frac{E}{2(1+v)} \quad \text{or} \quad 2G = \frac{E}{1+v} \tag{17.10}$$

The first equation of Eq. (17.6) can be rewritten as

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - v(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) + v\sigma_{xx}] = \frac{E}{1+v}\varepsilon_{xx} + \frac{3v}{1+v}\overline{\sigma} \tag{17.11}$$

Substituting for $3\overline{\sigma}$ from Eq. (17.9) into Eq. (17.11) and using the relation (17.10), we obtain

$$\sigma_{xx} = \frac{E}{1+v}\varepsilon_{xx} + \frac{v}{1+v}\frac{E\cdot e}{1-2v} = 2G\varepsilon_{xx} + \frac{2Gv}{1-2v}\cdot e \tag{17.12}$$

By subtracting $\overline{\sigma}$ from both sides of Eq. (17.12) we obtain

$$\sigma_{xx} - \overline{\sigma} = 2G\varepsilon_{xx} + \frac{2Gv}{1-2v}e - \overline{\sigma} \tag{17.13}$$

Using Eq. (17.8), Eq. (17.13) can be expressed as

$$\sigma_{xx} - \overline{\sigma} = 2G\varepsilon_{xx} + \frac{2Gv}{1-2v}e - \frac{E}{3(1-2v)}e$$

that is,

$$\sigma_{xx} - \overline{\sigma} = \left(2G\varepsilon_{xx} - \frac{e}{3}\right) \tag{17.14}$$

In a similar manner, the following relations can be derived:

$$\sigma_{yy} - \overline{\sigma} = 2G\left(\varepsilon_{yy} - \frac{e}{3}\right) \tag{17.15}$$

$$\sigma_{zz} - \overline{\sigma} = 2G\left(\varepsilon_{zz} - \frac{e}{3}\right) \tag{17.16}$$

The shear stress—shear strain relations can be written, from Eq. (17.6), as

$$\sigma_{xy} = G\varepsilon_{xy} \tag{17.17}$$

$$\sigma_{yz} = G\varepsilon_{yz} \tag{17.18}$$

$$\sigma_{zx} = G\varepsilon_{zx} \tag{17.19}$$

STRESS—RATE OF STRAIN RELATIONS FOR NEWTONIAN FLUIDS

Experimental results indicate that the stresses in a fluid are related to the time rate of strain instead of strain itself. Thus, the stress—rate of strain relations can be derived by analogy from Eqs. (17.14) to (17.19). As an example, consider Eq. (17.14). By replacing the shear modulus (G) by a quantity expressing its dimensions, we obtain

$$\sigma_{xx} - \overline{\sigma} = 2\left(\frac{F}{L^2}\right)\left(\varepsilon_{xx} - \frac{e}{3}\right) \tag{17.20}$$

where F is the force, and L is the length. Since the stresses are related to the time rates of strain for a fluid, Eq. (17.20) can be used to obtain the following relation valid for a Newtonian fluid:

$$\sigma_{xx} - \overline{\sigma} = 2\left(\frac{FT}{L^2}\right)\frac{\partial}{\partial t}\left(\varepsilon_{xx} - \frac{e}{3}\right) \tag{17.21}$$

In Eq. (17.21), the dimension of time (T) is added to the proportionality constant in order to preserve the dimensions. The proportionality constant in Eq. (17.21) is taken as the dynamic viscosity μ having dimensions FT/L^2. Thus, Eq. (17.21) can be expressed as

$$\sigma_{xx} - \overline{\sigma} = 2\mu\frac{\partial\varepsilon_{xx}}{\partial t} - \frac{2}{3}\frac{\partial e}{\partial t} \tag{17.22}$$

$$\sigma_{yy} - \overline{\sigma} = 2\mu\frac{\partial\varepsilon_{yy}}{\partial t} - \frac{2}{3}\frac{\partial e}{\partial t} \tag{17.23}$$

$$\sigma_{zz} - \overline{\sigma} = 2\mu\frac{\partial\varepsilon_{zz}}{\partial t} - \frac{2}{3}\frac{\partial e}{\partial t} \tag{17.24}$$

$$\sigma_{xy} = \mu\varepsilon_{xy} \tag{17.25}$$

$$\sigma_{yz} = \mu\varepsilon_{yz} \tag{17.26}$$

$$\sigma_{zx} = \mu\varepsilon_{zx} \tag{17.27}$$

If the coordinates of a point before deformation are given by x, y, z and after deformation by $x+\xi$, $y+\eta$, $z+\zeta$, the strains are given by

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial \xi}{\partial x}, \quad & \varepsilon_{yy} &= \frac{\partial \eta}{\partial y}, \quad & \varepsilon_{zz} &= \frac{\partial \zeta}{\partial z} \\ \varepsilon_{xy} &= \frac{\partial \xi}{\partial y}+\frac{\partial \eta}{\partial x}, \quad & \varepsilon_{yz} &= \frac{\partial \eta}{\partial z}+\frac{\partial \zeta}{\partial y}, \quad & \varepsilon_{zx} &= \frac{\partial \zeta}{\partial x}+\frac{\partial \xi}{\partial z} \end{aligned}\right\} \tag{17.28}$$

The rate of strain involved in Eq. (17.22) can be expressed as

$$\frac{\partial \varepsilon_{xx}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \tag{17.29}$$

where u is the component of velocity in x direction, and

$$\frac{\partial e}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}) = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \vec{\nabla}\cdot\vec{V} \tag{17.30}$$

The mean stress $\overline{\sigma}$ is generally taken as $-p$, where p is the mean fluid pressure. Thus, Eqs. (17.22) to (17.27) can also be expressed as

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V} \tag{17.31}$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V} \tag{17.32}$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V} \tag{17.33}$$

$$\sigma_{xy} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \tag{17.34}$$

$$\sigma_{yz} = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \tag{17.35}$$

$$\sigma_{zx} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \tag{17.36}$$

17.5.3 Equations of Motion

The equations of motion can be derived by applying Newton's second law to a differential volume ($dx\,dy\,dz$) of a fixed mass dm [17.1]. If the body forces acting on the fluid per unit mass are given by the vector

$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k} \tag{17.37}$$

the application of Newton's law in x direction gives

$$dF_x = dm\, a_x = (\rho\, dx\, dy\, dz)a_x \tag{17.38}$$

where dF_x is the differential force acting in x direction, and a_x is the acceleration of the fluid in x direction. Using a figure similar to that of Fig. 8.2, Eq. (17.38) can be rewritten as

$$\begin{aligned} dF_x =& (\rho dx\, dy\, dz)B_x - \sigma_{xx}dy\, dz + \left(\sigma_{xx}+\frac{\partial \sigma_{xx}}{\partial x}dx\right)dy\, dz - \sigma_{yx}dx\, dz \\ &+ \left(\sigma_{yx}+\frac{\partial \sigma_{yx}}{\partial y}dy\right)dx\, dz - \sigma_{zx}dx\, dy + \left(\sigma_{zx}+\frac{\partial \sigma_{zx}}{\partial z}dz\right)dx\, dy \end{aligned}$$

Dividing this equation throughout by the volume of the element gives

$$\rho B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = \rho a_x \tag{17.39a}$$

Similarly, we can obtain for the y and z directions,

$$\rho B_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = \rho a_y \tag{17.39b}$$

$$\rho B_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho a_z \tag{17.39c}$$

Eqs. (17.39a, b, c) are general and are applicable to any fluid with gravitational-type body forces. For Newtonian fluids with a single viscosity coefficient, we substitute Eqs. (17.31) to (17.36) into Eq. (17.39) and obtain the equations of motion in x, y, and z directions as

$$\rho a_x = \rho B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \tag{17.40a}$$

$$\rho a_y = \rho B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \tag{17.40b}$$

$$\underbrace{\rho a_z}_{\text{Inertia force}} = \underbrace{\rho B_z}_{\text{Body force}} - \underbrace{\frac{\partial p}{\partial z}}_{\text{Pressure force}} \underbrace{+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\left(2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}\right)\right]}_{\text{Viscous force}} \tag{17.40c}$$

Eq. (17.40) are called the *Navier–Stokes equations* for compressible Newtonian fluids in Cartesian form. For incompressible fluids, the Navier–Stokes equations of motion, Eq. (17.40), become

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = B_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{17.41a}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{17.41b}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{17.41c}$$

Furthermore, when viscosity μ is zero, Eqs. (17.41a, b, c) reduce to the *Euler equations*:

$$a_x = B_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{17.42a}$$

$$a_y = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{17.42b}$$

$$a_z = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{17.42c}$$

For steady flow, all derivatives with respect to time will be zero in Eqs. (17.40) to (17.42).

17.6 ENERGY, STATE, AND VISCOSITY EQUATIONS

17.6.1 Energy Equation

When the flow is nonisothermal, the temperature of the fluid will be a function of x, y, z, and t. Just as the continuity equation represents the law of conservation of mass and gives the velocity distribution in space, the energy equation represents the conservation of energy and gives the temperature distribution in space. To derive the energy equation, we consider a differential control volume of fluid of size $dx\, dy\, dz$ and write the energy balance equation as

$$\text{Energy input} = \text{energy output} + \text{energy accumulation} \tag{17.43}$$

The energy input to the element per unit time is given by

$$\Bigg\langle \underbrace{\left\{\rho u E - \frac{\partial}{\partial x}(\rho u E)\cdot\frac{dx}{2}\right\}}_{\text{Internal energy}} + \frac{1}{2}\underbrace{\left\{\rho u\left(u^2+v^2+w^2\right) - \frac{\partial}{\partial x}\left[\rho u\left(u^2+v^2+w^2\right)\right]\cdot\frac{dx}{2}\right\}}_{\text{Kinetic energy}}$$

$$+ \underbrace{\left\{pu - \frac{\partial}{\partial x}(pu)\frac{dx}{2}\right\}}_{\text{Pressure-volume work}} - \underbrace{\left\{k\frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\cdot\frac{dx}{2}\right\}}_{\text{Heat conduction}}\Bigg\rangle \cdot dy\, dz$$

$$+ \text{similar terms for } y \text{ and } z \text{ directions} + \frac{\partial Q}{\partial t} dx\, dy\, dz + \Phi\, dx\, dy\, dz$$

where T is the temperature, k is the thermal conductivity, Q is the heat generated in the fluid per unit volume, and Φ is the dissipation function—that is, time rate of energy dissipated per unit volume due to the action of viscosity.

Similarly, the energy output per unit time is given by

$$\Bigg\langle \left[\rho u E + \frac{\partial}{\partial x}(\rho u E)\cdot\frac{dx}{2}\right] + \frac{1}{2}\left\{\rho u\left(u^2+v^2+w^2\right) + \frac{\partial}{\partial x}\left[\rho u\left(u^2+v^2+w^2\right)\right]\cdot\frac{dx}{2}\right\}$$

$$+ \left[pu + \frac{\partial}{\partial x}(pu)\frac{dx}{2}\right] - \left\{k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\cdot\frac{dx}{2}\right\}\Bigg\rangle \cdot dy\, dz$$

$$+ \text{similar terms for } y \text{ and } z \text{ directions}$$

The energy accumulated in the element is given by

$$\left[\frac{\partial}{\partial x}(\rho E) + \frac{1}{2}\frac{\partial}{\partial t}\left\{\rho\left(u^2+v^2+w^2\right)\right\}\right] dx\, dy\, dz \tag{17.44}$$

By making the energy balance as per Eq. (17.43), we obtain, after some manipulation,

$$\begin{aligned}&\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \frac{\partial Q}{\partial t} + \Phi \\ &= \frac{\partial}{\partial x}(pu) + \frac{\partial}{\partial y}(pv) + \frac{\partial}{\partial z}(pw) + \frac{\rho}{2}\frac{D}{Dt}\left(u^2+v^2+w^2\right) + \rho\frac{D(E)}{Dt}\end{aligned} \tag{17.45}$$

By using the relation

$$c_v = \left.\frac{\partial E}{\partial T}\right|_{\text{at constant volume}} = \text{specific heat at constant volume}$$

we can substitute $c_v \cdot \frac{DT}{Dt}$ in place of $\frac{D(E)}{Dt}$ in Eq. (17.45).

For inviscid and incompressible fluids, $\vec{\nabla}\cdot\vec{V} = 0$, and the application of Eq. (17.43) leads to

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \frac{\partial Q}{\partial t} + \Phi \tag{17.46}$$

where c_v is the specific heat at constant volume, T is the temperature, k is the thermal conductivity, Q is the heat generated in the fluid per unit volume, and Φ is the dissipation function (i.e., time rate of energy dissipated per unit volume due to the action of viscosity) [17.2] given by

$$\begin{aligned}\Phi = &-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]\\ &+\mu\left[\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]\end{aligned} \tag{17.47}$$

It can be seen that Φ has a value of zero for inviscid fluids.

17.6.2 State and Viscosity Equations

The variations of density and viscosity with pressure and temperature can be stated in the form of equations of state and viscosity as

$$\rho = \rho(p, T) \tag{17.48}$$

$$\mu = \mu(p, T) \tag{17.49}$$

17.7 SOLUTION PROCEDURE

For a general three-dimensional flow problem, the continuity equation, the equations of motion, the energy equation, the equation of state, and the viscosity equation are to be satisfied. The unknowns are the velocity components (u, v, w), pressure (p), density (ρ), viscosity (μ), and the temperature (T). Thus, there are seven governing equations in seven unknowns, and hence the problem can be solved once the flow boundaries and the boundary and initial conditions for the governing equations are known. The general governing equations are valid at any instant of time and are applicable to laminar, transition, and turbulent flows. Note that the solution of the complete set of equations has not been obtained even for laminar flows. However, in many practical situations, the governing equations get simplified considerably, and hence the mathematical solution would not be very difficult. In a turbulent flow, the unknown variables fluctuate about their mean values randomly and the solution of the problem becomes extremely complex.

For a three-dimensional inviscid fluid flow, five unknowns, namely, u, v, w, p, and ρ, will be there. In this case, Eqs. (17.4) and (17.40) are used along with the equation of state (expressing ρ in terms of pressure p only) to find the unknowns. In the solution of these equations, constants of integration appear that must be evaluated from the boundary conditions of the specific problem.

EXAMPLE 17.1

Express the Navier–Stokes equations for the flow of an incompressible Newtonian fluid.

Solution

For an incompressible Newtonian fluid, the stresses vary linearly with the rate of deformation so that the normal stresses are given by

$$\sigma_{xx} = -p+2\mu\frac{\partial u}{\partial x},\quad \sigma_{yy} = -p+2\mu\frac{\partial v}{\partial y},\quad \sigma_{zz} = -p+2\mu\frac{\partial w}{\partial z} \tag{E.1}$$

and the shear stresses by

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad \sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\quad \sigma_{zx} = \sigma_{xz} = \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \tag{E.2}$$

By substituting Eqs. (E.1) and (E.2) and the expressions for the acceleration components of the fluid given by Eq. (17.3) into the equations of motion (obtained by considering the dynamic equilibrium of a small cube-type of element) given by Eq. (17.39), we

Continued

EXAMPLE 17.1 —cont'd

obtain the equations of motion (Navier—Stokes equations) in the x, y, and z directions (for incompressible flow of a Newtonian fluid) as

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}+\rho B_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{E.3}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}+\rho B_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{E.4}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}+\rho B_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{E.5}$$

EXAMPLE 17.2

Find an expression for the velocity of the fluid in a steady laminar flow between two fixed parallel plates.

Solution

Let the fluid flow between the parallel plates shown in Fig. 17.2A. Here a fluid particle moves in the x direction parallel to the plate, and hence it has no velocity in the y or z direction so that $v = w = 0$. Thus, the continuity equation, Eq. (17.5), becomes

$$\frac{\partial u}{\partial x} = 0 \tag{E.1}$$

In addition, there is no variation of u in the z direction by assuming the plates to be infinite in the z direction. For steady flow,

$$\frac{\partial u}{\partial t} = 0$$

Thus, u can be expressed as

$$u = u(y) \tag{E.2}$$

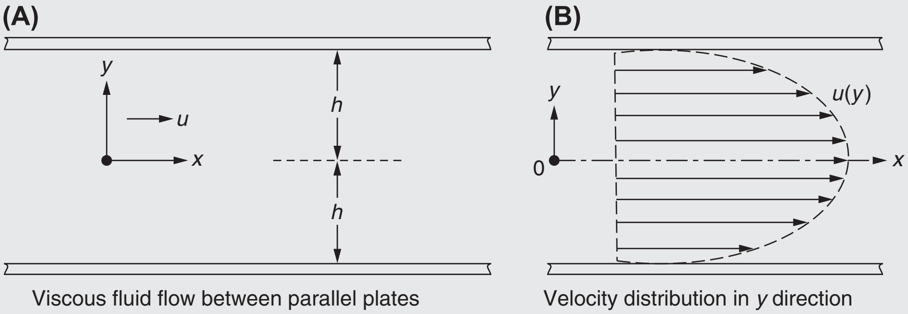

FIGURE 17.2 Viscous fluid flow between parallel plates.

Using these conditions, the Navier—Stokes equations given by Eqs. (E.3) to (E.5) of Example 17.1 reduce to

$$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} = 0 \tag{E.3}$$

$$-\frac{\partial p}{\partial y}-\rho\, g = 0 \tag{E.4}$$

$$-\frac{\partial p}{\partial z} = 0 \tag{E.5}$$

EXAMPLE 17.2 —cont'd

where $B_x = 0$, $B_y = -g$, $B_z = 0$ have been assumed (g = acceleration due to gravity). Eqs. (E.4) and (E.5) can be integrated to obtain

$$p = -\rho\, g\, y + f(x) \tag{E.6}$$

where $f(x)$ is some function of x. By rewriting Eq. (E.3) as

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{E.7}$$

and assuming $\frac{\partial p}{\partial x}$ to be a constant, Eq. (E.7) can be integrated to obtain

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + c_1 \tag{E.8}$$

where c_1 is a constant of integration. Integration of Eq. (E.8) yields

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) y^2 + c_1 y + c_2 \tag{E.9}$$

where c_2 is another constant of integration. Because the plates are fixed, $u = 0$ at $y = +h$ and $-h$. Using these boundary conditions in Eq. (E.9), we find the values of the constants as

$$c_1 = 0, \quad c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) h^2 \tag{E.10}$$

Thus, the velocity of the fluid flowing between the parallel plates is given by

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - h^2\right) \tag{E.11}$$

The variation of the velocity u in the y direction is shown in Fig. 17.2B.

17.8 INVISCID FLUID FLOW

In a large number of fluid flow problems (especially those with low-viscosity fluids, such as water and the common gases), the effect of viscosity will be small compared to other quantities such as pressure, inertia force, and field force; hence, the fluid can be treated as an inviscid fluid. Typical problems in which the effect of viscosity of the fluid can be neglected are flow through orifices, flow over weirs, flow in channel and duct entrances, and flow in converging and diverging nozzles. In these problems, the conditions very near the solid boundary, where the viscosity has a significant effect, are not of much interest; we would normally be interested in the movement of the main mass of the fluid. In any fluid flow problem, we would be interested in determining the fluid velocity and fluid pressure as a function of spatial coordinates and time. This solution will be greatly simplified if the viscosity of the fluid is assumed to be zero.

The equations of motion (Euler's equations) for this case are

$$\left.\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = B_x - \frac{1}{\rho}\frac{\partial p}{\partial x}\\
\frac{\mathrm{D}v}{\mathrm{D}t} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y}\\
\frac{\mathrm{D}w}{\mathrm{D}t} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}\right\} \tag{17.50}$$

The continuity equation is given by Eq. (17.4). Thus, the unknowns in Eqs. (17.50) and (17.4) are u, v, w, p, and ρ. Since the density ρ can be expressed in terms of the pressure p by using the equation of state, the four equations represented by Eqs. (17.50) and (17.4) are sufficient to solve for the four unknowns u, v, w, and p. While solving these equations, the constants of integration that appear are to be evaluated from the boundary conditions of the specific problem.

17.9 IRROTATIONAL FLOW

Let a point A and two perpendicular lines AB and AC be considered in a two-dimensional fluid flow. These lines, which are fixed to the fluid, are assumed to move with the fluid and assume the positions $A'B'$ and $A'C'$ after time Δt as shown in Fig. 17.3. If the original lines AB and AC are taken parallel to the x and y axes, the angular rotation of the fluid immediately adjacent to point A is given by $\frac{1}{2}(\beta_1 + \beta_2)$, and hence the rate of rotation of the fluid about the z axis (ω_z) is defined as

$$\omega_z = \frac{1}{2}\frac{\beta_1 + \beta_2}{\Delta t} \tag{17.51}$$

If the velocities of the fluid at the point A in x and y directions are u and v, respectively, the velocity components of the point C are $u + (\partial u/\partial y)\cdot\Delta y$ and $v + (\partial v/\partial y)\cdot\Delta y$ in x and y directions, respectively, where $\Delta y = AC$. Since β_2 is small,

$$\tan\beta_2 = \beta_2 = \frac{C'C_2}{A'C_2} = \frac{A_1A' - CC_1}{A'C_2} = \frac{u\Delta t - \left(u + \dfrac{\partial u}{\partial y}\Delta y\right)\Delta t}{\Delta y} = -\frac{\partial u}{\partial y}\Delta t \tag{17.52}$$

where it was assumed that $A'C_2 \approx AC = \Delta y$. Similarly,

$$\tan\beta_1 = \beta_1 = \frac{B_1B'}{A'B_1} = \frac{\left(v + \dfrac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t}{\Delta x} = \frac{\partial v}{\partial x}\Delta t \tag{17.53}$$

Thus, the rate of rotation (also called rotation) can be expressed as

$$\omega_z = \frac{1}{2}\left(\frac{\dfrac{\partial v}{\partial x}\Delta t - \dfrac{\partial u}{\partial y}\Delta t}{\Delta t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{17.54}$$

By proceeding in a similar manner, the rates of rotation about the x, y, and z axes in a three-dimensional fluid flow can be derived as

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{17.55a}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{17.55b}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{17.55c}$$

When the particles of the fluid are not rotating, the rotation is zero and the fluid is called *irrotational*. The physical meaning of irrotational flow can be seen in Fig. 17.4. In Fig. 17.4A, the particle maintains the same orientation everywhere along the streamline without rotation. Hence, it is called irrotational flow. On the other hand, in Fig. 17.4B, the particle

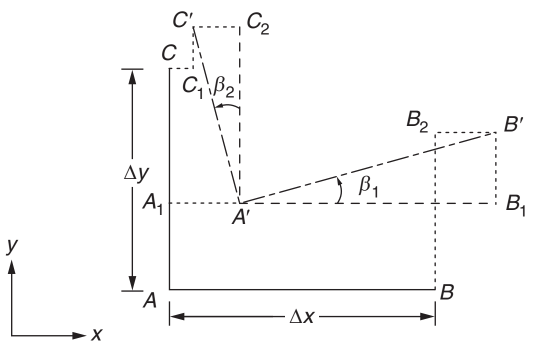

FIGURE 17.3 Angular rotation of fluid.

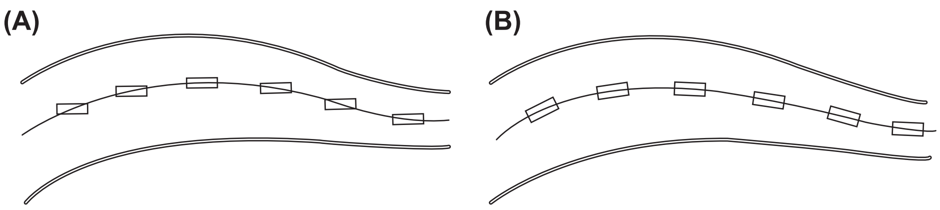

FIGURE 17.4 Irrotational and rotational flows.

rotates with respect to fixed axes and maintains the same orientation with respect to the streamline. Hence, this flow is not irrotational. The vorticity or fluid rotation vector $(\vec{\omega})$ is defined as the average angular velocity of any two mutually perpendicular line segments of the fluid whose x, y, and z components are given by Eq. (17.55).

Note

In both Fig. 17.4A and B, the particles can undergo deformation without affecting the analysis. For example, in the flow of a nonviscous fluid between convergent boundaries, the elements of the fluid deform as they pass through the channel, but there is no rotation about the z axis as shown in Fig. 17.5.

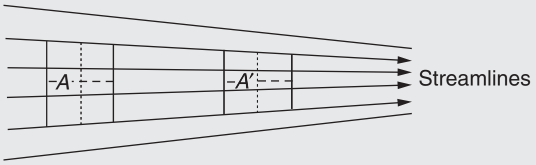

FIGURE 17.5 Irrotational flow between convergent boundaries.

17.10 VELOCITY POTENTIAL

It is convenient to introduce a function ϕ, called the potential function or velocity potential, in integrating Eq. (17.50). This function ϕ is defined in such a way that its partial derivative in any direction gives the velocity in that direction; that is,

$$\frac{\partial \phi}{\partial x} = u, \quad \frac{\partial \phi}{\partial y} = v, \quad \frac{\partial \phi}{\partial z} = w \tag{17.56}$$

Substitution of Eq. (17.56) into Eq. (17.4) gives

$$-\rho\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t} = \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{17.57}$$

For incompressible fluids, Eq. (17.57) becomes

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{17.58}$$

By differentiating u and v with respect to y and x, respectively, we obtain

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} \tag{17.59}$$

from which we can obtain

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{17.60}$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{17.61}$$

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \tag{17.62}$$

The terms on the left-hand side of Eqs. (17.60) to (17.62) are equal to twice the rates of rotation of the fluid element. Thus, the assumption of a velocity potential defined by Eq. (17.56) requires the flow to be irrotational.

17.11 STREAM FUNCTION

The motion of the fluid, at every point in space, can be represented by means of a velocity vector showing the direction and magnitude of the velocity. Since representation by vectors is unwieldy, we can use streamlines, which are the lines drawn tangent to the velocity vector at every point in space. Since the velocity vectors meet the streamlines tangentially for all points on a streamline, no fluid can cross the streamline.

For a two-dimensional flow, the streamlines can be represented in a two-dimensional plane. A stream function ψ may be defined (which is related to the velocity of the fluid) on the basis of the continuity equation and the nature of the streamlines. Let the streamlines AB and CD denote the stream functions ψ_1 and ψ_2, respectively, in Fig. 17.6. If a unit thickness of the fluid is considered, $\psi_2 - \psi_1$ is defined as the volume rate of fluid flow between the streamlines AB and CD. Let the streamline $C'D'$ lie at a small distance away from CD and let the flow between the streamlines CD and $C'D'$ be $\mathrm{d}\psi$. At a point P on CD, the distance between CD and $C'D'$ is denoted by the components of distance $-\mathrm{d}x$ and $\mathrm{d}y$. Let the velocity of the fluid at point P be u and v in x and y directions, respectively. Since no fluid crosses the streamlines, the volume rate of flow across the element $\mathrm{d}y$ is $u\,\mathrm{d}y$ and the volume rate of flow across the element $-\mathrm{d}x$ is $-v\,\mathrm{d}x$. If the flow is assumed to be incompressible, this volume rate of flow must be equal to $\mathrm{d}\psi$:

$$\therefore \mathrm{d}\psi = u\,\mathrm{d}y = -v\,\mathrm{d}x \tag{17.63}$$

Because ψ is a function of both x and y, we use partial derivatives and rewrite Eq. (17.63) as

$$\frac{\partial \psi}{\partial y} = u, \quad \frac{\partial \psi}{\partial x} = -v \tag{17.64}$$

Eq. (17.64) defines the stream function ψ for a two-dimensional incompressible flow. Physically, the stream function denotes the volume rate of flow per unit distance normal to the plane of motion between a streamline in the fluid and an arbitrary reference or base streamline. Hence, the volume rate of flow between any two adjacent streamlines is given by

$$Q = \psi_2 - \psi_1 \tag{17.65}$$

where ψ_1 and ψ_2 are the values of the adjacent streamlines, and Q is the flow rate per unit depth in the z direction. The streamlines also possess the property that there is no flow perpendicular to their direction. For a two-dimensional incompressible flow, the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17.66}$$

which is automatically satisfied by the stream function ψ—that is, by Eq. (17.64). If the flow is irrotational, the equation to be satisfied is

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{17.67}$$

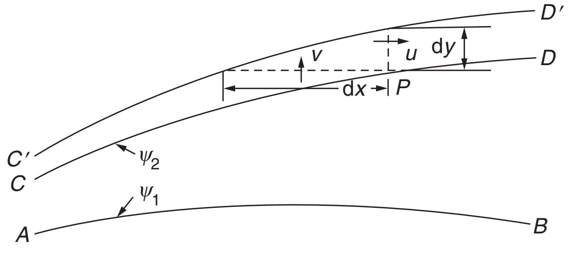

FIGURE 17.6 Streamlines corresponding to stream functions $\psi_1$and ψ_2.

By substituting Eq. (17.64) into Eq. (17.67), we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{17.68}$$

We can see that in a two-dimensional irrotational and incompressible flow, the solution of the Laplace equation gives either stream functions or velocity potentials depending on the choice.

EXAMPLE 17.3

Fig. 17.7A shows a uniform fluid flow with velocity 3 m/s. In this flow, the fluid particles move horizontally, hence the streamlines will be straight parallel lines. Find the stream and potential functions and show a few contours of the stream and potential functions.

Solution

Because the velocity components along the x and y directions are known to be $u = 3$ m/s and $v = 0$, the stream function (Ψ) can be found as

$$u = \frac{\partial \Psi}{\partial y} = 3\ \mathrm{m/s} \tag{E.1}$$

which after integration yields

$$\Psi = 3\ y\ \mathrm{m^2/s} \tag{E.2}$$

The potential function can be determined from the relation

$$u = \frac{\partial \Phi}{\partial x} = 3\ \mathrm{m/s} \tag{E.3}$$

By integrating (E.3), we find

$$\Phi = 3x\ \mathrm{m^2/s} \tag{E.4}$$

Three typical stream functions $\Psi_i = iy$, and three typical potential functions, $\Phi_i = ix$, $i = 1, 2, 3$, are shown in Fig. 17.7B.

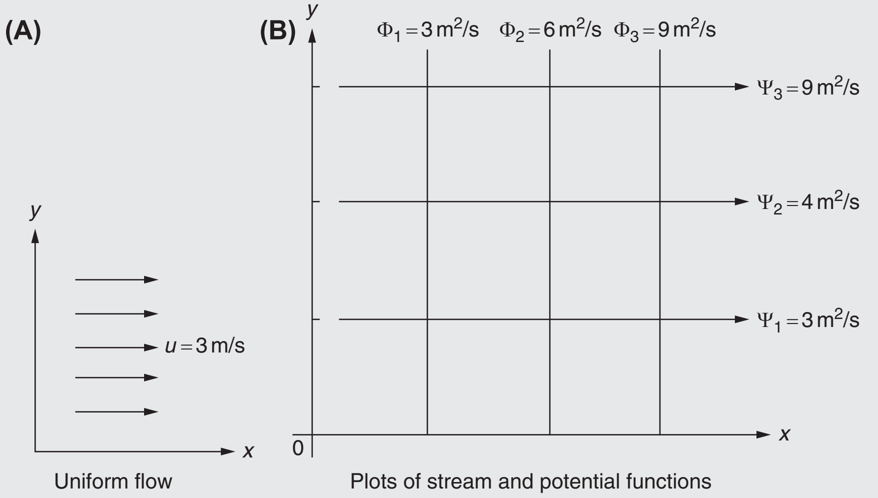

FIGURE 17.7 Stream and potential functions in uniform flow.

17.12 BERNOULLI EQUATION

The Bernoulli equation can be derived by integrating Euler's equations (17.50) with the help of Eqs. (17.56) and (17.60) to (17.62). By substituting the relations $\partial v/\partial x$ and $\partial w/\partial x$ for $\partial u/\partial y$ and $\partial u/\partial z$, respectively (from Eqs. 17.60 and 17.61), and $\partial \phi/\partial x$ for u (from Eq. 17.56) into the first equation of (17.50), and assuming that a body force potential (Ω) such as gravity exists, we have $B_x = -(\partial\Omega/\partial x)$, $B_y = -(\partial\Omega/\partial y)$, and $B_z = -(\partial\Omega/\partial z)$; hence

$$\frac{\partial^2 \phi}{\partial x \partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} + \frac{\partial \Omega}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{17.69}$$

By integrating Eq. (17.69) with respect to x, we obtain

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} + \Omega + \frac{p}{\rho} = f_1(y, z, t) \tag{17.70}$$

where f_1 cannot be a function of x because its partial derivative with respect to x must be zero. Similarly, the second and third equations of (17.50) lead to

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} + \Omega + \frac{p}{\rho} = f_2(x, z, t) \tag{17.71}$$

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} + \Omega + \frac{p}{\rho} = f_3(x, y, t) \tag{17.72}$$

Since the left-hand sides of Eqs. (17.70) to (17.72) are the same, we have

$$f_1(y, z, t) = f_2(x, z, t) = f_3(x, y, t) = f(t) \tag{17.73}$$

where $f(t)$ is a function of t alone. Since the magnitude of the velocity vector $\vec{V}$ is given by

$$|\vec{V}| = \left(u^2 + v^2 + w^2\right)^{1/2} \tag{17.74}$$

Eqs. (17.70) to (17.72) can be expressed as

$$\frac{\partial \phi}{\partial t} + \frac{|\vec{V}|^2}{2} + \Omega + \frac{p}{\rho} = f(t) \tag{17.75}$$

where $f(t)$ is a function of time. For a steady flow, Eq. (17.75) reduces to

$$\frac{|\vec{V}|^2}{2} + \Omega + \frac{p}{\rho} = \text{constant} \tag{17.76}$$

If the body force is due to gravity, $\Omega = gz$, where g is the acceleration due to gravity and z is the elevation. By substituting this expression of Ω into Eq. (17.76) and dividing throughout by g, we obtain a more familiar form of the Bernoulli equation for steady flows as

$$\underbrace{\frac{\vec{V}^2}{2g}}_{\substack{\text{Velocity}\\ \text{head}}} + \underbrace{z}_{\substack{\text{Elevation}\\ \text{head}}} + \underbrace{\frac{p}{\gamma}}_{\substack{\text{Pressure}\\ \text{head}}} = \text{constant} \tag{17.77}$$

where $\gamma = \rho g$.

EXAMPLE 17.4

Air flowing from a tank to the atmosphere through the nozzle of a hose is shown in Fig. 17.8. The diameters of the hose and the nozzle are 0.04 m and 0.01 m, respectively. If the pressure of air in the tank is 4.0 kPa (gage), which remains essentially constant, and the atmospheric conditions are standard with pressure equal to 14.696 psi (absolute) and temperature equal to 59 °F, determine the rate of flow and the pressure in the hose.

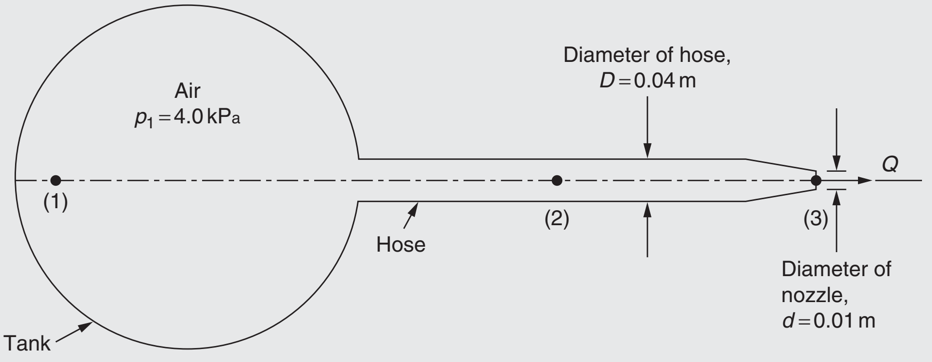

FIGURE 17.8 Flow through a nozzle.

Solution

Assuming the flow to be steady, inviscid, and incompressible, we can apply the Bernoulli equation to points (1), (2), and (3) of a streamline shown in Fig. 17.8 as

$$p_1 + \frac{1}{2}\rho v_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \gamma z_2 = p_3 + \frac{1}{2}\rho v_3^2 + \gamma z_3 \tag{E.1}$$

Because the tank is large, we can assume $v_1 = 0$ and also $p_3 = 0$ for a free jet. In addition, $z_1 = z_2 = z_3 = 0$ for a horizontal hose. Thus, considering the first and third terms in Eq. (E.1) to be equal, we obtain

$$v_3 = \left(\frac{2\,p_1}{\rho}\right)^{\frac{1}{2}} \tag{E.2}$$

By considering the first and second terms in Eq. (E.1) to be equal, we obtain

$$p_2 = p_1 - \frac{1}{2}\rho\, v_2^2 \tag{E.3}$$

Assuming standard absolute pressure and temperature conditions, the perfect gas law can be used to find the density of air in the tank as

$$\rho = \frac{p_1}{R\,T_1} = \frac{(4.0 + 101)(10^3)}{(286.9)(15 + 273)} = 1.2708\ \text{kg/m}^3 \tag{E.4}$$

Using this value in Eq. (E.2), we find

$$v_3 = \left(\frac{2(4.0 \times 10^3)}{1.2708}\right)^{\frac{1}{2}} = 79.3426\ \text{m/s} \tag{E.5}$$

Thus, the flow rate can be found as

$$Q = Q_3 = A_3 v_3 = \frac{\pi d^2}{4} v_3 = \frac{\pi}{4}(0.01)^2(79.3426) = 0.006231\ \text{m}^3/\text{s} \tag{E.6}$$

The pressure in the hose can be determined using the continuity equation

$$A_2 v_2 = A_3 v_3 \quad \text{or} \quad v_2 = \frac{A_3 v_3}{A_2} = \left(\frac{d}{D}\right)^2 v_3 = \left(\frac{0.01}{0.04}\right)^2 (79.3426) = 4.9589\ \text{m/s} \tag{E.7}$$

The pressure p_2 can be determined from Eq. (E.3) as

$$p_2 = 4.0 \times 10^3 - \frac{1}{2}(1.2708)(4.9589)^2 = 3984.3751\ \text{N/m}^2$$

EXAMPLE 17.5

A bent tube is used to siphon water from a water tank as shown in Fig. 17.9. The bent tube is located 2 m above the water level and the discharge from the tube occurs 4 m below the water level of the tank. Assuming the flow to be frictionless, determine the velocity of the fluid at discharge.

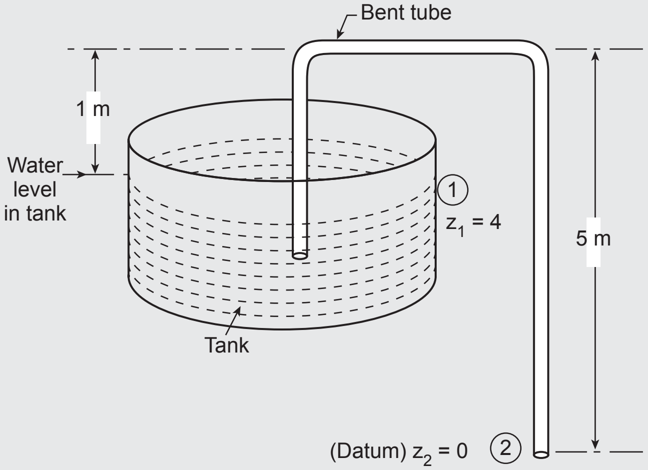

FIGURE 17.9 Siphoning water from a water tank.

Solution

The Bernoulli equation can be applied between points 1 and 2:

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + g\,z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g\,z_2 \tag{E.1}$$

where $v_1 = 0$ and $p_1 = p_2 = p_{atm}$. Thus Eq. (E.1) gives, by taking z_2 as datum (0),

$$\frac{p_{atm}}{\rho} + 0 + g(z_1 - z_2) = \frac{p_{atm}}{\rho} + \frac{v_2^2}{2} + g(z_2 - z_2) \tag{E.2}$$

or

$$v_2^2 = 2g(z_1 - z_2) = 2(9.81)(4) = 78.48\ \text{m}^2/\text{s}^2$$

Thus $v_2 = \sqrt{78.48} = 8.8589$ m/s.

REVIEW QUESTIONS

17.1 Give brief answers to the following questions.

1. Give two examples of non-Newtonian fluids.
2. What is an inviscid flow?
3. How is the position of a particle in a fluid flow represented in the Lagrangian method?
4. What is the Eulerian method?
5. How many stress components are needed in fluids?
6. What is the energy equation?
7. What are the equations of state?
8. How is stream function defined in terms of velocity components?
9. Define an irrotational flow.

17.2 Fill in the blank with a suitable word.

1. A Newtonian fluid is one in which ________________ stress is directly proportional to the rate of deformation.
2. The ratio of shear stress and the rate of deformation for a Newtonian fluid is called ________________________.
3. A fluid for which the stress is not proportional to the rate of deformation is called a ________________ fluid.
4. An example of a compressible fluid is ________.
5. A viscous flow is one in which the fluid has nonzero ________________.
6. The motion of a fluid can be described either using the Lagrangian or ________________ method.
7. Equations of motion are same as ________________ equations.
8. For a fluid at rest, all the normal stresses are ________________.
9. Bernoulli equation for a steady flow states that the sum of velocity, elevation, and ________ heads is a constant.

17.3 Indicate whether the following statement is true or false.

1. In general, liquids are incompressible fluids.
2. A vapor can be considered an incompressible fluid.
3. A streamline is an imaginary line that connects all fluid particles whose velocity vectors are tangential to the imaginary line.
4. The Lagrangian approach is more commonly used compared to the Eulerian approach in fluid mechanics.
5. The continuity equation is same as the principle of conservation of matter.
6. For a fluid at rest, all the shear stresses are zero.
7. For a two-dimensional irrotational and incompressible flow, the governing equation is the Laplace equation in terms of stream function only.

17.4 Select the most appropriate answer from the multiple choices given.

1. A fluid is a gas or liquid that deforms continuously under the action of:

(a) normal stress (b) shear stress (c) hydrostatic stress

2. The magnitude of surface stress in a fluid depends on

(a) angular deformation (b) linear deformation (c) both linear and angular deformations

17.5 Define the following terms.

1. Laminar flow
2. Turbulent flow
3. Transition flow

PROBLEMS

17.1 Derive the continuity equation in polar coordinates for an ideal fluid by equating the flow into and out of the polar element of area $r\,\mathrm{d}r\,\mathrm{d}\theta$.

17.2 If the x component of velocity in a two-dimensional flow is given by $u = x^2 + 2x - y^2$, find the y component of velocity that satisfies the continuity equation.

17.3 The potential function for a two-dimensional flow is given by $\phi = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$, where a_i $(i = 1-6)$ are constants. Find the expression for the stream function.

17.4 The velocity components in a two-dimensional flow are $u = -2x^2 + 3y$ and $v = 3x + 2y$. Determine whether the flow is incompressible or irrotational or both.

17.5 The potential function for a two-dimensional fluid flow is $\phi = 8xy + 6$. Determine whether the flow is incompressible or irrotational or both. Find the paths of some of the particles and plot them.

17.6 In the steady irrotational flow of a fluid at point P, the pressure is 15 kg/m^2 and the velocity is 10 m/s. At point Q, which is located 5 m vertically above P, the velocity is 5 m/s. If ρ of the fluid is 0.001 kg/cm^3, find the pressure at Q.

17.7 A nozzle has an inlet diameter of $d_1 = 4$ in, and an outlet diameter of $d_2 = 2$ in (Fig. 17.10). Determine the gage pressure of water required at the inlet of the nozzle in order to have a steady flow rate of 2 ft^3/s. Assume the density of water as 1.94 slug/ft^3.

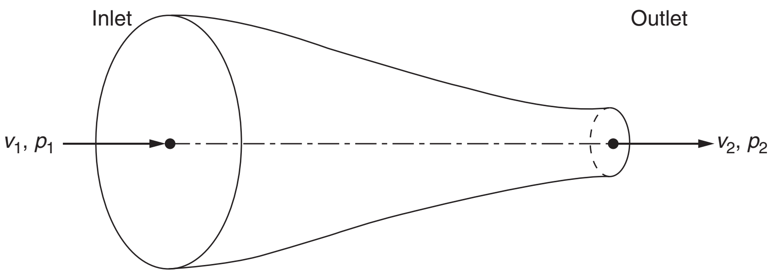

FIGURE 17.10 Tub with inflow and outflow of water.

17.8 The x and y components of velocity of a fluid in a steady, incompressible flow are given by $u = 4x$ and $v = -4y$. Find the stream function corresponding to this flow.

17.9 The velocity components of a three-dimensional fluid flow are given by

$$u = a_1x + a_2y + a_3z, \quad v = a_4x + a_5y + a_6z, \quad w = a_7x + a_8y + a_9z.$$

Determine the relationship between the constants $a_1, a_2, \ldots, a_9$ in order for the flow to be an incompressible flow.

17.10 The stream function corresponding to a fluid flow is given by $\Psi(x, y) = 10(x^2 - y^2)$.

a. Determine whether the flow is irrotational.
b. Find the velocity potential of the flow.
c. Find the velocity components of the flow.

17.11 Liquid from a tap is added steadily into a large tub of diameter $D = 0.25$ m while the liquid flows out of a nozzle of diameter $d = 0.01$ m as shown in Fig. 17.11. Find the flow rate of the liquid, Q, from the tap in order to maintain the depth of the liquid in the tub above the nozzle at a constant value of $h = 0.4$ m.

17.12 Find the volume rate of flow, Q, flowing between the parallel plates considered in Example 17.2 for unit width of plates in the z direction.

17.13 Fig. 17.7 shows a uniform fluid flow with velocity of 6 m/s. In this flow, the fluid particles move horizontally and hence the streamlines will be straight parallel lines. Find the stream and potential functions and plot a few contours of the stream and potential functions.

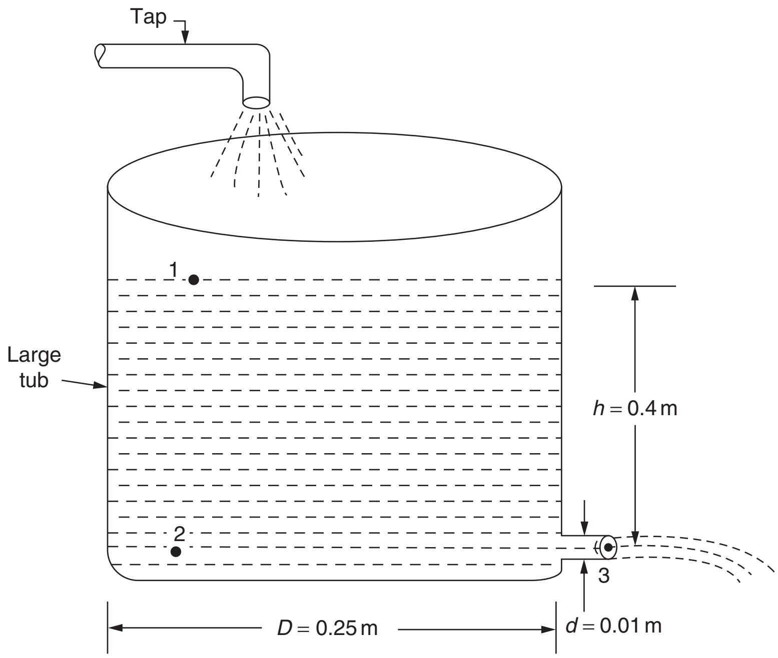

FIGURE 17.11 An orifice or Venturi meter.

17.14 The variation of velocity (v) of a Newtonian fluid flowing between two wide parallel plates is given by

$$v = \frac{3v_0}{2}\left\{1 - \left(\frac{y}{h}\right)^2\right\}$$

where v_0 is the mean velocity. For a fluid with viscosity, $\mu = 0.05$ lb-s/ft^2, $v_0 = 4$ ft/s, and $h = 0.3$ in, determine the following:

a. Shear stress in the fluid adjacent to the plate.

b. Shear stress in the fluid at the middle point ($y = 0$) in a plane parallel to that of the top and bottom plates.

17.15 The values of shear stress (τ) corresponding to six different values of the rate of shear strain ($\dot{\gamma}$) of a non-Newtonian fluid are observed experimentally and the following data are obtained:

τ (lb/ft^2)	0	20	40	60	80	100
$\dot{\gamma}$ (s^{-1})	0	0.8	2.1	3.7	5.9	8.4

Plot the relation between τ and $\dot{\gamma}$ and find a quadratic relation of the form $\tau = a + b\dot{\gamma} + c\dot{\gamma}^2$ using a least squares fit.

17.16 An orifice or Venturi meter, shown in Fig. 17.12, is used to measure the fluid flow rate in a pipe. In this device, the pressure difference between the high-pressure section (1) and the low-pressure section (2), $p_1 - p_2$, is measured to find the flow rate. If A_1 and A_2 denote the areas of cross section at sections (1) and (2) and ρ is the density of the fluid flowing through the device, find an expression for the fluid flow rate (Q) in terms of A_1, A_2, ρ, and $(p_1 - p_2)$.

17.17 A fluid of density $\rho = 900$ kg/m^3 flows through the Venturi meter shown in Fig. 17.12. If the diameters of the pipe sections (1) and (2) are 0.2 m and 0.05 m, respectively, and the flow rate varies between 0.015 m^3/s to 0.200 m^3/s, find the required range of the pressure difference, $p_1 - p_2$ for measuring all the possible flow rates in the stated range.

17.18 The equation of motion of the flow of a viscous fluid through a circular tube of radius R in the z direction (see Fig. 17.13) is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

where w is the velocity of the fluid in the z direction and μ is the dynamic viscosity of the fluid. Derive an expression for the velocity distribution in the radial direction, $w(r)$, in terms of μ, R, and $\frac{\partial p}{\partial z}$, where $\frac{\partial p}{\partial z}$ is the pressure gradient in the z direction (assumed to be a constant).

17.19 The components of velocity u, v, and w, respectively in the x, y, and z directions, of a flow field are given by

$$u = \frac{1}{2}(x^2 + y^2 + z^2), \quad v = xy + yz + zx, \quad w = -6xz - z^2 + 10$$

Find the volumetric dilation rate of the fluid.

17.20 The components of rotation ω_x, ω_y, and ω_z, respectively about the x, y, and z directions, of a fluid particle in a flow field are defined as

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

If the velocity components of a flow field in two dimensions are given by $u = x^2 - y^2$ and $v = -2\,xy$, determine the components of rotation about the x, y, and z axes. In addition, find whether the flow is irrotational.

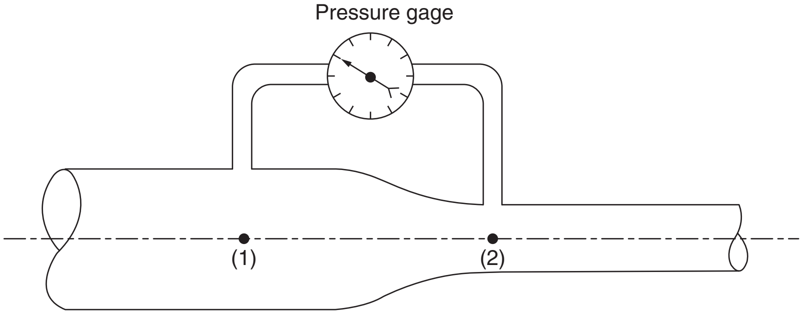

FIGURE 17.12 Flow in a horizontal circular tube.

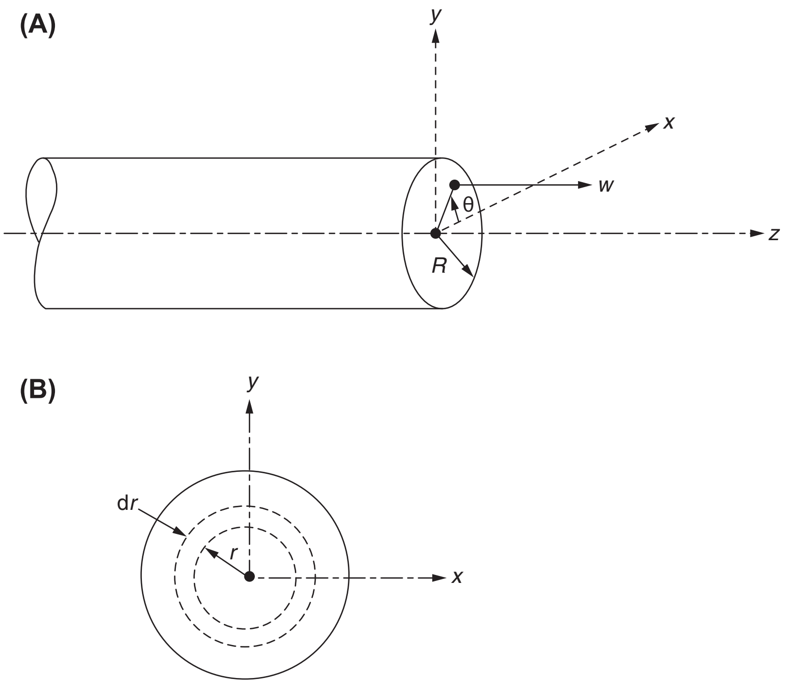

FIGURE 17.13 Siphoning water from a tank.

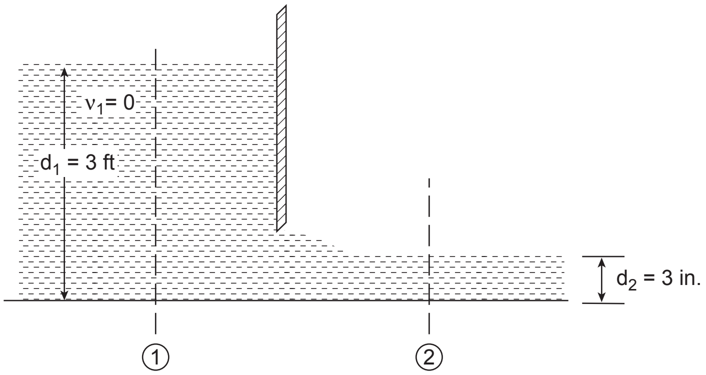

FIGURE 17.14 Water flow through a sluice gate.

17.21 The x and y components of velocity in a steady incompressible flow field are given by $u = \frac{1}{2}(x^2 + y^2 + z^2)$ and $v = xy + yz + zx$. Determine the component of velocity in the z direction, which satisfies the continuity equation.

17.22 The components of velocity in a two-dimensional flow are given by $u = y^2 - x^2 - x$ and $v = 2\ xy + y$.
a. Find whether the flow field satisfies the continuity equation.
b. Find whether the flow is irrotational.

17.23 The flow of water from a canal under a sluice gate is shown in Fig. 17.14. The water in the canal (upstream side) has a depth of 3 ft and the velocity of water is assumed to be zero. On the downstream side, the flow is laminar with negligible friction. Determine the flow velocity on the downstream side per unit width of the sluice gate.

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Chapter 18

Inviscid and Incompressible Flows

Chapter Outline

18.1 INTRODUCTION

In this chapter, we consider the finite element solution of ideal flow (inviscid incompressible flow) problems. Typical examples that may be found in this category are flow around a cylinder, flow out of an orifice, and flow around an airfoil. Two-dimensional potential flow (irrotational flow) problems can be formulated in terms of a velocity potential function (ϕ) or a stream function (ψ). In terms of the velocity potential, the governing equation for a two-dimensional problem is given by (obtained by substituting Eq. 17.56 into Eq. 17.5):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{18.1}$$

where the velocity components are given by:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \tag{18.2}$$

In terms of the stream function, the governing equation is (Eq. 17.68):

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{18.3}$$

and the flow velocities can be determined as:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{18.4}$$

In general, the choice between velocity and stream function formulations in finite element analysis depends on the boundary conditions, whichever is easier to specify. If the geometry is simple, there is no advantage of one over the other.

If the fluid is ideal, its motion does not penetrate into the surrounding body or separate from the surface of the body and leave empty space. This gives the boundary condition that the component of the fluid velocity normal to the surface must be equal to the component of the velocity of the surface in the same direction. Hence,

$$\vec{V} \cdot \vec{n} = \vec{V}_B \cdot \vec{n}$$

or

$$u l_x + v l_y = u_B l_x + v_B l_y \tag{18.5}$$

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00018-3

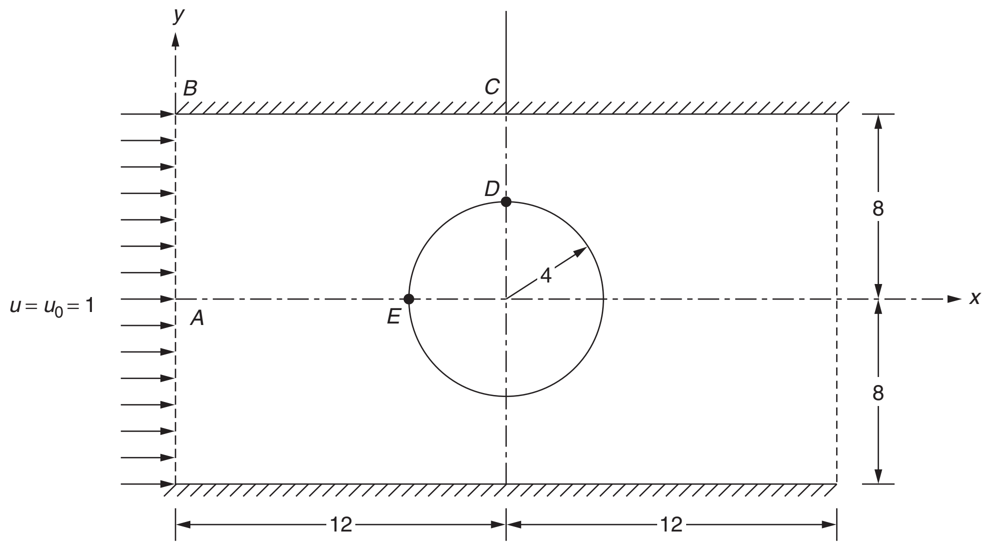

FIGURE 18.1 Confined flow around a cylinder.

where $\vec{V}$ is the velocity of the fluid, $\vec{V}_B$ is the velocity of the boundary, and $\vec{n}$ is the outward drawn normal to the boundary whose components (direction cosines) are l_x and l_y. If the boundary is fixed $\left(\vec{V}_B = \vec{0}\right)$, there will be no flow and hence no velocity perpendicular to the boundary. This implies that all fixed boundaries can be considered streamlines because there will be no fluid velocity perpendicular to a streamline. If there is a line of symmetry parallel to the direction of flow, it will also be a streamline. If $\vec{V}_B = \vec{0}$, Eqs. (18.2), (18.4), and (18.5) give the conditions:

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial y} l_x - \frac{\partial \psi}{\partial x} l_y = 0 \tag{18.6}$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} l_x + \frac{\partial \phi}{\partial y} l_y = 0 \tag{18.7}$$

Eq. (18.6) states that the tangential derivative of the stream function along a fixed boundary is zero, whereas Eq. (18.7) indicates that the normal derivative of the potential function (i.e., velocity normal to the fixed boundary) is zero.

The finite element solution of potential flow problems is illustrated in this chapter with reference to the problem of flow over a circular cylinder between two parallel plates, as shown in Fig. 18.1. Both potential and stream function formulations are considered.

18.2 POTENTIAL FUNCTION FORMULATION

The boundary value problem for potential flows can be stated as follows.

18.2.1 Differential Equation Form

Find the velocity potential $\phi(x, y)$ in a given region S surrounded by curve C such that:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ in } S \tag{18.8}$$

with the boundary conditions:

$$\text{Dirichlet condition: } \phi = \phi_0 \text{ on } C_1 \tag{18.9}$$

$$\text{Neumann condition: } V_n = \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} l_x + \frac{\partial \phi}{\partial y} l_y = V_0 \text{ on } C_2 \tag{18.10}$$

where $C = C_1 + C_2$, and V_0 is the prescribed value of the velocity normal to the boundary surface.

18.2.2 Variational Form

Find the velocity potential $\phi(x, y)$ that minimizes the functional:

$$I = \frac{1}{2}\iint_S \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]\cdot \mathrm{d}S - \int_{C_2} V_0 \phi \;\mathrm{d}C_2 \tag{18.11}$$

with the boundary condition:

$$\phi = \phi_0 \;\text{ on }\; C_1 \tag{18.12}$$

18.3 FINITE ELEMENT SOLUTION USING THE GALERKIN APPROACH

The finite element procedure using the Galerkin method can be stated by the following steps:

Step 1: Divide Region S into E finite elements of p nodes each.
Step 2: Assume a suitable interpolation model for $\phi^{(e)}$ in Element e as

$$\phi^{(e)}(x,y) = [N(x,y)]\vec{\Phi}^{(e)} = \sum_{i=1}^{p} N_i(x,y)\Phi_i^{(e)} \tag{18.13}$$

Step 3: Set the integral of the weighted residue over the region of the element equal to zero by taking the weights to be the same as the interpolation functions N_i. This yields:

$$\iint_{S^{(e)}} N_i\left[\frac{\partial^2\phi^{(e)}}{\partial x^2} + \frac{\partial^2\phi^{(e)}}{\partial y^2}\right]\mathrm{d}S = 0, \quad i = 1,2,\ldots,p \tag{18.14}$$

The integrals in Eq. (18.14) can be written as (see Appendix):

$$\iint_{S^{(e)}} N_i \frac{\partial^2\phi^{(e)}}{\partial x^2}\,\mathrm{d}S = -\iint_{S^{(e)}} \frac{\partial N_i}{\partial x}\frac{\partial \phi^{(e)}}{\partial x}\mathrm{d}S + \int_{C^{(e)}} N_i \frac{\partial \phi^{(e)}}{\partial x} l_x\,\mathrm{d}C \tag{18.15}$$

Similarly,

$$\iint_{S^{(e)}} N_i \frac{\partial^2\phi^{(e)}}{\partial y^2}\mathrm{d}S = -\iint_{S^{(e)}} \frac{\partial N_i}{\partial y}\frac{\partial \phi^{(e)}}{\partial y}\mathrm{d}S + \int_{C^{(e)}} N_i \frac{\partial \phi^{(e)}}{\partial y} l_y\,\mathrm{d}C \tag{18.16}$$

Thus, Eq. (18.14) can be expressed as:

$$-\iint_{S^{(e)}}\left(\frac{\partial N_i}{\partial x}\frac{\partial \phi^{(e)}}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial \phi^{(e)}}{\partial y}\right)\mathrm{d}S + \int_{C^{(e)}}\left(\frac{\partial \phi^{(e)}}{\partial x} l_x + \frac{\partial \phi^{(e)}}{\partial y} l_y\right)\mathrm{d}C = 0, \quad i = 1,2,\ldots,p \tag{18.17}$$

Because the boundary of Element $C^{(e)}$ is composed of ${C_1}^{(e)}$ and ${C_2}^{(e)}$, the line integral of Eq. (18.17) would be zero on ${C_1}^{(e)}$ (because $\phi^{(e)}$ is prescribed to be a constant ϕ_0 on $C_1^{(e)}$, the derivatives of $\phi^{(e)}$ with respect to x and y would be zero). On the boundary $C_2^{(e)}$, Eq. (18.10) needs to be satisfied. For this, the line integral of Eq. (18.17) can be rewritten as:

$$\int_{C_1^{(e)}+C_2^{(e)}} N_i\left(\frac{\partial \phi^{(e)}}{\partial x} l_x + \frac{\partial \phi^{(e)}}{\partial y} l_y\right)\;\mathrm{d}C = \int_{C_2^{(e)}} V_0 N_i\,\mathrm{d}C_2 \tag{18.18}$$

By using Eqs. (18.13) and (18.18), Eq. (18.17) can be expressed in matrix form as:

$$\left[K^{(e)}\right]\vec{\Phi}^{(e)} = \vec{P}^{(e)} \tag{18.19}$$

where:

$$\left[K^{(e)}\right] = \iint_{S^{(e)}} [B]^T[D][B]\cdot \mathrm{d}S \tag{18.20}$$

$$\vec{P}^{(e)} = -\int_{C_2^{(e)}} V_0 [N]^T \mathrm{d}C_2 \tag{18.21}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_p}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_p}{\partial y} \end{bmatrix} \tag{18.22}$$

and

$$[D] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18.23}$$

Step 4: Assemble the element Eq. (18.19) to obtain the overall equations as:

$$[\underset{\sim}{K}]\, \underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{P}} \tag{18.24}$$

Step 5: Incorporate the boundary conditions specified over C_1 and solve Eq. (18.24).

Note

Computation of the characteristic vector $\vec{P}^{(e)}$ using Eq. (18.21) requires the velocity of the fluid V_0 to be normal to the boundary $C_2^{(e)}$. For an arbitrary orientation of the boundary $C_2^{(e)}$, the computation of V_0 requires knowledge of the normal vector $\vec{n}$ to the boundary. The computational details are illustrated in Examples 18.1–18.3.

EXAMPLE 18.1

Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge ij of the triangular element ijk shown in Fig. 18.2.

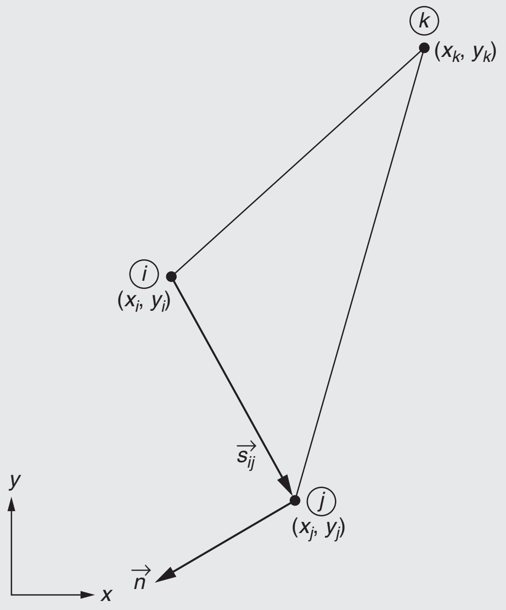

FIGURE 18.2 Outward normal to edge ij of a triangular element.

EXAMPLE 18.1 —cont'd

Solution

Let $\vec{s}_{ij}$ denote the line joining the node i to node j (position vector ij). The vector $\vec{s}_{ij}$ can be expressed in terms of the (x, y) coordinates of node i, (x_i, y_i), and node j, (x_j, y_j), as:

$$\vec{s}_{ij} = (x_j - x_i)\vec{i} + (y_j - y_i)\vec{j} + 0\,\vec{k} \tag{E.1}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors along the x, y, and z directions, respectively. The outward normal $\vec{n}$ to the edge ij can be expressed in terms of its direction cosines n_x and n_y as:

$$\vec{n} = n_x\vec{i} + n_y\vec{j} + 0\,\vec{k} \tag{E.2}$$

The normal vector $\vec{n}$ can be determined by noting that the cross-product of two vectors $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$ denotes another vector $\vec{c} = c_x\vec{i} + c_y\vec{j} + c_z\vec{k}$ given by:

$$\vec{c} = \frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} \tag{E.3}$$

where:

$$\vec{a}\times\vec{b} = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{bmatrix} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k} \tag{E.4}$$

The direction of $\vec{c}$ is defined by the right-hand screw rule while moving from vector $\vec{a}$ to vector $\vec{b}$. When vectors $\vec{a}$ and $\vec{b}$ are chosen to be $\vec{s}_{ij}$ and $\vec{k}$, respectively, we recognize that $\vec{c}$ denotes normal vector $\vec{n}$. Thus, Eqs. (E.1)–(E.3) yield the components of the outward normal to the edge ij, $\vec{n}$, as:

$$n_x = \frac{y_j - y_i}{s_{ij}}, \quad n_y = -\frac{x_j - x_i}{s_{ij}}, \quad n_z = 0 \tag{E.5}$$

where S_{ij} is the distance between nodes i and j (length of the edge ij):

$$S_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \tag{E.6}$$

EXAMPLE 18.2

If the (x, y) coordinates of nodes i, j, and k of a triangular element are given by $(x_i, y_i) = (2, 4)$ cm, $(x_j, y_j) = (9, 3)$ cm, and $(x_k, y_k) = (6, 7)$ cm, find vector $\vec{n}$ that denotes the outward normal to the edge ij.

Solution

Eq. (E.6) of Example 18.1 gives the length of edge ij as:

$$S_{ij} = \sqrt{(9-2)^2 + (3-4)^2} = 7.0711 \text{ cm} \tag{E.1}$$

The direction cosines of the outward normal to the edge ij are given by Eq. (E.5) of Example 18.1:

$$n_x = \frac{y_j - y_i}{S_{ij}} = \frac{3-4}{7.0711} = -0.1414, \quad n_y = -\frac{x_j - x_i}{S_{ij}} = \frac{9-2}{7.0711} = -0.9899, \quad n_z = 0 \tag{E.2}$$

EXAMPLE 18.3

The (x, y) components of velocity of a fluid on the edge ij of a triangular element ijk vary linearly from (u_i, v_i) at node i to (u_j, v_j) at node j, as shown in Fig. 18.3. If the (x, y) coordinates of nodes i and j are (x_i, y_i) and (x_j, y_j), respectively, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

Continued

EXAMPLE 18.3 —cont'd

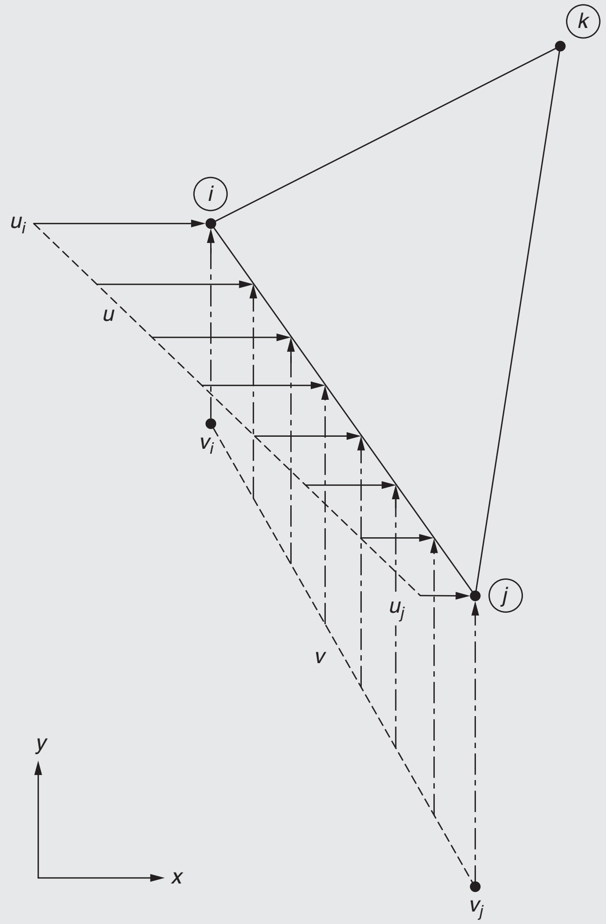

FIGURE 18.3 Velocity of fluid along edge *ij*.

Solution

The shape function N_k (the area coordinate L_k) is zero on the edge *ij*; hence, the linearly varying *x* and *y* velocity components on the edge *ij* can be expressed as:

$$u = L_i u_i + L_j u_j, \quad v = L_i v_i + L_j v_j \tag{E.1}$$

The characteristic vector $\vec{P}^{(e)}$ of the element given by Eq. (18.21) can be rewritten as:

$$\vec{P}^{(e)} = \int_{C_2^{(e)}} V_0 [N] dC_2 = \int_{S_{ij}} V_0 \begin{Bmatrix} L_i \\ L_j \\ 0 \end{Bmatrix} ds \tag{E.2}$$

where V_0 is the known or specified value of velocity of the fluid normal to the edge or boundary *ij*. Because the velocity of the fluid normal to the edge *ij* is not known directly (only the *x* and *y* components of velocity, *u* and *v*, are known), we write the unit vector along the outward normal to the edge *ij*, $\vec{n}$, as a vector with *x* and *y* components (direction cosines) as n_x and n_y, respectively, so that:

$$\vec{n} = n_x \vec{i} + n_y \vec{j} \tag{E.3}$$

where $\vec{i}$ and $\vec{j}$ are the unit vectors along the *x* and *y* directions, respectively. The velocity vector on the edge *ij*, $\vec{V}$, can be expressed as:

$$\vec{V} = u \vec{i} + v \vec{j} \tag{E.4}$$

The normal velocity of the fluid V_0 can be found as the projection of the velocity vector $\vec{V}$ along the normal $\vec{n}$ so that:

$$V_0 = \vec{V}^T \vec{n} = (u \; v) \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = u\, n_x + v\, n_y \tag{E.5}$$

EXAMPLE 18.3 —cont'd

In view of Eq. (E.5), Eq. (E.2) becomes:

$$\vec{P}^{(e)} = \int_{s_{ij}} V_0 \begin{Bmatrix} L_i \\ L_j \\ 0 \end{Bmatrix} ds = \int_{s_{ij}} (u\, n_x + v\, n_y) \begin{Bmatrix} L_i \\ L_j \\ 0 \end{Bmatrix} ds \tag{E.6}$$

Using Eq. (E.1), Eq. (E.6) can be rewritten as:

$$\vec{P}^{(e)} = \int_{s_{ij}} \{(u_i L_i + u_j L_j)n_x + (v_i L_i + v_j L_j)n_y\} \begin{Bmatrix} L_i \\ L_j \\ 0 \end{Bmatrix} ds$$

$$= \begin{Bmatrix} \int_{s_{ij}} \left(u_i n_x L_i^2 + u_j n_x L_i L_j + v_i n_y L_i^2 + v_j n_y L_j^2\right) ds \\ \int_{s_{ij}} \left(u_i n_x L_i L_j + u_j n_x L_j^2 + v_i n_y L_i L_j + v_j n_y L_j^2\right) ds \\ 0 \end{Bmatrix} \tag{E.7}$$

The integrals in Eq. (E.7) can be evaluated using Eq. (3.67):

$$\int_{s_{ij}} L_i^{\alpha} L_j^{\beta} ds = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} s_{ij} \tag{E.8}$$

so that $\vec{P}^{(e)}$ in Eq. (E.7) can be expressed as:

$$\vec{P}^{(e)} = \frac{s_{ij} n_x}{6} \begin{Bmatrix} 2u_i + u_j \\ u_i + 2u_j \\ 0 \end{Bmatrix} + \frac{s_{ij} n_y}{6} \begin{Bmatrix} 2v_i + v_j \\ v_i + 2v_j \\ 0 \end{Bmatrix} \tag{E.9}$$

EXAMPLE 18.4

Find the velocity distribution along the vertical center line *CD* in Fig. 18.1.

Solution

Owing to symmetry, we can consider only the portion *ABCDEA* in the finite element analysis. The boundary condition is that ϕ is constant along *CD*. This constant can be taken as zero for convenience.

Step 1: Idealize the solution region using triangular elements. In the current case, 13 elements are used to model the region, as shown in Fig. 18.4. Although it is crude in representing the cylindrical boundary, this idealization is considered for simplicity. The local corner numbers of the elements are labeled in an arbitrary manner. The information needed for subsequent calculations is given in Table 18.1.

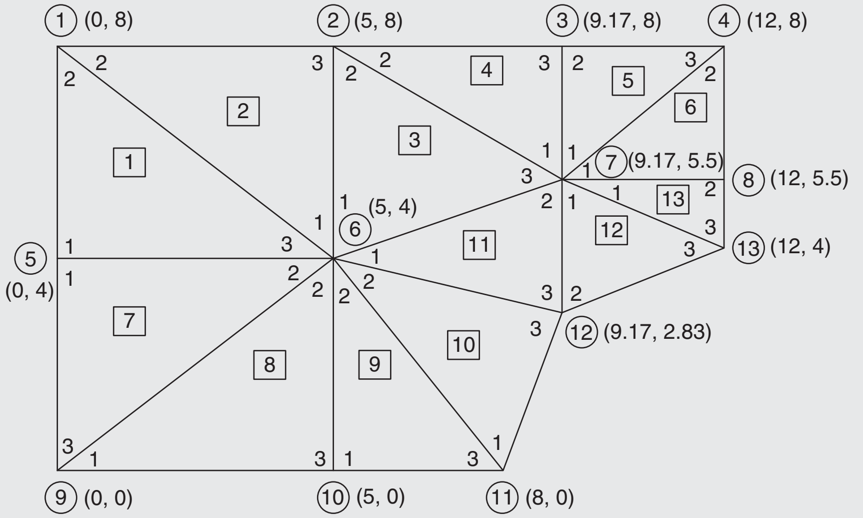

FIGURE 18.4 Finite element idealization of flow around a cylinder.

Continued

TABLE 18.1 Characteristics of Finite Elements

Node Number (i)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Global coordinates of node i (x_i, y_i)	(0, 8)	(5, 8)	(9.17, 8)	(12,8)	(0, 4)	(5, 4)	(9.17, 5.5)	(12, 5.5)	(0, 0)	(5, 0)	(8, 0)	(9.17, 2.83)	(12, 4)

Element Number (e)		**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Global node numbers i, j, and k corresponding to local nodes 1, 2, and 3	i	5	6	6	7	7	7	5	9	10	11	6	7	7
	j	1	1	2	2	3	4	6	6	6	6	7	12	8
	k	6	2	7	3	4	8	9	10	11	12	12	13	13

Element Number (e)	x_i	x_j	x_k	y_i	y_j	y_k	$c_k = x_j - x_i$	$c_i = x_k - x_j$	$c_j = x_i - x_k$	$b_i = y_j - y_k$	$b_j = y_k - y_i$	$b_k = y_i - y_j$	$A^{(e)} = \frac{1}{2}\left[x_{ij}y_{jk} - x_{jk}y_{ij}\right]$
1	0	0	5	4	8	4	0	5	−5	4	0	−4	$\frac{1}{2}\lvert 0 \times (-4) - 5 \times 4 \rvert = 10$
2	5	0	5	4	8	8	−5	5	0	0	4	−4	$\frac{1}{2}\lvert -5 \times 0 - 5 \times 4 \rvert = 10$
3	5	5	9.17	4	8	5.5	0	4.17	−4.17	2.5	1.5	−4	$\frac{1}{2}\lvert 0 \times (-2.5) - 4.17 \times 4 \rvert = 8.34$
4	9.17	5	9.17	5.5	8	8	−4.17	4.17	0	0	2.5	−2.5	$\frac{1}{2}\lvert -4.17 \times 0 - 4.17 \times 2.5 \rvert = 5.2125$
5	9.17	9.17	12	5.5	8	8	0	2.83	−2.83	0	2.5	−2.5	$\frac{1}{2}\lvert 0 \times 0 - 2.83 \times 2.5 \rvert = 3.5375$
6	9.17	12	12	5.5	8	5.5	2.83	0	−2.83	2.5	0	−2.5	$\frac{1}{2}\lvert 2.83 \times (-2.5) - 0 \times 2.5 \rvert = 3.5375$

7	0	5	0	4	4	0	5	−5	0	4	−4	0	$\frac{1}{2}\lvert 5\times(-4)-(-5)\times 0\rvert = 10$
8	0	5	5	0	4	0	5	0	−5	4	0	−4	$\frac{1}{2}\lvert 5\times(-4)-0\times 4\rvert = 10$
9	5	5	8	0	4	0	0	3	−3	4	0	−4	$\frac{1}{2}\lvert 0\times(-4)-3\times 4\rvert = 6$
10	8	5	9.17	0	4	2.83	−3	4.17	−1.17	1.17	2.83	−4	$\frac{1}{2}\lvert -3\times(-1.17)-4.17\times 4\rvert = 6.585$
11	5	9.17	9.17	4	5.5	2.83	4.17	0	−4.17	2.67	−1.17	−1.5	$\frac{1}{2}\lvert 4.17\times(-2.67)-0\times 1.5\rvert = 5.56695$
12	9.17	9.17	12	5.5	2.83	4	0	2.83	−2.83	−1.17	−1.5	2.67	$\frac{1}{2}\lvert 0\times 1.17-2.83\times(-2.67)\rvert = 3.77805$
13	9.17	12	12	5.5	5.5	4	2.83	0	−2.83	1.5	−1.5	0	$\frac{1}{2}\lvert 2.83\times(-1.5)-0\times 0\rvert = 2.1225$

EXAMPLE 18.4 —cont'd

Step 2: Determine the nodal interpolation functions. The variation of $\phi^{(e)}$ inside the element e is assumed to be linear as:

$$\phi^{(e)}(x,y) = [N(x,y)]\vec{\Phi}^{(e)} \tag{E.1}$$

where

$$[N(x,y)] = \begin{Bmatrix} N_i\ (x,y) \\ N_j\ (x,y) \\ N_k\ (x,y) \end{Bmatrix}^T = \begin{Bmatrix} (a_i + xb_i + yc_i)/2A^{(e)} \\ (a_j + xb_j + yc_j)/2A^{(e)} \\ (a_k + xb_k + yc_k)/2A^{(e)} \end{Bmatrix}^{(e)}$$

$$\vec{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{Bmatrix}$$

and the constants a_i, a_j, …, c_k are defined by Eq. (3.32). The information needed to compute $[N(x, y)]$ is given in Table 18.1 (the constants a_i, a_j, and a_k are not given because they are not required in the computations).

Step 3: Derive the element matrices using the known values of $A^{(e)}$, b_i, b_j, …, c_k. The element characteristic matrix is given by:

$$\left[K^{(e)}\right] = \iint_{A^{(e)}} [B]^T[D][B] \cdot \mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{4A^{(e)}}\begin{bmatrix} \left(b_i^2 + c_i^2\right) & (b_ib_j + c_ic_j) & (b_ib_k + c_ic_k) \\ & \left(b_j^2 + c_j^2\right) & (b_jb_k + c_jc_k) \\ \text{Symmetric} & & \left(b_k^2 + c_k^2\right) \end{bmatrix} \tag{E.2}$$

Thus, we obtain in this case,

$$\left[K^{(1)}\right] = \begin{array}{c} \begin{array}{ccc} 5 & 1 & 6 \end{array} \\ \begin{bmatrix} 1.025 & -0.625 & -0.400 \\ -0.625 & 0.625 & 0.000 \\ -0.400 & 0.000 & 0.400 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \end{array}$$

$$\left[K^{(2)}\right] = \begin{array}{c} \begin{array}{ccc} 6 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.625 & 0.000 & -0.625 \\ 0.000 & 0.400 & -0.400 \\ -0.625 & -0.400 & 1.025 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 1 \\ 2 \end{array}$$

$$\left[K^{(3)}\right] = \begin{array}{c} \begin{array}{ccc} 6 & 2 & 7 \end{array} \\ \begin{bmatrix} 0.7086 & -0.4088 & -0.2998 \\ -0.4088 & 0.5887 & -0.1799 \\ -0.2998 & -0.1799 & 0.4796 \end{bmatrix} \end{array} \begin{array}{c} \\ 6 \\ 2 \\ 7 \end{array}$$

EXAMPLE 18.4 —cont'd

$$[K^{(4)}] = \begin{array}{c} \begin{array}{ccc} 7 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.8340 & 0.0000 & -0.8340 \\ 0.0000 & 0.2998 & -0.2998 \\ -0.8340 & -0.2998 & 1.1338 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 2 \\ 3 \end{array}$$

$$[K^{(5)}] = \begin{array}{c} \begin{array}{ccc} 7 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.5660 & -0.5660 & 0.0000 \\ -0.5660 & 1.0077 & -0.4417 \\ 0.0000 & -0.4417 & 0.4417 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 3 \\ 4 \end{array}$$

$$[K^{(6)}] = \begin{array}{c} \begin{array}{ccc} 7 & 4 & 8 \end{array} \\ \begin{bmatrix} 0.4417 & 0.0000 & -0.4417 \\ 0.0000 & 0.5660 & -0.5660 \\ -0.4417 & -0.5660 & 1.0077 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 4 \\ 8 \end{array}$$

$$[K^{(7)}] = \begin{array}{c} \begin{array}{ccc} 5 & 6 & 9 \end{array} \\ \begin{bmatrix} 1.025 & -0.400 & -0.625 \\ -0.400 & 0.400 & 0.000 \\ -0.625 & 0.000 & 0.625 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 9 \end{array}$$

$$[K^{(8)}] = \begin{array}{c} \begin{array}{ccc} 9 & 6 & 10 \end{array} \\ \begin{bmatrix} 0.400 & 0.000 & -0.400 \\ 0.000 & 0.625 & -0.625 \\ -0.400 & -0.625 & 1.025 \end{bmatrix} \end{array} \begin{array}{c} 9 \\ 6 \\ 10 \end{array}$$

$$[K^{(9)}] = \begin{array}{c} \begin{array}{ccc} 10 & 6 & 11 \end{array} \\ \begin{bmatrix} 1.0417 & -0.3750 & -0.6667 \\ -0.3750 & 0.3750 & 0.0000 \\ -0.6667 & 0.0000 & 0.6677 \end{bmatrix} \end{array} \begin{array}{c} 10 \\ 6 \\ 11 \end{array}$$

$$[K^{(10)}] = \begin{array}{c} \begin{array}{ccc} 11 & 6 & 12 \end{array} \\ \begin{bmatrix} 0.7121 & -0.0595 & -0.6526 \\ -0.0595 & 0.3560 & -0.2965 \\ -0.6526 & -0.2965 & 0.9491 \end{bmatrix} \end{array} \begin{array}{c} 11 \\ 6 \\ 12 \end{array}$$

$$[K^{(11)}] = \begin{array}{c} \begin{array}{ccc} 6 & 7 & 12 \end{array} \\ \begin{bmatrix} 0.3201 & -0.1403 & -0.1799 \\ -0.1403 & 0.8424 & -0.7021 \\ -0.1799 & -0.7021 & 0.8819 \end{bmatrix} \end{array} \begin{array}{c} 6 \\ 7 \\ 12 \end{array}$$

Continued

EXAMPLE 18.4 —cont'd

$$\left[K^{(12)}\right]=\begin{array}{c} \begin{array}{ccc} 7 & 12 & 13 \end{array} \\ \begin{bmatrix} 0.6505 & -0.4138 & -0.2067 \\ -0.4138 & 0.6788 & -0.2650 \\ -0.2067 & -0.2650 & 0.4717 \end{bmatrix} \end{array}\begin{array}{c} 7 \\ 12 \\ 13 \end{array}$$

$$\left[K^{(13)}\right]=\begin{array}{c} \begin{array}{ccc} 7 & 8 & 13 \end{array} \\ \begin{bmatrix} 0.2650 & -0.2650 & 0.0000 \\ -0.2650 & 1.2084 & -0.9433 \\ 0.0000 & -0.9433 & 0.9433 \end{bmatrix} \end{array}\begin{array}{c} 7 \\ 8 \\ 13 \end{array}$$

To compute the element characteristic vectors, we use Eq. (18.21) and obtain:

$$\vec{P}^{(e)} = -\int_{C_2} V_0[N]^T \cdot dC_2 = -V_0\int_{s_i}^{s_j}\begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix} ds = -\frac{V_0 s_{ji}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{E.3}$$

if the velocity of the fluid leaving the edge ij is specified as V_0. Similarly, we obtain:

$$\vec{P}^{(e)} = -\frac{V_0 s_{kj}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \text{ if the velocity of the fluid leaving the edge } jk \text{ is specified as } V_0, \text{ and} \tag{E.4}$$

$$\vec{P}^{(e)} = -\frac{V_0 s_{ik}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \text{if the velocity of the fluid leaving the edge } ki \text{ is specified as } V_0. \tag{E.5}$$

In Eqs. (E.3)–(E.5), s_{ji}, s_{kj}, and s_{ik} denote the lengths of the edges ij, jk, and ki, respectively.
In the current case, the velocity entering the boundary AB is prescribed as $u_0 = 1$ (or $V_0 = -1$), and hence the vectors $\vec{P}^{(e)}$ will be nonzero only for elements 1 and 7. These nonzero vectors can be computed as follows:

$$\vec{P}^{(1)} = -\frac{1\times 4}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 \\ 0 \end{Bmatrix}\begin{array}{l} 5 \leftarrow \text{global node number} \\ 1 \\ 6 \end{array}$$

(For element 1, the specified velocity is along the edge ij, i.e., 12.)

$$\vec{P}^{(7)} = -\frac{1\times 4}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \\ 2 \end{Bmatrix}\begin{array}{l} 5 \leftarrow \text{global node number} \\ 6 \\ 9 \end{array}$$

(For element 7, the specified velocity is along the edge ik, i.e., 13.)

Step 4: Assemble the element matrices and vectors to obtain the overall system matrix and vector as follows:

EXAMPLE 18.4 —cont'd

$$\left[\underset{\sim}{K}\right] = \begin{array}{cc} \begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array} & \\ \begin{bmatrix} 1.0250 & -0.4000 & 0 & 0 & -0.6250 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4000 & 1.9135 & -0.2998 & 0 & 0 & -1.0338 & -0.1799 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.2998 & 2.1415 & -0.4417 & 0 & 0 & -1.4000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4417 & 1.0077 & 0 & 0 & 0 & -0.5660 & 0 & 0 & 0 & 0 & 0 \\ -0.6250 & 0 & 0 & 0 & 2.0500 & -0.800 & 0 & 0 & -0.6250 & 0 & 0 & 0 & 0 \\ 0 & -1.0338 & 0 & 0 & -0.800 & 3.8097 & -0.4401 & 0 & 0 & -1.0000 & -0.0595 & -0.4765 & 0 \\ 0 & -0.1799 & -1.4000 & 0 & 0 & -0.4401 & 4.0492 & -0.7067 & 0 & 0 & 0 & -1.1159 & -0.2067 \\ 0 & 0 & 0 & -0.5660 & 0 & 0 & -0.7067 & 2.2161 & 0 & 0 & 0 & 0 & -0.9433 \\ 0 & 0 & 0 & 0 & -0.6250 & 0 & 0 & 0 & 1.0250 & -0.4000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.0000 & 0 & 0 & -0.4000 & 2.0667 & -0.6667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0595 & 0 & 0 & 0 & -0.6667 & 1.3788 & -0.6526 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4764 & -1.1159 & 0 & 0 & 0 & -0.6526 & 2.5098 & -0.2650 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2067 & -0.9433 & 0 & 0 & 0 & -0.2650 & 1.4150 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array} \end{array}$$

Continued

EXAMPLE 18.4 —cont'd

$$\vec{\underset{\sim}{P}} = \begin{Bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 5: Solve the assembled equations:

$$[\underset{\sim}{K}]\vec{\underset{\sim}{\Phi}} = \vec{\underset{\sim}{P}} \tag{E.6}$$

after incorporating the boundary conditions specified on C_1. In the current case, the value of ϕ is set equal to zero along CD. Thus, the boundary conditions to be satisfied are $\Phi_4 = \Phi_8 = \Phi_{13} = 0$. One way to incorporate these boundary conditions is to delete the rows and columns corresponding to these degrees of freedom from Eq. (E.6). Another method is to modify the matrix $[\underset{\sim}{K}]$ and vector $\vec{\underset{\sim}{P}}$, as indicated in Section 6.5.

The solution to the resulting equations is given by:

$$\vec{\underset{\sim}{\Phi}} = \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \\ \Phi_{10} \\ \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \end{Bmatrix} = \begin{Bmatrix} 14.900 \\ 9.675 \\ 4.482 \\ 0.000 \\ 15.044 \\ 10.011 \\ 4.784 \\ 0.000 \\ 15.231 \\ 10.524 \\ 8.469 \\ 6.229 \\ 0.000 \end{Bmatrix}$$

From these nodal values of ϕ, the average value of the u component of velocity between nodes 7 and 8 can be computed as:

$$(u)_{7-8} = \frac{\partial \phi}{\partial x} \approx \frac{\Phi_8 - \Phi_7}{x_8 - x_7} = \frac{0.000 - 4.784}{9.17 - 12.00} = 1.690$$

18.4 STREAM FUNCTION FORMULATION

In the stream function formulation, the problem can be stated as follows.

18.4.1 Differential Equation Form

Find the stream function $\psi\ (x, y)$ in the given region S surrounded by the curve C such that:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \text{ in } S \tag{18.25}$$

with the following boundary conditions:

$$\text{Dirichlet condition;}\ \psi = \psi_0 \text{ on } C_1 \tag{18.26}$$

$$\text{Neumann condition;}\ V_s = -\frac{\partial\psi}{\partial n} = V_0 \text{ on } C_2 \tag{18.27}$$

where V_0 is the velocity of the fluid parallel to boundary C_2.

18.4.2 Variational Form

Find the stream function $\psi(x, y)$ that minimizes the functional:

$$I = \frac{1}{2}\iint_S \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] \mathrm{d}S - \int_{C_2} V_0\psi \mathrm{d}C_2 \tag{18.28}$$

with the boundary condition:

$$\psi = \psi_0 \text{ on } C_1 \tag{18.29}$$

18.4.3 Finite Element Solution

Because the governing equations (18.25) – (18.27) and (18.28, 18.29) are similar to Eqs. (18.8) – (18.10) and (18.11, 18.12), the finite element equations will also be similar. These can be expressed as:

$$\left[K^{(e)}\right]\vec{\Psi}^{(e)} = \vec{P}^{(e)} \tag{18.30}$$

where $[K^{(e)}]$ and $\vec{P}^{(e)}$ are given by Eqs. (18.20) and (18.21), respectively, and:

$$\vec{\Psi}^{(e)} = \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_p \end{Bmatrix} \tag{18.31}$$

EXAMPLE 18.5

Find the velocity distribution along the vertical center line CD in Fig. 18.1.

Solution

Here also, we consider only quadrant $ABCDEA$ for analysis. Boundaries AED and BC can be seen to be streamlines. We assume the value of the streamline along AED to be zero as a reference value. Because the velocity component entering the face AB is constant, we obtain:

$$\int_{\psi_A}^{\psi_B} \mathrm{d}\psi = \psi_B - \psi_A \equiv \int_{y=y_A}^{y=y_B} u_0\, \mathrm{d}y = u_0(y_B - y_A) \tag{E.1}$$

Continued

EXAMPLE 18.5 —cont'd

Because $u_0 = 1$ and the streamline passing through A, namely ψ_A, is taken as 0, Eq. (E.1) gives:

$$\psi_B - 0 = 1(4 - 0) \text{ or } \psi_B = 4 \tag{E.2}$$

Because u_0 is constant along AB, the value of ψ varies linearly along AB, and hence the value of ψ at node 5 (Ψ_5) will be equal to 2. Thus, the Dirichlet boundary conditions are given by:

$$\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 4, \quad \Psi_5 = 2, \quad \Psi_9 = \Psi_{10} = \Psi_{11} = \Psi_{12} = \Psi_{13} = 0 \tag{E.3}$$

Because no velocity is specified parallel to any boundary, the element characteristic vectors $\vec{P}^{(e)}$ will be $\vec{0}$ for all elements. The element characteristic matrices $[K^{(e)}]$ and the assembled matrix $\left[\underset{\sim}{K}\right]$ will be same as those derived in Example 18.4. Thus, the final matrix equations to be solved are given by:

$$\underset{13\times13}{\left[\underset{\sim}{K}\right]}\ \underset{13\times1}{\vec{\underset{\sim}{\Psi}}} = \underset{13\times1}{\vec{\underset{\sim}{P}}} \tag{E.4}$$

where $\left[\underset{\sim}{K}\right]$ is the same as the one derived in Example 18.4, $\vec{\underset{\sim}{\Psi}}^T = (\Psi_1 \ \Psi_2 \ \Psi_3 \ \ldots \ \Psi_{13})$ = vector of nodal unknowns, and $\vec{\underset{\sim}{P}}^T = (0 \ 0 \ 0\ldots0)$ = vector of nodal actions. The boundary conditions to be satisfied are given by Eq. (E.3). The solution of this problem can be obtained as:

$$\vec{\underset{\sim}{\Psi}} = \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \\ \Psi_9 \\ \Psi_{10} \\ \Psi_{11} \\ \Psi_{12} \\ \Psi_{13} \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 4.0 \\ 4.0 \\ 4.0 \\ 2.0 \\ 1.741 \\ 2.042 \\ 1.673 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix} \tag{E.5}$$

Because the stream function has been assumed to vary linearly within each element, the velocity will be a constant in each element. Thus, the u component of velocity, for example, can be computed between any two nodes i and j as:

$$u = \frac{\partial \Psi}{\partial y} \approx \frac{\Psi_j - \Psi_i}{y_j - y_i}$$

where y_i and y_j denote the y coordinates of nodes i and j, respectively. By using this formula, we can obtain the value of u along the line 4–8–13 as:

$$(u)_{4-8} = \frac{\Psi_8 - \Psi_4}{y_8 - y_4} = \frac{1.673 - 4.0}{5.5 - 8.0} = 0.9308$$

$$(u)_{8-13} = \frac{\Psi_{13} - \Psi_8}{y_{13} - y_8} = \frac{0 - 1.673}{4.0 - 5.5} = 1.1153$$

REVIEW QUESTIONS

18.1 Give brief answers to the following questions.

1. Give two examples of ideal flow.
2. State the potential equation in two dimensions.
3. Express the velocity components in terms of potential function.
4. State the two-dimensional ideal flow problem in terms of stream function.
5. Express the velocity components in terms of the stream function.
6. What is the Dirichlet condition in a potential function formulation?
7. What is the Dirichlet condition in a stream function formulation?
8. What is the Neumann condition in a stream function formulation?

18.2 Fill in the blank with a suitable word.

1. Inviscid incompressible flow is also called ________ ideal flow.
2. Two-dimensional potential flow problems can be formulated in terms of velocity potential function or a ________ function.
3. The Neumann condition in a potential function formulation is ________.

18.3 Indicate whether the following statement is true or false.

1. Potential flow is the same as irrotational flow.
2. The Galerkin method is a type of weighted residual method.
3. The least squares method is not applicable for deriving finite element equations.
4. It is necessary to incorporate the Neumann condition in the finite element solution of a two-dimensional potential flow problem.

PROBLEMS

18.1 Unsteady fluid flow through a porous medium (seepage flow) is governed by the equation:

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial \phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial \phi}{\partial y}\right]+\frac{\partial}{\partial z}\left[k_z\frac{\partial \phi}{\partial z}\right]+\dot{q}=\alpha\frac{\partial \phi}{\partial t} \tag{P.1}$$

with boundary conditions:

$$\phi=\phi_0 \text{ on } S_1 \tag{P.2}$$

$$k_x\frac{\partial \phi}{\partial x}l_x+k_y\frac{\partial \phi}{\partial y}l_y+k_z\frac{\partial \phi}{\partial z}l_z+q(t)=0 \text{ on } S_2 \tag{P.3}$$

where k_x, k_y, and k_z are the coefficients of permeability in the x, y, and z directions; $\dot{q}$ is the quantity of fluid added (recharge) per unit time; α is the specific storage (for a confined flow); ϕ is the fluid potential; l_x, l_y, l_z are the direction cosines of the outward normal to surface S_2; ϕ_0 is the specified value of ϕ on the boundary S_1; and $q(t)$ is the specified value of velocity of the fluid normal to the surfaces S_2. Derive the finite element equations of the flow using the Galerkin approach.

18.2 Consider the steady-state confined seepage through a rectangular soil mass subject to a specified fluid pressure head on the left side, as indicated in Fig. 18.5. Assuming the permeabilities of the soil in the horizontal and vertical directions to be $k_x = k_y = 2$ in/s, determine the distribution of the potential in the soil mass.
Hint: The governing equation is given by:

$$k_x\frac{\partial^2 \phi}{\partial x^2}+k_y\frac{\partial^2 \phi}{\partial y^2}=0$$

subject to $\phi_1 = \phi_4 = 6$ in.

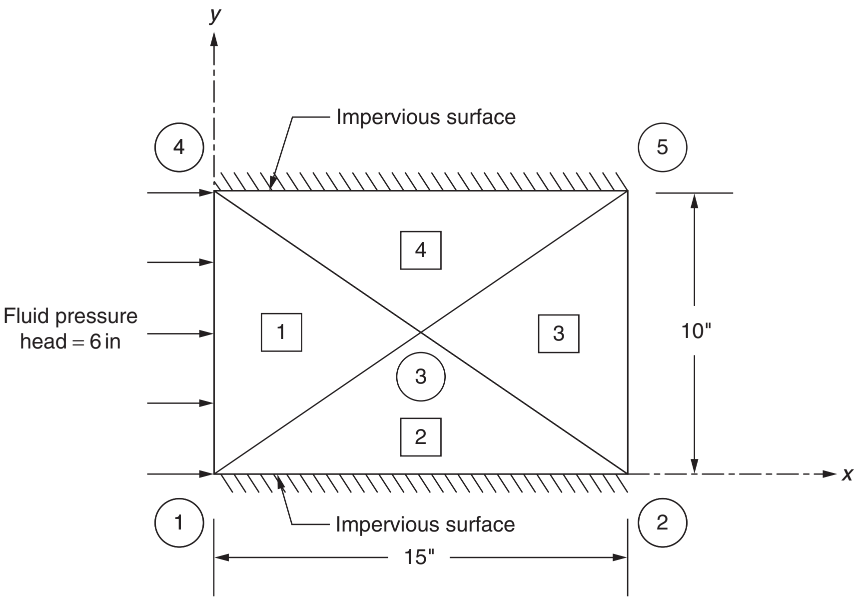

FIGURE 18.5 Seepage through a rectangular soil mass.

18.3 Determine the velocity components of the fluid for the seepage flow considered in Problem 18.2 using the nodal values of ϕ and Darcy's law:

$$u = -k_x \frac{\partial \phi}{\partial x}, \quad v = -k_y \frac{\partial \phi}{\partial y}$$

18.4 The dam shown in Fig. 18.6 retains water at a height of 12 ft on the upstream side and 2 ft on the downstream side. If the permeability of the soil, considered isotropic, is $k = 10$ ft/h, indicate a procedure to determine the following:

a. Equipotential lines

b. The quantity of water seeping into the soil per hour per 1-ft thickness of the dam (in the z direction)

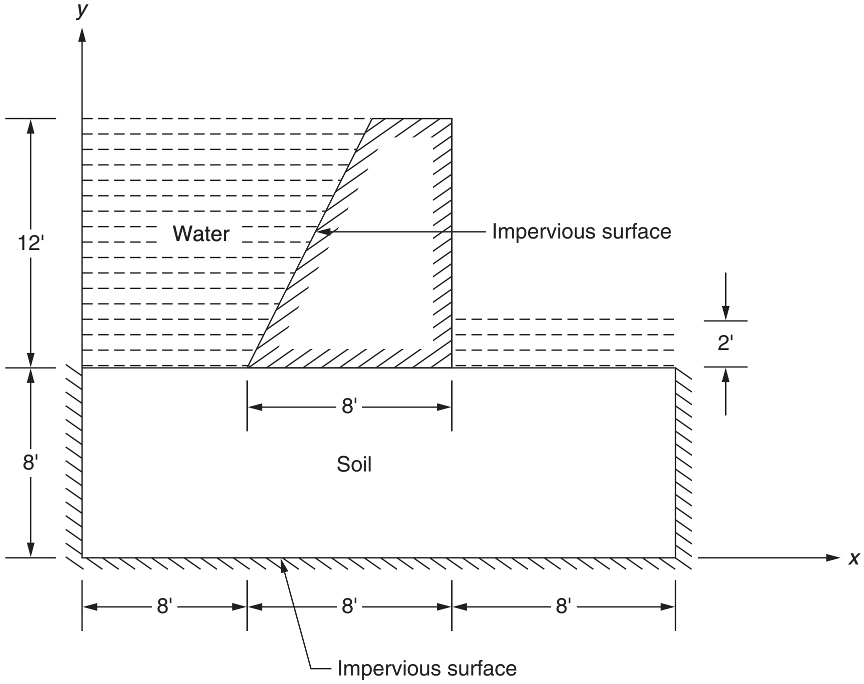

FIGURE 18.6 Dam retaining water.

18.5 In the finite element analysis of a two-dimensional flow using triangular elements, the velocity components u and v are assumed to vary linearly within an element (e) as:

$$u(x,y) = a_1 U_i^{(e)} + a_2 U_j^{(e)} + a_3 U_k^{(e)}$$

$$v(x,y) = a_1 V_i^{(e)} + a_2 V_j^{(e)} + a_3 V_k^{(e)}$$

where $\left(U_i^{(e)}, V_i^{(e)}\right)$ denote the values of (u, v) at node i. Find the relationship between $U_i^{(e)}, V_i^{(e)}, \ldots, V_k^{(e)}$ that is to be satisfied for the flow to be incompressible.

18.6 Develop the necessary finite element equations for the analysis of two-dimensional steady flow (seepage) toward a well using linear rectangular isoparametric elements.

18.7 The fluid flow in a duct is governed by the equation:

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + 1 = 0$$

with $W = 0$ on the boundary, and $W(x, y)$ is the nondimensional velocity of the fluid in the axial direction given by:

$$W(x,y) = \frac{w(x,y)}{2 w_0 f R_e}$$

where $w(x, y)$ is the axial velocity of the fluid, w_0 is the mean value of $w(x, y)$, f is the Fanning friction factor, R_e is the Reynolds number [$R_e = (w_0\, d_h/v)$], v is the kinematic viscosity of the fluid, and d_h is the hydraulic diameter of the cross-section [d_h = (4 times the area divided by perimeter)].

a. Determine the distribution of $W(x, y)$ in a rectangular duct using four linear triangles for idealization as shown in Fig. 18.7.

b. Suggest a method for finding the value of the Fanning friction factor f in each triangular element using the known nodal values of W.

c. Find the Fanning friction factor f for a flow with $R_e = 200$.

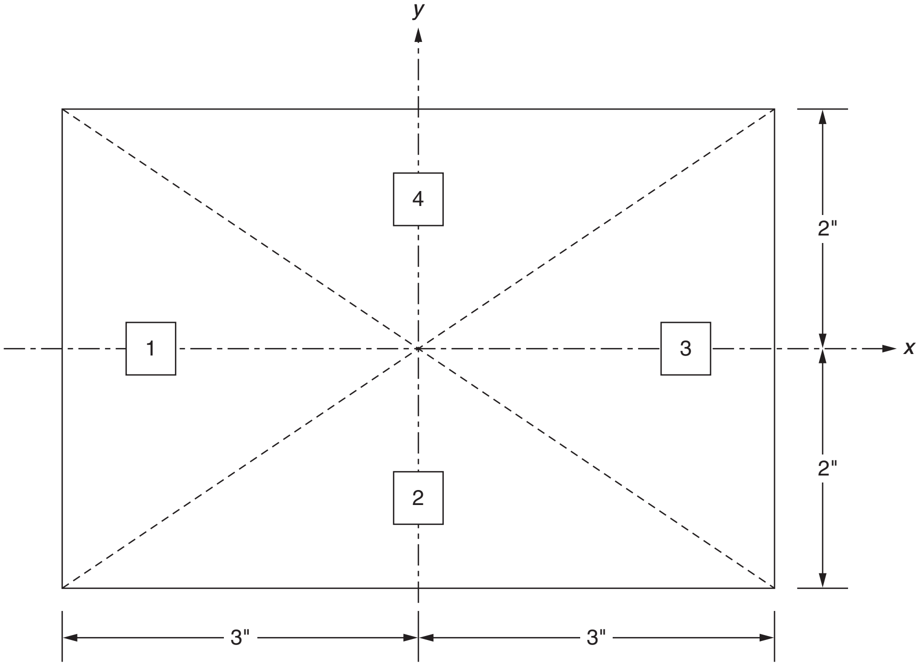

FIGURE 18.7 Fluid flow in a rectangular duct.

18.8 Find the velocity distribution along the vertical center line *CD* in Fig. 18.1 and 18.4 using a different finite element grid.

18.9 Find the direction cosines of the outward normal $\vec{n}$ of the edge *jk* of the triangular element shown in Fig. 18.2.

18.10 Find the direction cosines of the outward normal $\vec{n}$ of the edge *ki* of the triangular element shown in Fig. 18.2.

18.11 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *ij* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (4, 4)$ cm, $(x_j, y_j) = (10, 4)$ cm, and $(x_k, y_k) = (8, 10)$ cm.

18.12 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *jk* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (4, 4)$ cm, $(x_j, y_j) = (10, 4)$ cm, and $(x_k, y_k) = (8, 10)$ cm.

18.13 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *ki* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (4, 4)$ cm, $(x_j, y_j) = (10, 4)$ cm, and $(x_k, y_k) = (8, 10)$ cm.

18.14 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *ij* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (2, 4)$ cm, $(x_j, y_j) = (6, 7)$ cm, and $(x_k, y_k) = (-3, 9)$ cm.

18.15 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *jk* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (2, 4)$ cm, $(x_j, y_j) = (6, 7)$ cm, and $(x_k, y_k) = (-3, 9)$ cm.

18.16 Find the direction cosines n_x and n_y of the outward normal $\vec{n}$ of the edge *ki* of a triangular element *ijk* with nodal coordinates given by $(x_i, y_i) = (2, 4)$ cm, $(x_j, y_j) = (6, 7)$ cm, and $(x_k, y_k) = (-3, 9)$ cm.

18.17 The (x, y) components of velocity of a fluid on the edge *jk* of a triangular element *ijk* vary linearly from (u_j, v_j) at node *j* to (u_k, v_k) at node *k*. If the (x, y) coordinates of nodes *j* and *k* are (x_j, y_j) and (x_k, y_k), respectively, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

18.18 The (x, y) components of velocity of a fluid on the edge *ki* of a triangular element *ijk* vary linearly from (u_k, v_k) at node *k* to (u_i, v_i) at node *i*. If the (x, y) coordinates of nodes *k* and *i* are (x_k, y_k) and (x_i, y_i), respectively, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

18.19 The (x, y) components of velocity of a fluid on the edge *ij* of a triangular element *ijk* vary linearly from $(u_i, v_i) = (12, 8)$ cm/s at node *i* to $(u_j, v_j) = (17, 13)$ cm/s at node *j*. If the (x, y) coordinates of nodes *i* and *j* are $(x_i, y_i) = (4, 4)$ cm and $(x_j, y_j) = (10, 4)$ cm, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

18.20 The (x, y) components of velocity of a fluid on the edge *jk* of a triangular element *ijk* vary linearly from $(u_j, v_j) = (5, 10)$ cm/s at node *j* to $(u_k, v_k) = (2, 4)$ cm/s at node *k*. If the (x, y) coordinates of nodes *j* and *k* are $(x_j, y_j) = (10, 4)$ cm and $(x_k, y_k) = (8, 10)$ cm, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

18.21 The (x, y) components of velocity of a fluid on the edge *ki* of a triangular element *ijk* vary linearly from $(u_k, v_k) = (1, 5)$ cm/s at node *k* to $(u_i, v_i) = (5, 2)$ cm/s at node *i*. If the (x, y) coordinates of nodes *i* and *k* are $(x_i, y_i) = (4, 4)$ cm and $(x_k, y_k) = (8, 10)$ cm, determine the characteristic vector $\vec{P}^{(e)}$ of the element.

18.22 Derive finite element equations for solving inviscid and incompressible flow problems in terms of the potential function using a variational approach.

18.23 Derive finite element equations for solving inviscid and incompressible flow problems in terms of the stream function using a variational approach.

FURTHER READING

G.F. Pinder, W.G. Gray, Finite Element Simulation in Surface and Subsurface Hydrology, Academic Press, New York, 1977.

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Chapter 19

Viscous and Non-Newtonian Flows

Chapter Outline

19.1 INTRODUCTION

The basic equations governing two-dimensional steady incompressible Newtonian flow can be obtained from Eqs. (17.41) and (17.5):

$$\text{Conservation of momentum in } x \text{ direction}: \frac{\partial p}{\partial x} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = X + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{19.1}$$

$$\text{Conservation of momentum in } y \text{ direction}: \frac{\partial p}{\partial y} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = Y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{19.2}$$

$$\text{Continuity equation}: \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{19.3}$$

where X and Y denote, respectively, the x and y components of the body force per unit volume ($X = \rho B_x$, $Y = \rho B_y$). When the convective terms (terms involving ρ) in Eqs. (19.1) and (19.2) are neglected, we obtain:

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial p}{\partial x} + X = 0 \tag{19.4}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial p}{\partial x} + Y = 0 \tag{19.5}$$

which are known as Stokes equations. The boundary conditions for the problem may be specified in terms of the pressure, velocity, and velocity gradient. Three different formulations can be used to solve Eqs. (19.1)−(19.3): (1) the stream function formulation, in which stream function (ψ) is treated as the unknown function [19.1]; (2) the velocity−pressure formulation, in which u, v, and p are treated as unknowns [19.2]; and (3) the stream function−vorticity formulation, in which the stream function (ψ) and vorticity (ω) are taken as unknown field variables [19.3]. We consider all of these formulations in this chapter.

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19.2 STREAM FUNCTION FORMULATION (USING THE VARIATIONAL APPROACH)

By introducing the stream function ψ defined by Eq. (17.64), the continuity equation (19.3) can be satisfied exactly and the momentum equations can be combined to obtain a single equation in terms of ψ as (assuming the body forces to be zero):

$$\nu\nabla^4\psi + \frac{\partial\psi}{\partial x}\nabla^2\left(\frac{\partial\psi}{\partial y}\right) - \frac{\partial\psi}{\partial y}\nabla^2\left(\frac{\partial\psi}{\partial x}\right) = 0 \tag{19.6}$$

where $\nu = \mu/\rho$, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ = harmonic operator, and $\nabla^4 = (\partial^4/\partial x^4) + 2(\partial^4/\partial x^2\partial y^2) + (\partial^4/\partial y^4)$ = biharmonic operator. The nonlinear terms in Eq. (19.6), which come from the convective terms of Eqs. (19.1) and (19.2), are the ones that make this equation difficult to solve. However, approximate numerical solutions may be obtained using the finite element method if Eq. (19.6) is recast as an equivalent variational problem.

Although no universally accepted variational principle is available for Navier–Stokes equations or for Eq. (19.6), Olson [19.1] developed a pseudovariational principle to solve Eq. (19.6) using Cowper's 18–degrees of freedom triangular element (conforming bending element stated in Section 10.8). For a typical triangular element shown in Fig. 19.1, if the edge 1–2 is a boundary, the boundary conditions along the edge 1–2 will be

$$\psi = \text{constant or } \frac{\partial p}{\partial \xi} = 0 \tag{19.7}$$

and

$$\frac{\partial\psi}{\partial\eta} = \text{constant or } \sigma_{\xi\eta} = \mu\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right) = 0 \tag{19.8}$$

where (ξ, η) represent the local coordinate system and $\sigma_{\xi\eta}$ denotes the shear stress (for a Newtonian fluid). The functional I, which on minimization gives the governing differential Eq. (19.6) and the boundary conditions of Eqs. (19.7) and (19.8), is given by:

$$I(\psi) = \iint_{A^{(e)}}\left[\frac{\nu}{2}(\nabla^2\psi)^2 + \left(\underline{\frac{\partial\psi}{\partial\eta}\nabla^2\psi}\right)\frac{\partial\psi}{\partial\xi} - \left(\underline{\frac{\partial\psi}{\partial\xi}\nabla^2\psi}\right)\frac{\partial\psi}{\partial\eta}\right]\mathrm{d}\xi\,\mathrm{d}\eta + \int_{-b}^{a}\left[2\nu\underline{\frac{\partial^2\psi}{\partial\xi^2}}\frac{\partial\psi}{\partial\eta} - \left(\underline{\frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial\psi}{\partial\eta}\frac{\partial^2\psi}{\partial\xi\partial\eta}}\right)\psi\right]_{\eta=0}\mathrm{d}\xi \tag{19.9}$$

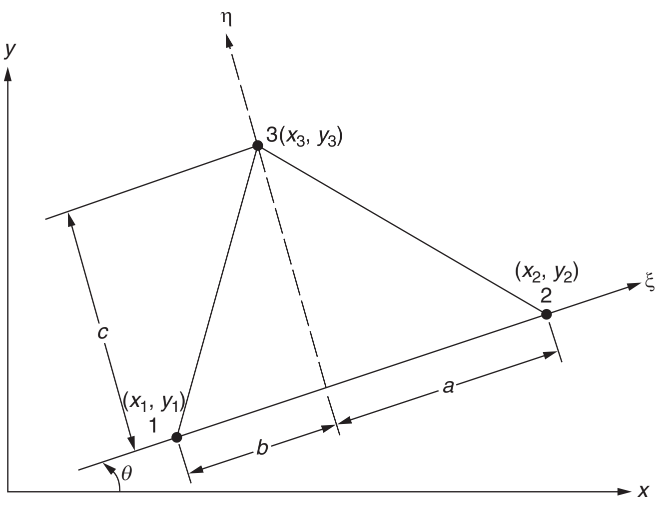

FIGURE 19.1 Triangular element.

where $A^{(e)}$ is the area of the triangular element and the underlined terms are to be taken as constants while taking the variation of I. Note that the boundary integral appears in Eq. (19.9) only when the triangular element (edge 1−2) lies on a boundary where some of the conditions of Eqs. (19.7) and (19.8) are to be satisfied.

Because the functional $I(\psi)$ contains derivatives up to an order of two, the continuity of ψ and its first derivatives between elements is to be maintained to guarantee convergence to the correct solution as the number of elements is increased. The triangular bending element owing to Cowper et al. [19.4] provides this continuity. This element employs the values of ψ, $(\partial\psi/\partial x)$, $(\partial\psi/\partial y)$, $(\partial^2\psi/\partial x^2)$, $(\partial^2\psi/\partial x\partial y)$, and $(\partial^2\psi/\partial y^2)$ at each of the three vertices as nodal variables. The interpolation function is taken as a full fifth-degree polynomial with 21 parameters as:

$$\begin{aligned}\psi(\xi,\eta) &= \alpha_1+\alpha_2\xi+\alpha_3\eta+\alpha_4\xi^2+\alpha_5\xi\eta+\alpha_6\eta^2+\alpha_7\xi^3+\alpha_8\xi^2\eta+\alpha_9\xi\eta^2+\alpha_{10}\eta^3+\alpha_{11}\xi^4+\alpha_{12}\xi^3\eta\\ &\quad+\alpha_{13}\xi^2\eta^2+\alpha_{14}\xi\eta^3+\alpha_{15}\eta^4+\alpha_{16}\xi^5+\alpha_{17}\xi^3\eta^2+\alpha_{18}\xi^2\eta^3+\alpha_{19}\xi\eta^4+\alpha_{20}\eta^5+\alpha_{21}\xi^4\eta\\ &=\sum_{i=1}^{21}\alpha_i\xi^{m_i}\eta^{n_i}\end{aligned} \tag{19.10}$$

Because the element has only 18 degrees of freedom (six at each node), the 21 constants (α_i) of Eq. (19.10) are evaluated by using 18 nodal conditions and three additional conditions. The additional conditions are chosen to ensure that the variation of the derivative of ψ normal to an edge (called normal slope) is a cubic function of the edgewise coordinate. The condition in which the normal slope $(\partial\psi/\partial\eta)$ is a cubic equation in ξ along the edge $\eta=0$ (edge 12) can be satisfied if we set $\alpha_{21}=0$ in Eq. (19.10). With this, Eq. (19.10) can be expressed as:

$$\underset{1\times1}{\psi}(\xi,\eta) = \underset{1\times20}{[\beta]}\ \underset{20\times1}{\vec{\alpha}} \tag{19.11}$$

where

$$[\beta] = \left[1\ \ \xi\ \ \eta\ \ \xi^2\ldots\eta^5\right] \tag{19.12}$$

and

$$\vec{\alpha}^T = \{\alpha_1\ \ \alpha_2\ldots\alpha_{20}\} \tag{19.13}$$

The following conditions for the cubic variation of a normal slope along the remaining two edges are more complicated [19.4].

For the cubic variation of a normal slope along edge 13:

$$5b^4c\alpha_{16}+\left(3b^2c^3-2b^4c\right)\alpha_{17}+\left(2bc^4-3b^3c^2\right)\alpha_{18}+\left(c^5-4b^2c^3\right)\alpha_{19}-5bc^4\alpha_{20} = 0 \tag{19.14}$$

For the cubic variation of a normal slope along edge 23:

$$5a^4c\alpha_{16}+\left(3a^2c^3-2a^4c\right)\alpha_{17}+\left(-2ac^4+3a^3c^2\right)\alpha_{18}+\left(c^5-4a^2c^3\right)\alpha_{19}+5ac^4\alpha_{20} = 0 \tag{19.15}$$

where dimensions a, b, and c (indicated in Fig. 19.1) are given by:

$$\left.\begin{aligned}a &= [(x_2-x_3)(x_2-x_1)+(y_2-y_3)(y_2-y_1)]/r\\ b &= [(x_3-x_1)(x_2-x_1)+(y_3-y_1)(y_2-y_1)]/r\\ c &= [(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)]/r\end{aligned}\right\} \tag{19.16}$$

where

$$r = \left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{1/2} \tag{19.17}$$

and (x_i, y_i) are the global (x, y) coordinates of node $i(i = 1, 2, 3)$. The 18 nodal unknowns of the element are given by:

$$\overrightarrow{\psi}^{(e)^T} = \{\psi_1 \ \psi_{\xi_1} \ \psi_{\eta_1} \ \psi_{\xi\xi_1} \ \psi_{\xi\eta_1} \ \psi_{\eta\eta_1} \ \psi_2 \ \psi_{\xi_2} \ \psi_{\eta_2} \ \psi_{\xi\xi_2} \ \psi_{\xi\eta_2} \ \psi_{\eta\eta_2} \ \psi_3 \ \psi_{\xi_3} \ \psi_{\eta_3} \ \psi_{\xi\xi_3} \ \psi_{\xi\eta_3} \ \psi_{\eta\eta_3}\} \tag{19.18}$$

where $\psi_i = \psi$ at node $i, \psi_{\xi\eta_i} = \left(\partial^2\psi/\partial\xi\partial\eta\right)$ at node i, and so on. Using Eqs. (19.11) and (19.18), we can obtain:

$$\underset{18\times1}{\overrightarrow{\psi}^{(e)}} = \underset{18\times20}{[\beta]} - \underset{20\times1}{\overrightarrow{\alpha}} \tag{19.19}$$

where $[\beta]$ can be derived without much difficulty. By using Eqs. (19.14), (19.15), and (19.19), we get 20 equations in 20 unknown coefficients α_i, $i = 1, 2, \ldots, 20$. These equations can be expressed in matrix form as:

$$\underset{20\times1}{\begin{Bmatrix} \overrightarrow{\psi}^{(e)} \\ 0 \\ 0 \end{Bmatrix}} = \underset{20\times20}{[\underset{\sim}{\beta}]} \ \underset{20\times1}{\overrightarrow{\alpha}} \tag{19.20}$$

where $[\underset{\sim}{\beta}]$ can be identified without much difficulty.

Eq. (19.20) can be inverted to obtain:

$$\underset{20\times1}{\overrightarrow{\alpha}} = \underset{20\times20}{[\underset{\sim}{\beta}]^{-1}} \underset{20\times1}{\begin{Bmatrix} \overrightarrow{\psi}^{(e)} \\ 0 \\ 0 \end{Bmatrix}} \tag{19.21}$$

and

$$\underset{20\times1}{\overrightarrow{\alpha}} = \underset{20\times18}{[R]} \ \underset{18\times1}{\overrightarrow{\psi}^{(e)}} \tag{19.22}$$

where the 20×18 matrix $[R]$ consists of the first 18 columns of $[\underset{\sim}{\beta}]^{-1}$. The element properties can be obtained first with respect to the polynomial coefficients α_i by substituting Eq. (19.10) into Eq. (19.9). For an interior element, this gives:

$$I = \sum_{i=1}^{21}\sum_{j=1}^{21} k_{ij}\alpha_i\alpha_j + \sum_{i=1}^{21}\sum_{j=1}^{21}\sum_{l=1}^{21} f_{ijl}\overline{\alpha}_i\overline{\alpha}_j\alpha_l \tag{19.23}$$

where

$$\begin{aligned} k_{ij} = \frac{\nu}{2}\{&m_i m_j(m_j-1)(m_i-1)g(m_i+m_j-4, n_i+n_j-1) \\ &+ [m_i n_j(m_i-1)(n_i-2) + m_j n_i(m_j-1)(n_j-2)]\cdot g(m_i+m_j-2, n_i+n_j-3) \\ &+ n_i n_j(n_i-2)(n_j-2)\cdot g(m_i+m_j, n_i+n_j-5)\} \end{aligned} \tag{19.24}$$

$$\begin{aligned} f_{ijl} = (n_i m_l - m_i n_l)[&m_j(m_j-1)\cdot g(m_i+m_j+m_l-3, n_i+n_j+n_l-3) \\ &+ n_j(n_j-2)g(m_i+m_j+m_l-1, n_i+n_j+n_l-5)] \end{aligned} \tag{19.25}$$

$$g(m,n) = \iint_{A^{(e)}} x^m y^n \mathrm{d}x\,\mathrm{d}y \tag{19.26}$$

and the bars over α_i in Eq. (19.23) indicate that those terms are not varied when the variation of I is taken. The element properties are transformed into the global coordinate system by using the basic transformation relation:

$$\vec{\psi}_i^{(e)} = [\lambda_i]\vec{\Psi}_i^{(e)} \tag{19.27}$$

where

$$\vec{\psi}_i^{(e)} = \begin{Bmatrix} \psi_i \\ \psi_{\xi_i} \\ \psi_{\eta_i} \\ \psi_{\xi\xi_i} \\ \psi_{\xi\eta_i} \\ \psi_{\eta\eta_i} \end{Bmatrix}, \qquad \vec{\Psi}_i^{(e)} = \begin{Bmatrix} \psi_i \\ \psi_{x_i} \\ \psi_{y_i} \\ \psi_{xx_i} \\ \psi_{xy_i} \\ \psi_{yy_i} \end{Bmatrix}$$

and

$$[\lambda_i] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2\theta & 2\sin\theta\cos\theta & \sin^2\theta \\ 0 & 0 & 0 & -\sin\theta\cos\theta & \left(\cos^2\theta - \sin^2\theta\right) & \sin\theta\cos\theta \\ 0 & 0 & 0 & \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \tag{19.28}$$

The final element equations can be expressed in the familiar form as:

$$\left[K^{(e)}\right]\vec{\Psi}^{(e)} + \left[F^{(e)}\left(\vec{\Psi}^{(e)}\right)\right]\vec{\Psi}^{(e)} = \vec{0} \tag{19.29}$$

where the elements of matrix $[F^{(e)}]$ are nonlinear functions of the nodal unknowns $\vec{\Psi}^{(e)}$ with

$$\vec{\Psi}^{(e)} = \begin{Bmatrix} \vec{\Psi}_1^{(e)} \\ \vec{\Psi}_2^{(e)} \\ \vec{\Psi}_3^{(e)} \end{Bmatrix} \tag{19.30}$$

The overall or assembled equations can be obtained as:

$$\left[\underset{\sim}{K}\right]\underset{\sim}{\vec{\Psi}} + \left[\underset{\sim}{F}\left(\underset{\sim}{\vec{\Psi}}\right)\right]\underset{\sim}{\vec{\Psi}} = \vec{0} \tag{19.31}$$

These equations represent a set of N nonlinear algebraic equations in the N global variables $\Psi_i, i = 1, 2, \ldots, N$. Olson [19.1] solved these equations using the Newton–Raphson method.

EXAMPLE 19.1

The finite element formulation presented previously is applied to solve for the flow over a circular cylinder, as shown in Fig. 19.2A [19.1]. The following boundary conditions are imposed on the problem:

along the upstream edge: $\psi = y, \frac{\partial\psi}{\partial x} = 0$
along the x axis: $\psi = 0$
along the upper edge: $\frac{\partial\psi}{\partial y} = 1$
along the downstream edge: $\frac{\partial\psi}{\partial x} = 0$
on the surface of the cylinder: $\psi = 0, \frac{\partial\psi}{\partial n} = 0$ (n denotes the normal to the cylinder)

Continued

EXAMPLE 19.1 —cont'd

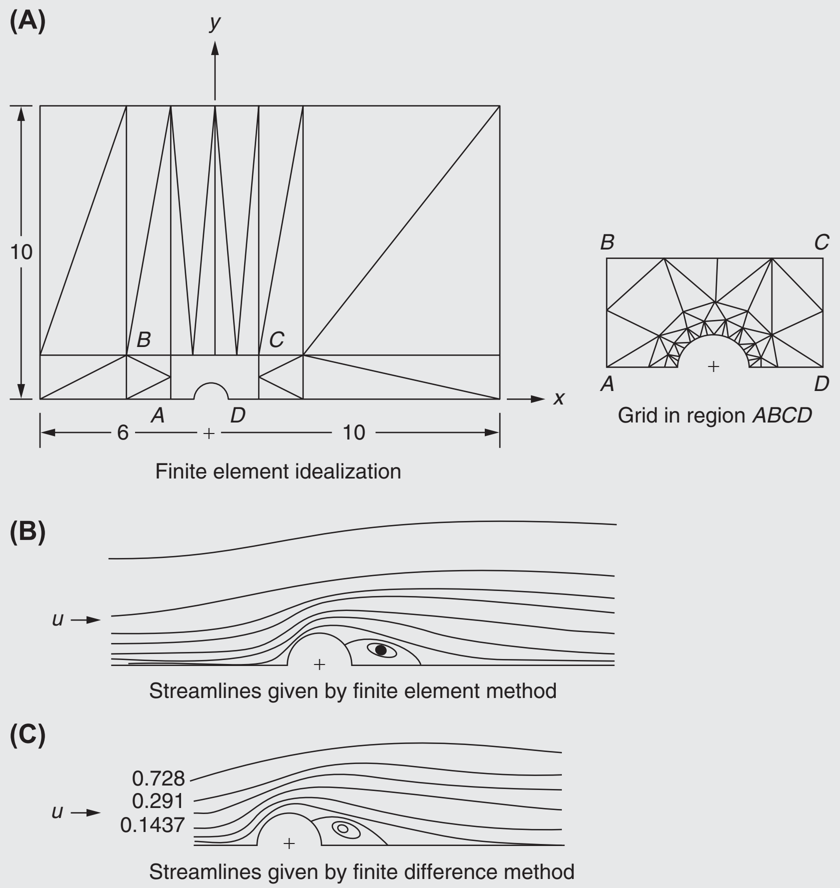

FIGURE 19.2 Flow over a circular cylinder ($R_e = 20$) [19.1].

The natural boundary conditions of zero shear stress along the symmetry line (bottom edge) and the zero-pressure gradient along the top and downstream edges are not imposed but are left for the program to approximate. The pattern of streamlines obtained from the solution for a Reynolds number of 20 is shown in Fig. 19.2B. These results compare well with those given by an accurate finite difference solution (shown in Fig. 19.2C). It can be seen that even the separation phenomenon behind the cylinder is predicted accurately.

19.3 VELOCITY–PRESSURE FORMULATION (USING THE GALERKIN APPROACH)

First, we consider the solution to the Stokes equations, Eqs. (19.4) and (19.5), and the continuity equation (19.3). We consider the pressure (p), the velocity component parallel to the x axis (u), and the velocity component parallel to the y axis (v) to be unknowns in the formulation. For a typical finite element inside the region S, the unknowns p, u, and v are assumed to vary as:

$$\begin{aligned}
p^{(e)}(x,y) &= \sum_i N_i^p(x,y)P_i^{(e)} = [N^p(x,y)]\vec{P}^{(e)} \\
u^{(e)}(x,y) &= \sum_i N_i^u(x,y)U_i^{(e)} = [N^u(x,y)]\vec{U}^{(e)} \\
v^{(e)}(x,y) &= \sum_i N_i^v(x,y)V_i^{(e)} = [N^v(x,y)]\vec{V}^{(e)}
\end{aligned} \tag{19.32}$$

where N_i^p, N_i^u, and N_i^v are the interpolation functions, which need not necessarily be of the same order, and $\overrightarrow{P}^{(e)}$, $\overrightarrow{U}^{(e)}$, and $\overrightarrow{V}^{(e)}$ are the vectors of nodal unknowns of element e. For simplicity, we assume $N_i^u = N_i^v \equiv N_i$ in the following development.

By using Galerkin's weighted residual method, we can write three sets of equations at any node i of element e. The first is given by:

$$\iint_{A^{(e)}} N_i(x,y)\left[X - \frac{\partial p^{(e)}}{\partial x} + \mu\left(\frac{\partial^2 u^{(e)}}{\partial x^2} + \frac{\partial^2 u^{(e)}}{\partial y^2}\right)\right]\mathrm{d}A^{(e)} = 0 \tag{19.33}$$

where the shape function corresponding to node i, $N_i(x, y)$, is used as the weighting function. Integrating Eq. (19.33) by parts, we obtain:

$$\iint_{A^{(e)}}\left[-N_iX - \frac{\partial N_i}{\partial x}p^{(e)} + \mu\frac{\partial N_i}{\partial x}\frac{\partial u^{(e)}}{\partial x} + \mu\frac{\partial N_i}{\partial y}\frac{\partial u^{(e)}}{\partial y}\right]\mathrm{d}A + \int_{C^{(e)}} N_i p^{(e)} l_x \cdot \mathrm{d}C - \int_{C^{(e)}} \mu N_i\left(\frac{\partial u^{(e)}}{\partial x}l_x + \frac{\partial u^{(e)}}{\partial y}l_y\right)\mathrm{d}C = 0 \tag{19.34}$$

where l_x and l_y indicate the direction cosines of the outward-drawn normal at the boundary of the element e, $C^{(e)}$. By substituting Eq. (19.32) into Eq. (19.34), we obtain:

$$\begin{aligned}&\iint_{A^{(e)}}\left[-N_iX - \frac{\partial N_i}{\partial x}[N^p]\overrightarrow{P}^{(e)} + \left(\mu\frac{\partial N_i}{\partial x}\frac{\partial [N]}{\partial x} + \mu\frac{\partial N_i}{\partial y}\frac{\partial [N]}{\partial y}\right)\overrightarrow{U}^{(e)}\right]\mathrm{d}A\\ &\quad - \int_{C^{(e)}} \mu N_i\left(\frac{\partial [N]}{\partial x}l_x + \frac{\partial [N]}{\partial y}l_y\right)\overrightarrow{U}^{(e)}\mathrm{d}C + \int_{C^{(c)}} N_i[N^p]\overrightarrow{P}^{(e)} l_x \mathrm{d}C = 0\end{aligned} \tag{19.35}$$

The second equation is similar to Eq. (19.35) and can be obtained by interchanging x and y, U and V, and X and Y in Eq. (19.35). By using N_i^p as the weighting function, the third equation arising from Eq. (19.3) is:

$$\iint_{A^{(e)}} N_i^p\frac{\partial [N]}{\partial x}\overrightarrow{U}^{(e)}\mathrm{d}A + \iint_{A^{(e)}} N_i^p\frac{\partial [N]}{\partial y}\overrightarrow{V}^{(e)}\mathrm{d}A = 0 \tag{19.36}$$

By assuming a quadratic variation for the velocity components u and v and a linear variation for the pressure inside a triangular element, the element equations can be written in matrix form as:

$$\left[K^{(e)}\right]\overrightarrow{\Phi}^{(e)} = \overrightarrow{\underline{P}}^{(e)} \tag{19.37}$$

where

$$\underset{15\times15}{\left[K^{(e)}\right]} = \begin{bmatrix} \underset{6\times6}{\left[K_1^{(e)}\right]} & \underset{6\times6}{[0]} & \underset{6\times3}{\left[K_2^{(e)}\right]} \\ \underset{6\times6}{[0]} & \underset{6\times6}{\left[K_1^{(e)}\right]} & \underset{6\times3}{\left[K_3^{(e)}\right]} \\ -\underset{3\times6}{\left[K_2^{(e)}\right]^T} & -\underset{3\times6}{\left[K_3^{(e)}\right]^T} & \underset{3\times3}{[0]} \end{bmatrix} \tag{19.38}$$

$$\underset{15\times1}{\overrightarrow{\Phi}^{(e)}} = \left\{U_1^{(e)} \quad U_2^{(e)}\ldots U_6^{(e)} \quad \middle| V_1^{(e)} \quad V_2^{(e)}\ldots V_6^{(e)} \quad \middle| -P_1^{(e)} \quad -P_2^{(e)} \quad -P_3^{(e)}\right\}^T \tag{19.39}$$

$$\underset{15\times1}{\overrightarrow{\underline{P}}^{(e)}} = \left\{\underline{P}_1^{(e)}\underline{P}_2^{(e)} \quad \ldots\underline{P}_6^{(e)} \quad \middle|\underline{P}_7^{(e)}\underline{P}_8^{(e)} \quad \ldots\underline{P}_{12}^{(e)} \quad \middle| 0 \ \ 0 \ \ 0\right\}^T \tag{19.40}$$

$$K_{1ij}^{(e)} = \iint_{A^{(e)}} \mu\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\mathrm{d}A \tag{19.41}$$

$$K_{2ij}^{(e)} = -\iint_{A^{(e)}} \frac{\partial N_i}{\partial x}N_j^p\mathrm{d}A \tag{19.42}$$

$$K_{3ij}^{(e)} = -\iint_{A^{(e)}} \frac{\partial N_i}{\partial y} N_j^p \,\mathrm{d}A \tag{19.43}$$

$$\underline{P}_i^{(e)} = \int_{C^{(e)}} N_i\left(\mu\frac{\partial u^{(e)}}{\partial n} - p^{(e)}\right) l_x \ \mathrm{d}C - \iint_{A^{(e)}} N_i X \ \mathrm{d}A, \quad i = 1, 2, \ldots, 6 \tag{19.44}$$

$$P_i^{(e)} = \int_{C^{(e)}} N_i\left(\mu\frac{\partial v^{(e)}}{\partial n} - p^{(e)}\right) l_y \,\mathrm{d}C - \iint_{A^{(e)}} N_i Y \,\mathrm{d}A, \quad i = 7, 8, \ldots, 12 \tag{19.45}$$

The element Eq. (19.37) can be assembled to obtain the overall equations as:

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{\mathrm{P}}} \tag{19.46}$$

where

$$[\underset{\sim}{K}] = \sum_{e=1}^{E}[K^{(e)}], \ \ \Phi_{\sim} = \sum_{e=1}^{E}\vec{\Phi}^{(e)}, \ \ P_{\sim} = \sum_{e=1}^{E}\underline{\vec{P}}^{(e)} \tag{19.47}$$

and E the number of elements.

19.4 SOLUTION TO NAVIER–STOKES EQUATIONS

To extend the solution of Stokes equations to the full Navier–Stokes equations, the following iterative procedure can be adopted. Let u_n, v_n, and p_n be an approximate solution (in n-th iteration) to the flow problem. Then we introduce this solution into the coefficients of convective terms of Eqs. (19.1) and (19.2) and write the set of equations as:

$$\left.\begin{aligned}
&\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\rho\left(u_n\frac{\partial u}{\partial x}+v_n\frac{\partial u}{\partial y}\right)-\frac{\partial p}{\partial x}+X = 0\\
&\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\rho\left(u_n\frac{\partial v}{\partial x}+v_n\frac{\partial v}{\partial y}\right)-\frac{\partial p}{\partial y}+Y = 0\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0
\end{aligned}\right\} \tag{19.48}$$

To start the iterative process, u_n, v_n, and p_n may be the solution to the Stokes equations. When the Galerkin procedure is applied to Eq. (19.48), we get:

$$\left.\begin{aligned}
&\iint_{A^{(e)}}\left[\mu\left(N_i\frac{\partial^2 u^{(e)}}{\partial x^2}+N_i\frac{\partial^2 u^{(e)}}{\partial y^2}\right)-\rho\left(u_n^{(e)}N_i\frac{\partial u^{(e)}}{\partial x}+v_n^{(e)}N_i\frac{\partial u^{(e)}}{\partial y}\right)-N_i\frac{\partial p^{(e)}}{\partial x}+N_iX\right]\mathrm{d}A = 0\\
&\iint_{A^{(e)}}\left[\mu\left(N_i\frac{\partial^2 v^{(e)}}{\partial x^2}+N_i\frac{\partial^2 v^{(e)}}{\partial y^2}\right)-\rho\left(u_n^{(e)}N_i\frac{\partial v^{(e)}}{\partial x}+v_n^{(e)}N_i\frac{\partial v^{(e)}}{\partial y}\right)-N_i\frac{\partial p^{(e)}}{\partial y}+N_iY\right]\mathrm{d}A = 0\\
&\iint_{A^{(e)}}\left(N_i\frac{\partial u^{(e)}}{\partial x}+N_i\frac{\partial v^{(e)}}{\partial y}\right)\mathrm{d}A = 0
\end{aligned}\right\} \tag{19.49}$$

We next integrate by parts all terms in Eq. (19.49) except those involving $u_n^{(e)}$ and $v_n^{(e)}$. In this manner, the natural boundary conditions are kept identical to those for the Stokes equations. This leads to element equations of the same form as Eq. (19.37) except that submatrix $\left[K_1^{(e)}\right]$ will be different in the current case. The elements of matrix $\left[K_1^{(e)}\right]$ in the current case are given by:

$$K_{1ij}^{(e)} = \iint_{A^{(e)}}\left[\mu\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)-\rho u_n^{(e)}N_i\frac{\partial N_j}{\partial x}-\rho v_n^{(e)}N_i\frac{\partial N_j}{\partial y}\right]\mathrm{d}A \tag{19.50}$$

Note that the total elemental matrix $[K^{(e)}]$ will be unsymmetrical because of the presence of the convective terms. The overall system equations can be expressed as:

$$\underset{\sim}{K}(u_n, v_n)\vec{\underset{\sim}{\Phi}}_{n+1} = \vec{\underset{\sim}{P}}_{n+1} \tag{19.51}$$

where the subscripts n and $n+1$ indicate the successive stages of iteration. Thus, Eq. (19.51) is to be solved successively for $\vec{\underset{\sim}{\Phi}}_{n+1}$ using the nodal values of u and v obtained in the previous iteration. This process is continued until the two vectors $\vec{\underset{\sim}{\Phi}}_n$ and $\vec{\underset{\sim}{\Phi}}_{n+1}$ are sufficiently close.

EXAMPLE 19.2

Yamada et al. [19.5] considered the problem of flow past a circular cylinder, to illustrate the previous procedure. As indicated in Fig. 19.3, the infinite field of flow is confined by a circle of radius $r/a = 8.0$ (or 20.0) and the region is divided into finite elements. The boundary conditions at these radii are specified as $u = 5.0$, $v = 0$, and $p = 0$ for a Reynolds number of 30.0. The boundary conditions on the surface of the cylinder ($r = a$) are taken as $u = 0$ and $v = 0$. The velocity distribution obtained by solving Stokes equations is given by Fig. 19.4. For the Navier–Stokes equations, the velocity distribution becomes unsymmetrical owing to the inclusion of convective terms, as shown in Fig. 19.5. The convergence of Navier–Stokes equations has been obtained in eight iterations starting from the Stokes solution.

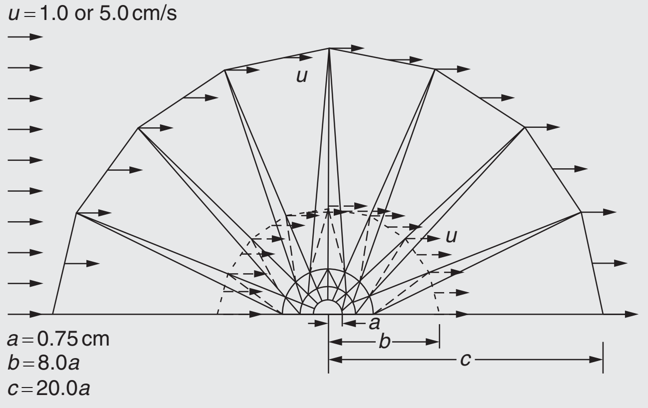

FIGURE 19.3 Boundary conditions and finite element mesh for the analysis of the flow past a circular cylinder [19.5].

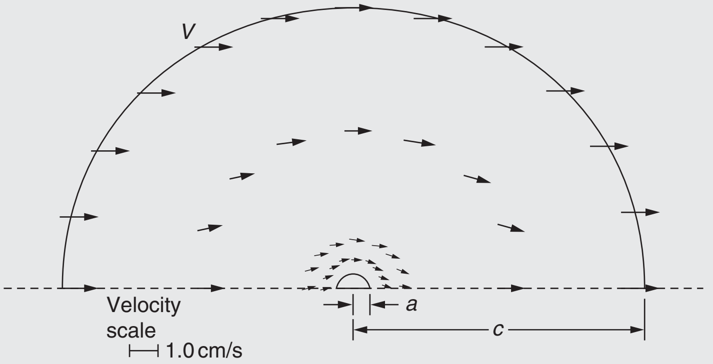

FIGURE 19.4 Velocity solution of Stokes equations [19.5].

Continued

EXAMPLE 19.2 —cont'd

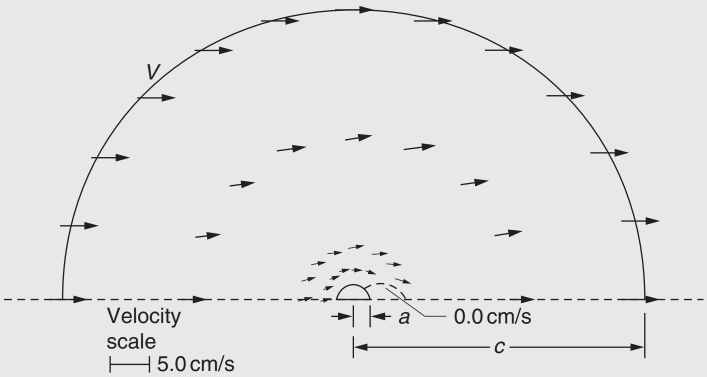

FIGURE 19.5 Velocity solution of Navier–Stokes equations ($R_e = 30$) [19.5].

19.5 STREAM FUNCTION–VORTICITY FORMULATION

19.5.1 Governing Equations

In the stream function formulation, the final equation governing the incompressible viscous flows is of the fourth order (see Eq. 19.6). If we choose the stream function (ψ) and vorticity (ω) as the unknowns, the governing equations will be two second-order equations coupled in ψ and ω,as shown subsequently [19.6]. From the definition of vorticity in two dimensions (see Eq. 17.54c),

$$\omega = \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{19.52}$$

By substituting the expressions of u and v from Eq. (17.64), Eq. (19.52) gives:

$$\omega = -\frac{1}{2}\nabla^2\psi \tag{19.53}$$

Eq. (19.6) can be rewritten as:

$$\nu\nabla^2\left(\nabla^2\psi\right) + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right) = 0 \tag{19.54}$$

By substituting Eq. (19.53) into Eq. (19.54), we obtain:

$$\nu\nabla^2\omega + \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} = 0 \tag{19.55}$$

For unsteady-state problems, Eq. (19.55) will be modified as:

$$\nu\nabla^2\omega + \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\omega}{\partial t} = 0 \tag{19.56}$$

Thus, Eqs. (19.53) and (19.56) represent two coupled second-order equations (Eq. 19.56 is nonlinear) governing the transient incompressible viscous flows.

19.5.2 Finite Element Solution (Using a Variational Approach)

Cheng [19.3] presented an iterative procedure based on a quasivariational approach to the solution of the differential Eqs. (19.53) and (19.56). In this method, the solution at the n-th time step (ψ_n, ω_n) is assumed to be known and the solution at the $n+1$-th time step is determined by solving the following set of linear differential equations:

$$-\nabla^2 \psi_{n+1} = \omega_n \tag{19.57}$$

$$\nu \nabla^2 \omega_{n+1} + \frac{\partial \psi_{n+1}}{\partial x} \frac{\partial \omega_n}{\partial y} - \frac{\partial \psi_{n+1}}{\partial y} \frac{\partial \omega_n}{\partial x} - \frac{\partial \omega_n}{\partial t} = 0 \tag{19.58}$$

The functionals I_1 and I_2, whose Euler–Lagrange equations yield Eqs. (19.57) and (19.58), respectively, are given by:

$$I_1 = \frac{1}{2} \iint_S \left[\left(\frac{\partial \psi_{n+1}}{\partial x} \right)^2 + \left(\frac{\partial \psi_{n+1}}{\partial y} \right)^2 - 2\omega_n \psi_{n+1} \right] \mathrm{d}S \tag{19.59}$$

$$I_2 = \frac{1}{2} \iint_S \left[\nu \left\{ \left(\frac{\partial \omega_{n+1}}{\partial x} \right)^2 + \left(\frac{\partial \omega_{n+1}}{\partial y} \right)^2 \right\} + 2 \left(-\frac{\partial \psi_{n+1}}{\partial x} \frac{\partial \omega_n}{\partial y} + \frac{\partial \psi_{n+1}}{\partial y} \frac{\partial \omega_n}{\partial x} + \underline{\frac{\partial \omega_n}{\partial t}} \right) \omega_{n+1} \right] \mathrm{d}S \tag{19.60}$$

where the underlined term in Eq. (19.60) is to be taken as a constant while taking the variation of I_2. Because the functions I_1 and I_2 involve only the first derivatives of ψ and ω, the interpolation functions need to satisfy only the C^0 continuity. By assuming the variations of ψ and ω inside an element e at the time step $n+1$ as:

$$\psi_{n+1}^{(e)}(x, y, t) = [N(x, y)] \vec{\Psi}_{n+1}^{(e)}(t) \tag{19.61}$$

$$\omega_{n+1}^{(e)}(x, y, t) = [N(x, y)] \vec{\Omega}_{n+1}^{(e)}(t) \tag{19.62}$$

the element equations can be derived from the conditions $\delta I_1 = 0$ and $\delta I_2 = 0$ as:

$$\left[K_1^{(e)}\right] \vec{\Psi}_{n+1}^{(e)} + \vec{P}_{1n}^{(e)} = \vec{0} \tag{19.63}$$

$$\left[K_2^{(e)}\right] \dot{\vec{\Omega}}_{n+1}^{(e)} + \left[K_3^{(e)}\right] \vec{\Omega}_{n+1}^{(e)} + \vec{P}_{2n}^{(e)} = \vec{0} \tag{19.64}$$

where

$$K_{1ij}^{(e)} = \iint_{A^{(e)}} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \mathrm{d}A \tag{19.65}$$

$$K_{2ij}^{(e)} = \iint_{A(e)} N_i N_j \cdot \mathrm{d}A \tag{19.66}$$

$$K_{3ij}^{(e)} = \nu K_{1ij}^{(e)} \tag{19.67}$$

$$P_{1n_i} = -\iint_{A(e)} \omega_n^{(e)} N_i \, \mathrm{d}A \tag{19.68}$$

$$P_{2n_i}^{(e)} = \iint_{A(e)} N_i \left(\frac{\partial \psi_{n+1}^{(e)}}{\partial y} \frac{\partial \omega_n^{(e)}}{\partial x} - \frac{\partial \psi_{n+1}^{(e)}}{\partial x} \frac{\partial \omega_n}{\partial y} \right) \cdot \mathrm{d}A \tag{19.69}$$

and $A^{(e)}$ denotes the area of element e. Eqs. (19.63) and (19.64) can be assembled in the usual manner to obtain the overall system equations as:

$$\left[\underset{\sim}{K_1}\right]\underset{\sim}{\vec{\Psi}}_{n+1}+\underset{\sim}{\vec{P}}_{1n}=\vec{0} \tag{19.70}$$

$$\left[\underset{\sim}{K_2}\right]\dot{\underset{\sim}{\vec{\Omega}}}_{n+1}+\left[\underset{\sim}{K_3}\right]\underset{\sim}{\vec{\Omega}}_{n+1}+\underset{\sim}{\vec{P}}_{2n}=\vec{0} \tag{19.71}$$

By approximating the vector of time derivatives $\dot{\underset{\sim}{\vec{\Omega}}}_{n+1}$ in Eq. (19.71) as:

$$\dot{\underset{\sim}{\vec{\Omega}}}_{n+1}=\begin{Bmatrix}\dfrac{\partial\omega}{\partial t}(\text{at node } 1)\\ \dfrac{\partial\omega}{\partial t}(\text{at node } 2)\\ \vdots\end{Bmatrix}_{n+1}=\frac{1}{\Delta t}\left(\underset{\sim}{\vec{\Omega}}_{n+1}-\underset{\sim}{\vec{\Omega}}_{n}\right) \tag{19.72}$$

where the time interval Δt is given by $\Delta t=t_{n+1}-t_n$, Eq. (19.71) can be expressed as:

$$\left(\left[\underset{\sim}{K_3}\right]+\frac{1}{\Delta t}\left[\underset{\sim}{K_2}\right]\right)\underset{\sim}{\vec{\Omega}}_{n+1}=\frac{1}{\Delta t}\left[\underset{\sim}{K_2}\right]\underset{\sim}{\vec{\Omega}}_{n}-\underset{\sim}{\vec{P}}_{2n} \tag{19.73}$$

Eqs. (19.70) and (19.73) represent the final matrix equations to be solved after applying the known boundary conditions. The solution procedure starts with known initial values of $\underset{\sim}{\vec{\Omega}}_n$ (for $n=0$). Once $\underset{\sim}{\vec{\Omega}}_n$ is known, $\underset{\sim}{\vec{P}}_{1n}$ can be evaluated using Eq. (19.68), and then Eq. (19.70) can be solved for $\underset{\sim}{\vec{\Psi}}_{n+1}$. From the known values of $\underset{\sim}{\vec{\Omega}}_n$ and $\underset{\sim}{\vec{\Psi}}_{n+1}$ the vector $\underset{\sim}{\vec{P}}_{2n}$ can be determined from Eq. (19.69), and hence Eq. (19.73) can be solved for $\underset{\sim}{\vec{\Omega}}_{n+1}$. This recursive procedure is continued until the solution at the specified final time is found.

19.6 FLOW OF NON-NEWTONIAN FLUIDS

19.6.1 Governing Equations

FLOW CURVE CHARACTERISTIC

Many practical fluid flows do not follow Newton's law of viscosity and the assumption of a constant viscosity independent of temperature, density, and shear rate does not hold. Such types of fluids are called non-Newtonian fluids. The flow of many crude oils, especially at low temperatures, as well as industrial wastes, slurries, and suspensions of all kinds fall under the category of non-Newtonian fluids. The shear stress τ of a Newtonian fluid in uniaxial flow is given by (Fig. 19.6):

$$\tau=-\frac{\mu}{g}\frac{\mathrm{d}u}{\mathrm{d}y} \tag{19.74}$$

where μ is the coefficient of viscosity [units = mass/(length × time)], g is the acceleration owing to gravity, and $(\mathrm{d}u/\mathrm{d}y)$ is the velocity gradient that is equivalent to the shear rate. Here, the quantity (μ/g) is a constant and is independent of the shear rate. Eq. (19.74) is represented as a linear curve in Fig. 19.7. Certain fluids, known as Bingham plastic fluids, behave as a rigid solid until a certain level of shear stress (τ_0) is attained; they behave as a Newtonian fluid afterward.

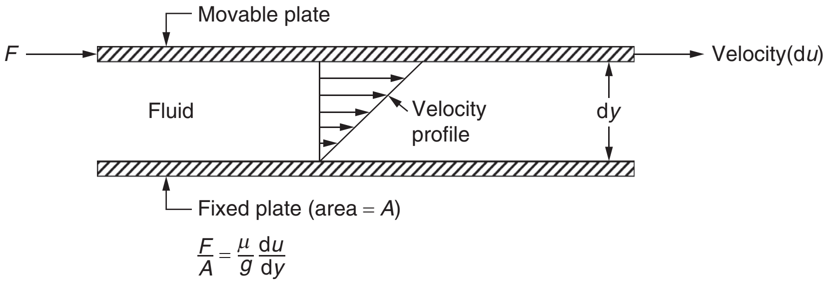

FIGURE 19.6 Definition of viscosity for a Newtonian fluid.

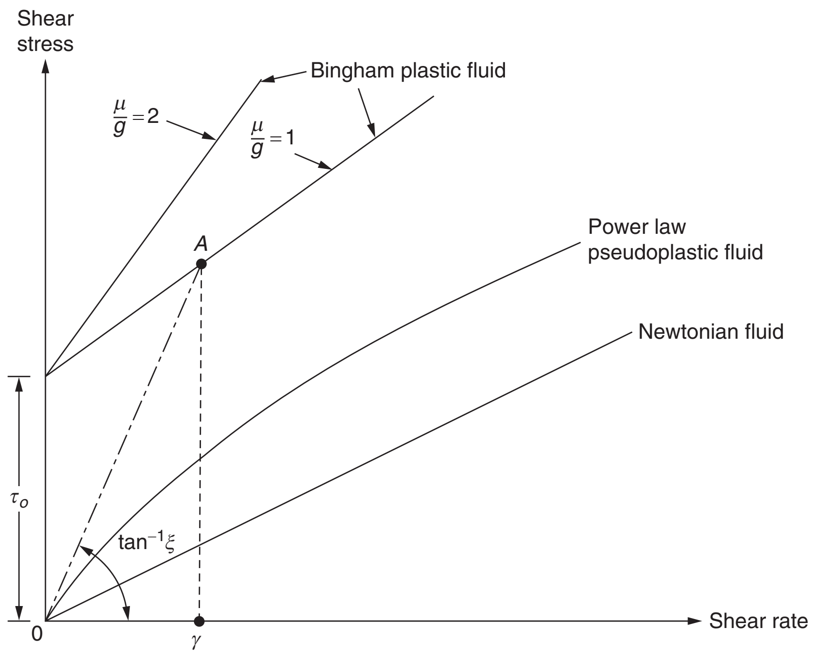

FIGURE 19.7 Flow curves for uniaxial flow.

Thus, they can be described by:

$$\tau = \begin{cases} -\dfrac{\mu}{g}\dfrac{du}{dy} + \tau_0 & \text{for} \quad |\tau| > \tau_0 \\ 0 & \text{for} \quad |\tau| < \tau_0 \end{cases} \tag{19.75}$$

True Bingham fluids, which follow Eq. (19.75), are not encountered in practice, but most fluids exhibit pseudoplastic characteristics. Hence, many empirical equations have been developed to represent their behavior. One widely used such equation, known as the power law, is given by:

$$\tau = -\frac{k}{g}\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy} \tag{19.76}$$

where k is the consistency index and n is the flow behavior index ($n < 1$). Eqs. (19.75) and (19.76) are also plotted in Fig. 19.7.

EQUATION OF MOTION

By assuming fluid flow to be incompressible and time independent but possessing a variable viscosity, the steady-state equation of motion for flow through a parallel-sided conduit can be stated in the form of a nonlinear Poisson equation as [19.7,19.8]:

$$\frac{\partial}{\partial x}\left(\frac{\mu}{g}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu}{g}\frac{\partial u}{\partial y}\right) + \rho g - \frac{\partial p}{\partial z} = 0 \tag{19.77}$$

where u is the velocity, ρ is the density, and $\partial p/\partial z$ is the pressure gradient. If the pressure gradient is known, the solution of Eq. (19.77) enables us to find the velocity distribution, $u(x, y)$, and the quantity of flow.

19.6.2 Finite Element Equations Using the Galerkin Method

In the Galerkin method, the weighted residue must be zero over the region of flow, S. Thus,

$$\iint_S \left[\frac{\partial}{\partial x}\left(\frac{\mu}{g}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu}{g}\frac{\partial u}{\partial y}\right) + Q\right] W \cdot dS = 0 \tag{19.78}$$

where

$$Q = \rho g - \frac{\partial p}{\partial z} \tag{19.79}$$

and W is the weighting function. By applying the Green–Gauss theorem provided in Appendix B, Eq. (19.78) can be expressed as:

$$-\iint_S \frac{\mu}{g}\left[\frac{\partial u}{\partial x}\frac{\partial W}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial W}{\partial y}\right]\cdot \mathrm{d}S + \iint_S WQ\cdot \mathrm{d}S + \int_C W\frac{\mu}{g}\left(\frac{\partial u}{\partial x}l_x + \frac{\partial u}{\partial y}l_y\right)\cdot \mathrm{d}C \tag{19.80}$$

where C denotes the boundary of the region S, and l_x and l_y represent the direction cosines of the normal to this boundary at any point. The velocity u is assumed to vary within an element e as:

$$u(x,y) = \sum_i N_i U_i^{(e)} = [N]\vec{U}^{(e)} \tag{19.81}$$

where N_i is the shape function corresponding to the i-th nodal degree of freedom $U_i^{(e)}$ of element e. It can be seen from Eq. (19.80) that the shape functions N_i need to satisfy only the C^1 continuity. By substituting Eq. (19.81) into Eq. (19.80) and by taking $W = N_i$, we obtain:

$$\begin{aligned}&-\iint_{A(e)} \frac{\mu}{g}\left\{\frac{\partial N_i}{\partial x}\frac{\partial}{\partial x}[N]\cdot\vec{U}^{(e)} + \frac{\partial N_i}{\partial y}\frac{\partial}{\partial y}[N]\cdot\vec{U}^{(e)}\right\}\mathrm{d}A + \iint_{A(e)} N_i Q \mathrm{d}A\\ &+\int_{C^{(e)}} \frac{\mu}{g}N_i\frac{\partial [N]}{\partial n}\vec{U}^{(e)}\cdot \mathrm{d}C = 0\end{aligned} \tag{19.82}$$

The applicable boundary conditions are:

$$(1)\ \ u = u_0 \ \text{ on } \ C_1 \tag{19.83}$$

(known value of velocity on boundary C_1):

$$(2)\ \ \alpha u = -\frac{\mu}{g}\frac{\partial u}{\partial n} \ \text{ on } \ C_2 \tag{19.84}$$

where α is the coefficient of "sliding friction" between the fluid and the boundary (fluid slippage at the boundary is proportional to the velocity gradient normal to the boundary C_2). With the help of Eq. (19.84), the last term on the left-hand side of Eq. (19.82) can be replaced by $-\int_{C_2}\alpha N_i[N]\cdot \mathrm{d}C$. Using Eq. (19.82), the element equations can be stated in matrix form as:

$$\left(\left[K_1^{(e)}\right] + \left[K_2^{(e)}\right]\right)\vec{U}^{(e)} = \vec{P}^{(e)} \tag{19.85}$$

where the elements of matrices $\left[K_1^{(e)}\right]$ and $\left[K_2^{(e)}\right]$ and vector $\vec{P}^{(e)}$ are given by:

$$K_{1ij}^{(e)} = \iint_{A(e)} \frac{\mu}{g}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right]\mathrm{d}A \tag{19.86}$$

$$K_{2ij}^{(e)} = \int_{C_2^{(e)}} \alpha N_i[N]\cdot \mathrm{d}C \tag{19.87}$$

$$P_i^{(e)} = \int_{A^{(e)}} QN_i\cdot \mathrm{d}A \tag{19.88}$$

Note that matrix $\left[K_2^{(e)}\right]$ denotes the contribution of slippage at the boundary and is applicable only to elements with such a boundary condition. Assembly of the element in Eq. (19.85) leads to the overall equations:

$$[\underset{\sim}{K}]\,\vec{\underset{\sim}{U}} \equiv \left[[\underset{\sim}{K}_1] + [\underset{\sim}{K}_2]\right]\vec{\underset{\sim}{U}} = \vec{\underset{\sim}{P}} \tag{19.89}$$

19.6.3 Solution Procedure

The elements of matrix $\left[\underset{\sim}{K}_1\right]$ are functions of viscosity μ and hence are functions of the derivatives of the unknown velocity $u(x, y)$. The viscosity–velocity relationship is nonlinear and is given by Eq. (19.75) or Eq. (19.76). Hence, Eq. (19.89) represents a set of simultaneous nonlinear equations. These equations can be solved by using an iterative procedure. The procedure starts with the assumption of an apparent viscosity and Eqs. (19.75) and (19.76) are written in the form:

$$\tau = -[\xi(\dot{\gamma})]\dot{\gamma} \tag{19.90}$$

This apparent viscosity at any point A shown in Fig. 19.7 will be the slope of the secant OA and can be expressed as:

$$\frac{\xi A}{g} = \frac{\tau A}{\dot{\gamma} A} \tag{19.91}$$

Thus, for Bingham plastic fluids, apparent viscosity can be expressed as:

$$\xi(\dot{\gamma}) = \frac{\frac{\mu}{g}\dot{\gamma} + \tau_0}{\dot{\gamma}} = \frac{\tau_0}{\dot{\gamma}} + \frac{\mu}{g} \tag{19.92}$$

and for pseudoplastic fluid as:

$$\tau = -\frac{k}{g}|\dot{\gamma}|^{n-1}\dot{\gamma} \tag{19.93}$$

or:

$$\xi(\dot{\gamma}) = \frac{k}{g}|\dot{\gamma}|^{n-1} \tag{19.94}$$

Then Eq. (19.92) or Eq. (19.94) is substituted into Eq. (19.86) to obtain:

$$K_{1ij}^{(e)} = \iint_{A(e)} \xi(\dot{\gamma})\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\mathrm{d}A \tag{19.95}$$

where:

$$\dot{\gamma} = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{1}{2}} = \left[\left(\frac{\partial [N]}{\partial x}\vec{U}^{(e)}\right)^2 + \left(\frac{\partial [N]}{\partial y}\vec{U}^{(e)}\right)^2\right]^{\frac{1}{2}} \tag{19.96}$$

Usually, the initial approximation to the solution of Eq. (19.89) is based on the Newtonian velocity distribution. Eq. (19.89) can be written as:

$$\underset{\sim}{\vec{U}}_{n+1} = -\left[\underset{\sim}{K}\left(\underset{\sim}{\vec{U}}_n\right)\right]^{-1}\underset{\sim}{\vec{P}} \tag{19.97}$$

where $\underset{\sim}{\vec{U}}_n$ and $\underset{\sim}{\vec{U}}_{n+1}$ denote the solutions of $\underset{\sim}{\vec{U}}$ in the n-th and $n+1$-*th* iterations. The solution procedure can be summarized thus [19.9]:

1. Solve Eq. (19.89) for the Newtonian velocity distribution after applying the boundary conditions of Eq. (19.83).
2. Compute the shear rate and apparent viscosity.
3. Calculate the matrix $[K_n]$.
4. Solve Eq. (19.89) for $\underset{\sim}{\vec{U}}_{n+1}$ after applying the boundary conditions of Eq. (19.83).
5. Test for the convergence of the process using the criterion

$$\left|\frac{(U_i)_n - (U_i)_{n-1}}{(U_i)_n}\right| < \varepsilon, \quad i = 1, 2, \ldots \tag{19.98}$$

where $\varepsilon \approx 0.01$ for each nodal velocity U_i.

If convergence of Eq. (19.98) is not satisfied, repeat Steps 2–5.

EXAMPLE 19.3

The problem of Bingham flow between parallel plates was considered by Lyness et al. [19.7]. The problem is shown in Fig. 19.8, where the plug dimensions are given by:

$$h_p = \tau_0 \bigg/ \frac{\mathrm{d}p}{\mathrm{d}z} \tag{E.1}$$

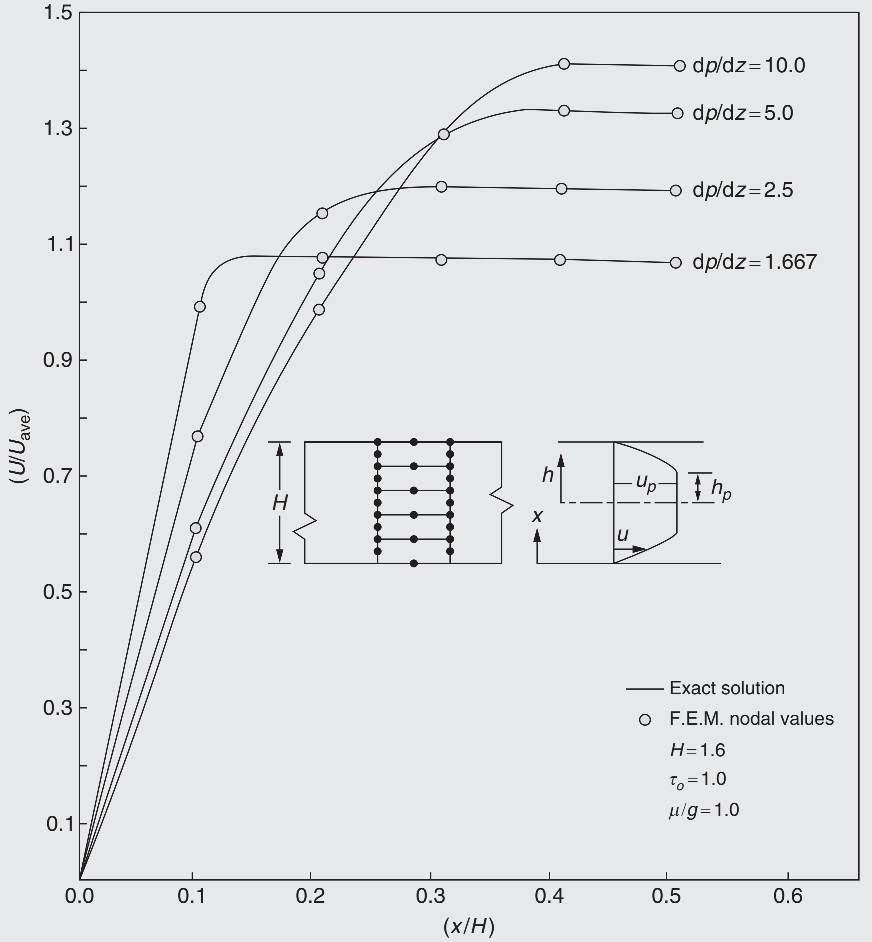

FIGURE 19.8 Bingham flow between parallel plates [19.7]. *F.E.M.*, finite element method.

where h_p defines the extent of the solid plug. For this problem, the exact solution for the velocity is given by:

$$u_p = \frac{1}{\mu}\left[\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}\left(\frac{H^2}{4} - h_p^2\right) - \tau_0\left(\frac{H}{2} - h_p\right)\right] \quad \text{for} \quad h < h_p \tag{E.2}$$

$$u = \frac{1}{\mu}\left[\frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}z}\left(\frac{H^2}{4} - h^2\right) - \tau_0\left(\frac{H}{2} - h\right)\right] \quad \text{for} \quad h > h_p \tag{E.3}$$

The problem was solved for different values of true viscosity and plug sizes. Five quadratic rectangular elements were used for the idealization, as shown in Fig. 19.8. The finite element (velocity) solution is compared with the exact solution given by Eqs. (E.2) and (E.3) in Fig. 19.8. It can be seen that the finite element solution compares well with the exact solution. The flow rate through the section can be computed by numerically integrating the velocity over an element and summing over all of the elements. The error in the flow rates predicted by the finite element method was less than 1% compared with those obtained by explicit integration of Eqs. (E.2) and (E.3).

19.7 OTHER DEVELOPMENTS

As can be observed in this chapter and Chapter 18, most finite element applications to fluid mechanics problems employed one of the weighted residual criteria to derive the equations. Lynn [19.10] studied the problem of laminar boundary layer flow using the least squares criterion and Popinski and Baker [19.11] used the Galerkin approach. Transient compressible flow in pipelines was studied by Bisgaard et al. [19.12]. A study of penalty elements for incompressible laminar flow was conducted by Dhatt and Hubert [19.13]. An optimal control finite element approximation for a penalty variational formulation of three-dimensional Navier–Stokes problem was presented by Li et al. [19.14]. Numerical experiments with several finite elements were conducted to study viscous incompressible flow [19.15]. The problems of finite element mesh generation for arbitrary geometry, an efficient solution to equations, the derivation of new elements, and the development of efficient codes for fluid flow problems have also been investigated [19.16]. An overview of the application of finite elements in computational fluid dynamics was presented by Lohner [19.17].

REVIEW QUESTIONS

19.1. Give brief answers to the following questions.

1. State the names of equations that govern two-dimensional steady incompressible Newtonian flow.
2. What are Stokes equations?
3. What are Navier–Stokes equations?
4. What is vorticity?
5. What is a Bingham plastic fluid?
6. What is the significance of the Green–Gauss theorem?

19.2. Indicate whether the following statement is true or false.

1. Vorticity can be expressed in terms of stream function.
2. Vorticity (ω) in a two-dimensional problem is defined as $\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$.
3. Bingham fluids are encountered in several practical applications.

PROBLEMS

19.1 Consider an incompressible viscous flow using the stream function–vorticity formulation of Section 19.5.1. Derive the corresponding finite element equations using the Galerkin method.

19.2 A fully developed laminar forced flow of a Newtonian, incompressible fluid between two parallel plates is shown in Fig. 19.9. Consider the energy balance of an infinitesimal element, dx dy, including the energy transported by the fluid motion, and derive the equation governing the temperature distribution in the fluid, $T(x, y)$, as:

$$\rho c u(y)\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right] \tag{P.1}$$

where ρ is the density, c is the specific heat at constant pressure, u is the x component of velocity, and k is the thermal conductivity.

19.3 Using the Galerkin method, derive the finite element equations corresponding to Eq. (P.1) of Problem 19.2 by assuming the temperature variation in an element as:

$$T(x,y) = [N(x,y)]\vec{T}^{(e)} \tag{P.2}$$

Note that the components of velocity of the fluid parallel to the y and x axes are given by 0 and $u(y) = (3/2)$ $u_0\{1 - [y/d]^2\}$, respectively, where u_0 is the average velocity.

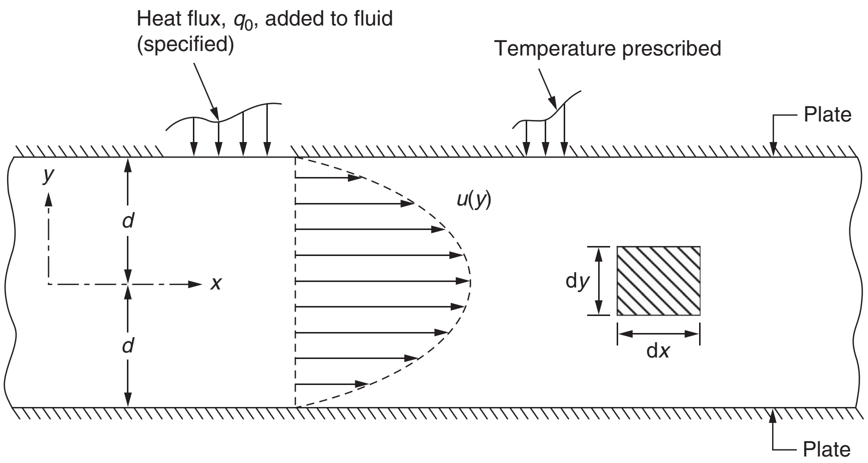

FIGURE 19.9 Fully developed laminar flow between parallel plates.

19.4 Consider a fully developed laminar forced flow of a Newtonian, incompressible fluid in a circular duct. Axisymmetric temperature and heat fluxes are prescribed on different parts of the boundary (walls) of the duct. Consider the energy balance of an annular element of volume $2\pi r\ \delta r\ \delta z$, including the energy transported by the fluid motion along the axial (z) direction, and derive the equation governing the temperature distribution of the fluid, $T(r, z)$, as:

$$\rho c v(r)\frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] + \frac{\partial}{\partial z}\left[k\frac{\partial T}{\partial z}\right] \tag{P.3}$$

where ρ is the density, c is the specific heat at constant pressure, v is the z component of velocity, and k is the thermal conductivity.

19.5 Using the Galerkin method, derive the finite element equations corresponding to Eq. (P.3) of Problem 19.4 by assuming the temperature variation in an element as:

$$T(r,z) = [N(r,z)]\vec{T}^{(e)} \tag{P.4}$$

Note that the components of velocity of the fluid parallel to the r and z axes are given by 0 and $v(r) = 2v_0$ (1 − $[r/d]^2$), respectively, where v_0 is the average velocity and d is the radius of the duct.

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Part VI

Solution and Applications of Quasi-Harmonic Equations

Chapter 20

Solution of Quasi-Harmonic Equations

Chapter Outline

20.1 INTRODUCTION

A significant class of physical problems representing phenomena such as heat conduction, the torsion of shafts, and the distribution of electric potential are known as field problems. These field problems have common characteristics in that they are all governed by similar partial differential equations in terms of the field variable of concern. This permits us to discuss the solution of the governing partial differential equation without identifying the field variable ϕ as a particular physical quantity. The steady-state (time-independent) field problems are governed by the quasi-harmonic equation:

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial \phi}{\partial z}\right)+c = 0 \tag{20.1}$$

where ϕ is an unknown function or field variable (assumed to be single valued in the domain), and k_x, k_y, k_z, and c are known functions of x, y, and z.

The physical interpretation of k_x, k_y, k_z, c, and ϕ depends on the particular physical problem. Table 20.1 lists some typical field problems along with the significance of ϕ and other parameters for each problem. Eq. (20.1) assumes that the medium is inhomogeneous and/or anisotropic, and coordinates x, y, and z coincide with the principal coordinates. If the medium is homogeneous, k_x, k_y, and k_z will be constants, and if it is isotropic, $k_x = k_y = k_z = k$ constant. The general boundary conditions for Eq. (20.1) are given by:

$$\phi = \overline{\phi}; \quad \text{value of } \phi \text{ prescribed on part of the boundary, } S_1 \text{ (Dirichlet condition)} \tag{20.2}$$

and

$$k_x\frac{\partial \phi}{\partial x}l_x + k_y\frac{\partial \phi}{\partial y}l_y + k_z\frac{\partial \phi}{\partial z}l_z + q + r\phi = 0$$
$$\text{on the remaining part of the boundary, for example, } S_2 \text{ (Cauchy condition)} \tag{20.3}$$

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TABLE 20.1 Typical Steady-State Field Problems Governed by Eq. (20.1)

Serial Number	Physical Problem	Field Variable, ϕ	Significance of k_x, k_y, k_z	C	Remarks
1	Heat conduction	Temperature	Thermal conductivities	Rate of internal heat generation	$q =$ boundary heat generation $r =$ convective heat transfer coefficient
2	Seepage flow	Pressure	Permeability coefficients	Internal flow source	—
3	Torsion of prismatic shafts	Stress function	$1/G$, where G is shear modulus	$c = 2\theta$, where θ is angle of twist per unit length	—
4	Irrotational flow of ideal fluids	Velocity potential or stream function	—	$c = 0$	$q =$ boundary velocity, $r = 0$
5	Fluid film lubrication	Pressure	k_x and k_y are functions of film thickness and viscosity, $k_z = 0$	Net flow owing to various actions	$q =$ boundary flow
6	Distribution of electric potential	Electric potential (voltage)	Specific conductivities	Internal current source	$q =$ externally applied boundary current
7	Electrostatic field	Electric force field intensity	Permittivities	Internal current source	—
8	Magnetostatics	Magnetomotive force	Magnetic permeabilities	Internal magnetic field source	$q =$ externally applied magnetic field

where l_x, l_y, and l_z are direction cosines of the outward drawn normal to the surface. If $q = r = 0$, the Cauchy boundary condition becomes a Neumann condition. The field problem stated in Eqs. (20.1) – (20.3) is called an elliptic[1] mixed boundary value problem. It is mixed because some portion of boundary S has Dirichlet conditions, whereas the remaining one has Cauchy conditions.

If $k_x = k_y = k_z = k =$ constant and $q = r = 0$, Eq. (20.1) reduces to:

$$\nabla^2\phi = -\frac{c(x, y, z)}{k} \tag{20.4}$$

which is called Poisson's equation. In this case, the boundary condition of Eq. (20.3) reduces to:

$$\frac{\partial\phi}{\partial n} = 0 \quad (\text{nonconducting boundary on } S_2) \tag{20.5}$$

where n indicates the direction of outward drawn normal to the surface. Furthermore, if $c = 0$, Eq. (20.4) becomes:

$$\nabla^2\phi = 0 \tag{20.6}$$

which is known as Laplace equation.

1. Partial differential equations (of the second order) can be classified as parabolic, elliptic, hyperbolic, or some combination of these three categories, such as elliptically parabolic, hyperbolically parabolic, and so on. To indicate the method of classification, consider the following general partial differential equation in two independent variables:

$$A\frac{\partial^2\phi}{\partial x^2} + 2B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = D\left(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, x, y\right)$$

where A, B, and C are functions of x and y, and D is a function of x, y, ϕ, $(\partial\phi/\partial x)$, and $(\partial\phi/\partial y)$. The nature of expression for D will decide whether the partial differential equation is linear or nonlinear. Irrespective of the form of D, the partial differential equation is called parabolic if $B^2 - AC = 0$, elliptic if $B^2 - AC < 0$, and hyperbolic if $B^2 - AC > 0$. Usually, the solution domains are defined by closed boundaries for elliptic problems and by open domains for parabolic and hyperbolic problems. Most finite element applications so far have been directed toward solving elliptic boundary value problems with irregular solution domains.

20.2 FINITE ELEMENT EQUATIONS FOR STEADY-STATE PROBLEMS

The finite element solution of differential Eq. (20.1) subject to boundary conditions Eqs. (20.2) and (20.3) was presented in Sections 13.5.1 and 13.5.2 using the variational and Galerkin methods, respectively, in connection with the solution of heat transfer problems. The overall system equations can be stated as:

$$\left[\underset{\sim}{K}\right] \underset{\sim}{\vec{\Phi}} = \underset{\sim}{\vec{P}} \tag{20.7}$$

where the element characteristic matrices $[K^{(e)}]$ and characteristic vectors $\vec{P}^{(e)}$, whose assembly yields Eq. (20.7), can be obtained similar to Eqs. (13.30), (13.31), and (13.33) as:

$$K_{1_{ij}}^{(e)} = \iiint_{V^{(e)}} \left(k_x \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + k_y \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + k_z \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right) dV \tag{20.8}$$

$$K_{2_{ij}}^{(e)} = \iint_{S_2^{(e)}} rN_iN_j \mathrm{d}S_2 \tag{20.9}$$

$$P_i^{(e)} = \iiint_{S^{(e)}} cN_i \mathrm{d}V - \iint_{S_2^{(e)}} qN_i \mathrm{d}S_2 \tag{20.10}$$

$$K_{ij}^{(e)} = K_{1_{ij}}^{(e)} + K_{2_{ij}}^{(e)} \tag{20.11}$$

The vector of nodal unknowns of element e is denoted by $\vec{\Phi}^{(e)} = \left\{\Phi_1^{(e)}\Phi_2^{(e)}\cdots\right\}^T$ and the vector $\underset{\sim}{\vec{\Phi}}$ is given by the assemblage of $\vec{\Phi}^{(e)}$. After incorporating the boundary conditions given by Eq. (20.2), the solution of Eq. (20.7) gives the nodal values of the field variable ϕ. Once the nodal values Φ_i are known, if required, the element resultants can be evaluated without much difficulty.

20.3 SOLUTION OF POISSON'S EQUATION

Poisson's equation (Eq. 20.4) is a special case of the general field Eq. (20.1). We consider the solution of Poisson's equation in the context of the torsion of prismatic shafts in this section.

20.3.1 Derivation of the Governing Equation for the Torsion Problem

Consider a solid prismatic shaft with any arbitrary cross-section, as shown in Fig. 20.1. When this shaft is subjected to a twisting moment, M_z, it is usual to assume that all stresses except shear stresses σ_{xz} and σ_{yz} are zero [20.1]. Thus,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0 \tag{20.12}$$

For this case, the equilibrium equations given by Eq. (8.4) reduce, in the absence of body forces, to:

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \tag{20.13}$$

$$\frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{20.14}$$

$$\frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{20.15}$$

Eqs. (20.14) and (20.15) indicate that stresses σ_{xz} and σ_{yz} do not vary with coordinate z (i.e., along a line taken parallel to the axis of the shaft). Now a stress function ϕ, known as Prandtl's stress function, is defined as:

$$\left.\begin{aligned} \sigma_{xz} &= \frac{\partial \phi}{\partial y} \\ \sigma_{yz} &= -\frac{\partial \phi}{\partial x} \end{aligned}\right\} \tag{20.16}$$

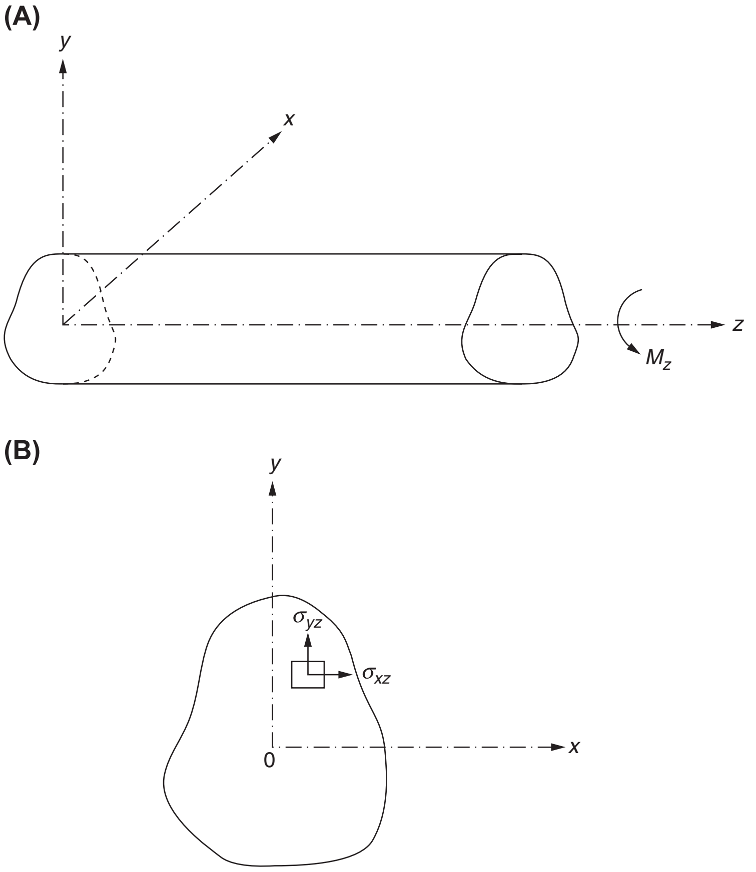

FIGURE 20.1 Prismatic shaft under torsion.

so that Eq. (20.13) is satisfied automatically. If we consider strains, Hooke's law (Eq. 8.7) indicates that only ε_{xz} and ε_{yz} will be nonzero. Thus,

$$\left.\begin{aligned} \varepsilon_{xz} &= \frac{\sigma_{xz}}{G} \\ \varepsilon_{yz} &= \frac{\sigma_{yz}}{G} \end{aligned}\right\} \tag{20.17}$$

where G is the modulus of rigidity. Clearly, these shear strains must be independent of coordinate z. If the displacement components are taken as [20.2]:

$$u = -\theta yz \tag{20.18a}$$

$$v = \theta xz \tag{20.18b}$$

$$\frac{\partial w}{\partial x} = \frac{1}{G}\frac{\partial \phi}{\partial y} + \theta y \tag{20.18c}$$

$$\frac{\partial w}{\partial y} = -\frac{1}{G}\frac{\partial \phi}{\partial x} - \theta x \tag{20.18d}$$

where θ denotes the angle of twist per unit length, the stress–strain relations, Eq. (20.17), can be satisfied. To ensure continuity of axial displacement w, we have, by differentiating Eqs. (20.18c) and (20.18d),

$$\frac{\partial}{\partial x}\left(\frac{1}{G}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{G}\frac{\partial \phi}{\partial y}\right) = -2\theta \tag{20.19}$$

If G is a constant, Eq. (20.19) reduces to the Poisson equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta \tag{20.20}$$

Because the shear stress normal to the external boundary must be zero, it follows from Eq. (20.16) that the stress function on this boundary must have a constant value. This constant can be arbitrarily fixed as zero. Thus, a sufficient and necessary boundary condition on the external boundary becomes:

$$\phi = 0 \tag{20.21}$$

To find the torque acting on section (M_z), we integrate the moment owing to shear stresses as:

$$M_z = \iint_S (x\sigma_{yz} - y\sigma_{xz})\,\mathrm{d}x\,\mathrm{d}y \tag{20.22}$$

where S denotes the cross-section of the shaft. By substituting Eqs. (20.16) and (20.21), Eq. (20.22) becomes:

$$M_z = 2\iint_S \phi\,\mathrm{d}x\,\mathrm{d}y \tag{20.23}$$

By defining a set of nondimensional quantities as:

$$x' = \frac{x}{l},\quad y' = \frac{y}{l},\quad \phi' = \frac{\phi}{G\theta l^2} \tag{20.24}$$

where l is the length of the shaft, Eqs. (20.20) and (20.21) reduce to:

$$\nabla^2\phi' = -2 \equiv -c \tag{20.25}$$

with

$$\phi' = 0 \text{ on external boundary} \tag{20.26}$$

Note

The torsion problem can be formulated using a different approach. In this approach, the warping function $\psi(x, y)$, which represents the movement of the cross-section in the z direction per unit twist, is treated as the unknown function. Finally, a Laplace equation needs to be solved to determine $\psi(x, y)$. This approach was used by Herrmann [20.3] for the torsion analysis of irregular shapes.

20.3.2 Finite Element Solution

The functional corresponding to Eq. (20.25) can be written as:

$$I(\phi') = \frac{1}{2}\iint_S \left[\left(\frac{\partial\phi'}{\partial x'}\right)^2 + \left(\frac{\partial\phi'}{\partial y'}\right)^2\right]\mathrm{d}x'\,\mathrm{d}y' + \iint_S c\phi'\,\mathrm{d}x'\,\mathrm{d}y' \tag{20.27}$$

Let the cross-section S be idealized by using triangular elements and let the nodal values of ϕ', namely $\Phi'_1, \Phi'_2, \ldots, \Phi'_M$, be taken as unknowns. By choosing a suitable form of variation of ϕ' within each element as:

$$\phi'(x', y') = [N(x', y')]\vec{\Phi}'^{(e)} \tag{20.28}$$

the element matrices and vectors can be obtained, as indicated in Eqs. (20.8)−(20.11). The resulting element equations are then assembled to obtain the overall system equations as in Eq. (20.7) and are solved to obtain $\underset{\sim}{\vec{\Phi}}$. It will be necessary to apply the boundary conditions $\phi' = 0$ on the outer periphery of S before solving Eq. (20.7).

Once $\overrightarrow{\underset{\sim}{\Phi}}'$ and hence $\overrightarrow{\underset{\sim}{\Phi}}'^{(e)}$ and $\phi'(x, y)$ within all of the elements are known, we can find θ by using Eq. (20.23), as:

$$M_z = 2\iint_S \phi \, dx \, dy = 2G\theta l^2 \iint_S \phi'(x', y') dx' dy' \tag{20.29}$$

or

$$\theta = \frac{M_z}{2Gl^2 \iint_S \phi' \, dx' \, dy'} = \frac{M_z}{2Gl^2 \sum_{e=1}^{E} (\text{area of triangle } e \times \text{average of three nodal values of } \phi' \text{ of element } e)} \tag{20.30}$$

Finally, the shear stresses within any element can be computed as:

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = G\theta l^2 \begin{Bmatrix} \dfrac{\partial \phi'}{\partial y'} \\ -\dfrac{\partial \phi'}{\partial x'} \end{Bmatrix} \tag{20.31}$$

where the derivatives of ϕ' can be obtained by differentiating Eq. (20.28).

EXAMPLE 20.1

Find the stresses developed in a 4 × 4-cm square shaft when the angle of twist is 2 degrees in a length of 100 cm. The value of G is 0.8×10^6 N/cm^2.

Solution

Step 1: Idealize the region by finite elements. Because the shaft has four axes of symmetry, we consider only one-eighth of the total cross-section for analysis. The idealization using four elements is shown in Fig. 20.2. The information needed for subsequent computations is given in Table 20.2.

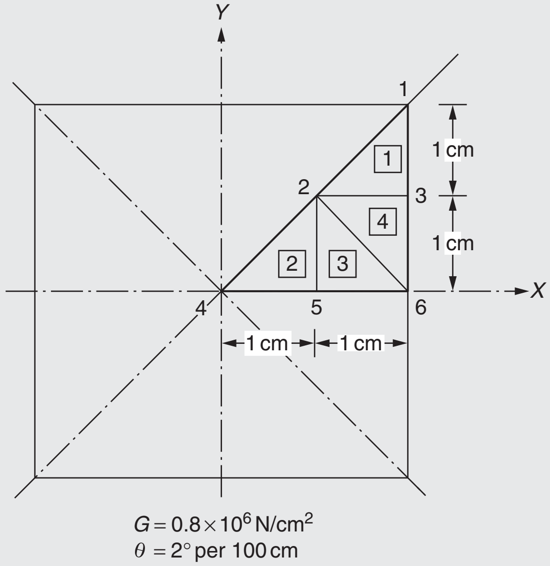

FIGURE 20.2 Rectangular shaft under torsion.

EXAMPLE 20.1 —cont'd

Step 2: Assume a suitable interpolation model for field variable ϕ (to solve Eq. 20.20). By assuming linear variation within any element, $\phi^{(e)}(x, y)$ can be expressed as:

TABLE 20.2 Information on Elements

Node	1	2	3	4	5	6
$X(x)$ coordinate (cm)	2.0	1.0	2.0	0.0	1.0	2.0
$Y(y)$ coordinate (cm)	2.0	1.0	1.0	0.0	0.0	0.0

Element number e		1	2	3	4
Global nodes corresponding to local nodes	i	2	4	5	6
	j	3	5	6	3
	k	1	2	2	2

Element Number e	x_i	x_j	x_k	y_i	y_j	y_k	$c_k = (x_j - x_i)$	$c_i = (x_k - x_j)$	$c_j = (x_i - x_k)$
1	1.0	2.0	2.0	1.0	1.0	2.0	1.0	0.0	−1.0
2	0.0	1.0	1.0	0.0	0.0	1.0	1.0	0.0	−1.0
3	1.0	2.0	1.0	0.0	0.0	1.0	1.0	−1.0	0.0
4	2.0	2.0	1.0	0.0	1.0	1.0	0.0	−1.0	1.0

Element Number e	$b_i = (y_j - y_k)$	$b_j = (y_k - y_i)$	$b_k = (y_i - y_j)$	$A(e) = \frac{1}{2}\lvert x_{ij}y_{jk} - x_{jk}y_{ij}\rvert$
1	−1.0	1.0	0.0	0.5
2	−1.0	1.0	0.0	0.5
3	−1.0	1.0	0.0	0.5
4	0.0	1.0	−1.0	0.5

$$\phi^{(e)}(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y \equiv N_i\Phi_i + N_j\Phi_j + N_k\Phi_k \tag{E.1}$$

where α_1, α_2, and α_3 are given by Eq. (3.30), and N_i, N_j, and N_k by Eq. (3.35):

$$\left.\begin{aligned} \alpha_1 &= (a_i\Phi_i + a_j\Phi_j + a_k\Phi_k)/(2A^{(e)}) \\ \alpha_2 &= (b_i\Phi_i + b_j\Phi_j + b_k\Phi_k)/(2A^{(e)}) \\ \alpha_3 &= (c_i\Phi_i + c_j\Phi_j + c_k\Phi_k)/(2A^{(e)}) \end{aligned}\right\} \tag{E.2}$$

$$\left.\begin{aligned} N_i &= (a_i + b_i x + c_i y)/(2A^{(e)}) \\ N_j &= (a_j + b_j x + c_j y)/(2A^{(e)}) \\ N_k &= (a_k + b_k x + c_k y)/(2A^{(e)}) \end{aligned}\right\} \tag{E.3}$$

and Φ_i, Φ_j, and Φ_k denote the values of ϕ at nodes i, j, and k of element e, respectively.

Continued

EXAMPLE 20.1 —cont'd

Step 3: Derive element characteristic matrices and vectors. The finite element equations corresponding to Eq. (20.20) can be derived as a special case of Eq. (20.7) with:

$$K_{ij}^{(e)} = \iint_{A^{(e)}} \left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right] dA = \frac{1}{4A^{(e)}} \begin{bmatrix} (b_i^2 + c_i^2) & (b_i b_j + c_i c_j) & (b_i b_k + c_i c_k) \\ & \left(b_j^2 + c_j^2\right) & (b_j b_k + c_j c_k) \\ \text{Symmetric} & & (b_k^2 + c_k^2) \end{bmatrix} \tag{E.4}$$

and

$$P_i^{(e)} = \iint_{A^{(e)}} c N_i dA = \frac{cA^{(e)}}{3} \tag{E.5}$$

where $c = 2G\theta$. Thus, we can compute the element matrices and vectors as:

$$\left[K^{(1)}\right] = \left[K^{(2)}\right] = \left[K^{(4)}\right] = \frac{1}{2}\begin{bmatrix} 1.0 & -1.0 & 0.0 \\ -1.0 & 2.0 & -1.0 \\ 0.0 & -1.0 & 1.0 \end{bmatrix} \tag{E.6}$$

$$\left[K^{(3)}\right] = \frac{1}{2}\begin{bmatrix} 2.0 & -1.0 & -1.0 \\ -1.0 & 1.0 & 0.0 \\ -1.0 & 0.0 & 1.0 \end{bmatrix} \tag{E.7}$$

$$\vec{P}^{(1)} = \vec{P}^{(2)} = \vec{P}^{(3)} = \vec{P}^{(4)} = \frac{G\theta}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.8}$$

Step 4: Derive the overall equations. By using the nodal connectivity information of Step 1, the overall matrix $[\underset{\sim}{K}]$ and vector $\underset{\sim}{\vec{P}}$ can be derived as:

$$\left[\underset{\sim}{K}\right] = \frac{1}{2}\begin{bmatrix} 1.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 4.0 & -2.0 & 0.0 & -2.0 & 0.0 \\ -1.0 & -2.0 & 4.0 & 0.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & -1.0 & 0.0 \\ 0.0 & -2.0 & 0.0 & -1.0 & 4.0 & -1.0 \\ 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 2.0 \end{bmatrix} \tag{E.9}$$

$$\underset{\sim}{\vec{P}} = \frac{G\theta}{3}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 1 \\ 2 \\ 2 \end{Bmatrix} \tag{E.10}$$

Step 5: Solve the system equations after applying the boundary conditions. The boundary conditions are given by $\phi = 0$ on the external boundary; that is, $\Phi_1 = \Phi_3 = \Phi_6 = 0$. By eliminating these variables, Φ_1, Φ_3, and Φ_6, the system equations can be written as:

$$[K]\vec{\Phi} = \vec{P} \tag{E.11}$$

where

$$[K] = \frac{1}{2}\begin{bmatrix} 4.0 & 0.0 & -2.0 \\ 0.0 & 1.0 & -1.0 \\ -2.0 & -1.0 & 4.0 \end{bmatrix}, \quad \vec{P} = \frac{G\theta}{3}\begin{Bmatrix} 4 \\ 1 \\ 2 \end{Bmatrix}, \quad \text{and} \quad \vec{\Phi} = \begin{Bmatrix} \Phi_2 \\ \Phi_4 \\ \Phi_5 \end{Bmatrix}$$

EXAMPLE 20.1 —cont'd

The solution of Eq. (E.11) gives:

$$\Phi_1 = \Phi_3 = \Phi_6 = 0, \quad \Phi_2 = 3G\theta/2 = 418.8, \quad \Phi_4 = 7G\theta/3 = 651.5,$$
$$\Phi_5 = 5G\theta/3 = 465.3$$

Step 6: Compute the element resultants.
The shear stresses induced are given by Eq. (20.16):

$$\sigma_{xz} = \frac{\partial \phi}{\partial y} = \alpha_3 = \left(c_i\Phi_i + c_j\Phi_j + c_k\Phi_k\right)/\left(2A^{(e)}\right) \tag{E.12}$$

$$\sigma_{yz} = -\frac{\partial \phi}{\partial x} = -\alpha_2 = -\left(b_i\Phi_i + b_j\Phi_j + b_k\Phi_k\right)/\left(2A^{(e)}\right) \tag{E.13}$$

For $e = 1$: $i = 2$, $j = 3$, and $k = 1$:

$$\sigma_{xz} = -\Phi_3 + \Phi_1 = 0, \quad \sigma_{yz} = \Phi_2 - \Phi_3 = 418.8\ \mathrm{N/cm^2}$$

For $e = 2$: $i = 4$, $j = 5$, and $k = 2$:

$$\sigma_{xz} = -\Phi_5 + \Phi_2 = -46.5\ \mathrm{N/cm^2}, \quad \sigma_{yz} = \Phi_4 - \Phi_5 = 186.2\ \mathrm{N/cm^2}$$

For $e = 3$: $i = 5$, $j = 6$, and $k = 2$:

$$\sigma_{xz} = -\Phi_5 + \Phi_2 = -46.5\ \mathrm{N/cm^2}, \quad \sigma_{yz} = \Phi_5 - \Phi_6 = 465.3\ \mathrm{N/cm^2}$$

For $e = 4$: $i = 6$, $j = 3$, and $k = 2$:

$$\sigma_{xz} = -\Phi_6 + \Phi_3 = 0.0, \quad \sigma_{yz} = -\Phi_3 + \Phi_2 = 418.8\ \ \mathrm{N/cm^2}$$

COMPUTATION OF THE TWISTING MOMENT (M_z)

The twisting moment acting on the shaft can be computed, using Eq. (20.23), as:

$$\text{Twisting moment} = 2\iint_S \phi\ \mathrm{d}x\ \mathrm{d}y \simeq \sum_{e=1}^{4} \iint_{A^{(e)}} 2\phi^{(e)} \mathrm{d}x\ \mathrm{d}y \tag{20.32}$$

Because $\phi^{(e)}$ is given by Eq. (E.1) of Example 20.1, the integral in Eq. (20.32) can be evaluated to obtain:

$$\text{Twisting moment} = \sum_{e=1}^{4} \frac{2A^{(e)}}{3}\left(\Phi_i + \Phi_j + \Phi_k\right)^{(e)}$$

$$= \frac{1}{3}[(418.8 + 0.0 + 0.0) + (651.5 + 465.3 + 418.8) + (465.3 + 0.0 + 418.8) + (0.0 + 0.0 + 418.8)]$$

$$= 1085.67\ \mathrm{N\ cm}$$

Because the region of finite elements is only one-eighth the total cross-section, the total twisting moment (M_z) is given by:

$$M_z = 8(1085.67) = 8685.36\ \mathrm{N\ cm}$$

The exact solution for a square shaft ($2a \times 2a$) [20.1] is given by:

$$M_z = 0.1406\ \ G\theta(2a)^4$$
$$= 0.1406\left(0.8 \times 10^6\right)\left(\frac{2}{100 \times 57.3}\right)(4)^4 = 10,046.0\ \mathrm{N\ cm}$$

Thus, the finite element solution can be seen to be in error by 13.54%. This error can be reduced either by increasing the number of elements or by using higher-order elements for idealization.

20.4 TRANSIENT FIELD PROBLEMS

20.4.1 Governing Equations

Whenever a field problem involves time as an independent parameter, it is called a propagation, transient, dynamic, or time-dependent problem. Transient field problems are governed by the quasi-harmonic equation with time differentials. For example, in three dimensions, we have:

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial\phi}{\partial z}\right)-c-\alpha\frac{\partial\phi}{\partial t}-\beta\frac{\partial^2\phi}{\partial t^2}=0 \tag{20.33}$$

where k_x, k_y, k_z, c, α, and β will be, in general, functions of x, y, z, and time t. If time is not considered a variable, Eq. (20.33) can be seen to reduce to the steady-state quasi-harmonic equation considered in the previous section. The boundary conditions associated with Eq. (20.33) are:

$$\phi=\overline{\phi}\quad\text{for}\quad t>0\ \text{on}\ S_1 \tag{20.34}$$

and

$$k_x\frac{\partial\phi}{\partial x}l_x+k_y\frac{\partial\phi}{\partial y}l_y+k_z\frac{\partial\phi}{\partial z}l_z+q+r\phi=0\quad\text{for}\quad t>0\ \text{on}\ S_2 \tag{20.35}$$

Because this is a time-dependent problem, the initial conditions also have to be specified as

$$\phi(x,y,z,t=0)=\phi_0(x,y,z)\ \text{in}\ V \tag{20.36}$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}(x,y,z,t=0)=\dot{\phi}_0(x,y,z)\ \text{in}\ V \tag{20.37}$$

Eq. (20.33) represents a general damped wave equation and has application in phenomena such as electromagnetic, acoustic, and surface waves. No variational principle (functional I) exists for the problem stated by Eqs. (20.33)−(20.37).

20.4.2 Finite Element Solution

We present the finite element solution of the problem according to the weighted residual (Galerkin) method in this section.

Step 1: Discretize the domain V into E three-dimensional finite elements with p nodes each.

Step 2: Assume the variation of the field variable in a typical element e as:

$$\phi(x,y,z,t)=\sum_{i=1}^{p}N_i(x,y,z)\Phi_i^{(e)}=[N]\,\vec{\Phi}^{(e)} \tag{20.38}$$

where N_i is the interpolation function corresponding to the nodal unknown $\Phi_i^{(e)}$ of element e. The nodal unknowns $\Phi_i^{(e)}$ are assumed to be functions of time.

Step 3: Derive the finite element equations using the Galerkin method.

In this method, the criterion to be satisfied at any instant in time is given by:

$$\iiint_{V^{(e)}}N_i\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial\phi}{\partial z}\right)-c-\alpha\frac{\partial\phi}{\partial t}-\beta\frac{\partial^2\phi}{\partial t^2}\right]\mathrm{d}V=0,\quad i=1,2,\ldots,p \tag{20.39}$$

Each of the first three terms in the brackets in Eq. (20.39) can be integrated by parts using the Green−Gauss theorem of the Appendix:

$$\begin{aligned}\iiint_{V^{(e)}}N_i\frac{\partial}{\partial x}\left(k_x\frac{\partial\phi}{\partial x}\right)\mathrm{d}V&=-\iiint_{V^{(e)}}\frac{\partial N_i}{\partial x}k_x\frac{\partial\phi}{\partial x}\,\mathrm{d}V+\iint_{S^{(e)}}N_ik_x\frac{\partial\phi}{\partial x}\mathrm{d}y\ \mathrm{d}z\\&=-\iiint_{V^{(e)}}\frac{\partial N_i}{\partial x}k_x\frac{\partial\phi}{\partial x}\,\mathrm{d}V+\iint_{S^{(e)}}N_ik_x\frac{\partial\phi}{\partial x}l_x\,\mathrm{d}S\end{aligned} \tag{20.40}$$

where l_x is the x-direction cosine of the outward normal. Thus, Eq. (20.39) can be written as:

$$-\iiint_{V^{(e)}}\left[k_x\frac{\partial N_i}{\partial x}\frac{\partial \phi}{\partial x}+k_y\frac{\partial N_i}{\partial y}\frac{\partial \phi}{\partial y}+k_z\frac{\partial N_i}{\partial z}\frac{\partial \phi}{\partial z}\right]\mathrm{d}V+\iint_{S^{(e)}}N_i\left[k_x\frac{\partial \phi}{\partial x}l_x+k_y\frac{\partial \phi}{\partial y}l_y+k_z\frac{\partial \phi}{\partial z}l_z\right]\mathrm{d}S$$
$$-\iiint_{V^{(e)}}N_i\left(c+\alpha\frac{\partial \phi}{\partial t}+\beta\frac{\partial^2\phi}{\partial t^2}\right)\mathrm{d}V=0,\quad i=1,2,\ldots,p \tag{20.41}$$

Because the boundary of the elements, $S^{(e)}$, is composed of $S_1^{(e)}$ and $S_2^{(e)}$, the surface integral in Eq. (20.41) over $S_1^{(e)}$ would be zero (because ϕ is prescribed to be a constant $\overline{\phi}$ on $S_1^{(e)}$, the derivative of ϕ would be zero). On the surface $S_2^{(e)}$, the boundary condition given by Eq. (20.35) has to be satisfied. For this, we express the surface integral in Eq. (20.41) over $S_2^{(e)}$ in equivalent form as:

$$\iint_{S_2^{(e)}}N_i\left[k_x\frac{\partial \phi}{\partial x}l_x+k_y\frac{\partial \phi}{\partial y}l_y+k_z\frac{\partial \phi}{\partial z}l_z\right]\mathrm{d}S_2=\iint_{S_2^{(e)}}N_i[-q-r\phi]\ \mathrm{d}S_2 \tag{20.42}$$

By using Eqs. (20.38) and (20.42), Eq. (20.41) can be expressed in matrix form as:

$$\left[K^{(e)}\right]\vec{\Phi}^{(e)}+\left[K_1^{(e)}\right]\dot{\vec{\Phi}}^{(e)}+\left[K_2^{(e)}\right]\ddot{\vec{\Phi}}^{(e)}+\left[K_3^{(e)}\right]\vec{\Phi}^{(e)}+\vec{P}^{(e)}=\vec{0} \tag{20.43}$$

where the elements of the various matrices in Eq. (20.43) are given by:

$$K_{ij}^{(e)}=\iiint_{V^{(e)}}\left(k_x\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+k_y\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}+k_z\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right)\mathrm{d}V \tag{20.44}$$

$$K_{1_{ij}}^{(e)}=\iiint_{V^{(e)}}\alpha N_iN_j\mathrm{d}V \tag{20.45}$$

$$K_{2_{ij}}^{(e)}=\iiint_{V^{(e)}}\beta N_iN_j\ \mathrm{d}V \tag{20.46}$$

$$K_{3_{ij}}^{(e)}=\iint_{S_2^{(e)}}rN_iN_j\ \mathrm{d}S_2 \tag{20.47}$$

$$P_i^{(e)}(t)=\iiint_{V^{(e)}}cN_i\ \mathrm{d}V+\iint_{S_2^{(e)}}qN_i\ \mathrm{d}S_2 \tag{20.48}$$

Step 4: Assemble the element equations and obtain the overall equations. Eq. (20.43) represents the element equations, and the assembly of these equations leads to the following type of ordinary differential equations:

$$[\underset{\sim}{K}]\,\underset{\sim}{\vec{\Phi}}+[\underset{\sim}{K}_1]\,\dot{\underset{\sim}{\vec{\Phi}}}+[\underset{\sim}{K}_2]\,\ddot{\underset{\sim}{\vec{\Phi}}}+[\underset{\sim}{K}_3]\,\underset{\sim}{\vec{\Phi}}+\underset{\sim}{\vec{P}}=\vec{0} \tag{20.49}$$

where a dot over $\underset{\sim}{\Phi}$ represents the time derivative.

Step 5: Solve the assembled equations. The system of Eq. (20.49) can be solved for $\underset{\sim}{\vec{\Phi}}(t)$ with the discretized form of initial conditions stated in Eqs. (20.36) and (20.37) and by incorporating the boundary conditions of Eq. (20.34). The solution procedures outlined in Sections 7.4 and 12.5 can be used to solve Eq. (20.49).

Step 6: Find the element resultants. From the known values of the nodal values of ϕ, the required element resultants can be computed with the help of Eq. (20.38).

20.4.3 Space–Time Finite Elements

In a general time-dependent or propagation problem, one time and three spatial parameters will be involved. Usually, we first use the finite element method to formulate the solution in physical space. Next, we use a different method such as finite differences to find the solution over a period of time. Thus, this procedure involves idealization of the field variable ϕ (x, y, z, t) in any element e (in three-dimensional space) as:

$$\phi(x, y, z, t) = [N_1(x, y, z)]\overrightarrow{\Phi}^{(e)}(t) \tag{20.50}$$

where $[N_1]$ is the matrix of interpolation or shape functions in space and $\overrightarrow{\Phi}^{(e)}$ is the vector of time-dependent nodal variables. By using Eq. (20.50) and the specified initial conditions, we use a finite difference scheme such as:

$$\overrightarrow{\Phi}^{(e)}(t) = [N_2(t, \Delta t)]\overrightarrow{\Phi}^{(e)}(t - \Delta t) \tag{20.51}$$

where $[N_2]$ indicates the matrix of interpolation functions in the time domain.

Instead of solving the problem using Eqs. (20.50) and (20.51), finite elements can be constructed in four dimensions (x, y, z, and t) and the field variable can be expressed as follows [20.4–20.6]:

$$\phi(x, y, z, t) = [N(x, y, z, t)]\overrightarrow{\Phi}^{(e)} \tag{20.52}$$

where $[N]$ represents the matrix of shape functions in x, y, z, and t, and $\overrightarrow{\Phi}^{(e)}$ is the vector of nodal values of element e. In this case, the time-dependent problem can be solved directly without using special techniques.

REVIEW QUESTIONS

20.1 Give brief answers to the following questions.

1. What is the difference between harmonic and quasi-harmonic equations?
2. What is the difference between Laplace and Poisson equations?
3. What is a transient field problem?
4. What is a space–time finite element?
5. State two approaches that can be used to derive the finite element equations of a transient field problem.

20.2 Fill in the blank with a suitable word.

1. Solution domains of parabolic and hyperbolic problems are usually defined by _______ domains.
2. Second-order partial differential equations are classified as parabolic, elliptic, and _______ equations.

20.3 Indicate whether the following statement is true or false.

1. Solution domains of elliptic problems are usually defined by an open boundary.

20.4 Match the given physical problems with their corresponding significance associated with k_x, k_y, and k_z:

1. Heat conduction	a. Reciprocal of shear modulus
2. Seepage flow	b. Permittivities
3. Torsion of a shaft	c. Thermal conductivities
4. Electrostatic field	d. Specific conductivities
5. Distribution of electric potential	e. Permeability coefficients

PROBLEMS

Identify whether the given partial differential equation is parabolic, elliptic, or hyperbolic in Problems 20.1–20.6:

20.1 Steady-state fluid (seepage) flow under a dam:

$$k_x\frac{\partial^2 h}{\partial x^2} + k_y\frac{\partial^2 h}{\partial y^2} = 0$$

20.2 Laminar flow heat exchanger equation:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{\rho uc}{k}\frac{\partial T}{\partial z}$$

20.3 Ion exchange equation (for flow of a solution through a packed column containing an ion exchange resin):

$$\frac{\partial u}{\partial t}+\alpha(x,\ t)\frac{\partial u}{\partial x}=p(x,\ t)$$

20.4 Transient heat conduction in two dimensions:

$$\rho c\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}+k\frac{\partial^2 T}{\partial y^2}+Q(x,y)$$

20.5 Torsion of a prismatic shaft (Poisson's equation):

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+p(x,y)=0$$

20.6 Vibration of a membrane:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial t^2}$$

20.7 A steel shaft with an elliptic cross-section and length of 1 m is twisted by an angle of $\theta = 2$ degrees (Fig. 20.3). Assuming the shear modulus to be $G = 80$ GPa, determine the maximum shear stress and the torque (M_z) in the shaft by modeling one-quarter of the ellipse using linear triangular finite elements. Compare the finite element solution with the following exact solution:

$$\tau_{\max}=\frac{2M_z}{\pi ab^2}\quad \text{at}\quad x=0,\quad y=\pm b$$

$$M_z=\frac{G\theta\pi a^3 b^3}{(a^2+b^2)}$$

20.8 Consider the torsion of a uniform shaft with an I section (Fig. 20.4). Using the finite element method, determine the stresses in the elements and the torque on the shaft for the following data:

$$\text{Length}=1\text{ m},\quad G=80\text{ GPa},\quad \theta=1^\circ$$

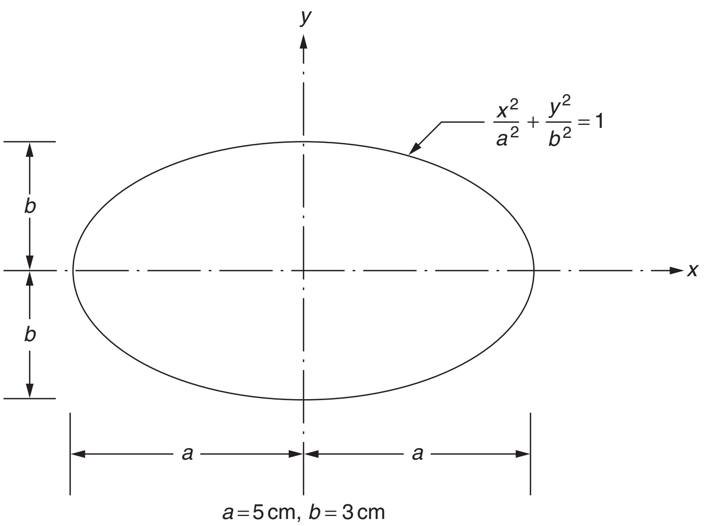

FIGURE 20.3 Elliptic cross-section.

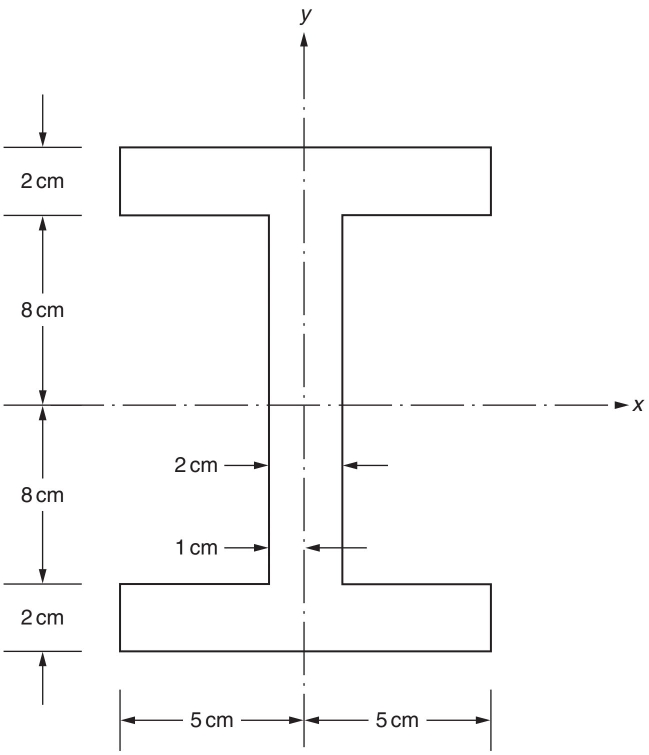

FIGURE 20.4 I Section.

20.9 The torsion of a prismatic shaft is governed by the equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2 = 0$$

in the interior cross-section of the shaft subject to $\psi = 0$ on the boundary of the cross-section. Derive the corresponding finite element equations using the Galerkin method.

20.10 In the electrostatic problem, the electric potential (ϕ) is governed by the equation (Fig. 20.5):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\rho}{\varepsilon} = 0$$

with the boundary conditions:

$$\phi = c_1 \text{ on } S_1; \quad \phi = c_2 \text{ on } S_2$$

where ρ is the charge density (Coulomb/m^3) and ε is the permittivity of the dielectric medium (Farad/m). Derive the finite element equations using a variational approach.

20.11 Determine the distribution of voltage in the circular coaxial cable shown in Fig. 20.6 if a voltage of 200 V is applied at the inner surface of the dielectric insulator. Assume the voltage at the outer surface to be zero.

20.12 Consider the following transient field problem:

$$c_1 \frac{\partial^2 \phi}{\partial x^2} + c_2 \frac{\partial \phi}{\partial t} = 0; \quad 0 \le x \le L; \quad 0 \le t \le T \tag{P.1}$$

where $\phi = \phi_{x0}$ or $(\mathrm{d}\phi/\mathrm{d}x) = 0$ at $x = 0$ and $x = L$, and $\phi = \phi_{t0}(x)$ at $t = 0$. Using $(\partial\phi/\partial t) = (\phi_{i+1} - \phi_i)/\Delta t$ with $\phi_i = \phi(t_i)$ and $\phi_{i+1} = \phi(t_{i+1} = t_i + \Delta t)$, derive the finite element equations using the Galerkin approach.

20.13 Consider the transient field problem described in Problem 20.12. Using linear triangular elements in a space−time coordinate system, derive the finite element equations using the Galerkin method.

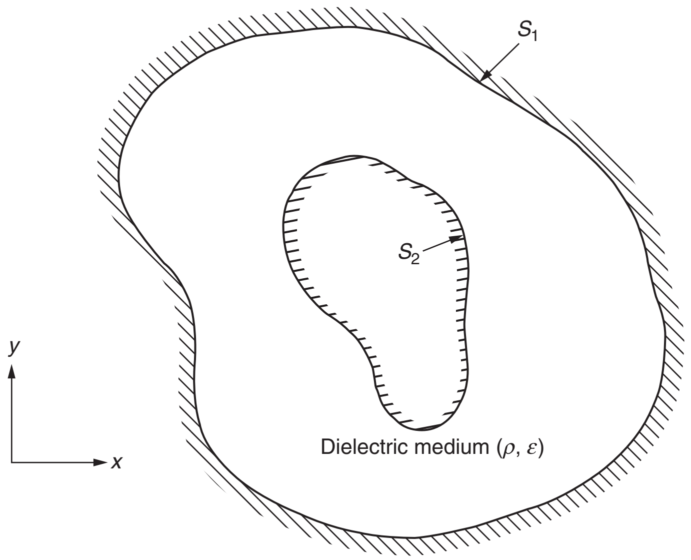

FIGURE 20.5 Electrostatic problem.

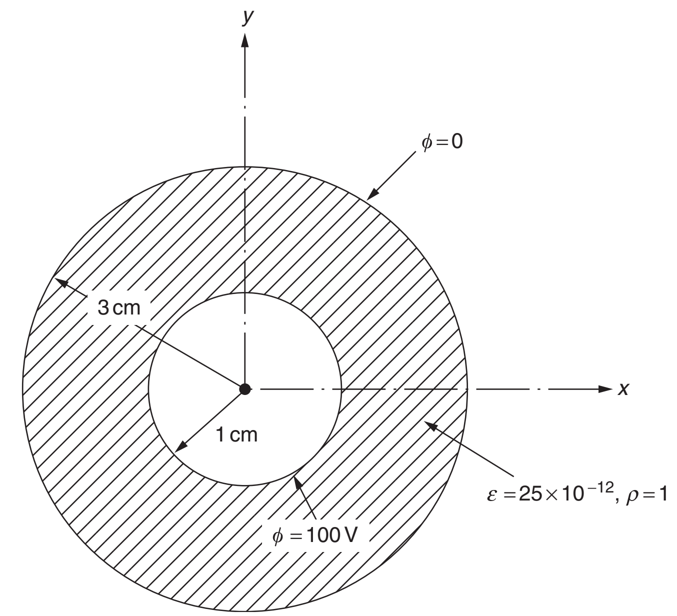

FIGURE 20.6 Circular coaxial cable.

20.14 Find the stresses induced in a shaft subjected to torsion. The cross-section of the shaft is in the form of an equilateral triangle, as shown in Fig. 20.7. The data follow: $a = 10$ in, length $(L) = 100$ in, shear modulus $(G) = 11.5 \times 10^6$ psi, and angle of twist $(\theta) = 0.01$ degrees/in $= 1.7453 \times 10^{-4}$ rad/in.
Compare your solution with the exact solution given by:

$$\phi(x, y) = -G\ \theta \left\{ \frac{1}{2}\left(x^2 + y^2\right) - \frac{1}{2a}\left(x^3 - 3\ xy^2\right) - \frac{2}{27}a^2 \right\}$$

where the origin of the (x, y) coordinates is at the centroid of the triangle and $\tau_{\max} = \frac{G\,\theta\,a}{2}$ at the middle of each side of the triangle.

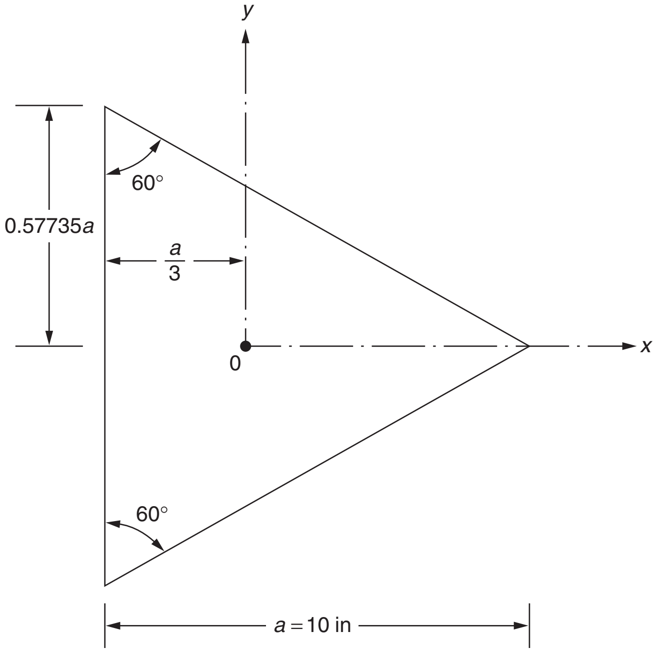

FIGURE 20.7 Equilateral triangle section.

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Part VII

ABAQUS and ANSYS Software and MATLAB® Programs for Finite Element Analysis

Chapter 21

Finite Element Analysis Using ABAQUS*

Chapter Outline

21.1 INTRODUCTION

ABAQUS finite element software has strong capabilities for solving specifically nonlinear problems and was developed by Hibbitt, Karlsson, and Sorenson, Inc. The solution to a general problem by ABAQUS involves three stages: an ABAQUS preprocessor, an ABAQUS solver, and an ABAQUS postprocessor. ABAQUS/CAE or another suitable preprocessor provides a compatible input file to ABAQUS. ABAQUS/Standard or ABAQUS/Explicit can be used as an ABAQUS solver to solve problems. ABAQUS/Standard, which is based on implicit algorithms, is good for static, strongly nonlinear problems. ABAQUS/Explicit, which is based on explicit algorithms, is intended for dynamic problems. Both ABAQUS/Standard and ABAQUS/Explicit can be executed under ABAQUS/CAE. ABAQUS/CAE or another suitable postprocessor can be used to display the output (results) of the problem.

ABAQUS/CAE provides a complete ABAQUS environment that offers a simple, consistent interface for creating, submitting, monitoring, and evaluating results from ABAQUS/Standard and ABAQUS/Explicit simulations. ABAQUS/CAE is divided into modules in which each one defines a logical aspect of the modeling process: for example, to define the geometry, define the material properties, and generate a mesh. As we move from one module to another, we build the model from which ABAQUS/CAE generates an input file that we can submit to ABAQUS/Standard or ABAQUS/Explicit to carry the analysis. After completing the analysis, the unit (ABAQUS/Standard or ABAQUS/Explicit) sends the information to ABAQUS/CAE to allow us to monitor the progress of the job and generates an output database. Finally, we use the visualization module of ABAQUS/CAE (also licensed separately as ABAQUS/Viewer) to read the output database and view the results of analysis. ABAQUS/Viewer provides graphical displays of ABAQUS finite element models and results. It obtains information about the model and results from the output database. We can control the output information displayed. For example, we can obtain plots such as the undeformed shape, deformed shape, contours, *x-y* data, and time history animation from ABAQUS/Viewer.

21.2 EXAMPLES

EXAMPLE 21.1

Input File (Fig. 21.1)

```
*HEADING
FINITE ELEMENT ANALYSIS OF FIXED-FIXED BEAM
*NODE, NSET=NODE_ALL
```

Continued

* Abaqus Finite Element Method software is marketed by SIMULIA, Rising Sun Mills, 166 Valley Street, Providence, RI 02909-2499.

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00021-3

EXAMPLE 21.1 —cont'd

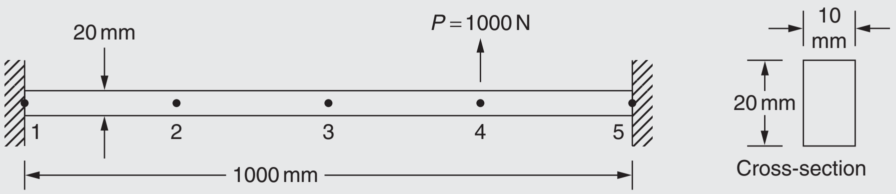


FIGURE 21.1 A fixed–fixed beam.

```
1      0.0,      0.0
2    250.00,     0.0
3    500.00,     0.0      → Five nodes and their coordinates
4    750.00,     0.0
5   1000.00,     0.0
*ELEMENT, TYPE=B23, ELSET=ELE_ALL
    1,  1,  2
    2,  2,  3
    3,  3,  4      → Four beam elements and their end node numbers
    4,  4,  5
*BEAM SECTION, ELSET=ELE_ALL, MATERIAL=MAT_A, SECTION=RECT
    10.,  20.
    0.,   0.,   -1.      → Beam cross-section dimensions
*MATERIAL, NAME=MAT_A
*ELASTIC
    70.  0e3,  0.0 → Young's modulus and Poisson's ratio (set to zero as it is not required for beams)
*BOUNDARY
    1,  1,  2
    1,  6,  6
    5,  1,  2      → Boundary conditions (end nodes 1, 5 fixed dof 1, 2, 6 zero)
    5,  6,  6
*STEP
*Static
    1.,  1.,  1e-05,  1.
*CLOAD
    4,  2,  1000.00 → Load applied in direction 2 to node 4
*output, field, variable=all
*EL PRINT, ELSET=ELE_ALL
SM1
*NODE PRINT, NSET=NODE_ALL
U
*END STEP
```

Output

The following table is printed at the integration points for Element Type B23 and Element Set ELE_ ALL.

EXAMPLE 21.1 —cont'd

Element	PT Footnote	SM1
1	1	−4.2473E+04
1	2	−2.7344E+04
1	3	−1.2215E+04
2	1	−3410.
2	2	1.1719E+04
2	3	2.6848E+04
3	1	3.5652E+04
3	2	5.0781E+04
3	3	6.5910E+04
4	1	4.6539E+04
4	2	−3.5156E+04
4	3	−1.1685E+05
Maximum		6.5910E+04
Element		3
Minimum		−1.1685E+05
Element		4

Node Output

The following table is printed for nodes belonging to Node Set NODE_ALL.

Node Footnote	U1	U2	UR3
2	0.000	2.267	1.4648E-02
3	0.000	5.580	8.3705E-03
4	0.000	4.708	−1.8834E02
Maximum	0.000	5.580	1.4648E-02
At node	1	3	2
Minimum	0.000	0.000	−1.8834E02
At node	1	1	4

The analysis has been completed (Figs. 21.2 and 21.3).

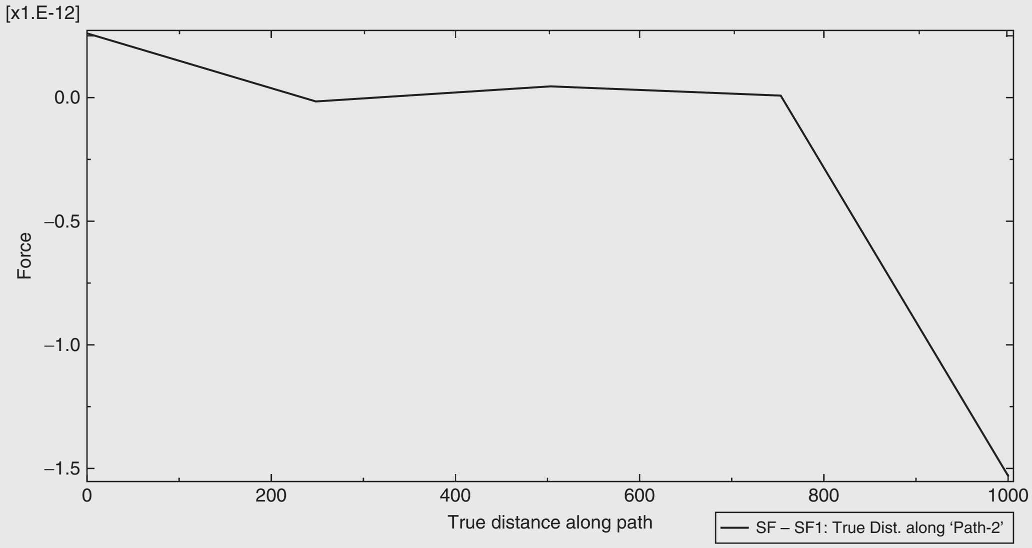


FIGURE 21.2 Shear force distribution along the beam.

Continued

EXAMPLE 21.1 —cont'd

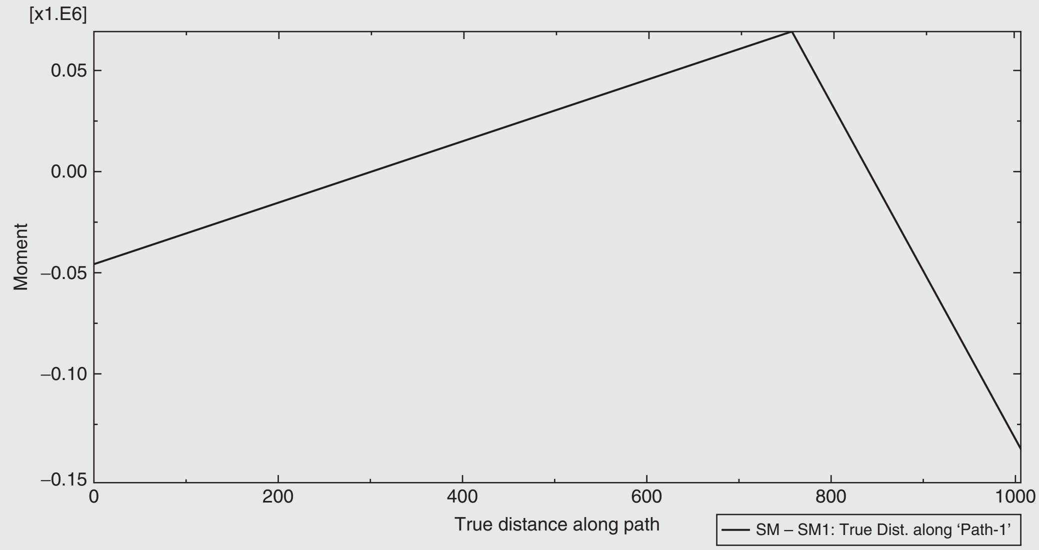


FIGURE 21.3 Bending moment distribution along the beam.

EXAMPLE 21.2

Data (Fig. 21.4):

$E = 10^4$ ksi, $A_i = 2$ in^2; $i = 1, 2, \ldots, 10$, $P_4 = P_8 = 100$ kip

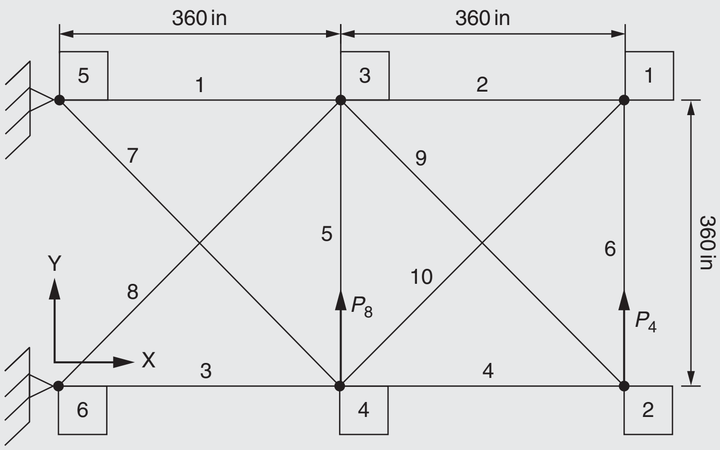


FIGURE 21.4 A 10-bar planar truss.

Input File

```
*HEADING
FINITE ELEMENT ANALYSIS OF A 10-BAR PLANAR TRUSS
** UNIT SYSTEM
** LENGTH: in
** FORCE: klb
** Modulus: ksi
*NODE
1, 720, 360,
2, 720, 0,
3, 360, 360,
4, 360, 0,
5, 0, 360,
6, 0, 0,
```

EXAMPLE 21.2 —cont'd

```
* NSET, NSET=NODE_LOAD
2, 4
*NSET,
NSET=NODE_FREE
1, 2, 3, 4
*ELEMENT,TYPE=T2D2,ELSET=ELE_ALL
```

1,	5,	3,
2,	3,	1,
3,	6,	4,
4,	4,	2,
5,	4,	3,
6,	2,	1,
7,	5,	4,
8,	6,	3,
9,	3,	2,
10,	4,	1,

```
*MATERIAL, NAME = MAT_1
*ELASTIC
1.0E4
*SOLID SECTION, ELSET=ELE_ALL, MATERIAL=MAT_1
2.0
*BOUNDARY
5, 1, 2
6, 1, 2
*STEP
*STATIC
*CLOAD
NODE_LOAD, 2, 100
*NODE PRINT, NSET=NODE_FREE
U
*EL PRINT, ELSET=ELE_ALL
S
*END STEP
```

Output

Element Output

The following table is printed at the integration points for Element Type T2D2 and Element Set TRUSS

Element	PT Footnote	S11
1	1	−97.68
2	1	−20.06
3	1	102.3
4	1	29.94
5	1	−17.74
6	1	−20.06
7	1	−73.99
8	1	67.43
9	1	−42.34
10	1	28.37
Maximum		102.3
Element		3
Minimum		−97.68
Element		1

Continued

EXAMPLE 21.2 —cont'd

Node Output

The following table is printed for nodes belonging to Node Set NODE_FREE (Fig. 21.5).

Node Footnote	U1	U2
1	−4.239	18.98
2	4.761	19.70
3	−3.517	8.372
4	3.683	9.011
Maximum	4.761	19.70
At node	2	2
Minimum	−4.239	8.372
At node	1	3

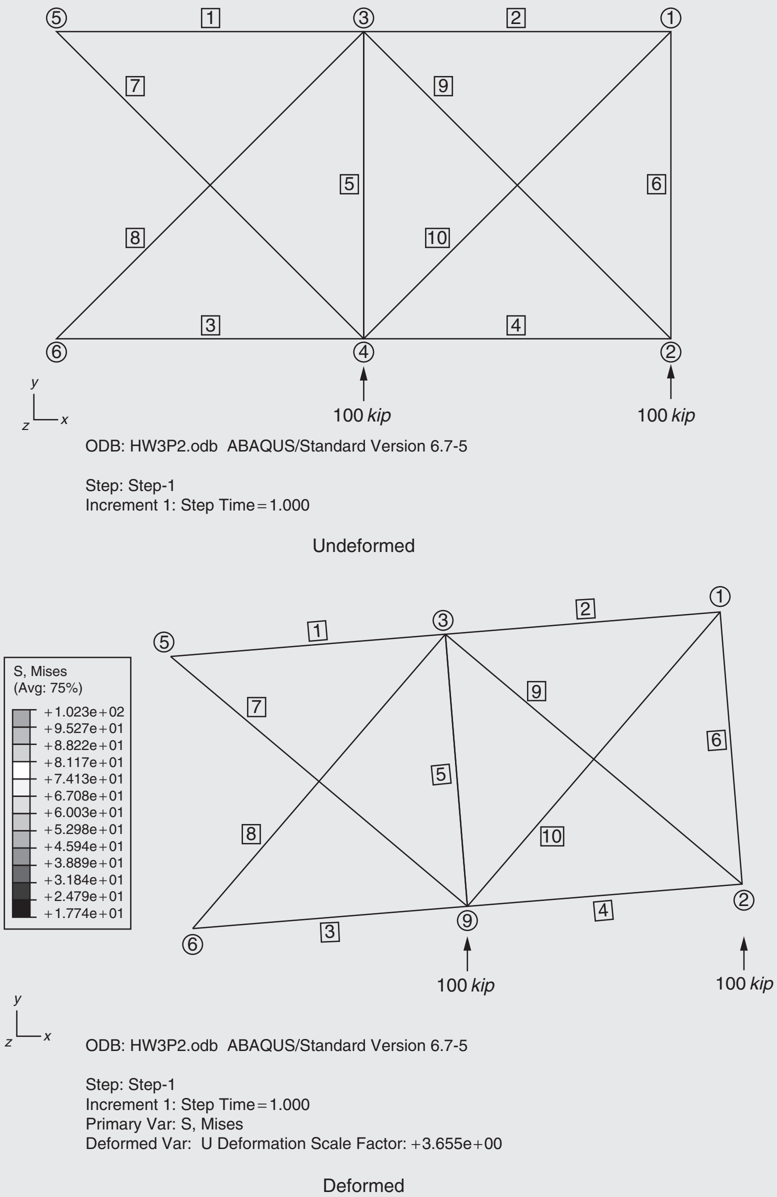


FIGURE 21.5 Undeformed and deformed plots. *Avg*, average; *Var*, variable.

EXAMPLE 21.3

Data (Fig. 21.6):

$E = 2 \times 10^7$ N/cm^2, $A = 2$ cm^2 for all members

All dimensions are in centimeters; all base nodes are fixed.

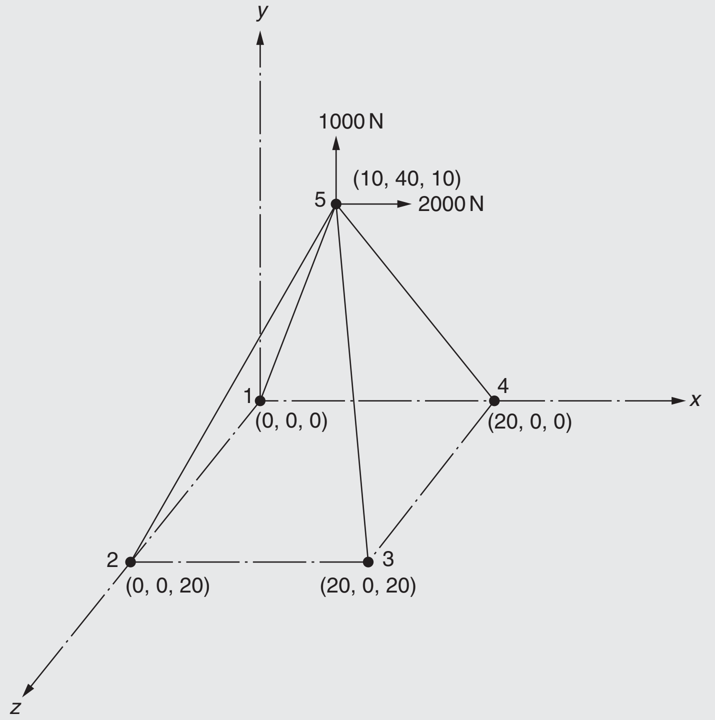


FIGURE 21.6 A four-bar space truss.

Input File

```
*HEADING
FINITE ELEMENT ANALYSIS OF A 4-BAR SPACE TRUSS
** UNIT SYSTEM
** LENGTH: cm
** FORCE: N
** Modulus: N/cm^2
*NODE
```

1,	0.0,	0.0,	0.0,
2,	0.0,	0.0,	20.0,
3,	20.0,	0.0,	20.0,
4,	20.0,	0.0,	0.0,
5,	10.0,	40.0,	10.0,

```
*NSET, NSET=NODE_FREE
5,
*ELEMENT, TYPE=T3D2, ELSET=ELE_1
```

1,	1,	5,
2,	2,	5,
3,	3,	5,
4,	4,	5,

```
*ELSET, ELSET=ELE_ALL
ELE_1
*MATERIAL, NAME=MAT_1
*ELASTIC
2.0E7
*SOLID SECTION, ELSET=ELE_ALL,MATERIAL=MAT_1
2.0
*BOUNDARY
```

Continued

EXAMPLE 21.3 —cont'd

1,	1,	3
2,	1,	3
3,	1,	3
4,	1,	3

```
*STEP
*STATIC
*CLOAD
5, 1, 2000
5, 2, 1000
*NODE PRINT, NSET=NODE_FREE
U
*EL PRINT, ELSET=ELE_ALL
S11
*END STEP
```

Output

The following table us printed at the integration points for Element Type T3D2 and Element Set ELE_ALL.

Element	PT Footnote	S11
1	1	1193.
2	1	1193.
3	1	−928.1
4	1	−928.1
Maximum		1193.
Element		1
Minimum		−928.1
Element		3

Node Output

The following table is printed for nodes belonging to Node Set NODE_FREE.

	Node Footnote	U1	U2	U3
	5	9.5459E-03	2.9831E-04	0.000
Maximum		9.5459E-03	2.9831E-04	0.000
At node		5	5	5
Minimum		9.5459E-03	2.9831E-04	0.000
At node		5	5	5

The analysis has been completed.
ANALYSIS COMPLETE (Fig. 21.7)

S.S11 at Loc 1	Node Label	U.Magnitude at Loc 1	U.U1 at Loc 1	U.U2 at Loc 1	U.U3 at Loc 1
1.19324E+03	1	2.38649E-33	562.5E-36	2.25E-33	562.5E-36
1.19324E+03	2	2.38649E-33	562.5E-36	2.25E-33	−562-5E- 36
−928.078	3	1.85616E-33	437.5E-36	−1.75E-33	437.5E-36
−928.078	4	1.85616E-33	437-5E-36	−1.75E-33	−437.5E-36
132.583	5	9.5506E-03	9.54594E-03	298.311E-06	0.
Minimum −928.078		1.85616E-33	437.5E-36	−1.75E-33	−562.5E-36
At Node 3		4	4	4	2
Maximum 1.19324E+03		9.5506E-03	9.54594E-03	298.311E-06	562.5E-36
At Node 2		5	5	5	1
Total 662.913		9.55060E-03	9.54594E-03	298.311E-06	0.

EXAMPLE 21.3 —cont'd

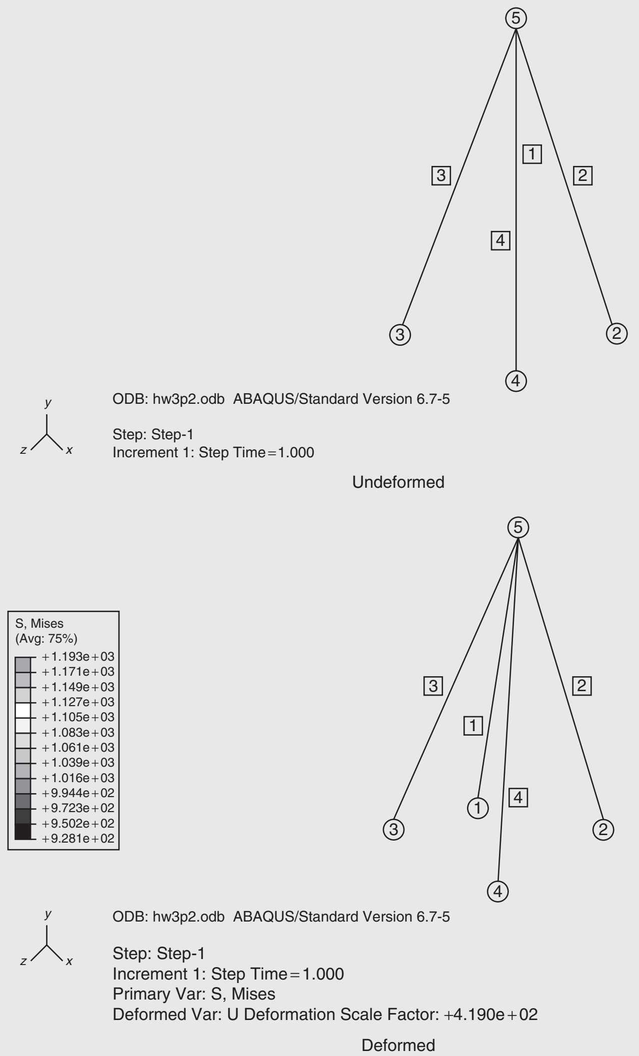


FIGURE 21.7 Undeformed and deformed plots.

EXAMPLE 21.4

Data (Fig. 21.8):

Finite element to be used: T3D2 = three-dimensional (3D) two-node truss element
Dimensions in figure are in millimeters; area of cross-section of each bar: 3225.8 mm^2
Material: Aluminum; $E = 69$ GPa
Load applied: Vertical load of 10,000 N at Node 1

Input File

```
** UNIT
** FORCE: N
** LENGTH: m
*HEADING:
FINITE ELEMENT ANALYSIS OF A 25-BAR SPACE TRUSS
*********** The coordinates of nodes
*NODE, NSET=NODE_ALL
```

Continued

EXAMPLE 21.4 —cont'd

1,	−0.9525,	0.0,	5.0800
2,	0.9525,	0.0,	5.0800
3,	−0.9525,	.9525,	2.5400
4,	.9525,	.9525,	2.5400
5,	.9525,	−.9525,	2.5400
6,	−.9525,	−.9525,	2.5400
7,	−2.5400,	2.5400,	0.0
8,	2.5400,	2.5400,	0.0
9,	2.5400,	−2.5400,	0.0
10,	−2.5400,	−2.5400,	0.0

```
********** The connectivity of elements
********** The first term in each data line is the number of the element
********** The last two terms in each data line are the numbers of nodes
*ELEMENT, TYPE=T3D2, ELSET=ELE_ALL
```

1,	1,	2,
2,	1,	4,
3,	2,	3,
4,	1,	5,
5,	2,	6,
6,	2,	4,
7,	2,	5,
8,	1,	3,
9,	1,	6,
10,	3,	6,
11,	4,	5,
12,	3,	4,
13,	6,	5,
14,	3,	10,
15,	6,	7,
16,	4,	9,
17,	5,	8,
18,	4,	7,
19,	3,	8,
20,	5,	10,
21,	6,	9,
22,	6,	10,
23,	3,	7,
24,	4,	8,
25,	5,	9,

```
********** Elements are grouped into several element sets, which facilitates assigning element
attributes
*ELSET, ELSET=SET1
1
*ELSET, ELSET= SET2, GENERATE
2,5,1
*ELSET, ELSET=SET3, GENERATE
6,9,1
*ELSET, ELSET=SET4, GENERATE
10,11,1
*ELSET, ELSET=SET5, GENERATE
12,13,1
```

EXAMPLE 21.4 —cont'd

```
*ELSET, ELSET=SET6, GENERATE
14,17,1
*ELSET, ELSET=SET7, GENERATE
18,21,1
*ELSET, ELSET=SET8, GENERATE
22,25,1*
************* Assign cross-section area to members of each element set
*SOLID SECTION, ELSET=SET1, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET2, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET3, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET4, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET5, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET6, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET7, MATERIAL=BAR_MAT
0.0032258
*SOLID SECTION, ELSET=SET8, MATERIAL=BAR_MAT
0.0032258
************* Material properties definition
*MATERIAL, NAME=BAR_MAT
************ Young's modulus
*ELASTIC
6.9E10
************ Density
*DENSITY
2.77E3
************ Constrain the displacement boundary conditions
*NSET, NSET=FIXED
7, 8, 9, 10
************ Define the force boundary conditions
*NSET, NSET=LOAD_NODE
1
*BOUNDARY
FIXED, 1, 3
************ The comment lines are a static analysis of the structure
*STEP
*STATIC
1, 1.0, 1.0E-5, 1.0
*CLOAD
LOAD_NODE, 3, 10,000.0
*NODE· PRINT, NSET=NODE_ALL
U, RF
*EL PRINT, ELSET=ELE_ALL
S
*END STEP
```

Continued

EXAMPLE 21.4 —cont'd

Output

Element Output

The following table is printed at the integration points for Element Type T3D2 and Element Set ELE_ALL.

Element	PT Footnote	S11
1	1	−3.6221E+05
2	1	3.1513E+05
3	1	3.1513E+05
4	1	3.1513E+05
5	1	3.1513E+05
6	1	−2.5789E+05
7	1	−2.5789E+05
8	1	1.3975E+06
9	1	1.3975E+06
10	1	−2.2164E+05
11	1	−3.4060E+04
12	1	−4.0399E+04
13	1	−4.0399E+04
14	1	5.5171E+05
15	1	5.5171E+05
16	1	3.0848E+04
17	1	3.0848E+04
18	1	−1.2744E+05
19	1	7.4997E+05
20	1	−1.2744E+05
21	1	7.4997E+05
22	1	1.1096E+06
23	1	1.1096E+06
24	1	7.1166E+04
25	1	7.1166E+04
Maximum		1.3975E+06
Element		8
Minimum		−3.6221E+05
Element		1

Node Output

The following table is printed for nodes belonging to Node Set NODE_ALL.

Node Footnote	U1	U2	U3	RF1	RF2	RF3
1	1.4758E-04	0.000	1.3624E-04	0.000	0.000	0.000
2	1.3758E-04	0.000	−1.2548E-5	0.000	0.000	0.000
3	−8.9143E-06	−3.0596E.06	7.6418E-05	0.000	0.000	0.000
4	−1.0030E-05	−4.7017E-07	−1.8960E-06	0.000	0.000	0.000
5	−1.0030E-05	4.7017E-07	−1.8960E-06	0.000	0.000	0.000
6	−8.9143E-06	3.0596E-06	7.6418E-05	0.000	0.000	0.000
7	0.000	0.000	0.000	−1978.	2885.	−3438.
8	0.000	0.000	0.000	1978.	1018.	−1562.
9	0.000	0.000	0.000	1978.	−1018.	−1562.
10	0.000	0.000	0.000	−1978.	−2885.	−3438.
Maximum	1.4758E-04	3.0596E-06	1.3624E-04	1978.	2885.	0.000
At node	1	6	1	8	7	1
Minimum	−1.0030E-05	−3.0596E-06	−1.2548E-05	−1978.	−2885.	−3438.
At node	4	3	2	10	10	10

EXAMPLE 21.4 —cont'd

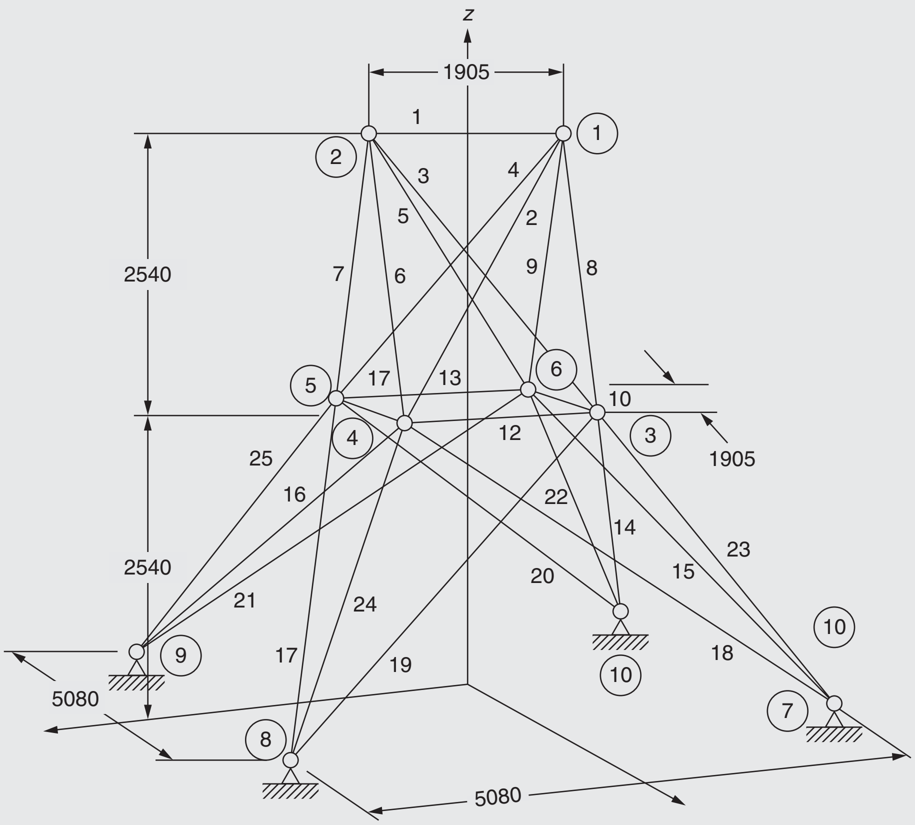


FIGURE 21.8 A 25-bar space truss.

EXAMPLE 21.5

Data (Fig. 21.9):

Plate is fixed at one edge and supported by rollers at the opposite edge

Other two edges are free (no support)

A concentrated transverse force of 100 N is applied at center

Material: $E = 10^3$ N/mm^2, Poisson ratio = 0.3

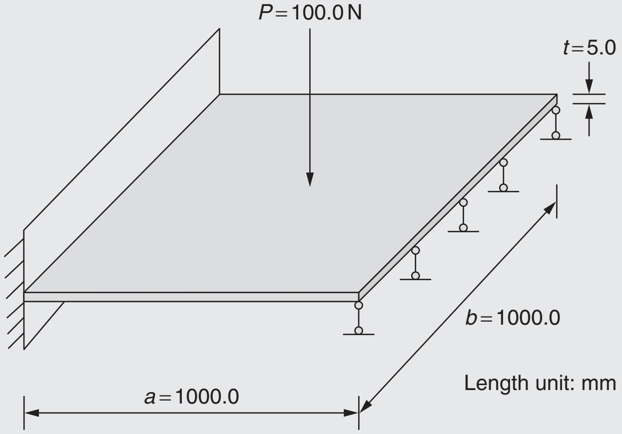


FIGURE 21.9 Plate subjected to transverse load.

Continued

EXAMPLE 21.5 —cont'd

Input File

```
*Heading
** Job name: plate model name: Model-1
*Preprint, echo=NO, model=NO, history=NO, contact=NO
**
**PARTS
**
*Part, name=Part-1
*End Part
**
**
** ASSEMBLY
**
*Assembly, name=Assembly
**
*Instance, name = Part-1-1, part = Part-1
*Node
```

1,	0.,	−500.,	0.
2,	0.,	0.,	0.
3,	−500.,	0.,	0.
4,	−500.,	−500.,	0.
5,	0.,	500.,	0.
6,	500.,	0.,	0.
7,	500.,	500.,	0.
8,	−500.,	500.,	0.
9,	500.,	−500.,	0.
10,	0.,	−250.,	0.
11,	−250.,	0.,	0.
12,	−500.,	−250.,	0.
13,	−250.,	−500.,	0.
14,	0.,	250.,	0.
15,	250.,	0.,	0.
16,	500.,	250.,	0.
17,	250.,	500.,	0.
18,	−250.,	500.,	0.
19,	−500.,	250.,	0.
20,	250.,	−500.,	0.
21,	500.,	−250.,	0.
22,	−250.,	−250.,	0.
23,	250.,	250.,	0.
24,	−250.,	250.,	0.
25,	250.,	−250.,	0.

Node definition and coordinates

Format: Node label *i*, *xi*, *yi*, *zi*

```
*Element, type=S4R
```

EXAMPLE 21.5 —cont'd

1,	12,	22,	11,	3
2,	11,	22,	10,	2
3,	4,	13,	22,	12
4,	1,	10,	22,	13
5,	16,	23,	15,	6
6,	15,	23,	14,	2
7,	7,	17,	23,	16
8,	5,	14,	23,	17
9,	19,	24,	18,	8
10,	18,	24,	14,	5
11,	3,	11,	24,	19
12,	2,	14,	24,	11
13,	21,	25,	20,	9
14,	20,	25,	10,	1
15,	6,	15,	25,	21
16,	2,	10,	25,	15

Element definition and connectivity

```
Format: Element label, node label i, node label j, node label k, node label l
*Nset, nset=_PickedSet2, internal, generate
1, 25, 1
*Elset, elset=_PickedSet2, internal, generate
1, 16, 1
```

Node set definition

Format: The first node label, the last node label, label number increment

Element set definition

Format: The first element label, the last element label, label increment

```
** Section: Section-1
*Shell Section, elset=_PickedSet2, material=Material-1
5., 5
Plate thickness, Number of integration points
*End Instance
**
*Nset, nset=_PickedSet2, internal, generate, instance=Part-1-1
1, 25, 1
*Elset, elset=_PickedSet2, internal, generate, instance=Part-1-1
1, 16, 1
*Nset, nset=_PickedSet7, internal, instance=Part-1-1
2,
*Nset, nset=_PickedSet11, internal, instance=Part-1-1
3, 4, 8, 12, 19
*Elset, elset=_PickedSet11, internal, instance=Part-1-1
1, 3, 9, 11
*Nset, nset=_PickedSet12, internal, instance=Part-1-1
6, 7, 9, 16, 21
*Elset, elset=_PickedSet12, internal, instance=Part-1-1
5, 7, 13, 15
*End Assembly
**
** MATERIALS
**
```

Continued

EXAMPLE 21.5 —cont'd

```
Material definition
Format: Young's modulus, Poisson ratio
*Material, name=Material-1
*Elastic
10,000., 0.3
**
**
** STEP: Step-1
**
*Step, name=Step-1
*Static
1., 1., 1e-05, 1.
**
** BOUNDARY CONDITIONS
**
** Name: BC-1 Type: Displacement/Rotation
*Boundary
Boundary condition (BC) definition
Format: Node set applied BCs., the first degree of freedom (dof), the last dof, the magnitude (default is
zero)
_PickedSet11, 1, 1
_PickedSet11, 2, 2
_PickedSet11, 3, 3
_PickedSet11, 4, 4
_PickedSet11, 5, 5
_PickedSet11, 6, 6
** Name: BC-2 Type: Displacement/Rotation
*Boundary
_PickedSet12, 3, 3
**
** LOADS
**
** Name: Load-1 Type: Concentrated force
Concentrated load definition
Format: Node set applied load, dof, the magnitude
*Cload
_PickedSet7, 3, -100.
**
** OUTPUT REQUESTS
**
*Restart, write, frequency=0
**
**FIELD OUTPUT: F-Output-1
**
*Output, field, variable=ALL
**
** HISTORY OUTPUT: H-Output-1
**
*Output, history, variable=PRESELECT
*NODE PRINT, NSET=_PickedSet2
U
*EL PRINT, ELSET=_PickedSet2
SF
*End Step
```

EXAMPLE 21.5 —cont'd

Output

Element Output

The following table is printed at the integration points for Element Type S4R and Element Set ASSEMBLY_PICKEDSET2.

Element	PT Footnote	SF1	SF2	SF3	SF4	SF5	SF6	SM1	SM2	SM3
1	1	0.000	0.000	0.000	−9.0928E-02	−2.1040E-02	0.000	12.25	2.649	0.7926
2	1	0.000	1.1102E-16	−5.5511E-17	-0.1123	−5.7767E-02	0.000	−10.76	−8.031	2.114
3	1	0.000	1.3878E-17	5.5511E-17	−4.5811E-02	2.3770E-02	0.000	7.399	0.9524	2.547
4	1	−1.1102E-16	−6.9389E-18	−1.3878E-17	−2.4416E-02	−7.0566E-03	0.000	−3.778	0.3697	1.132
5	1	0.000	−2.7756E-17	2.7756E-17	3.4932E-02	−2.2192E-02	0.000	−3.033	−2.439	1.176
6	1	−2.2204E-16	0.000	5.5511E-17	7.8370E-02	9.5066E-02	0.000	−14.98	9.796	1.730
7	1	0.000	0.000	0.000	2.8329E-02	4.9182E-03	0.000	−4.874	−1.830	2.631
8	1	0.000	−9.7145E-17	0.000	−1.5109E-02	1.2125E-02	0.000	−8.743	−0.2302	1.048
9	1	0.000	1.3878E-17	−5.5511E-17	−4.5811E-02	−2.3770E-02	0.000	7.399	0.9524	−2.547
10	1	−1.1102E-16	6.9389E-18	0.000	−2.4416E-02	7.0566E-03	0.000	−3.778	0.3697	−1.132
11	1	0.000	0.000	0.000	−9.0928E-02	2.1040E-02	0.000	12.25	2.649	−0.7926
12	1	0.000	0.000	5.5511E-17	−0.1123	5.7767E-02	0.000	−10.76	−8.031	−2.114
13	1	0.000	−5.5511E-17	0.000	2.8329E-02	−4.9182E-03	0.000	−4.874	−1.830	−2.631
14	1	0.000	−1.0755E-16	0.000	−1.5109E-02	−1.2125E-02	0.000	−8.743	−0.2302	−1.048
15	1	0.000	−2.7756E-17	−2.7756E-17	3.4932E-02	2.2192E-02	0.000	−3.033	−2.439	−1.176
16	1	−2.2204E-16	0.000	2.7756E-17	7.8370E-02	−9.5066E-02	0.000	−14.98	−9.796	−1.730

Node Output

The following table is printed for nodes belonging to Node Set ASSEMBLY_PICKEDSET2 (Fig. 21.10).

Node Footnote	U1	U2	U3	UR1	UR2	UR3
1	4.0947E-19	1.0701E-18	−3.800	−4.8347E-03	1.2127E-02	0.000
2	3.6196E-19	0.000	−8.940	0.000	1.1545E-02	0.000
5	4.0947E-19	−1.0701E-18	−3.800	4.8347E-03	1.2127E-02	0.000
6	1.5953E-18.	0.000	0.000	0.000	−2.1182E-02	0.000
7	2.1991E-18	−1.6027E-19	0.000	−2.7621E-05	−1.4155E-02	0.000
9	2.1991E-18	1.6027E-19	0.000	2.7621E-05	−1.4155E-02	0.000
10	−1.0011E-18	−8.9511E-20	−6.672	−1.8132E-02	3.3478E-03	0.000
11	1.1197E-18	0.000	−3.745	0.000	2.9973E-02	0.000
13	−9.6191E-19	−5.4061E-19	−1.140	−1.1003E-02	9.1408E-03	0.000
14	−1.0011E-18	8.9511E-20	−6.672	1.8132E-02	3.3478E-03	0.000
15	−4.4973E-19	0.000	−6.512	0.000	−3.0934E-02	0.000
16	−1.9575E-19	1.1251E-19	0.000	2.1357E-05	−3.2770E-02	0.000
17	−2.9004E-19	−1.8432E-19	−3.541	9.1281E-03	−1.4193E-02	0.000
18	9.6191E-19	5.4061E-19	−1.140	1.1003E-02	9.1408E-03	0.000
20	−2.9004E-19	1.8432E-19	−3.541	−9.1281E-03	−1.4193E-02	0.000
21	−1.9575E-19	−1.1251E-19	0.000	−2.1357E-05	−3.2770E-02	0.000
22	−5.6097E-19	−2.2925E-20	−3.130	−4.9207E-03	2.5002E-02	0.000
23	1.5307E-18	−1.0232E-19	−5.596	7.3157E-03	−1.1969E-02	0.000
24	−5.6097E-19	2.2925E-20	−3.130	4.9207E-03	2.5002E-02	0.000
25	1.5307E-18	1.0232E-19	−5.596	−7.3157E-03	−1.1969E-02	0.000

Continued

EXAMPLE 21.5 —cont'd

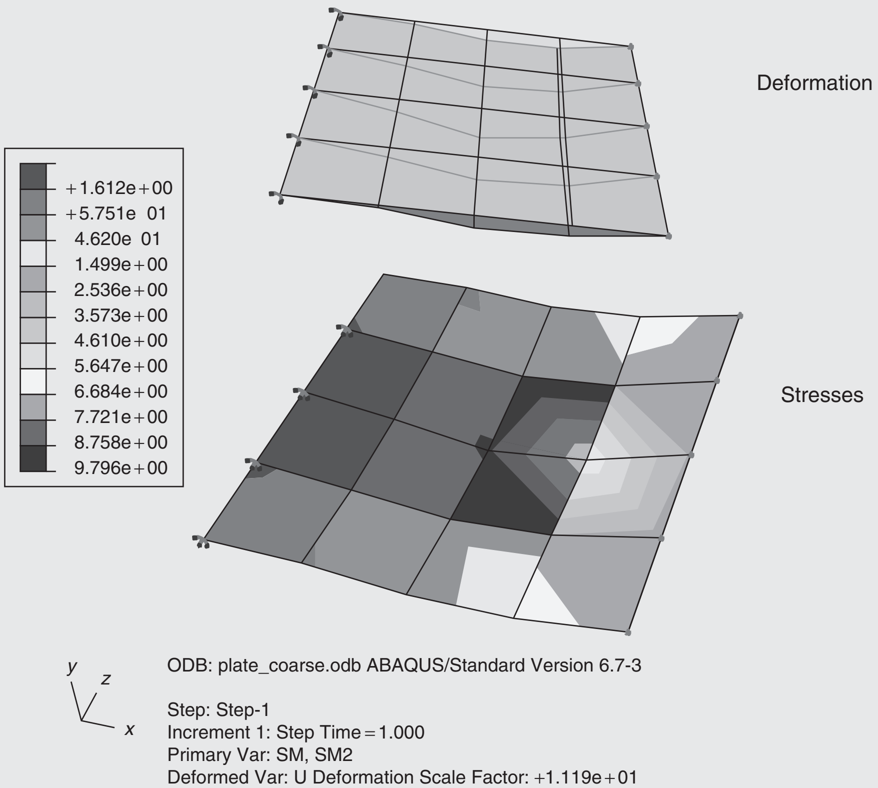


FIGURE 21.10 Deformation of and stresses in the plate.

EXAMPLE 21.6

Problem Description

The thin "L-shaped" part shown is exposed to 20°C on the two surfaces of the inner corner and 120°C on the two surfaces of the outer corner (Fig. 21.11). A heat flux of 10 W/m^2 is applied to the top surface. Treat the remaining surfaces as insulated.

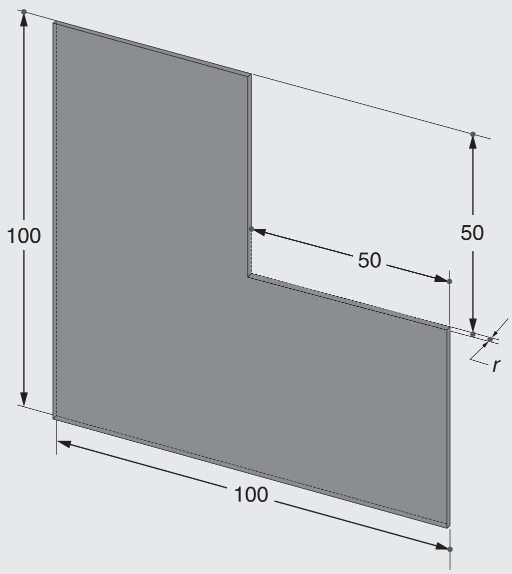


FIGURE 21.11 An L-shaped plate.

EXAMPLE 21.6 —cont'd

Analysis Steps

1. Start ABAQUS and choose to create a new model database.
2. In the model tree, double-click on the Parts node (or right-click on Parts and select Create) (Fig. 21.12).

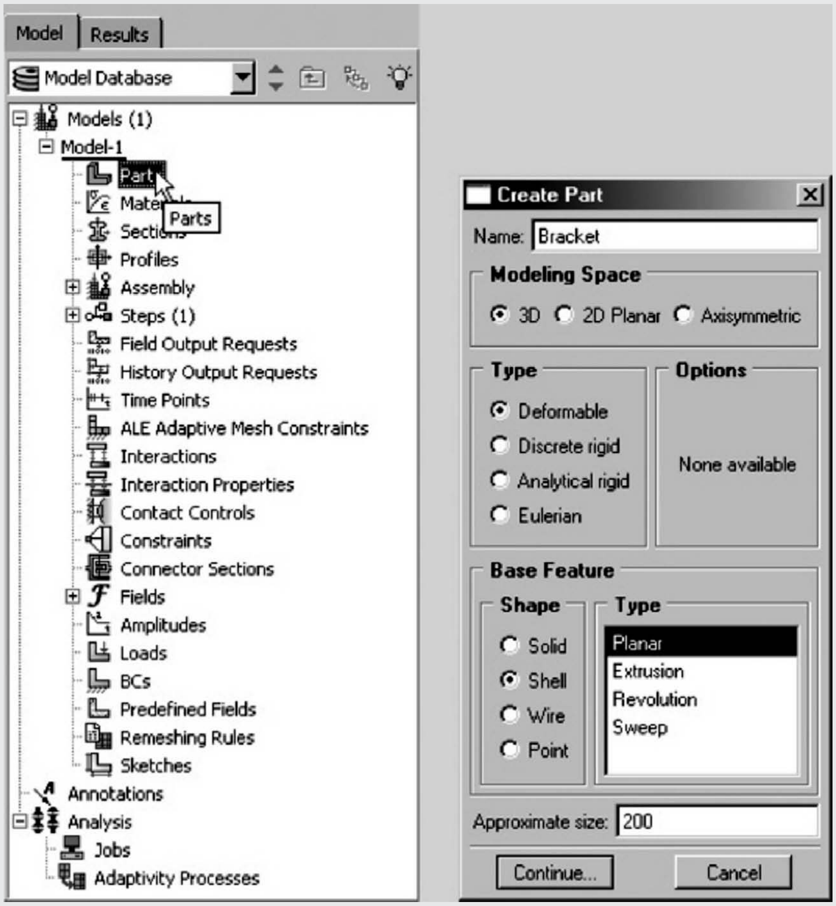


FIGURE 21.12 Screen display for Start ABAQUS and Create Part.

3. In the Create Part dialog box (shown in Fig. 21.12), name the part and proceed as follows:
   - **a.** Select 3D.
   - **b.** Select Deformable.
   - **c.** Select Shell.
   - **d.** Select Planar.
   - **e.** Set approximate size = 20.
   - **f.** Click Continue …
4. Create the geometry shown in Fig. 21.13.
5. Double-click on the Materials node in the model tree (Fig. 21.14).
   - **a.** Name the new material and give it a description.
   - **b.** Click on the Thermal tab → Conductivity.
   - **c.** Define the thermal conductivity (use International System of Units).

     Warning:

     There is no predefined system of units within ABAQUS, so the user is responsible for ensuring that the correct values are specified.
   - **d.** Click OK (Fig. 21.15).

Continued

EXAMPLE 21.6 —cont'd

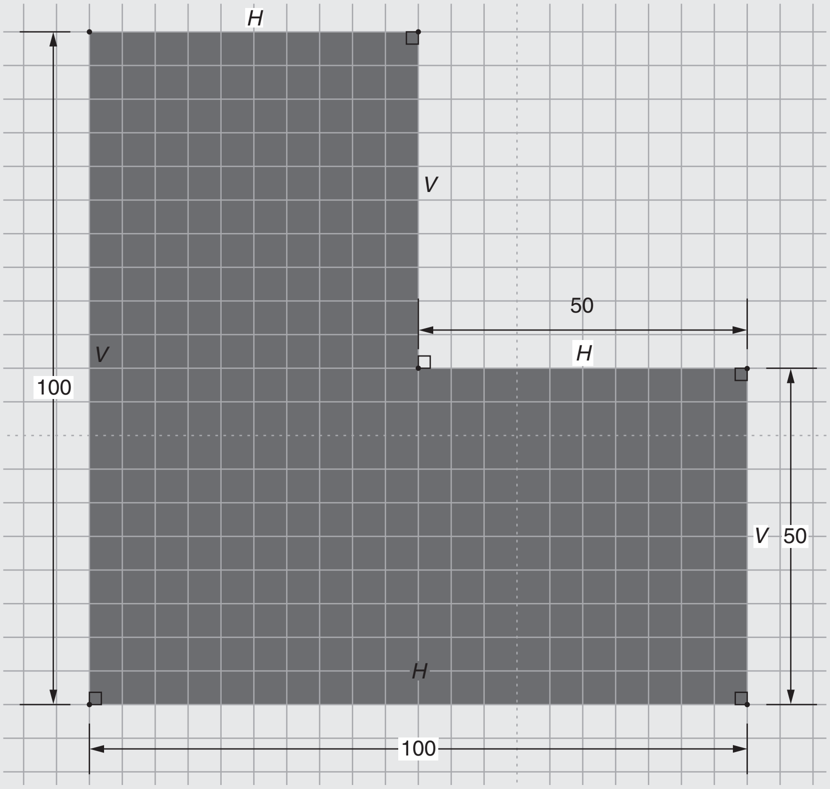


FIGURE 21.13 Screen for Create Geometry.

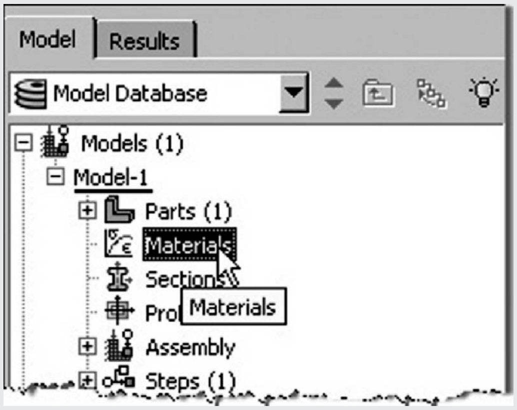


FIGURE 21.14 Materials node in the model database.

6. Double-click on the Sections node in the model tree.
   - **a.** Name the section *ShellProperties* and select Shell for the category and Homogeneous for the type.
   - **b.** Click Continue …
   - **c.** Select the material created above (Steel) and set the thickness to 1.
   - **d.** Click OK (Fig. 21.16).

EXAMPLE 21.6 —cont'd

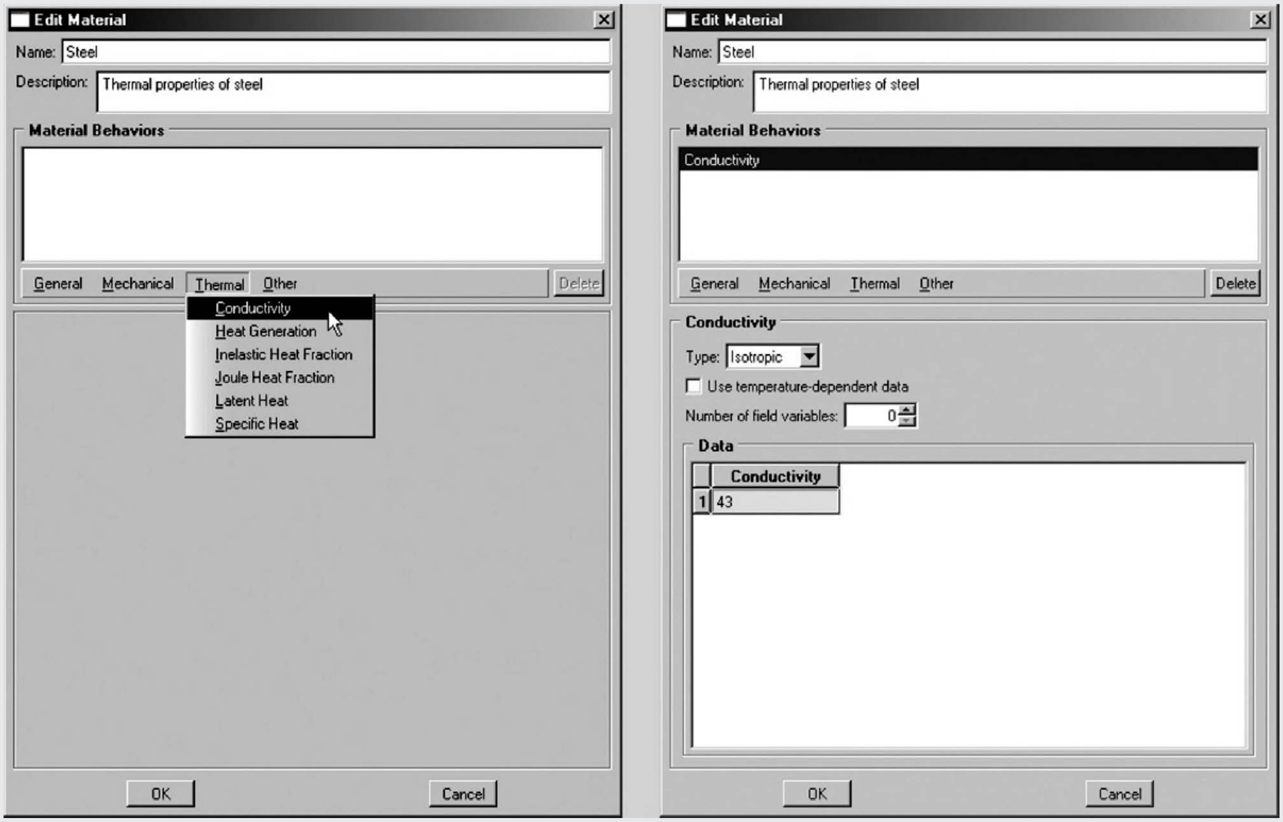


FIGURE 21.15 Specification of conductivity in material dialogue box.

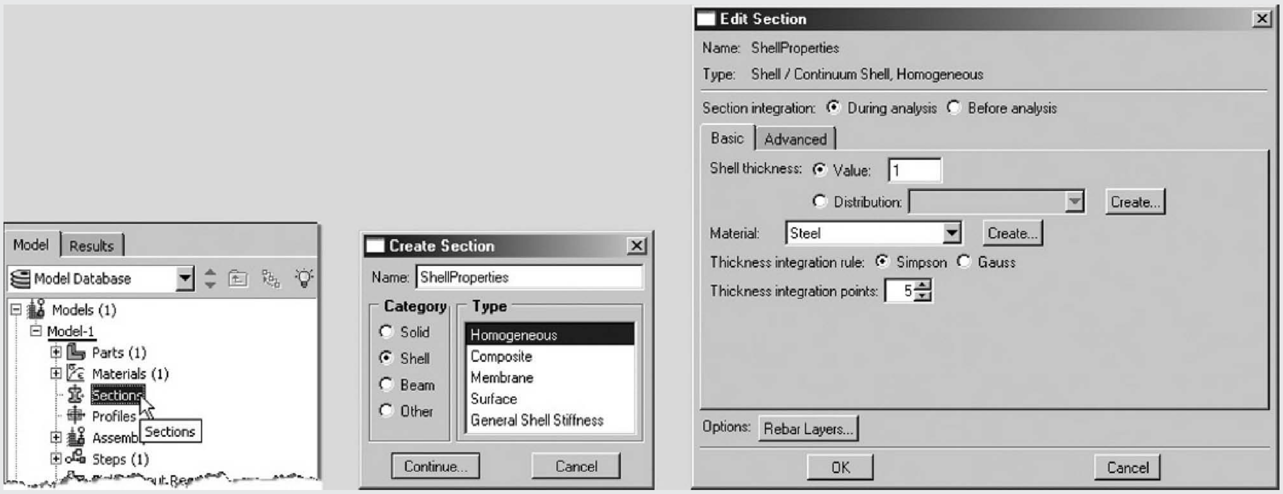


FIGURE 21.16 Specification of section properties in model dialogue box.

7. Expand the Parts node in the model tree, expand the node of the part just created, and double-click on Section Assignments.
   a. Select the surface geometry in the viewport and press Done in the prompt area.
   b. Select the section just created (ShellProperties).
   c. Click OK (Fig. 21.17).

Continued

EXAMPLE 21.6 —cont'd

FIGURE 21.17 Section assignment in model dialogue box.

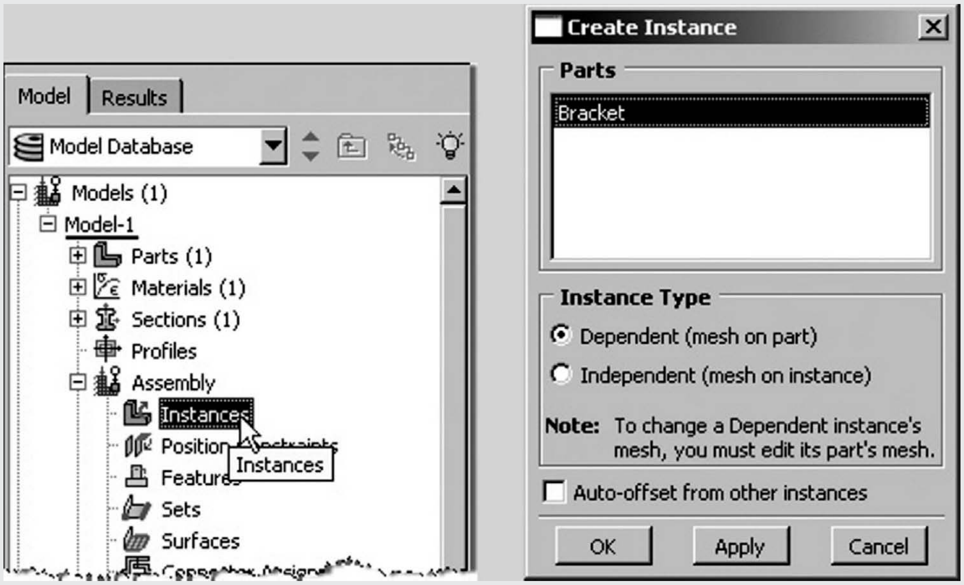


FIGURE 21.18 Selection of instance type in model dialogue box.

8. Expand the Assembly node in the model tree and then double-click on Instances.
   a. Select Dependent for the instance type.
   b. Click OK (Fig. 21.18).
9. In the model tree, under the expanded Assembly node, double-click on Sets.
   a. Name the set OutsideTemp.
   b. Click Continue …
   c. On the selection toolbar, from the drop-down menu select Edges.
   d. Select the two surfaces on the outside of the corner (left and bottom edges) in the viewport and press Done in the prompt area (Fig. 21.19).
   e. Create another set named *InsideTemp*.
   f. Select the two surfaces on the inside of the corner in the viewport and press Done in the prompt area.

EXAMPLE 21.6 —cont'd

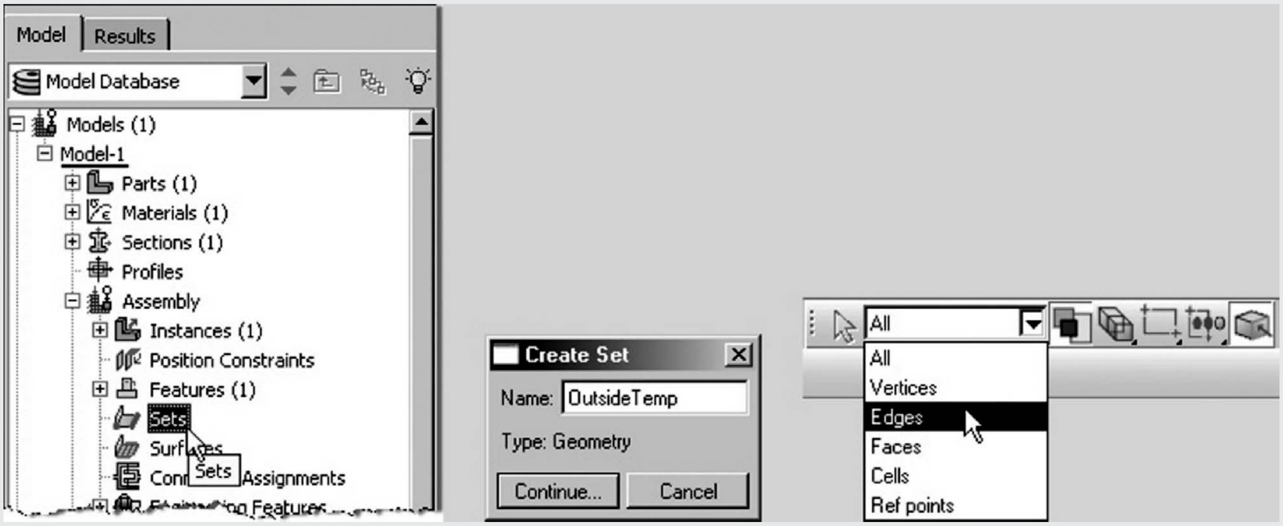


FIGURE 21.19 Selection of outside temperature and edges.

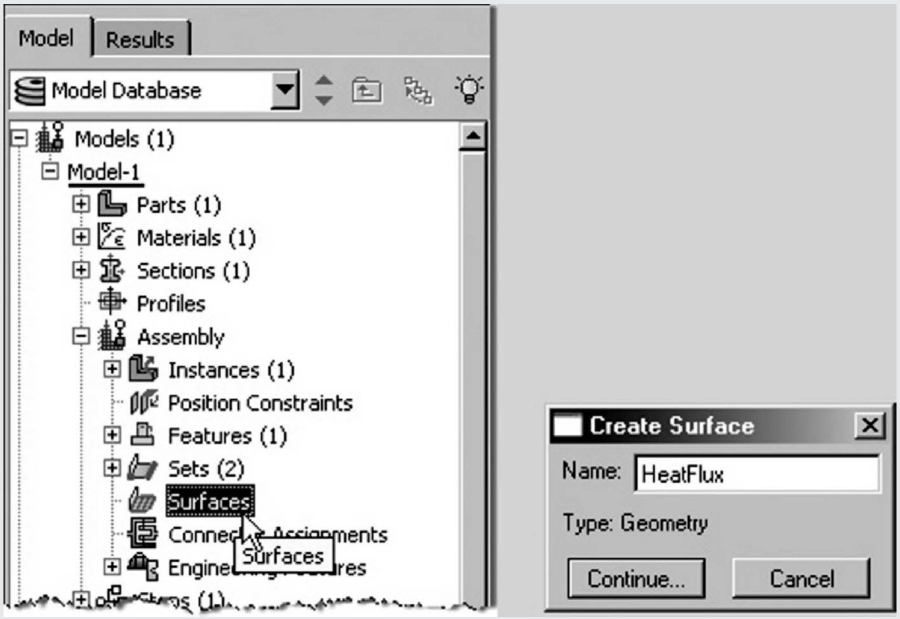


FIGURE 21.20 Specification of surfaces for heat flux.

10. In the model tree, under the expanded Assembly node, double-click on Surfaces.
  - **a.** Name the surface *HeatFlux*.
  - **b.** Click Continue …
  - **c.** Select the surface in the viewport and press Done in the prompt area.
  - **d.** Choose the Brown side (Fig. 21.20).

11. Double-click on the Steps node in the model tree.
  - **a.** Name the step, set the procedure to General, and select Heat Transfer.
  - **b.** Click Continue …
  - **c.** Give the step a description.
  - **d.** Set the response to Steady state.
  - **e.** Click OK (Fig. 21.21).

Continued

EXAMPLE 21.6 —cont'd

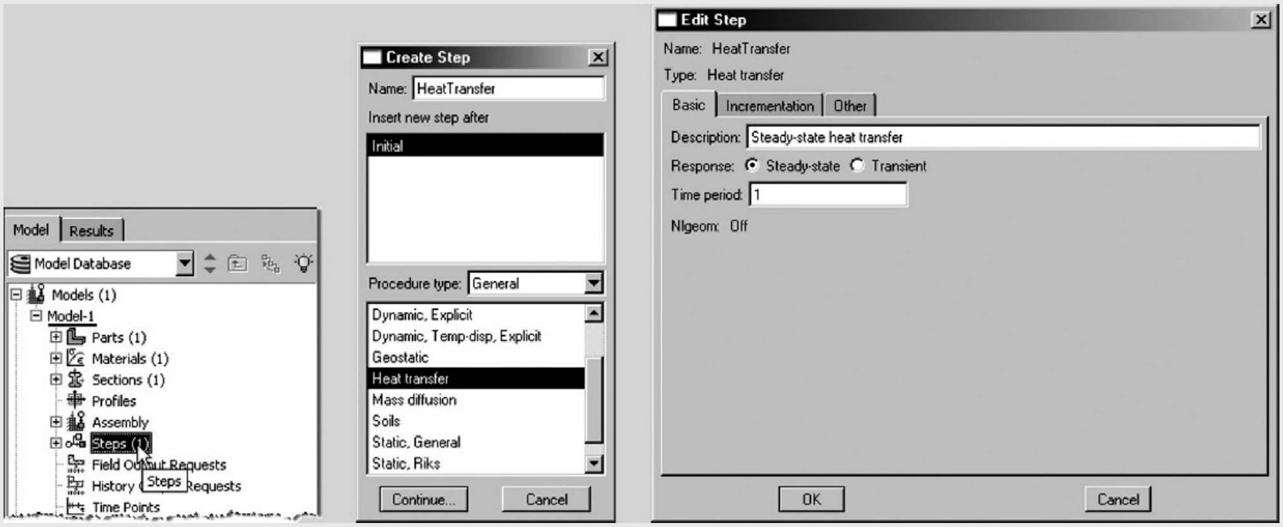


FIGURE 21.21 Specification of heat transfer analysis as steady state.

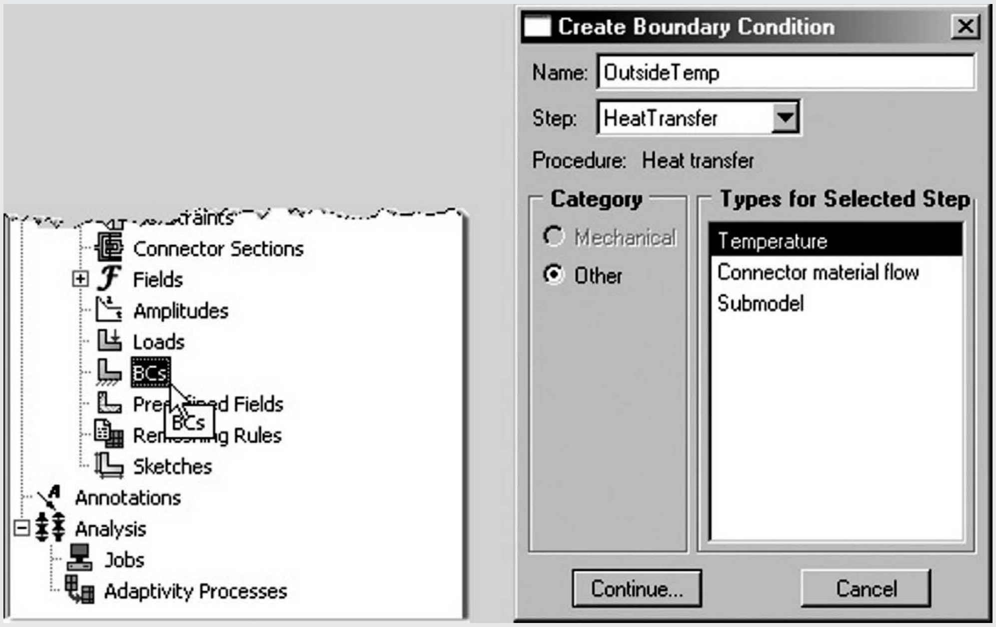


FIGURE 21.22 Creation of boundary conditions (BCs) as outside temperature.

12. Double-click on the BCs node in the model tree.
  a. Name the boundary condition *OutsideTemp* and select Temperature for the type.
  b. Click Continue … (Fig. 21.22)
  c. In the prompt area click on the Sets button.
  d. Select the set named OutsideTemp.
  e. Click Continue …
  f. Set the magnitude to 293.
  g. Click OK (Fig. 21.23).
  h. Repeat the procedure for the inside temperature using the set named InsideTemp, setting the magnitude to 393 (120°C).

13. Double-click on the Loads node in the model tree.
  a. Name the load *HeatFlux* and select Surface heat flux as the type.
  b. Click OK (Fig. 21.24).
  c. Select the surface named HeatFlux.

EXAMPLE 21.6 —cont'd

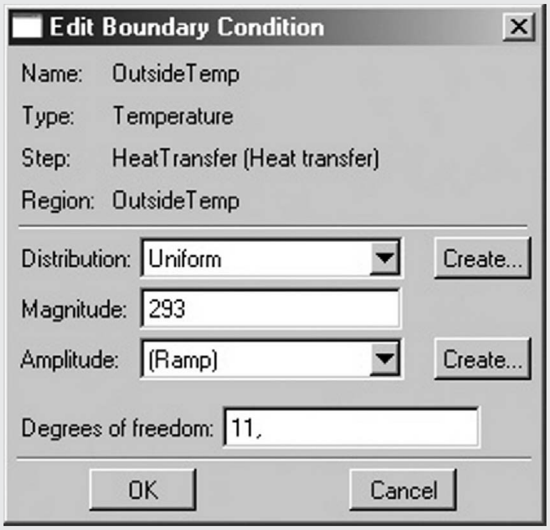


FIGURE 21.23 Specification of outside temperature as 293 (20°C).

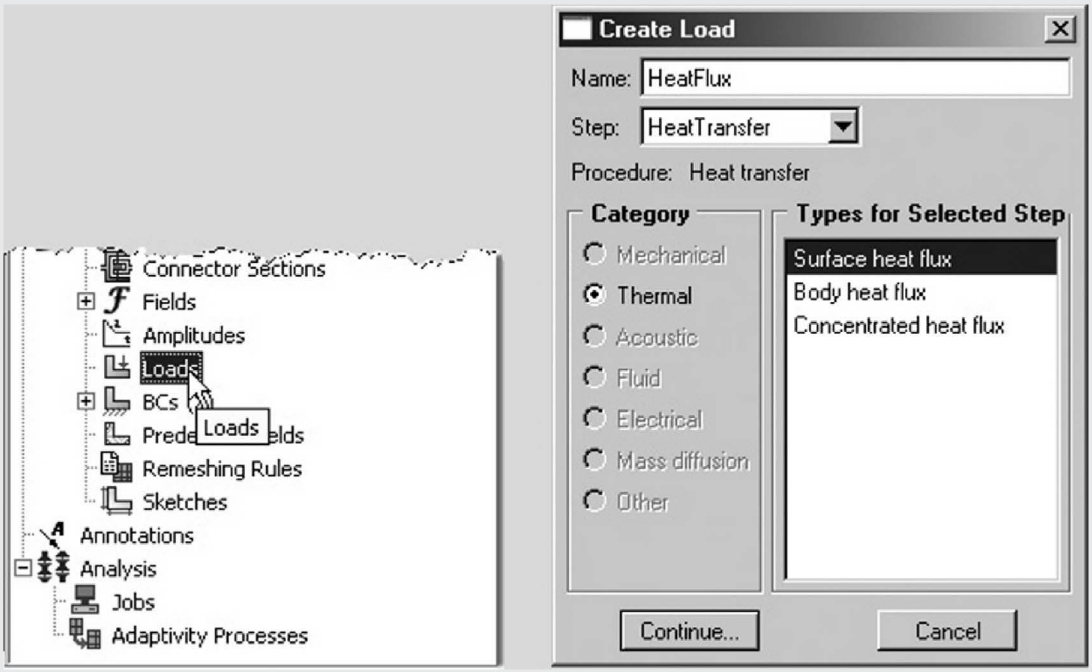


FIGURE 21.24 Selection of load as surface heat flux.

- **d.** For the magnitude enter 10.
- **e.** Click OK (Fig. 21.25).
- **f.** Note that any edge or surface without a boundary condition or load is treated as insulated.

14. Expand the Parts node in the model tree, expand the node of the Bracket part, and double-click on Mesh.

15. In the toolbox area, click on the Assign Element Type icon.

- **a.** Select Standard for the element type.
- **b.** Select Linear for the geometric order.
- **c.** Select HeatTransfer for the family.
- **d.** Note that the name of the element (DS4) and its description are given below the element controls.
- **e.** Click OK (Fig. 21.26).

Continued

EXAMPLE 21.6 —cont'd

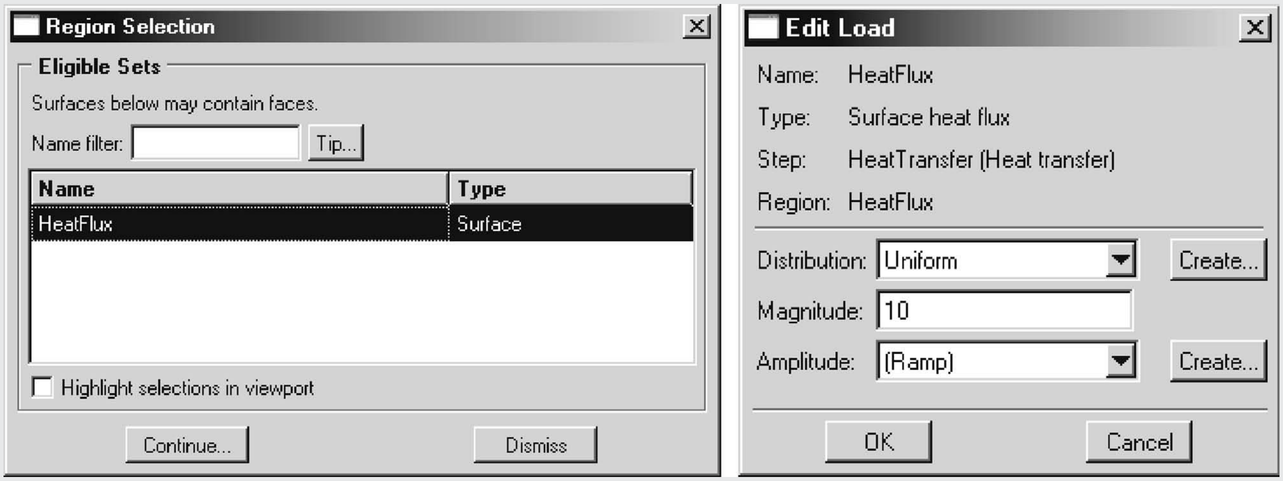


FIGURE 21.25 Specification of heat flux magnitude as 10.

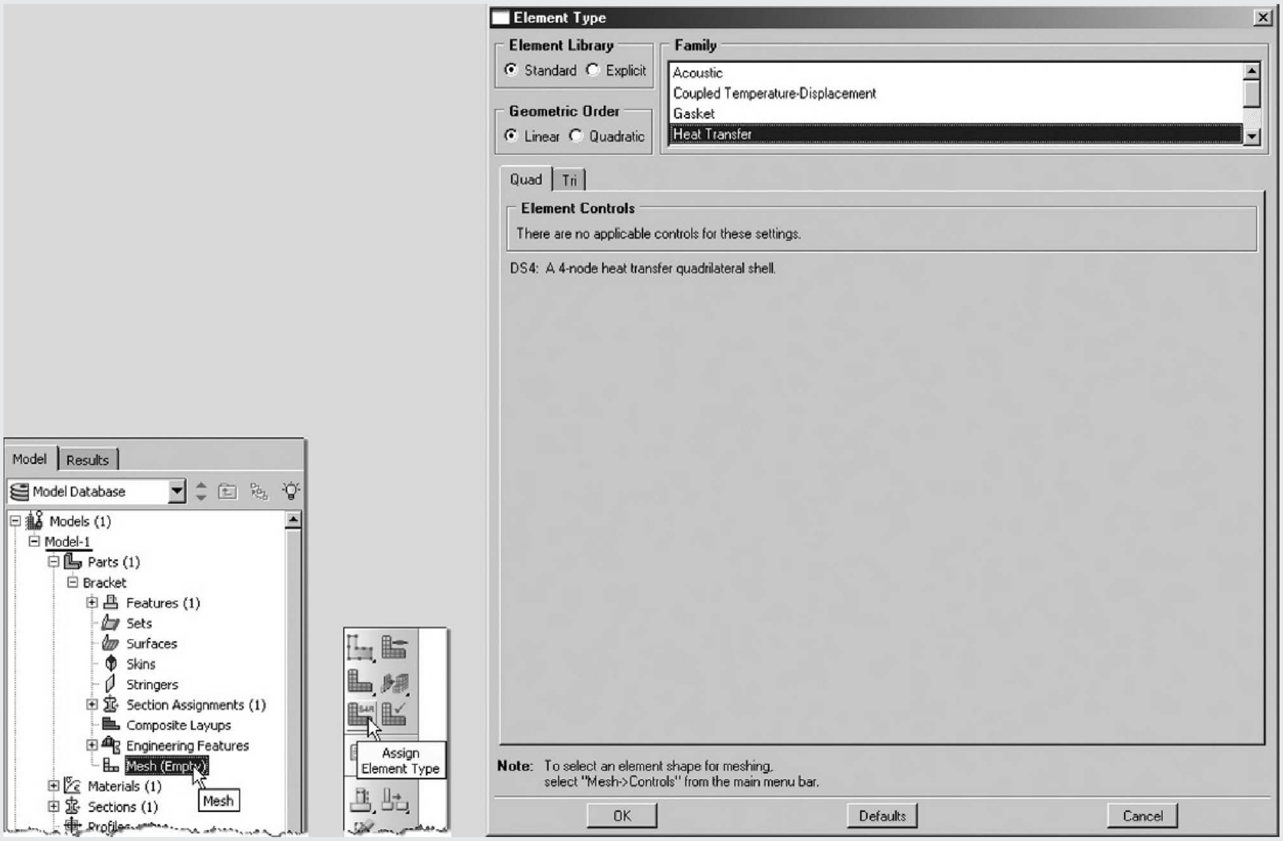


FIGURE 21.26 Assigning mesh with Element Type DS4.

16. In the toolbox area, click on the Assign Mesh Controls icon.
    a. Change the element shape to Quad.
    b. Change the algorithm to Medial axis to produce a more uniform mesh for this geometry.
17. In the toolbox area, click on the Seed Part icon.
    a. Set the approximate global size to 5.
    b. Click OK.
18. In the toolbox area, click on the Mesh Part icon.

EXAMPLE 21.6 —cont'd

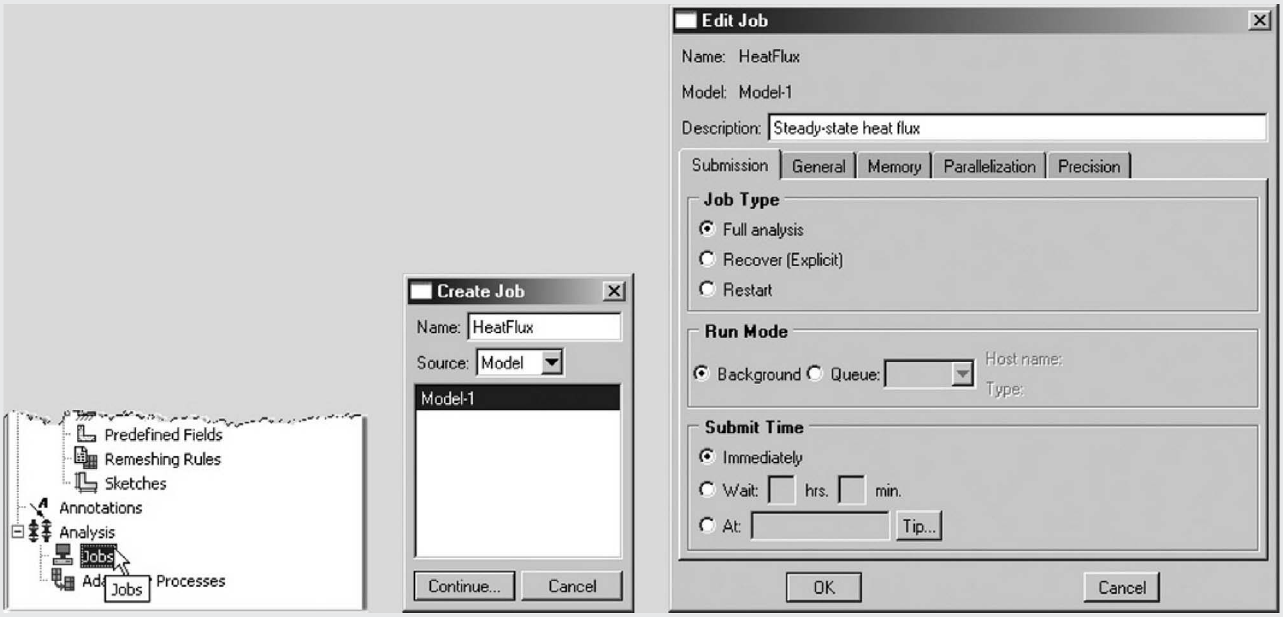


FIGURE 21.27 Specification of job name as heat flux with all default parameters.

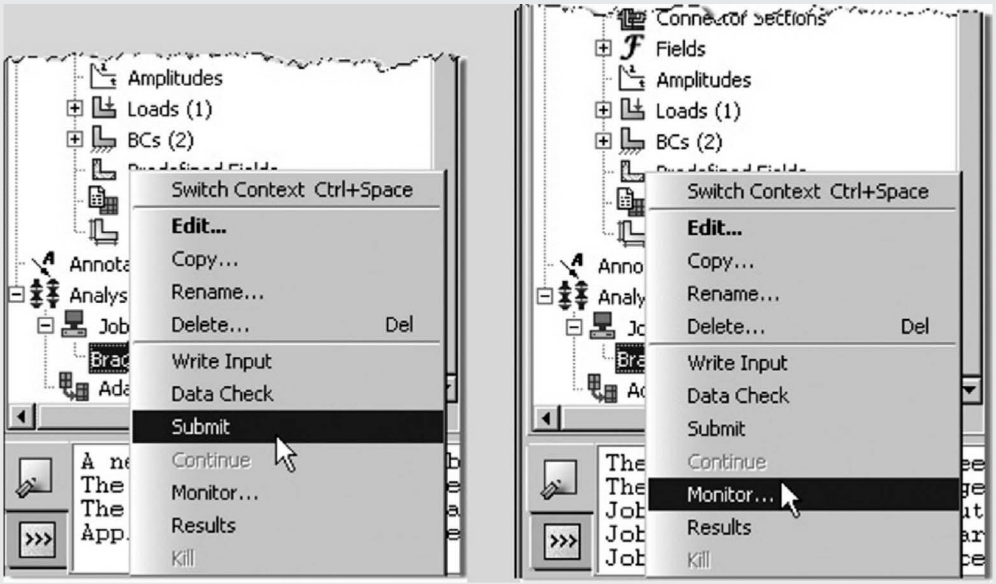


FIGURE 21.28 Submit the job and monitor it.

19. In the model tree, double-click on the Job node.
- **a.** Name the job *HeatFlux*.
- **b.** Click Continue …
- **c.** Give the job a description and accept all default parameters.
- **d.** Click OK (Fig. 21.27).

20. In the model tree, right-click on the job just created (HeatFlux) and select Submit.
- **a.** While ABAQUS is solving the problem, right-click on the job submitted (HeatFlux), and select Monitor (Fig. 21.28).
- **b.** In the Monitor window, check to confirm that there are no errors or warnings.
  - **i.** If there are errors, investigate the cause(s) before resolving.
  - **ii.** If there are warnings, determine whether the warnings are relevant; some warnings can be safely ignored (Fig. 21.29).

Continued

EXAMPLE 21.6 —cont'd

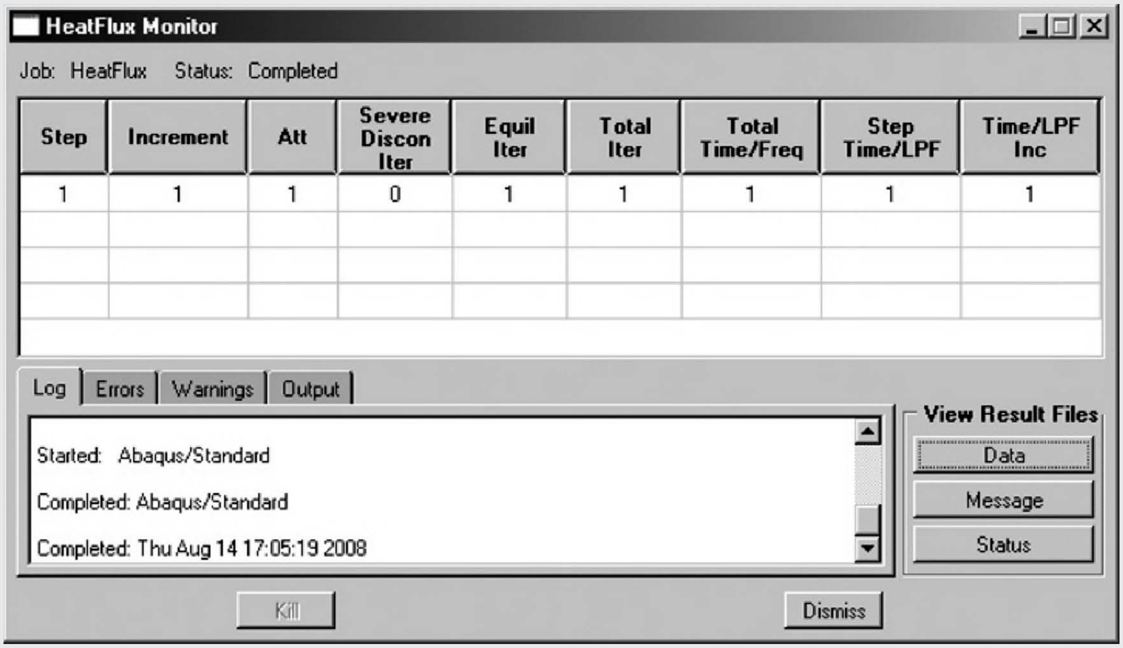


FIGURE 21.29 Monitor window.

21. In the model tree, right-click on the submitted and successfully completed job (HeatFlux), and select Results.
22. In the menu bar, click on Viewport → Viewport Annotations Options.
  a. Uncheck the Show compass option.
  b. Click OK.
23. To change the output being displayed, in the menu bar click on Results → Field Output.
  a. Select NT11 Nodal temperature at nodes.
  b. Click OK (Fig. 21.30).
24. Display the contour of the temperatures.
  a. In the toolbox area, click on the Plot Contours on Deformed Shape icon (Fig. 21.31).
25. To determine the temperature values, from the menu bar click Tools → Query.
  a. Change the probe option to Nodes.
  b. Check the boxes labeled Node ID and NT11.
  c. In the viewport, mouse over the node of interest.
  d. When done, click Cancel (Fig. 21.32).
26. To create a text file containing the nodal temperatures, in the menu bar click on Report → Field Output.
  a. Change the position to Unique Nodal.
  b. For the output variable, select NT11: Nodal temperature.
  c. On the Setup tab, specify the name and the location for the text file.
  d. Uncheck the Column totals option.
  e. Click OK (Fig. 21.33).
27. Open the .rpt file with any text editor (Fig. 21.34).

EXAMPLE 21.6 —cont'd

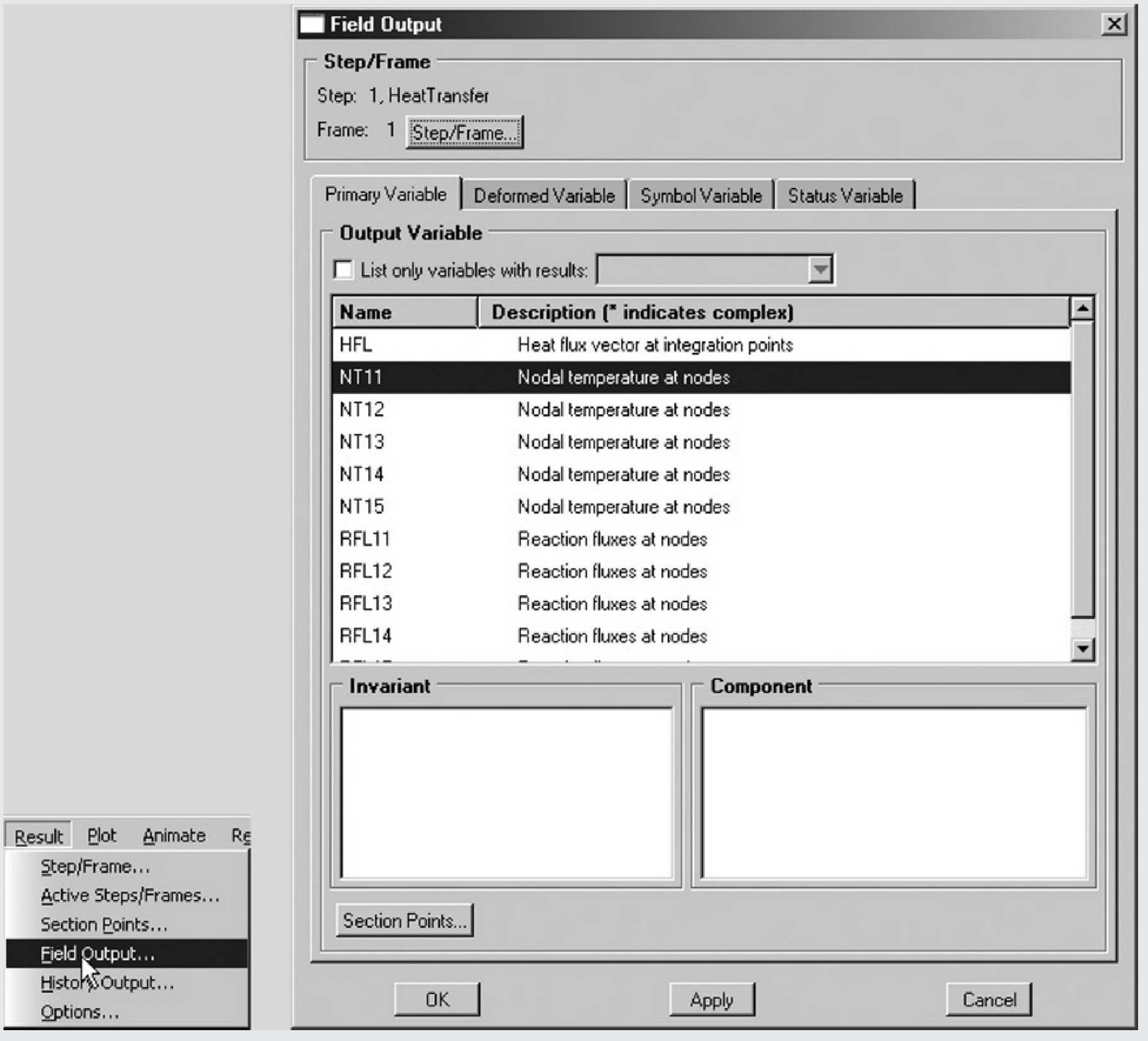


FIGURE 21.30 Selection of field output and NT11 nodal temperature at nodes.

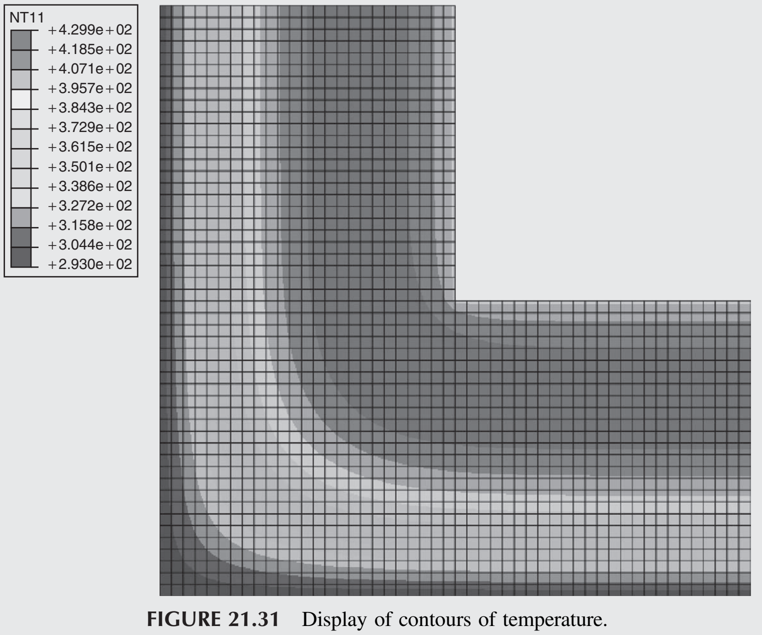


FIGURE 21.31 Display of contours of temperature.

Continued

EXAMPLE 21.6 —cont'd

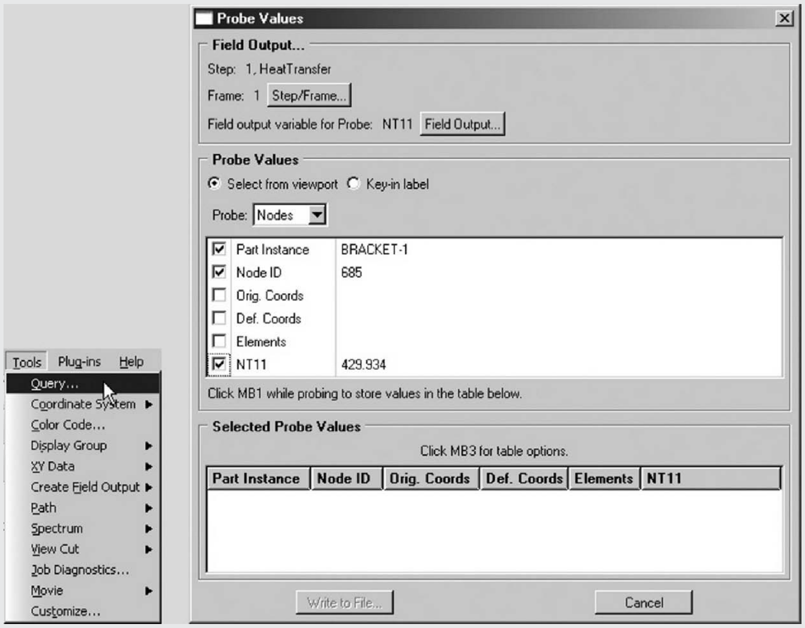


FIGURE 21.32 Operations to determine temperature values.

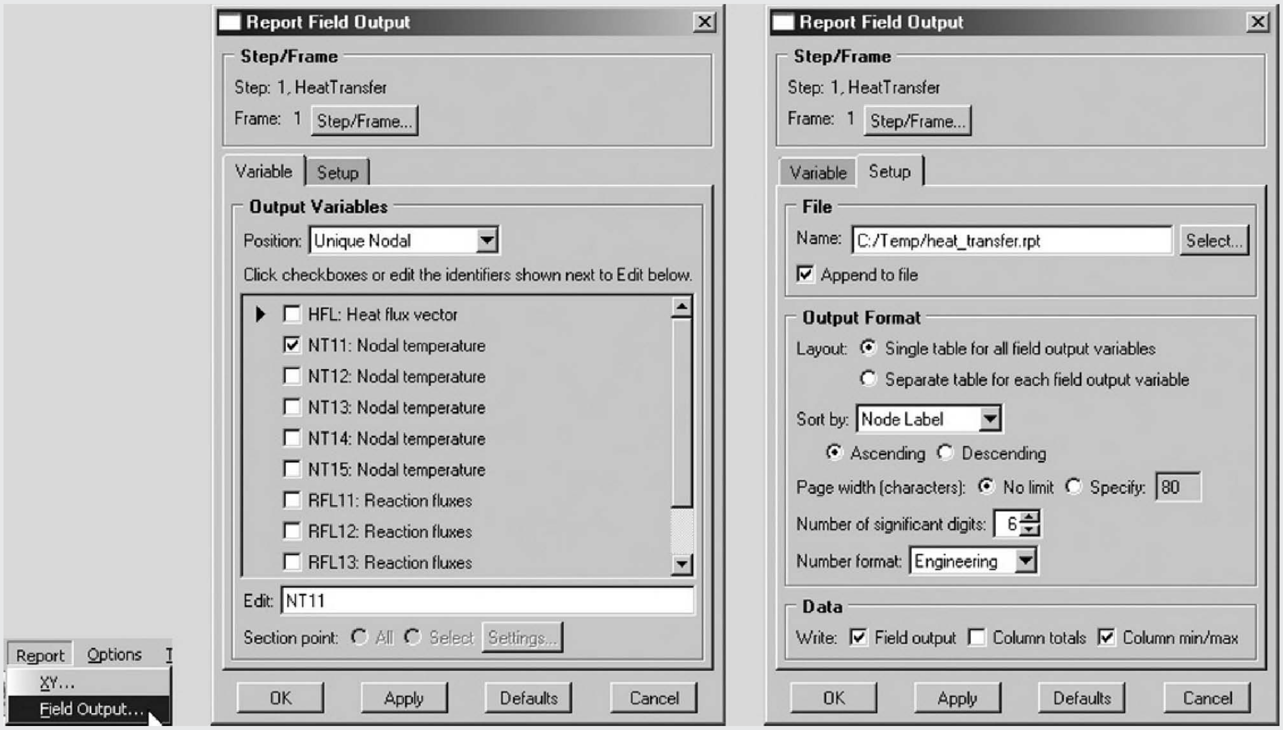


FIGURE 21.33 Creation of text file of nodal temperatures.

EXAMPLE 21.6 —cont'd

```
********************************************************
Field Output Report, written Fri Aug 15 11:48:28 2008

Source 1
---------

   ODB: C:/Temp/HeatFlux.odb
   Step: HeatTransfer
   Frame: Increment      1: Step Time =    1.000

Loc 1 : Nodal values from source 1

Output sorted by column "Node Label".

Field Output reported at nodes for part: BRACKET-1

            Node           NT11
           Label          @Loc 1
---------------------------------
               1           293.
               2           293.
               3           293.
               4           393.
               :             :
               :             :
               :             :
            1974        414.022
            1975        408.502
            1976        402.754


  Minimum                  293.

         At Node            224
  Maximum               429.934

         At Node            685
```

FIGURE 21.34 Display of nodal temperatures.

PROBLEMS

Find the deflections and stresses in the beams described in the following problems using the finite element method with the following data:

$P = 1000$ N, $L = 1$ m, $E = 70 \times 10^9$ Pa, $M_0 = 50{,}000$ N cm, cross-section of the beam: rectangular with a depth of 2 cm and width (perpendicular to the page) of 1 cm.

Use ABAQUS to solve the problems.

21.1 A uniform cantilever beam of length L subject to a concentrated transverse load P at the free end as shown in Fig. 1.20.

21.2 A uniform cantilever beam of length L subject to a concentrated bending moment M_0 at the free end as shown in Fig. 1.21.

21.3 A uniform fixed−simply supported beam of length L subject to a concentrated transverse load P at the middle as shown in Fig. 1.22.

21.4 A uniform fixed−simply supported beam of length L subject to a concentrated bending moment M_0 at the middle as shown in Fig. 1.23.

21.5 A uniform fixed−fixed beam of length L subject to a concentrated transverse load P at a distance of $(3\,L/4)$ from the left end as shown in Fig. 1.24.

Chapter 22

Finite Element Analysis Using ANSYS*

Chapter Outline

22.1 INTRODUCTION

ANSYS is a general-purpose finite element software that includes preprocessing (to create the geometry and generating mesh), solver, and postprocessing modules in a unified graphical user interface (GUI) environment. ANSYS commonly refers to ANSYS Mechanical or ANSYS Multiphysics. In ANSYS, a problem can be solved in either a batch or interactive mode. In batch mode, an input file is to be created and executed from the command line. In interactive mode, GUI is used and the operations to be performed are either chosen from a menu or typed in a graphics window. Examples are presented to illustrate both modes in this chapter.

22.2 GRAPHICAL USER INTERFACE LAYOUT IN ANSYS

(See Fig. 22.1)

22.3 TERMINOLOGY

22.3.1 Database and Files

The database refers to the data ANSYS maintains in memory as users build, solve, and postprocess models. The database stores both input data and ANSYS results data:

input data: information that users must enter, such as dimensions, material properties, and loads
results data: quantities that ANSYS calculates, such as displacements, stresses, and temperatures

22.3.2 Defining the Jobname

Utility Menu > File > Change Jobname

* ANSYS FEM software is marketed by ANSYS, Inc., Southpointe, 275 Technology Drive, Canonsburg, PA 15317.

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00022-5

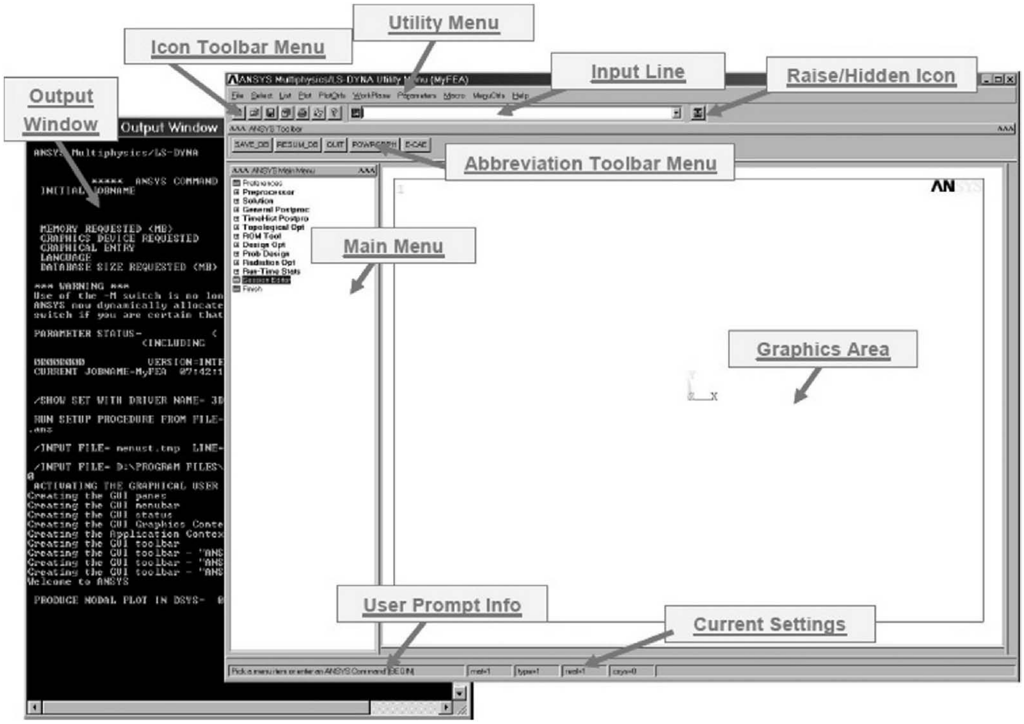


FIGURE 22.1 ANSYS graphical user interface.

The jobname is a name up to 32 characters that identifies the ANSYS job. When we define a jobname for analysis, the jobname becomes the first part of the name of all files the analysis creates. (The extension or suffix for these file names is a file identifier such as .DB.) By using a jobname for each analysis, we ensure that no files are overwritten.

22.3.3 File Management Tips

- Run each analysis in a separate working directory.
- Use different jobnames to differentiate analysis runs.
- We should keep the following files after any ANSYS analysis:

log file (log); database file (.db); results files (.rst, .rth, …); load step files, if any (.s01, .s02, …)

22.3.4 Defining an Analysis Title

Utility Menu > File > Change Title

This will define a title for the analysis. ANSYS includes the title on all graphics displays and on the solution output.

22.3.5 Save and Resume

Because the database is stored in the computer's memory (random access memory), it is a good practice to save it to a disk frequently so that we can restore the information in the event of a computer crash or power failure. The SAVE operation copies the database from memory to a file called the database file (db file).

The easiest way to save is to click on: Toolbar > SAVE_DB

or use:

- Utility Menu > File > Save as Jobname.db
- Utility Menu > File > Save as…

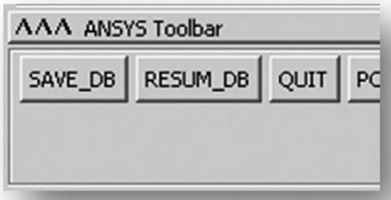


FIGURE 22.2 Solving database (DB) file.

To restore the database from the db file back into memory, use the RESUME operation.
Toolbar > RESUME_ DB (Fig. 22.2), or use:

- Utility Menu > File > Resume Jobname.db
- Utility Menu > File > Resume from…

22.3.6 Tips on SAVE and RESUME

- Periodically, we need to save the database as we progress through an analysis. ANSYS does not do automatic saves.
- We should definitely SAVE the database before attempting an unfamiliar operation (such as a Boolean or meshing) or an operation that may cause major changes (such as a delete).
- RESUME can then be used as an "undo" if we do not like the results of that operation.
- SAVE is also recommended before doing a solver.

22.4 FINITE ELEMENT DISCRETIZATION

Finite element discretization or meshing is the process used to "fill" the solid model with nodes and elements: that is, to create the finite element analysis model. Remember, we need nodes and elements for the finite element solution, not just the solid model. The solid model in computer-aided design does not participate in the finite element solution.

22.4.1 Element Type

The element type is an important choice that determines the following element characteristics:

- Degrees of freedom (dof) set. A thermal element type, for example, has 1 dof: TEMP, whereas a structural element type may have up to 6 dof: UX, UY, UZ, ROTX, ROTY, and ROTZ
- Element shape: brick, tetrahedron, quadrilateral, triangle, and so on
- Dimensionality: two-dimensional (2D) solid (X−Y plane only), or three-dimensional (3D) solid
- Assumed displacement shape: linear versus quadratic

22.4.2 To Define an Element

Main Menu > Preprocessor > Element Type > Add/Edit/Delete > Add

22.4.3 Meshing Methods

Free and Mapped Meshing Methods (Fig. 22.3)

The free mesh

- has no element shape restrictions
- does not follow a pattern
- is suitable for complex-shaped areas and volumes

Volume meshes consist of high-order tetrahedral (10 nodes) and a large number of degrees of freedom

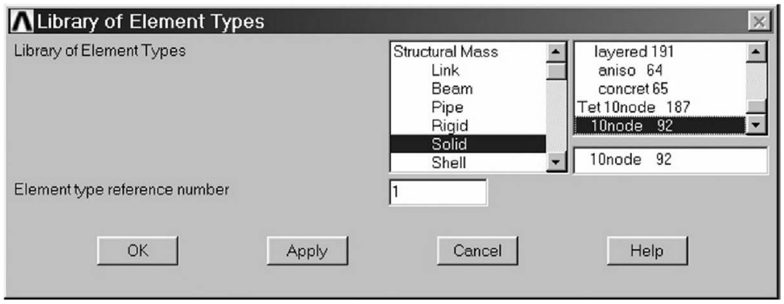


FIGURE 22.3 Specification of element type.

The mapped mesh

- restricts element shapes to quadrilaterals (areas) and hexahedra (volume)
- typically has a regular pattern with obvious rows of elements
- is suitable only for "regular" shapes such as rectangles and bricks

22.4.4 Mesh Density Control

ANSYS provides many tools to control mesh density on both global and local levels:

- Global controls: SmartSizing; Global element sizing; Default sizing 6
- Local controls: Keypoint sizing; Line sizing; Area sizing

  To access the MeshTool:

- Main Menu > Preprocessor > Meshing > MeshTool
- SmartSizing: Turn on SmartSizing and set the desired size level. Size level ranges from 1 (fine) to 10 (coarse), and defaults to 6. Then mesh all volumes (or all areas) at once rather than one by one.
- Advanced SmartSize controls such as mesh expansion and transition factors are available via Main Menu > Preprocessor > Meshing > Size Cntrls > SmartSize > Adv Opts.
- Global Element Sizing: Allows you to specify a maximum element edge length for the entire model (or number of divisions per line): Go to Size Controls, Global, and click [Set] or Main Menu > Preprocessor > Meshing > Size Cntrls > ManualSize > Global > Size.

22.4.5 Material Properties

Every analysis requires material property input: Young's modulus (EX), Poisson's ratio (PRXY) for structural elements, thermal conductivity (KXX) for thermal elements, and so on.

To define the material properties: Main Menu > Preprocessor > Material Props > Material Models.

More than one set of material properties can be defined when needed (Fig. 22.4).

22.5 SYSTEM OF UNITS

The ANSYS program does not assume a system of units for the analysis (except in magnetic field analyses). We can use any system of units as long as we make sure that we use that system for all of the data we enter (units must be consistent for all input data).

It is suggested to use the International system of Units whenever possible to avoid confusion.

1. ANSYS has no built-in unit system.
2. The units must be consistent, as indicated in the table.

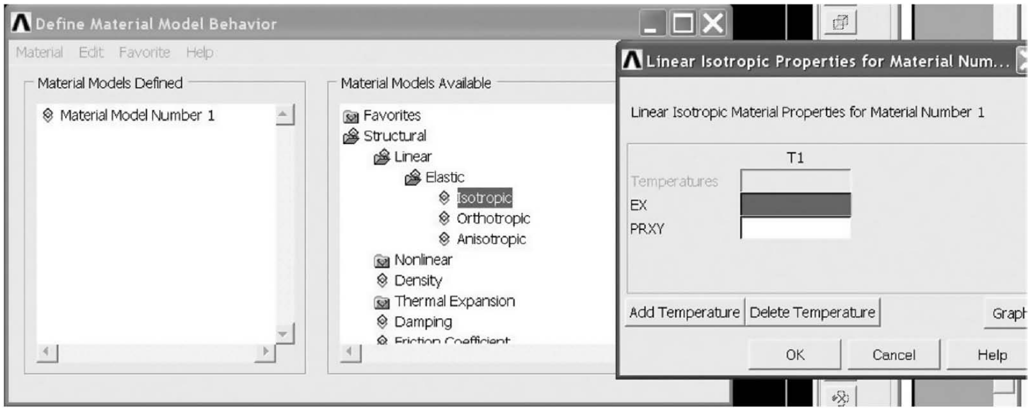


FIGURE 22.4 Specification of material properties.

Quantity	International Unit	International Unit (mm)	US Unit (ft)	US Unit (in)
Length	m	mm	ft	in
Force	N	N	lbf	lbf
Mass	kg	ton (10^3 kg)	slug	lbf S^2/in
Time	s	s	s	s
Stress	Pa (N/m^2)	MPa (N/mm^2)	lbf/ft^2	psi (lbf/in^2)
Energy	J	MJ (10^{-3} J)	ft lbf	in lbf
Density	kg/m^3	ton/mm^3	slug/ft^3	lbf s^2/in^4

22.6 STAGES IN SOLUTION

ANSYS is a general-purpose finite element modeling package for numerically solving a wide variety of mechanical problems. These problems include static/dynamic structural analysis (both linear and nonlinear), heat transfer and fluid problems, as well as acoustic and electromagnetic problems.

In general, a finite element solution may be divided into the following three stages. This is a general guideline that can be used to set up any finite element analysis.

Preprocessing

Preprocessing consists of defining the problem. Major steps include:
- Define keypoints/lines/areas/volumes.
- Define element type and material/geometric properties.
- Define mesh lines/areas/volumes as required.

The amount of detail required will depend on the dimensionality of the analysis (i.e., 1D, 2D, axisymmetric, 3D).

Solution

The solution stage is composed of assigning loads, constraints, and solving.
- Specify the loads (point or pressure).
- Specify the constraints (translational and rotational).
- Solve the resulting set of equations.

Postprocessing

This stage includes additional processing and viewing of results, such as:

- lists of nodal displacements
- element forces and moments
- deflection plots
- stress contour diagrams

EXAMPLE 22.1

Analysis of a 2D Truss

ANSYS 7.0 is used to solve the following 2D Truss problem.

Problem Description

Determine the nodal deflections, reaction forces, and stresses for the truss shown in Fig. 22.5 ($E = 200$ GPa, $A = 3250$ mm^2).

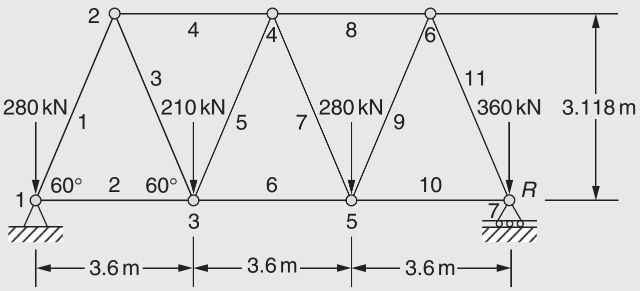


FIGURE 22.5 Two-dimensional truss.

Preprocessing: Defining the Problem

1. **Input a Title** (such as Bridge Truss Tutorial).
   In the **Utility menu bar** select **File > Change Title**:
   The following window will appear (Fig. 22.6):
   Enter the title and click OK. This title will appear in the bottom left corner of the Graphics window once we begin. Note: to get the title to appear immediately, select **Utility Menu > Plot > Replot**.

FIGURE 22.6 Giving a title to the problem.

2. **Enter Keypoints.**
   The overall geometry is defined in ANSYS using keypoints, which specify various principal coordinates to define the body. For this example, these keypoints are the ends of each truss.
   - We define seven keypoints for the simplified structure as given in the table.

	Coordinate	
Keypoint	***x***	***y***
1	0	0
2	1800	3118
3	3600	0
4	5400	3118
5	7200	0
6	9000	3118
7	10,800	0

EXAMPLE 22.1 —cont'd

- These keypoints are depicted as node numbers in Fig. 22.5.
- From the ANSYS Main Menu, select:

Preprocessor > Modeling > Create > Keypoints > In Active CS
The window in Fig. 22.7 will then appear.

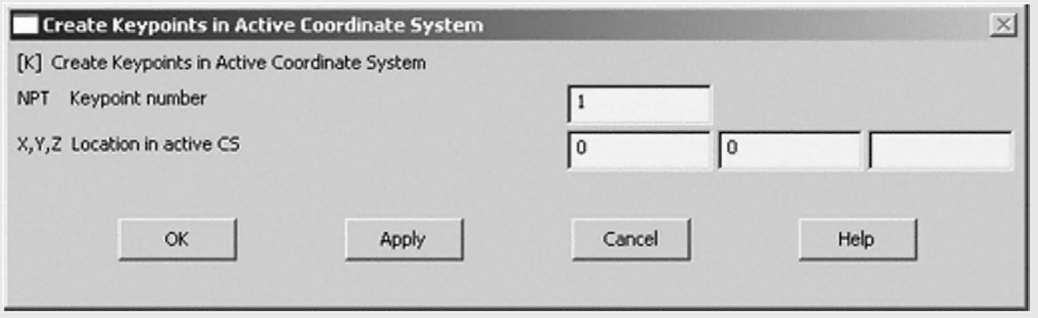


FIGURE 22.7 Specifying coordinates of Keypoint (Node) 1.

- To define the first keypoint, which has the coordinates $x = 0$ and $y = 0$, enter keypoint 1 in the appropriate box, and enter the x,y coordinates 0, 0 in their appropriate boxes (as shown in Fig. 22.7). Click Apply to accept what we have typed.
- Enter the remaining keypoints using the same method.

Note
When entering the final data point, click on OK to indicate that we are finished entering keypoints. If we first press Apply and then OK for the final keypoint, we will have defined it twice.
If we did press Apply for the final point, simply press Cancel to close this dialog box.

Units
Note that the units of measure (i.e., millimeters) were not specified. It is the responsibility of the user to ensure that a consistent set of units is used for the problem; make conversions where necessary.

Correcting Mistakes
When defining keypoints, lines, areas, volumes, elements, constraints, and loads, we are bound to make mistakes. Fortunately, these are easily corrected so that we do not need to begin from scratch every time an error is made. Every Create menu for generating these various entities also has a corresponding Delete menu for fixing things.

3. Form Lines

The keypoints must now be connected.
We will use the mouse to select the keypoints to form the lines.

- In the main menu, select: **Preprocessor > Modeling > Create > Lines > Lines > In Active Coord**
- Use the mouse to pick keypoint 1 (i.e., click on it). It will now be marked by a small yellow box.
- Now move the mouse toward keypoint 2. A line will now show up on the screen joining these two points. Left click and a permanent line will appear.
- Connect the remaining keypoints using the same method.
- When you are done, click on OK in the Lines in Active Coord window; minimize the Lines menu and the Create menu. Your ANSYS Graphics window should look similar to Fig. 22.8.

Disappearing Lines
Please note that any lines we have created may disappear throughout our analysis. However, most likely they have not been deleted. If this occurs at any time, from the **Utility Menu** select:
Plot > Lines

4. Define the Type of Element

It is now necessary to create elements. This is called *meshing*. ANSYS first needs to know what kind of elements to use:

- From the Preprocessor Menu, select: **Element Type > Add/Edit/Delete.**
- Click on the Add … button. The window in Fig. 22.9 will appear.
- For this example, we will use the 2D spar element as selected in the figure. Select the element shown and click OK. We should see Type 1 LINK1 in the Element Types window.
- Click on Close in the Element Types dialog box.

Continued

EXAMPLE 22.1 —cont'd

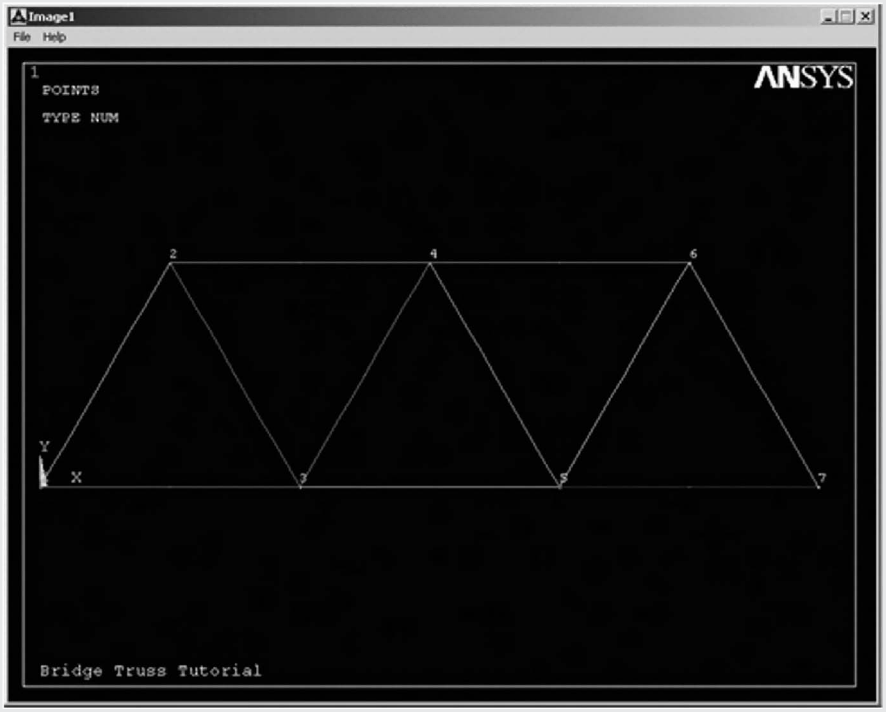


FIGURE 22.8 Connecting lines between keypoints (nodes).

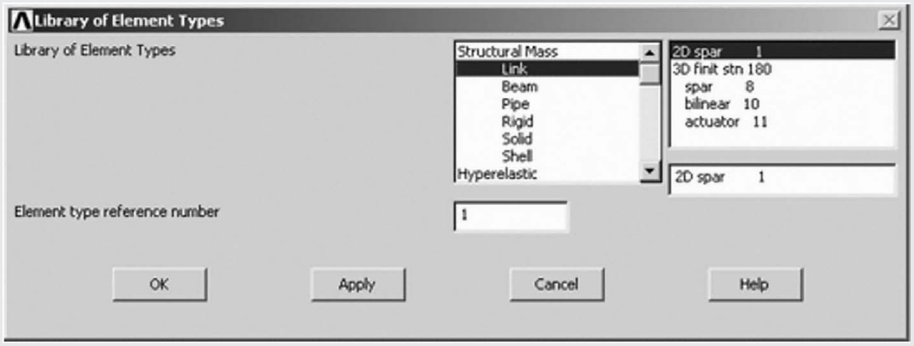


FIGURE 22.9 Window for library of element types.

5. **Define Geometric Properties**

   We now need to specify geometric properties for our elements:
   - In the Preprocessor menu, select **Real Constants > Add/Edit/Delete.**
   - Click **Add** ... and select Type 1 LINK1 (actually it is already selected). Click on OK. Fig. 22.10 will appear.
   - As shown in the window, enter the cross-sectional area (3250 mm).
   - Click on OK.
   - Set 1 now appears in the dialog box. Click on Close in the Real Constants window.
6. **Element Material Properties**

   We then need to specify material properties. In the Preprocessor menu, select **Material Props > Material Models**. Double-click on **Structural > Linear > Elastic > Isotropic** (Fig. 22.11).

   We are going to give the properties of steel. Enter the following field:

   EX 200,000

EXAMPLE 22.1 —cont'd

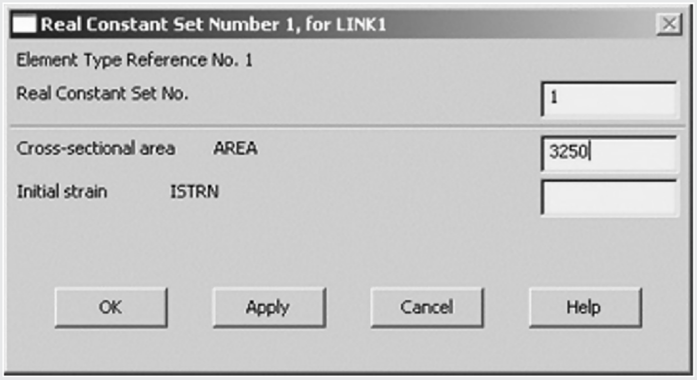


FIGURE 22.10 Specification of area of cross-section of members.

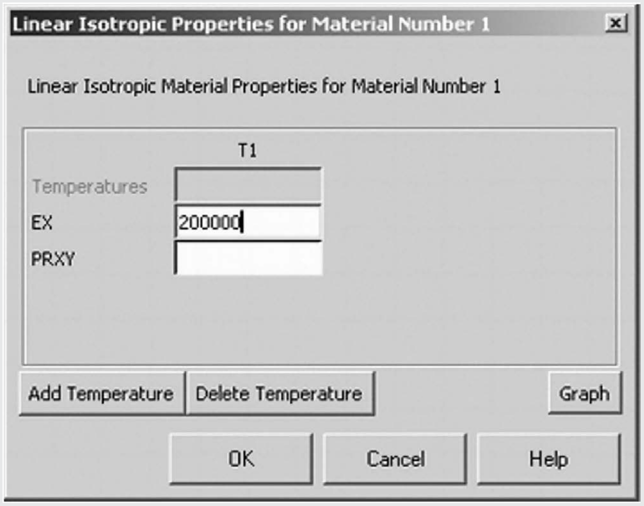


FIGURE 22.11 Specification of value of Young's modulus.

- Set these properties and click on OK. (Note: We may obtain the note that PRXY will be set to 0.0. This is Poisson's ratio and is not required for this element type. Click OK on the window to continue. Close Define Material Model Behavior by clicking on the X box in the upper right corner.)

7. Mesh Size

The last step before meshing is to tell ANSYS what size the elements should be. There are a variety of ways to do this, but we will deal with just one method for now.

- In the Preprocessor menu, select **Meshing > Size Cntrls > ManualSize > Lines > All Lines**.
- In the size NDIV field, enter the desired number of divisions per line. For this example, we want only one division per line; therefore, enter 1 and then click OK. Note that we have not yet meshed the geometry; we have simply defined the element sizes.

8. Mesh

Now the frame can be meshed.

- In the Preprocessor menu, select **Meshing > Mesh > Lines** and click Pick All in the Mesh Lines Window. Your model should now appear similar to the one shown in Fig. 22.8.

Plot Numbering

To show the line numbers, keypoint numbers, node numbers:

- From the **Utility Menu** (top of screen) select **PlotCtrls > Numbering** …
- Fill in the window as shown in Fig. 22.12 and click OK.

Continued

EXAMPLE 22.1 —cont'd

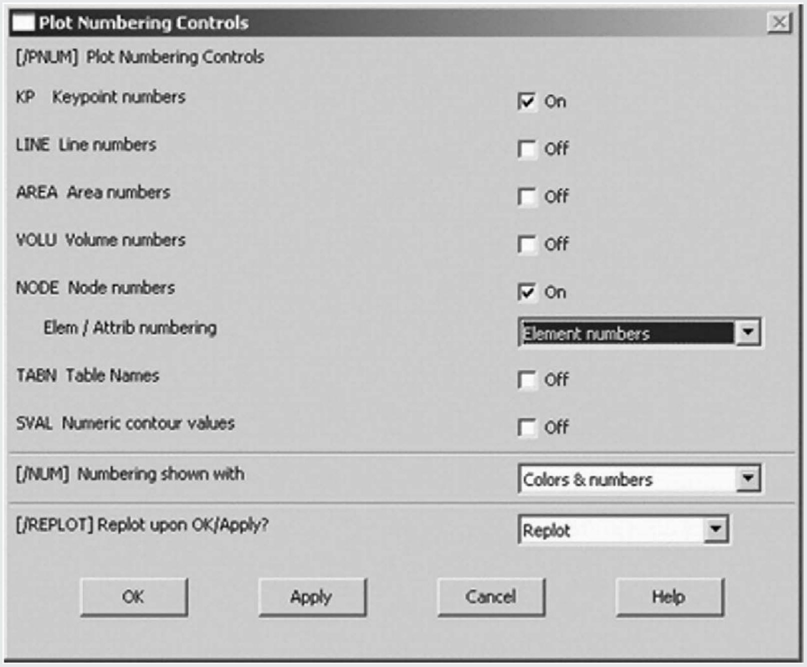


FIGURE 22.12 Showing line and keypoint (node) numbers.

Now we can turn numbering on or off at our discretion.

Saving Our Work

Save the model at this time, so that if we make some mistakes later on, at least we will be able to come back to this point. To do this, on the **Utility Menu** select **File > Save as** … Select the name and location where we want to save your file.

It is a good idea to save your job at different times throughout the building and analysis of the model to back up your work in case of a system crash.

Solution Phase: Assigning Loads and Solving

We have now defined our model. It is time to apply the load(s) and constraint(s) and solve the resulting system of equations. Open up the Solution menu (from the same ANSYS Main Menu).

1. **Define Analysis Type**

First, we must tell ANSYS how we want it to solve this problem:

- From the **Solution** Menu, select **Analysis Type > New Analysis**.
- Ensure that Static is selected; that is, we are going to do a static analysis on the truss, as opposed to a dynamic analysis, for example.
- Click OK.

2. **Apply Constraints**

It is necessary to apply constraints to the model; otherwise the model is not tied down or grounded and a singular solution will result. In mechanical structures, these constraints will typically be fixed, pinned, and roller-type connections. As shown earlier, the left end of the truss bridge is pinned whereas the right end has a roller connection.

- In the **Solution** menu, select **Define Loads > Apply > Structural > Displacement > On Keypoints**
- Select the left end of the bridge (Keypoint 1) by clicking on it in the Graphics Window and click on OK in the Apply U,ROT on KPs window (Fig. 22.13).
- This location is fixed, which means that all translational and rotational dofs are constrained. Therefore, select All DOF by clicking on it, enter 0 in the Value field, and click OK.

We will see some blue triangles in the graphics window indicating the displacement constraints.

- Using the same method, apply the roller connection to the right end (UY constrained). Note that more than one dof constraint can be selected at a time in the Apply U,ROT on KPs window. Therefore, we may need to deselect the All DOF option to select just the UY option.

EXAMPLE 22.1 —cont'd

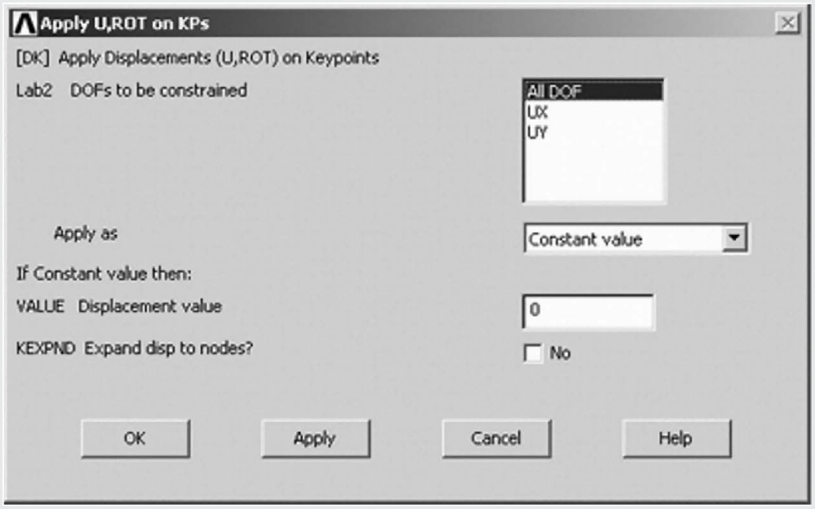


FIGURE 22.13 Specifying UX and UY as displacement degrees of freedom at keypoints (nodes).

3. Apply Loads

As shown in the diagram, there are four downward loads of 280, 210, 280, and 360 kN at keypoints 1, 3, 5, and 7, respectively.

- Select **Define Loads > Apply > Structural > Force/Moment > on Keypoints**.
- Select the first Keypoint (left end of the truss) and click OK in the Apply F/M on KPs window.
- Select FY in the Direction of force/mom. This indicates that we will be applying the load in the *y* direction
- Enter a value of −280,000 in the Force/moment value box and click OK. Note that we are using units of *N* here, which is consistent with the previous values input.
- The force will appear in the graphics window as a red arrow.
- Apply the remaining loads in the same manner (Fig. 22.14).

The applied loads and constraints should now appear as shown in Fig. 22.15.

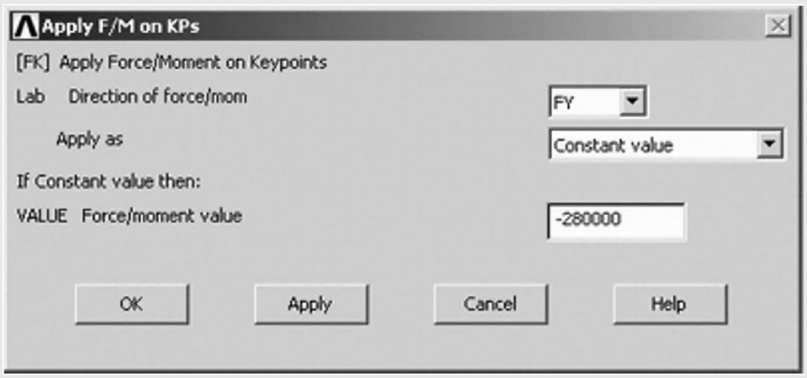


FIGURE 22.14 Specifying the values of loads at nodes.

4. Solving the System

We now tell ANSYS to find the solution:

- In the Solution menu select **Solve > Current LS.** This indicates that we desire the solution under the current Load Step (LS).
- The windows in Fig. 22.16 will appear. Ensure that our solution options are the same as shown earlier and click OK.
- Once the solution is done, a window will pop up displaying "Solution is done" (Fig. 22.16).

Continued

EXAMPLE 22.1 —cont'd

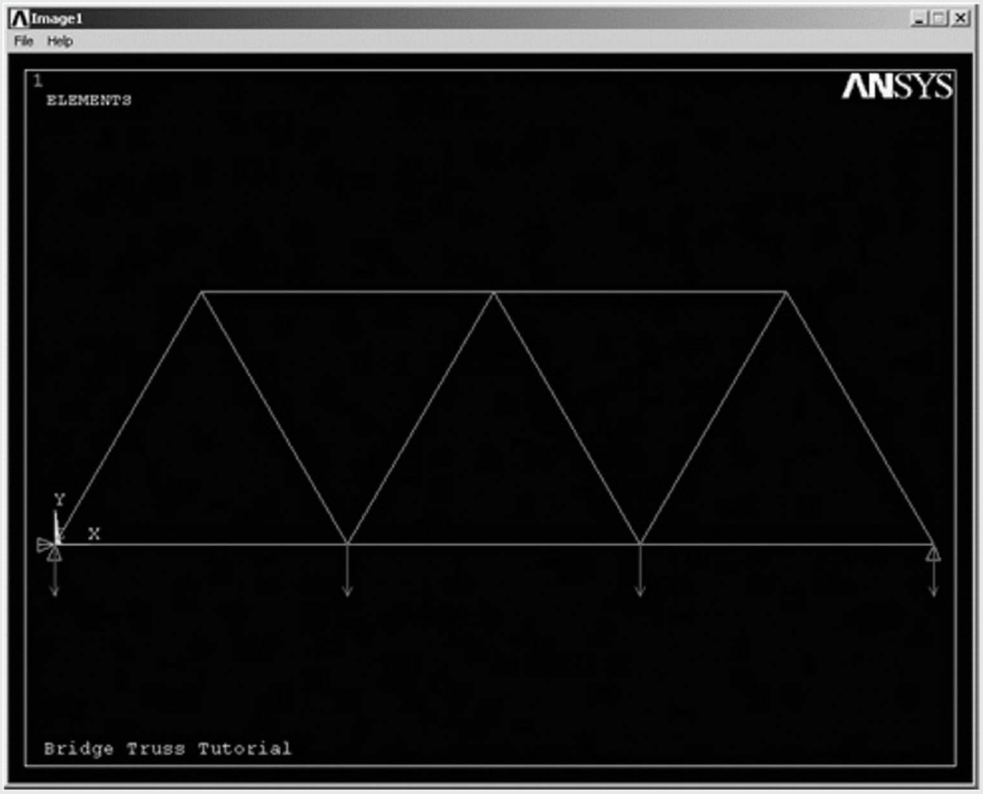


FIGURE 22.15 Display of loads and constraints.

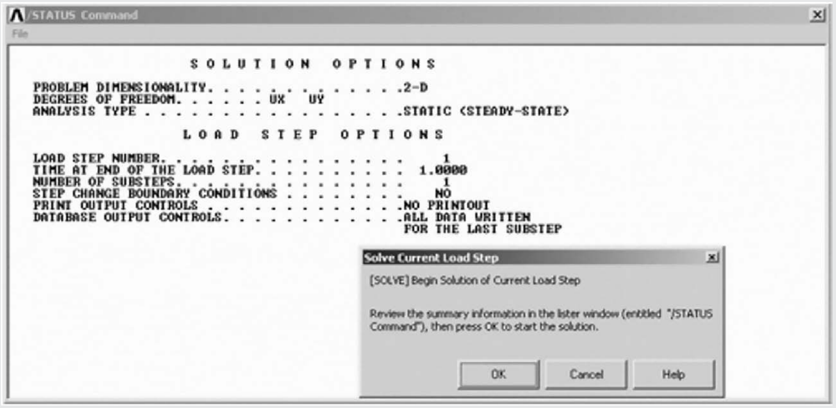


FIGURE 22.16 Display of solution options.

Postprocessing: Viewing the Results

1. Hand Calculations

We will first calculate the forces and stress in element 1 (as labeled in the problem description).

$$\sum M_1 = 0 = .210\text{ kN}(3.6\text{ m}) - 280\text{ kN}(7.2\text{ m}) - 360\text{ kN}(10.8\text{ m}) + F_7(10.8\text{ m})$$

$$F_7 = \frac{210\text{ kN}(3.6\text{ m}) + 280\text{ kN}(7.2\text{ m}) + 360\text{ kN}(10.8\text{ m})}{10.8\text{ m}} = 617\text{ kN}$$

EXAMPLE 22.1 —cont'd

$$\uparrow \sum F_y = 0 = -280\text{ kN} - 210\text{ kN} - 280\text{ kN} - 360\text{ kN} + 617\text{ kN} + F_1$$
$$F_1 = 280\text{ kN} + 210\text{ kN} + 280\text{ kN} + 360\text{ kN} - 617\text{ kN} = 513\text{ kN}$$

Element 1 Forces/Stress

$$F_{E_1} = \frac{513\text{ kN} - 280\text{ kN}}{\cos(30)} = 269\text{ kN}$$
$$\sigma_{E_1} = \frac{F_{E_1}}{A} = \frac{269\text{ kN}}{3250\text{ mm}^2} = 82.8\text{ MPa}$$

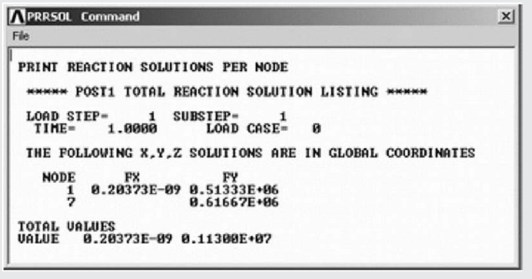
2. Results Using ANSYS

Reaction Forces

A list of the resulting reaction forces can be obtained for this element:

- From the Main Menu, select General Postproc > List Results > Reaction Solu.
- Select All struc forc F and click OK (Fig. 22.17).

These values agree with the reaction forces calculated by hand previously.



```
PRRSOL Command
File
PRINT REACTION SOLUTIONS PER NODE

 ***** POST1 TOTAL REACTION SOLUTION LISTING *****

 LOAD STEP=     1  SUBSTEP=     1
  TIME=    1.0000      LOAD CASE=   0

 THE FOLLOWING X,Y,Z SOLUTIONS ARE IN GLOBAL COORDINATES

    NODE       FX           FY
       1   0.20373E-09  0.51333E+06
       7                0.61667E+06

TOTAL VALUES
VALUE    0.20373E-09  0.11300E+07
```

FIGURE 22.17 Display of reaction forces.

Deformation

In the General Postproc menu, select **Plot Results > Deformed Shape**.

- Select Def + undef **edge and click OK to view** both the undeformed and the deformed object (Fig. 22.18).
- Observe the value of the maximum deflection in the upper left corner (DMX = 7.409). One should also observe that the constrained dofs appear to have a deflection of 0 (as expected).

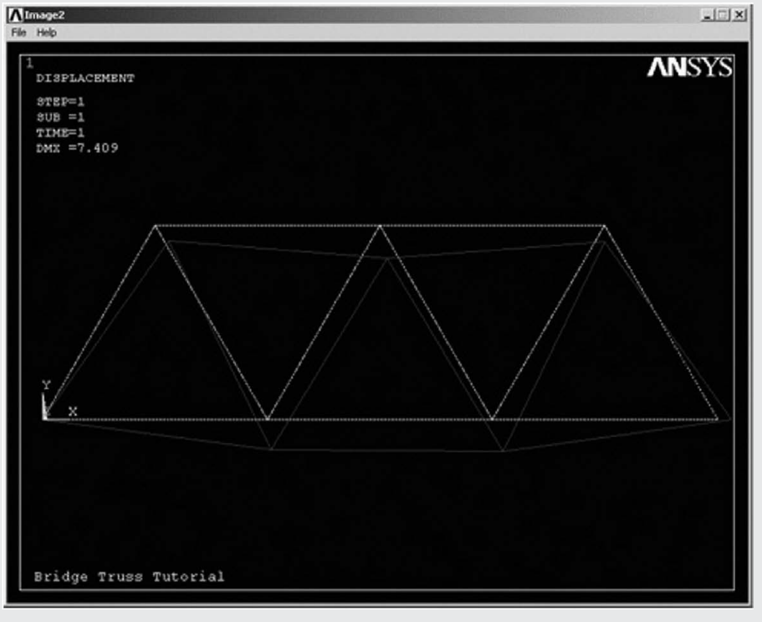


FIGURE 22.18 Display of undeformed system.

Continued

EXAMPLE 22.1 —cont'd

Deflection

For a more detailed version of the deflection of the beam:

- From the General Postproc menu select **Plot results > Contour Plot > Nodal Solution**.
- Select DOF solution and USUM in the display window. Leave the other selections as the default values. Click OK (Fig. 22.19).

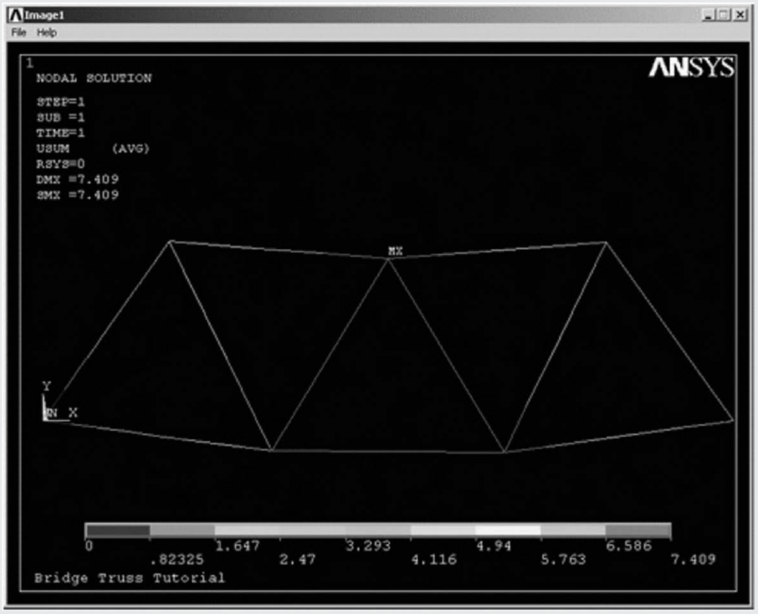
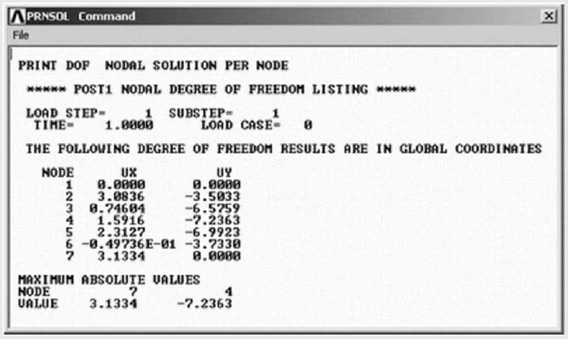


FIGURE 22.19 Display of deformed system.

- The deflection can also be obtained as a list as shown below. **General Postproc > List Results > Nodal Solution** select DOF Solution and ALL DOFs from the lists in the List Nodal Solution window and click OK. This means that we want to see a listing of all degrees of freedom from the solution (Fig. 22.20).
- Are these results what we expected? Note that all dofs were constrained to zero at node 1 whereas UY was constrained to zero at node 7.


PRNSOL Command

File

PRINT DOF NODAL SOLUTION PER NODE

***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

LOAD STEP= 1 SUBSTEP= 1
TIME= 1.0000 LOAD CASE= 0

THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN GLOBAL COORDINATES

NODE	UX	UY
1	0.0000	0.0000
2	3.0836	-3.5033
3	0.74604	-6.5759
4	1.5916	-7.2363
5	2.3127	-6.9923
6	-0.49736E-01	-3.7330
7	3.1334	0.0000

MAXIMUM ABSOLUTE VALUES

NODE	7	4
VALUE	3.1334	-7.2363

FIGURE 22.20 Window with nodal displacements.

- If we want to save these results to a file, select File within the results window (at the upper left corner of this list window) and select Save as.

Axial Stress

For line elements (i.e., links, beams, spars, and pipes), you will often need to use the **Element Table** to gain access to derived data (i.e., stresses, strains). For this example, we should obtain axial stress to compare with the hand calculations.

EXAMPLE 22.1 —cont'd

- From the **General Postprocessor** menu select **Element Table > Define Table**.
- Click on Add ...
- Enter SAXL in the Lab box. This specifies the name of the item that we are defining. Next, in the Item, Comp boxes, select By sequence number and LS. Then enter 1 after LS in the selection box.
- Click on OK and close the Element Table Data window.
- Plot the Stresses by selecting **Element Table > Plot Elem Table**.
- A window will appear. Ensure that SAXL is selected and click OK.
- Because we changed the contour intervals for the Displacement plot to User Specified, we need to switch this back to Auto Calculated to obtain new values for VMIN/VMAX.

  Utility Menu > PlotCtrls > Style > Contours > Uniform Contours ... (Fig. 22.21)

  Again, we may wish to select more appropriate intervals for the contour plot.
- List the Stresses

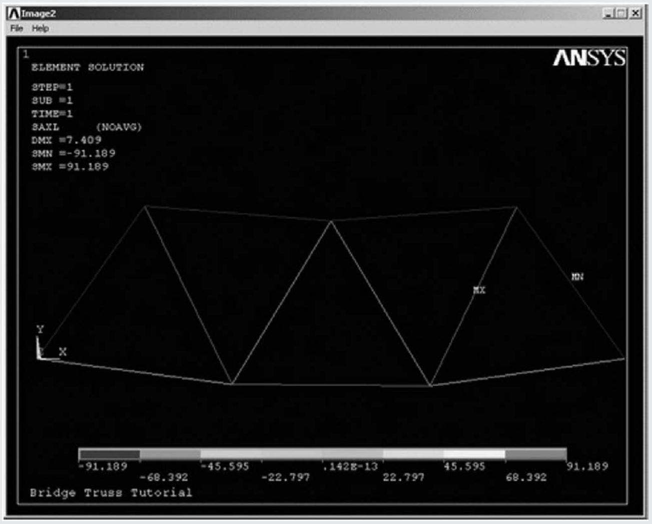


FIGURE 22.21 Display of stresses (contours).

PRETAB Command

File

```
PRINT ELEMENT TABLE ITEMS PER ELEMENT

 ***** POST1 ELEMENT TABLE LISTING *****

   STAT     CURRENT
   ELEM     SAXL
      1   -82.900
      2    41.447
      3    82.900
      4   -82.893
      5   -8.2900
      6    87.038
      7    8.2900
      8   -91.183
      9    91.189
     10    45.591
     11   -91.189

MINIMUM VALUES
ELEM          11
VALUE    -91.189

MAXIMUM VALUES
ELEM           9
VALUE     91.189
```

FIGURE 22.22 Table of element stresses.

- From the Element Table menu, select **List Elem Table**.
- From the List Element Table Data window that appears, ensure that SAXL is highlighted.
- Click OK (Fig. 22.22).

EXAMPLE 22.2

Heat Transfer in a Steel Rod (Link)

A steel link with no internal stresses is pinned between two solid structures at a reference temperature of 0°C (273K) (Fig. 22.23). One of the solid structures is heated to 75°C (348K). As heat is transferred from the solid structure into the link, the link will attempt to expand. However, because it is pinned this cannot occur; as such, stress is created in the link. A steady-state solution of the resulting stress will be found to simplify the analysis. Loads will not be applied to the link, only a temperature change of 75°C. The link is steel with a modulus of elasticity of 200 GPa, a thermal conductivity of 60.5 W/m K, and a thermal expansion coefficient of 12e-6/K.

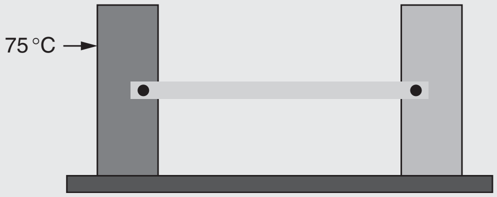


FIGURE 22.23 Heat transfer in a steel rod (link).

As in the case of Example 22.1, the following steps are used in the solution process:

Preprocessing: Construct geometry by defining key points and lines, Specify element type and element constants, Specify the material, Generate the mesh, Apply the boundary conditions.

Solution: Solve with current LS (load step), Display solution.

Postprocessing: Plot the solution, List the nodal/element solution (Figs. 22.24–22.27).

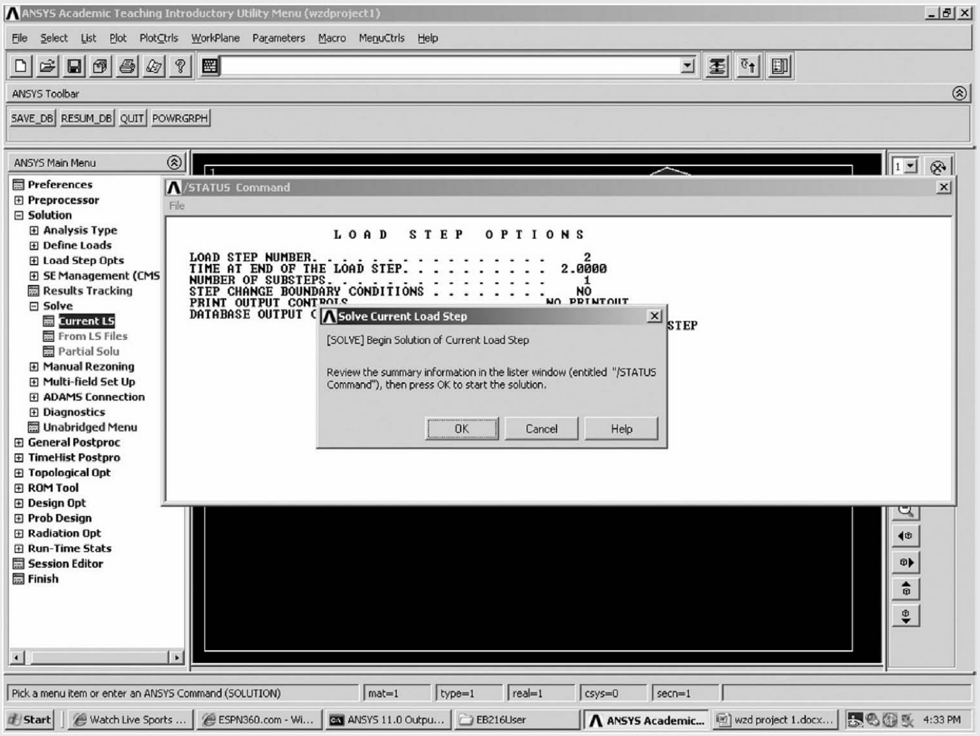


FIGURE 22.24 Begin to solve with current load step.

EXAMPLE 22.2 —cont'd

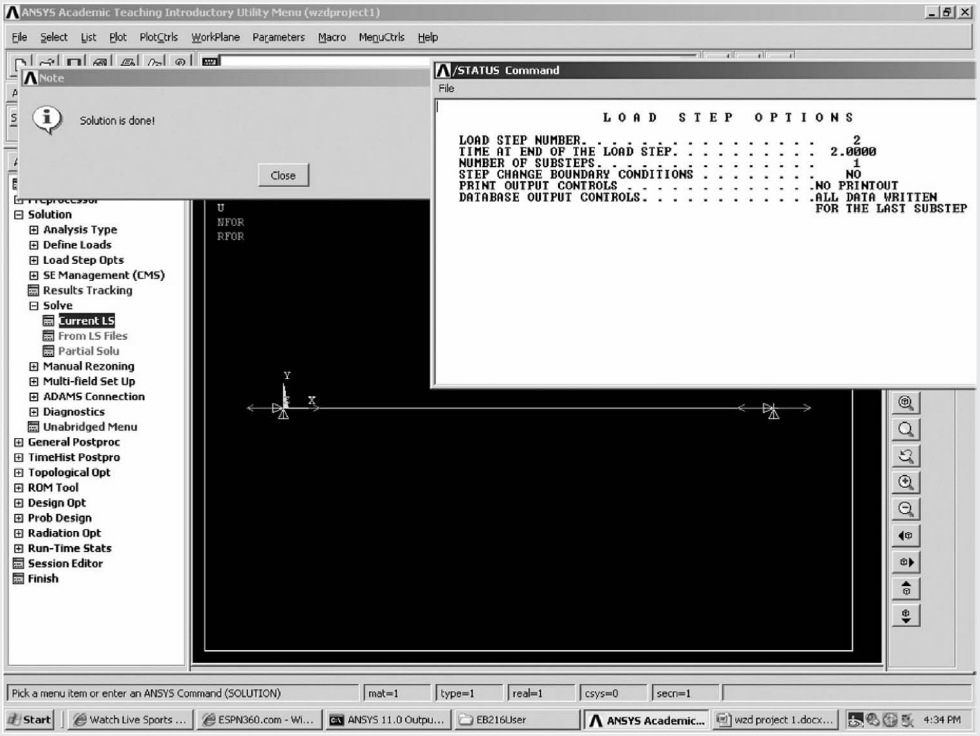


FIGURE 22.25 Completion of solution.

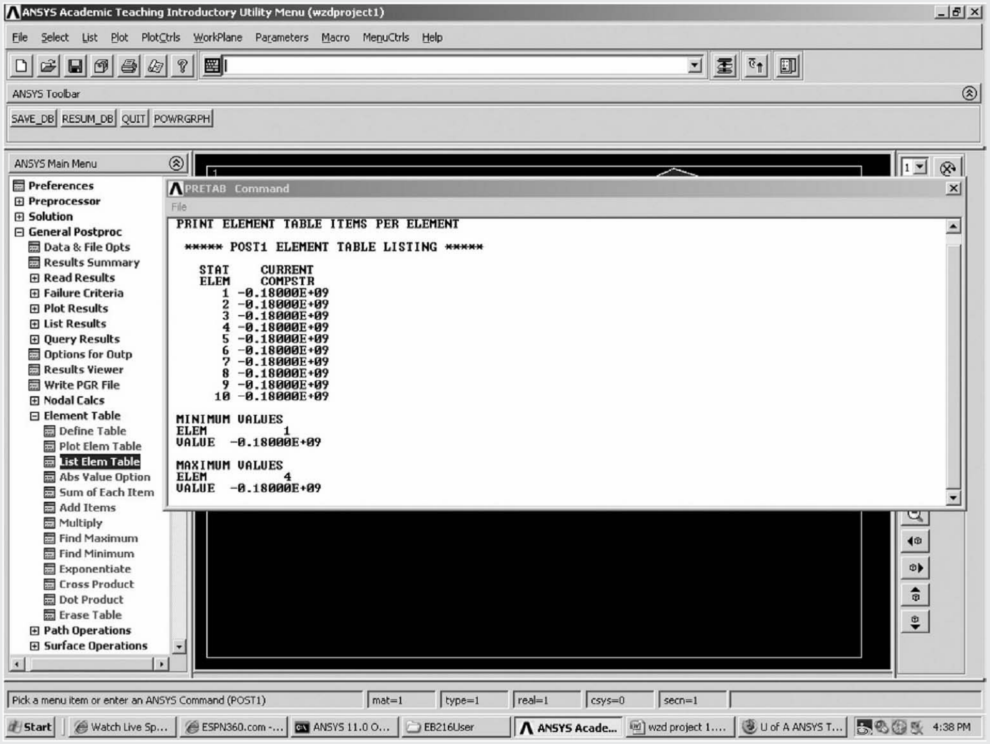


FIGURE 22.26 Listing of thermal stresses in elements.

Continued

EXAMPLE 22.2 —cont'd

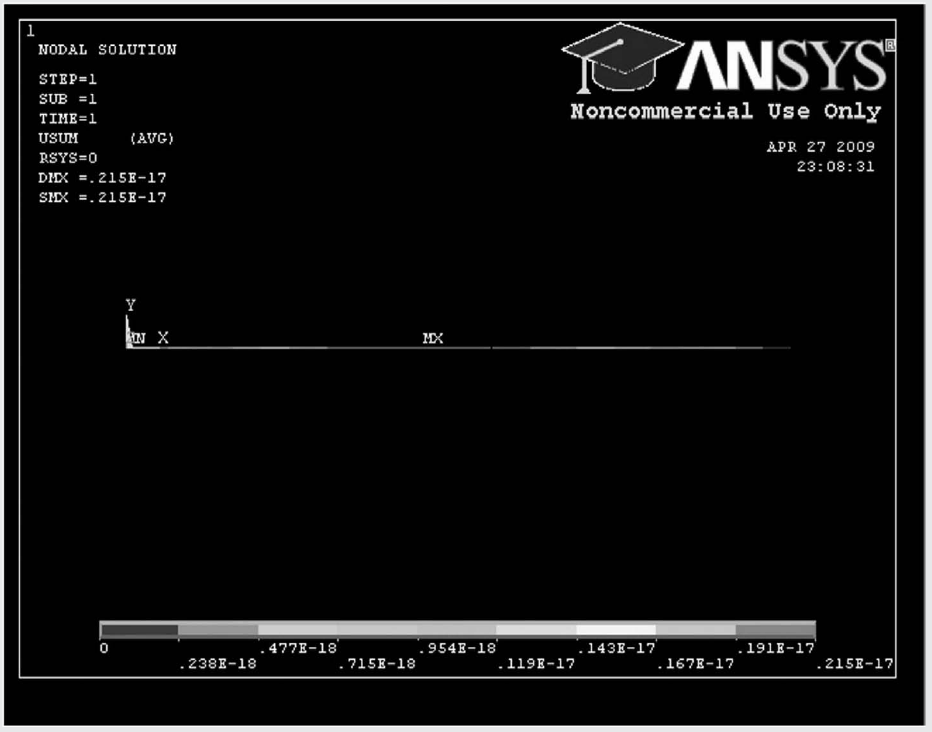


FIGURE 22.27 Display of deformation shape.

Comparison

Expansion caused by thermal stress in a link can be calculated using:

$$\delta = \alpha \Delta T L$$

Expansion owing to structural forces can be determined using:

$$\delta = \frac{PL}{EA}$$

Solving for the structural forces caused by thermal expansion:

$$P = \alpha \Delta T E A$$

or

$$\sigma = \frac{F}{A} = \alpha \Delta T E$$

Therefore, in this example,

$$\sigma = (0.000012/\text{K})(348\text{K} - 273\text{K})(200e3\ \text{MPa}) = 180\ \text{MPa}$$

PROBLEMS

Find the deflections and stresses in the beams described in the following problems using the finite element method with the following data:

$P = 1000$ N, $L = 1$ m, $E = 70 \times 10^9$ Pa, $M_0 = 50{,}000$ N cm, cross-section of the beam: rectangular with a depth of 2 cm and width (perpendicular to the page) of 1 cm.

Use ANSYS to solve the problems.

22.1 A uniform cantilever beam of length L subject to a concentrated transverse load P at the free end as shown in Fig. 1.20.

22.2 A uniform cantilever beam of length L subject to a concentrated bending moment M_0 at the free end as shown in Fig. 1.21.

22.3 A uniform fixed, simply supported beam of length L subject to a concentrated transverse load P at the middle as shown in Fig. 1.22.

22.4 A uniform fixed-simply supported beam of length L subject to a concentrated bending moment M_0 at the middle as shown in Fig. 1.23.

22.5 A uniform fixed-fixed beam of length L subject to a concentrated transverse load P at a distance of (3 L/4) from the left end as shown in Fig. 1.24.

Chapter 23

MATLAB Programs for Finite Element Analysis

Chapter Outline

MATLAB[1] is popular software used to solve a variety of engineering analysis problems. This chapter presents 10 MATLAB programs for the finite element analysis of different types of problems discussed in Chapters 1−20. The programs are given in the form of m-files. Each program requires subprograms, which are also in the form of m-files, to solve a problem. In addition, depending on the problem to be solved, the user needs to create a main program in the form of an m-file to specify the input data, call the relevant program (m-file), and display the results. Sample main programs are given for all 10 programs described in this chapter. The m-files of the programs and all of the subprograms needed to execute the programs are available on the website for the book. The programs and purpose or use of the programs are given in the subsequent table.

Program Number	Program Name	Relevant Location in the Book (Section Number)	Use of the Program
1	choleski.m	7.2.2	To solve a symmetric system of algebraic equations with several right-side vectors, $[A]\vec{X} = \left[\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M\right]$; $[A]$ can be stored in banded form
2	adjust.m	6.3	To incorporate boundary conditions (specified values to boundary degrees of freedom) in matrix equations, $[A]\vec{X} = \vec{b}$ where the matrix $[A]$ can be stored in banded form
3	truss3D.m	9.2	Analysis of three-dimensional (3D) trusses (to find displacements of nodes and stresses in bar elements)
4	cst.m	10.3.1	Displacement and stress analysis of plates lying in XY plane subjected to in-plane loads using linear triangular (CST) elements

(*Continued*)

1. MATLAB is a registered trademark of The MathWorks, Inc., 3 Apple Hill Drive, Natick, MA 01760-2098.

The Finite Element Method in Engineering. https://doi.org/10.1016/B978-0-12-811768-2.00023-7

5	CST3D.m	10.3.3	Displacement and stress analysis of three-dimensional box-type of structures subjected to nodal loads using linear triangular (CST) elements
6	heat1.m	14.2	Temperature distribution in 1D fins (by considering conduction and convection)
7	radiat.m	14.6	Temperature distribution in 1D fins by considering radiation heat transfer (along with conduction and convection)
8	heat2.m	15.2	Temperature distribution in 2D plates with arbitrary geometry using linear triangular elements (by considering conduction and convection)
9	phiflo.m	18.3	Confined fluid flow around a cylinder (or any shaped body) for inviscid and incompressible fluids using potential function approach using linear triangular elements
10	torson.m	20.3	Stress function variation in a cross-section of a solid prismatic shaft subjected to twisting moment using linear triangular elements

23.1 SOLUTION OF LINEAR SYSTEM OF EQUATIONS USING CHOLESKI METHOD

Two MATLAB subprograms called decomp.m and solve.m were developed to solve linear algebraic equations using the Choleski decomposition method (details are given in Section 7.2.2):

$$[A]\vec{X} = \vec{b} \tag{23.1}$$

where $[A]$ is a symmetric banded matrix of order N. It is assumed that the elements of the matrix $[A]$ are stored in band form in the first N rows and NB columns of the array A, where NB denotes the semibandwidth of the matrix $[A]$. Thus, the diagonal terms a_{ii} of $[A]$ occupy the locations $A(I, 1)$.

The subprogram decomp.m decomposes the matrix $[A]$ (stored in the form of an array A) into $[A] = [U]^T[U]$ and the elements of the upper triangular matrix $[U]$ are stored in the array A. The subprogram solve.m solves Eq. (23.1) by using the decomposed coefficient matrix $[A]$. This subprogram can solve Eq. (23.1) for several right hand−side vectors $\vec{b}$. If $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M$ indicate the right hand−side vectors[2] for which the corresponding solutions $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_M$ are to be found, all vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M$ are stored column-wise in the array B. Thus, the j-th element of $\vec{b}_i$ will be stored as $B(J, I)$, $J = 1, 2, \ldots, N$. The equations to be solved for any right hand−side vector $\vec{b}$ can be expressed as $[A]\vec{X} = [U]^T[U]\vec{X} = \vec{b}$. These equations can be solved as:

$$[U]\vec{X} = \left([U]^T\right)^{-1}\vec{b} \equiv \vec{Z} \tag{23.2}$$

and

$$\vec{X} = [U]^{-1}\vec{Z}$$

In the subprogram, solve.m, the vectors $\vec{Z}_i$ for different $\vec{b}_i$ are found in the forward pass and are stored in the array B. The solutions $\vec{X}_i$ corresponding to different $\vec{b}_i$ are found in the backward pass and are stored in the array B. Thus, the columns of the array B returned from solve.m will give the desired solutions $\vec{X}_i$, $i = 1, 2, \ldots, M$. A main program named main_choleski.m is created to provide the input data, call the subprograms decomp.m and solve.m, and obtain the results. The procedure is illustrated in the following example.

2. The right-hand-side vectors $\vec{b}_1, \vec{b}_2, \ldots,$ represent different load vectors (corresponding to different load conditions) in a static structural or solid mechanics problem.

EXAMPLE 23.1

Solve the following system of equations:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \vec{X}_i = \vec{b}_i \tag{E.1}$$

where

$$\vec{X}_i = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix}_i, \quad \vec{b}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \vec{b}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \text{and} \quad \vec{b}_3 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Here the number of equations $= N = 5$, the semibandwidth of $[A] = NB = 2$, and the number of vectors $\vec{b}_i = M = 3$.

Solution

The program main_choleski.m created to solve the system of Eq. (E.1) and the results obtained are shown:

```
clear all; close all; clc;
% ---------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ---------------------------------------
% Input data
A = [1 2 2 2 2; -1 -1 -1 -1 0]';
B = [1 0 0 0 0; 0 0 0 0 1; 1 1 1 1 1]';
N = 5; NB = 2; M = 3;
% End of input data
[A, DIFF] = decomp (N, NB, A);
P = solve (N, NB, M, A, B, DIFF);

for j = 1:M
fprintf ('%s', 'solution')
fprintf ('%2.0f', j')
fprintf ('%s\n',':')
fprintf ('%8.0f %8.0f %8.0f %8.0f %8.0f\n', P(: , j)')
end
Solution 1:
```

5	4	3	2	1

```
Solution 2:
```

1	1	1	1	1

```
Solution 3:
```

15	14	12	9	5

23.2 INCORPORATION OF BOUNDARY CONDITIONS

To incorporate the boundary conditions for solving the equations $[A]\vec{X} = \vec{B}$, a MATLAB program called adjust.m was created. This program assumes that the global characteristic or stiffness matrix $[A]$ is stored in a band form. If the degree of freedom or the component "II" of the vector $\vec{X}$ is specified as a constant value CONST, the following statement modifies matrix A and vector B to incorporate the stated boundary condition:

```
function [A, B] = adjust [A,B,NN,NB,II,CONST]
```

where NN is the total number of degrees of freedom, NB is the bandwidth of A, B is the global vector of nodal actions or load (size: NN), and A is the global characteristic or stiffness matrix (size: $NN \times NB$).

Notes

1. If the prescribed values of several degrees of freedom are to be incorporated, the program adjust.m is to be used once for each prescribed degree of freedom.
2. The program adjust.m incorporates the boundary conditions using the procedure described in Method 2 of Section 6.3.
3. The program works even when matrix [A] is not stored in band form. If [A] is not stored in band form, the bandwidth of A can be defined as $NB = NN$ and the full original matrix [A] can be treated as if it were in band form.

The following example illustrates the use of the program adjust.m.

EXAMPLE 23.2

Consider the following system of equations:

$$\begin{bmatrix} 1.9 & 2.1 & -5.7 & 0.0 & 0.0 \\ 2.1 & 3.4 & 1.5 & 3.3 & 0.0 \\ -5.7 & 1.5 & 2.2 & 4.5 & 2.8 \\ 0.0 & 3.3 & 4.5 & 5.6 & -1.8 \\ 0.0 & 0.0 & 2.8 & -1.8 & 4.7 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

By renaming the characteristic matrix [A] as [GK] and the characteristic vector $\vec{B}$ as $\vec{P}$, arrays [GK] and $\vec{P}$ can be identified from Eq. (E.1) as:

$$[GK] = \begin{bmatrix} 1.9 & 2.1 & -5.7 \\ 3.4 & 1.5 & 3.3 \\ 2.2 & 4.5 & 2.8 \\ 5.6 & -1.8 & 0.0 \\ 4.7 & 0.0 & 0.0 \end{bmatrix}, \quad \vec{P} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

with $NN = 5$ and $NB = 3$.

Modify the arrays GK and P to incorporate the boundary condition $\Phi_3 = \Phi_3^* = 2.0$.

Solution

The stated boundary condition gives II = 3 and CONST = 2.0. The main program named main_adjust.m in which the sub-program is called adjust.m, as well as the results (modified arrays GK and P), are shown subsequently:

```
clear all; close all; clc;
% ----------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ----------------------------------------
GK = [1.9 2.1 5.7; 3.4 1.5 3.3; 2.2 4.5 2.8; 5.6 -1.8 0.0; 4.7 0.0 0.0];
P = [0 0 0 0 0]';
NN = 5;
NB = 3;
M = 3;
CONST = 2;
[GK, P] = Adjust (GK, P, NN, NB, M, CONST);

fprintf('     Modified characteristic matrix (GK) \n')
for i = 1:NN
fprintf ('% 12.4f % 12.4f %12.4f\n', GK (i, 1), GK(i, 2), GK(i, 3))
end
fprintf('     Modified characteristic vector (P) \n')
for j = 1:NN
fprintf ('% 12.4f\n', P(j))
end
```

EXAMPLE 23.2 —cont'd

```
Modified characteristic matrix (GK)
```

1.9000	2.1000	0.0000
3.4000	0.0000	3.3000
1.0000	0.0000	0.0000
5.6000	−1.8000	0.0000
4.7000	0.0000	0.0000

```
Modified characteristic vector (P)
```

−11.4000
−3.0000
2.0000
−9.0000
−5.6000

23.3 ANALYSIS OF SPACE TRUSSES

A MATLAB program called truss3D.m was developed to analyze space trusses (details are given in Section 9.2). The program requires the data in the subsequent table.

nn	Number of nodes
ne	Number of bar elements
nd	Number of degrees of freedom of the truss, including the fixed degrees of freedom
nb	Bandwidth of the system stiffness matrix
nfix	Number of fixed degrees of freedom
e	Young's modulus
ifix	Degree of freedom numbers that are fixed
ifree	Number of free degrees of freedom; nfree = nd − nfix
loc	Nodal connectivity array of size ne × 2; loc (i, j) denotes the node number of end j (j = 1, 2) of bar element i (i = 1, 2, …, ne)
x	Array of size nn; $x(i)$ is the x-coordinate of node i
y	Array of size nn; $y(i)$ is the y-coordinate of node i
z	Array of size nn; $z(i)$ is the z-coordinate of node i
A	Array of size ne; $A(i)$ is the cross-sectional area of bar element i
p	Array of size nfree; $p(i,1)$ is the load applied at i-th degree of freedom in load condition 1
p1	Array of size nfree; $p1(i,1)$ is the load applied i-th degree of freedom in load condition 2

Note: The fixed nodes are to be numbered after the free nodes. Thus, the last nfix degrees of freedom of the nd degrees of freedom of the system represent fixed degrees of freedom.

The following example illustrates the application of the program truss3D.m.

EXAMPLE 23.3

Find the deflections of nodes and the axial stresses developed in the space truss shown in Fig. 21.8 under the following two load conditions:

Load condition 1: Nonzero components of the load vector are $p(2,1) = 88{,}960$ N, $p(3,1) = -22{,}240$ N, $p(5,1) = -88{,}960$ N, $p(6,1) = -22{,}240$ N

Continued

EXAMPLE 23.3 —cont'd

Load condition 2: Nonzero components of the load vector are $p1(1,1) = 4448$ N, $p1(2,1) = 44{,}480$ N, $p1(3,1) = -22{,}240$ N, $p1(5,1) = 44{,}480$ N, $p1(6,1) = -22{,}240$ N, $p1(7,1) = 2224$ N, $p1(16,1) = 2224$ N
Other data are $E = 6.9 \times 10^{10}$ Pa, and area of cross-section $= 1$ mm^2 for every bar element.

Solution

Each bar of the truss shown in Fig. 21.8 is modeled as a bar element in 3D space. A main program called main_truss3D.m is created to define the input data, call the program truss3D.m, and display the results. The program main_truss3D.m and the results given by the program are shown subsequently:

```
% FEM analysis of space truss 25-bar truss
% ----------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ----------------------------------------

clear all; close all; clc;
% ****************
% Input Parameters
% ****************
% units (MKS meter (distance), Newton (load))
nn=10;      % number of nodes
ne=25;      % number of elements (bars)
nd=30;      % total number of system degrees of freedom
nb=30;
nfix=12;    % number of fixed degrees of freedom
e=6.9*10ˆ10;% Young's modulus of elasticity (Pa)
rho=2770;   % density of the material (kg/m3)
ifix=[19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]; % fixed degrees of freedom
loc=[1 2; 1 4; 2 3; 1 5; 2 6; 2 4; 2 5 ; 1 3 ; 1 6 ;...
3 6; 4 5; 3 4 ;6 5 ;3 10 ;6 7 ;4 9 ;5 8 ;4 7 ;3 8 ;...
5 10; 6 9 ;6 10 ;3 7 ;4 8 ;5 9];% nodal connectivity matrix
x=[-952.5 952.5 -952.5 952.5 952.5 -952.5 -2540 2540 -2540]*le-3;
% nodal coordinates
y=[0 0 952.5 952.5 -952.5 -952.5 2540 2540 -2540 -2540]*le-3;
z=[5080 5080 2540 2540 2540 2540 0 0 0 0]*le-3;
A=ones (1, ne); % areas of cross-section
% ***************
% load vector (p)
% condition 1
% ***************
p=zeros (18, 1);
p(2, 1)=88,960;
p(3, 1)=-22,240;
p(5, 1)=-88,960;
p(6, 1)=-22,240;
p;
% *************
% condition 2
% *************
p1=zeros (18, 1);
p1(1, 1)=4448;
p1(2, 1)=44,480;
p1(3, 1)=-22,240;
p1(5, 1)=44,480;
p1(6, 1)=-22,240;
p1(7, 1)=2224;
p1(16, 1)=2224;
p1;
```

EXAMPLE 23.3 —cont'd

```
[D,D1, st, st1]=truss3D (nn, ne, nd, nb, nfix, e, ifix, loc, x, y, z, A, p, p1) ;
fprintf ( '    Load condition 1:    \n ') ;
fprintf ( '    Nodal displacements:    \n ') ;
fprintf ( ' Node    X-disp    Y-disp     Z-disp\n ') ;
for i=1 : 6
fprintf ( '%1d%4.4e%4.4e%4.4e\n ', i, D (3*i−2), D
(3*i−1), D(3*i) ) ;
end
for i=7: nn
fprintf ( '%1d%4.4e%4.4e%4.4e\n ', i, 0, 0, 0) ;
end
fprintf ( '    \n ' ) ;
fprintf ( '    Load condition 2:    \n ' ) ;
fprintf ( '    Nodal displacements:\n ' ) ;
fprintf ( 'Node    X-disp Y-disp Z-disp\n') ;
for i=1 : 6
fprintf ( '% 1d% 4.4e    % 4.4e    % 4.4e\n',i,D1 (3*i−
2), D1 (3*i−1), D1 (3*i−1)) ;
end
for i=7 : nn
fprintf ( '% 1d % 4.4e    % 4.4e    % 4.4e\n ', i, 0, 0, 0) ;
end
fprintf ( '    \n ' ) ;
fprintf ( '    Load condition 1:    \n ' ) ;
fprintf ( '    Element Stress:     \n ' ) ;
fprintf ( 'Element    Axial stress \n ' ) ;
for i=1 : ne
fprintf ( '% 1d % 4.4e \n', i, st (i)) ;
end
fprintf ( '    \n ' ) ;
fprintf ( '    Load condition 2:    \n ' ) ;
fprintf ( '    Element Stress:     \n ' ) ;
fprintf ( 'Element    Axial stress \n ' ) ;
for i=1 : ne
fprintf ( ' %1d %4.4e \n ', i, st1 (i) ) ;
end
Load condition 1:
Nodal displacements:
```

Node	X Displacement	Y Displacement	Z Displacement
1	−7.1742e−008	1.2450e−005	−8.8742e−007
2	7.1745e−008	−1.2450e−005	−8.8742e−007
3	2.9731e−006	−5.2279e−007	−2.2515e−006
4	2.9891e−006	5.7343e−007	1.1822e−006
5	−2.9731e−006	5.2279e−007	−2.2515e−006
6	−2.9891e−006	−5.7343e−007	1.1822e−006
7	0.0000e+000	0.0000e+000	0.0000e+000
8	0.0000e+000	0.0000e+000	0.0000e+000
9	0.0000e+000	0.0000e+000	0.0000e+000
10	0.0000e+000	0.0000e+000	0.0000e+000

Continued

EXAMPLE 23.3 —cont'd

Load condition 2:
Nodal displacements:

Node	X Displacement	Y Displacement	Z Displacement
1	6.5910e−007	1.2726e−005	−6.8846e−007
2	7.5028e−007	1.2726e−005	−1.0704e−006
3	3.2594e−008	8.4982e−007	−3.1324e−006
4	2.1198e−007	8.7459e−007	−3.3721e−006
5	2.6689e−008	8.0020e−007	2.0590e−006
6	2.1789e−007	8.2497e−007	2.2987e−006
7	0.0000e+000	0.0000e+000	0.0000e+000
8	0.0000e+000	0.0000e+000	0.0000e+000
9	0.0000e+000	0.0000e+000	0.0000e+000
10	0.0000e+000	0.0000e+000	0.0000e+000

Load condition 1:
Element Stress:

Element	Axial stress
1	5.1971e+003
2	−6.7431e+004
3	5.8388e+004
4	5.8388e+004
5	−6.7431e+004
6	6.7020e+004
7	−8.3372e+004
8	−8.3372e+004
9	6.7020e+004
10	1.8344e+003
11	1.8344e+003
12	5.7966e+002
13	5.7966e+002
14	−9.2069e+003
15	8.4817e+002
16	8.4817e+002
17	−9.2069e+003
18	4.0847e+004
19	−4.9780e+004
20	−4.9780e+004
21	4.0847e+004
22	−1.5928e+004
23	−1.0143e+003
24	−1.5928e+004
25	−1.0143e+003

EXAMPLE 23.3 —cont'd

```
Load condition 2:
Element Stress:
```

Element	Axial stress
1	3.3027e+003
2	−3.3429e+004
3	−2.9559e+004
4	1.9943e+004
5	2.3812e+004
6	−5.1025e+004
7	3.1976e+004
8	−4.7858e+004
9	3.5143e+004
10	9.0003e+002
11	2.6945e+003
12	6.4976e+003
13	−6.9254e+003
14	−1.6090e+004
15	1.0767e+004
16	−1.9058e+004
17	7.7990e+003
18	−3.0030e+004
19	−3.0701e+004
20	2.1491e+004
21	2.0819e+004
22	4.4997e+004
23	−5.5561e+004
24	−6.1784e+004
25	3.8774e+004

```
EDU>>
```

23.4 ANALYSIS OF PLATES SUBJECTED TO IN-PLANE LOADS USING CST ELEMENTS

A program called cst.m was developed for the deflection and stress analysis of plates (in the $X-Y$ plane) using CST elements. The program requires the following data:

NN = the total number of nodes (including the fixed nodes)
NE = the number of triangular elements
ND = the total number of degrees of freedom (including the fixed degrees of freedom). Two degrees of freedom (one parallel to the X axis and the other parallel to the Y axis) are considered at each node
NB = the bandwidth of the overall stiffness matrix
M = the number of load conditions
LOC = an array of size NE × 3. LOC(I, J) denotes the global node number corresponding to the J-th corner of element I
CX,CY = vector arrays of size NN each. CX (I) and CY (I) denote the global X and Y coordinates of node I
E = Young's modulus of the material

ANU = Poisson's ratio of the material
T = the thickness of the plate
NFIX = the number of fixed degrees of freedom (zero displacements)
IFIX = a vector array of size NFIX. IFIX (I) denotes the I-th fixed degree of freedom number
P = an array of size ND $\times$ M representing the global load vectors

The array P returned from the program cst.m to the main program represents the global displacement vectors, with P(I, J) denoting the I-th component of global load in input (or displacement in output) vector in the J-th load condition.

The program cst.m requires the following subprograms: lambda.m, decomp.m, and solve.m. The following example illustrates the application of the program cst.m.

EXAMPLE 23.4

Find the nodal deflections and element stresses in the plate under tension described in Section 10.3.1 and shown in Fig. 10.4A using CST elements.

Solution

Because of the double symmetry of the plate and the loading, only a quadrant of the plate is used for idealization. The finite element idealization and the node numbers used are indicated in Fig. 10.4B. Two degrees of freedom are considered at each node, as shown in Fig. 10.4C.

The boundary (symmetry) conditions are:

$Q_5 = Q_7 = Q_9 = 0$ (the X component of displacement of nodes 3, 4, and 5 is zero)

$Q_{10} = Q_{12} = Q_{14} = 0$ (the Y component of displacement of nodes 5, 6, and 7 is zero)

The only load condition is:

```
P(6, 1) = 1000.0 N
P(4, 1) = 2000.0 N
P(2, 1) = 1000.0 N
```

Note

The node numbering scheme used in Fig. 10.4B leads to a high bandwidth (NB = 18). We can reduce NB to 10 by relabeling the nodes 8, 9, 4, 7, 6, and 5 of Fig. 10.4B as 4, 5, 6, 7, 8, and 9, respectively.

A main program called main_cst.m was created to define the input data, call the program cst.m, and display the results. The program, main_cst.m, and the results of the analysis are shown subsequently:

```
% Main_CST
% ----------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ----------------------------------------
clear all; close all; clc;
format short
NN = 9;   % Input data
NE = 8;
ND = 18;
NB = 18;
NFIX = 6;
M = 1;
E = 2e6;
ANU = 0.1;

LOC = [9, 1, 9, 3, 9, 5, 9, 7; 1, 9, 3, 9, 5, 9, 7, 9; 8, 2, 2, 4, 4, 6, 6, 8]';
CX = [20 10 0 0 0 10 20 20 10]';
CY = [20 20 20 10 0 0 0 10 10]';
T = 0.1;
IFIX = [5 7 9 10 12 14]';
```

EXAMPLE 23.4 —cont'd

```
P = zeros (18, 1);
P(6, 1) = 1000;
P(4, 1) = 2000;
P(2, 1) = 1000;

% End of input data
[P, DIFF, SLOC, STRES] = CST (NN, NE, ND, NB, M, LOC, CX, CY, E, ANU, T, NFIX, IFIX, P);

% output results
fprintf('   Nodal displacements:   \n');
fprintf('NodeX-component   Y-component \n');
for i=1: NN
fprintf('%1d   %6.4f%6.4f\n', i, P(2*i-1), P(2*i));
end
fprintf('\n');
fprintf('Stresses in elements:\n');
fprintf('Element   sigma xx   sigma yy   sigma xy\n');
for i = 1:NE
fprintf('%1d %6.4f %6.4f % 6.4f\n', i, STRES
(i, 1), STRES (i, 2), STRES (i, 3));
end
Nodal displacements:
```

Node	X Component	Y Component
1	−0.0020	0.0200
2	−0.0010	0.0200
3	−0.0000	0.0200
4	−0.0000	0.0100
5	−0.0000	0.0000
6	−0.0010	0.0000
7	−0.0020	0.0000
8	−0.0020	0.0100
9	−0.0010	0.0100

Stresses on elements:

Element	Sigma xx	Sigma yy	Sigma xy
1	0.0000	2000.0000	−0.0000
2	0.0000	2000.0000	−0.0000
3	−0.0000	2000.0000	0.0000
4	−0.0000	2000.0000	0.0000
5	−0.0000	2000.0000	0.0000
6	−0.0000	2000.0000	−0.0000
7	−0.0000	2000.0000	−0.0000
8	0.0000	2000.0000	−0.0000

The results given by the program cst.m are identical to the results obtained by hand computations (shown in Table 10.1).

23.5 ANALYSIS OF THREE-DIMENSIONAL STRUCTURES USING CST ELEMENTS

A program called CST3D.m was developed for the deflection and stress analysis of 3D box-type structures using CST elements. The program requires the data in the table.

neltot	Total number of triangular elements
nelm	Number of membrane elements (used in top and bottom surfaces)
nels	Number of shear elements (used in vertical side panels)
nnel	Number of nodes per element
ndof	Number of degrees of freedom per node
nnode	Total number of nodes in the structure
sdof	Total number of degrees of freedom in the structure (=nnode * ndof)
effdof	Total number of free degrees of freedom (excluding fixed dof.)
topdof	Number of free dof. on top surface
edof	Number of degrees of freedom per element (=nnel * ndof)
emodule	Young's modulus
poisson	Poisson's ratio
NB	Number of elements near root (in which stresses are required)
FF	Applied loads; FF(I, J) = load applied along the I-th degree of freedom in the J-th load condition
CX, CY, CZ	Global X, Y, Z coordinates of nodes; CX(I), CY(I), CZ(I) = global X, Y, Z coordinates of node I
gcoord	Global X, Y, Z coordinates of nodes 1, 2, ..., nnode
nodes	Nodal connectivity array; nodes(i, j) = global node number of the j-th local node (corner) of element i
ISTRES	ISTRES(I) = I-th element number in which stress is required
bcdof	Fixed degrees of freedom numbers
nmode	Number of eigenvalues required (not computed in this program)

CST elements with three degrees of freedom per node as shown in Fig. 10.2 are considered in the model. To illustrate the application of the program CST3D.m, the following example is considered.

EXAMPLE 23.5

Find the nodal deflections and element stresses in the box beam described in Section 10.3.3 and shown in Fig. 10.7 using CST elements.

Solution

The finite element idealization using CST elements is shown in Fig. 10.8. A main program called main_cst3D.m was created to define the input data, call the program cst3D.m, and display the results. The program main_CST3D.m and the results of the analysis are shown subsequently:

```
% FEM analysis of 3_D STRUCTURE USING CST ELEMENT
% MAIN_CST3D.m
% SUBPART: CST3D.m
% ------------------------------------------------------------
% WRITTEN BY SINGIRESU S. RAO
% THE FINITE ELEMENT METHOD IN ENGINEERING
% ------------------------------------------------------------
% dof. ARE 3 PER NODE
% INPUT PARAMETER DATA
```

EXAMPLE 23.5 —cont'd

```
clear all; close all; clc;

%units: inch (distance) and Ib (force)
nelotot = 40;% total no. of elements
nelm=20;     % no. of membrane elements
nemp=10;     % element of top plane surfaces
nels=20;     % no. of shear elements
nnel= 3;     % no. of nodes per element
ndof=3;      % no. of dofs per node
nnode=24;    % total number of nodes in system
sdof=nnode*ndof;  % total system dofs
effdof=60;   % total no. of free dofs
topdof=30;   % no. of free dofs on top surface
edof=nnel*ndof;   % dof per element
emodule=30e6;     % Young's modulus
poisson=0.3;% Poisson's ratio

for ii=1: nelm;
    tt (ii) =1;   % thickness
end
for ii= (nelm+1) :neltot;
    tt (ii) =1;   % thickness
end
rho=0.28/384;     % weight density (unit weight/gravitational constant)
nmode=3;
NB=4;% No. of elements near root (for stress)

FF=zeros (effdof, 1);     % applied forces along free dofs
FF (3, 1) = -5000;
FF (6, 1) = -5000;
%
% FF (i, 1) = Load applied along dof i (1 = load condition number)

%
% -----------------------------------
% input data for nodal coordinates (only CST elements)
% -----------------------------------
% model 1
% CX = X coordinates of all nodes
% CY = Y coordinates of all nodes
% CZ = Z coordinates of all nodes
CX=
[60, 60, 48, 48, 36, 36, 24, 24, 12, 12, 0, 0, 60, 60, 48, 48, 36, 36, 24, 24, 12, 12, 0, 0];
CY= [18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0, 18, 0];
CZ= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12];
%
% -----------------------------------------------------
% input data for nodal coordinates in vector from-------
% gcood = (X, Y, Z) coordinates of nodes 1, 2..., nnode
% ----------------------------------------------------------------
gcoord=[60 18 0;60 0 0;48 18 0;48 0 0;36 18 0;36 0 0;24 18 0;24 0 0;12
18 0;12 0 0;0 18 0;0 0 0;...
```

Continued

EXAMPLE 23.5 —cont'd

```
60 18 12; 60 0 12; 48 18 12; 48 0 12;36 18 12;36 0 12;24 18 12;24 0
12; ...
12 18 12;12 0 12;0 18 12;0 0 12];
% -----------------------------------------------------------------
% input data for nodal connectivity for each element
% nodes (i, j), i=element no., j=global nodes of local corners 1, 2, 3 of
% elements 1, 2, ..., neltot
% -----------------------------------------------------------------
nodes = [3 2 1; 2 3 4; 5 4 3;4 5 6;7 6 5; 6 7 8; 9 8 7;8 9 10; 11 10 9; 10
11 12; ...
15 14 13; 14 15 16;17 16 15; 16 17 18; 19 18 17; 18 19 20; 21 20
19; 20 21 22; 23 22 21;22 23 24; ...
4 14 2;14 4 16; 6 16 4;16 6 18; 8 18 6; 18 8 20; 10 20 8; 20 10
22;12 22 10;22 12 24; ...
3 13 1;13 3 15;5 15 3;15 5 17;7 17 5;17 7 19;9 19 7; 19 9 21;
11 21 9; 21 11 23];
% ISTRES=array of element numbers in which stresses are to be found
ISTRES=[10 20 30 40];
% -------------------------------------------
% input data for boundary conditions
% -------------------------------------------
bcdof= [31 32 33 34 35 36 67 68 69 70 71 72];
% bedof = fixed dof numbers------
% end of data-------------------------

[disp, STRESS] =CST3D
(neltot, nelm, nemp, nels, nnel, ndof, nnode, sdof, effdof, topdof, edof, emodule, p
oisson, tt, ...
rho, nmode, NB, FF, CX, CY, CZ, gcoord, nodes, ISTRES, bcdof) ;
% ************************
% OUTPUT:=
fprintf ('     Nodal displacements     \n');
fprintf ('NodeX-dispY-dispZ-disp\n');
for i=1:1:20
fprintf ('%1d  %6.8f %6.8f %6.8f\n',
i, disp (3* (i-1)+1), disp (3*(i-1)+2), disp (3* (i-1)+3));
end
fprintf('     \n');

fprintf ('     Stress in the structure     \n');
fprintf ('Element     Principal Stress 1     Principal Stress 2
\n');
for IJK=1:1:NB;
ii = ISTRES (IJK); %%% ISTRES is the array of element numbers in which
stresses are to be found
for jj=1:1:3;
stressp (IJK, jj) = STRESS (jj, IJK);
end
A1= (STRESS (1, IJK) + STRESS (2, IJK) ) /2;
A2= sqrt (((STRESS (1, IJK) -STRESS (2, IJK))/2)^2+STRESS (3, IJK)^2);
boxbeam_stress (IJK, 1) = A1+A2;
boxbeam_stress (IJK, 2) = A1-A2;
boxbeam_stress;
fprintf ('  %1d     % 6.8f% 6.8f\n', ii,
boxbeam_stress (IJK, 1) , boxbeam_stress (IJK, 2));
end
```

EXAMPLE 23.5 —cont'd

Nodal displacements

Node	X Displacement	Y Displacement	Z Displacement
1	−0.00143909	0.00009891	−0.01149128
2	−0.00142851	0.00005433	−0.01128080
3	−0.00135326	0.00009942	−0.00824795
4	−0.00133865	0.00003701	−0.00805697
5	−0.00118348	0.00010738	−0.00530461
6	−0.00115996	0.00000364	−0.00514629
7	−0.00090429	0.00011121	−0.00280302
8	−0.00087101	−0.00003803	−0.00269616
9	−0.00051266	0.00009874	−0.00096246
10	−0.00047439	−0.00007322	−0.00091898
11	0.00137522	−0.00008099	−0.01131236
12	0.00136999	−0.00004809	−0.01109938
13	0.00131777	−0.00008645	−0.00820368
14	0.00130824	−0.00003279	−0.00801115
15	0.00115913	−0.00009939	−0.00526670
16	0.00114000	−0.00000073	−0.00510831
17	0.00088969	−0.00010641	−0.00276502
18	0.00086045	0.00003853	−0.00266030
19	0.00050770	−0.00009675	−0.00092457
20	0.00047142	0.00007256	−0.00089349

Stress in the structure

Element	Principal Stress 1	Principal Stress 2
10	−799.42490921	−1313.09508435
20	1304.82999848	794.41165622
30	1295.10699450	850.47289482
40	1394.79009474	897.81264906

The results given by the program CST3D.m are identical to those shown in Table 10.3.

23.6 TEMPERATURE DISTRIBUTION IN ONE-DIMENSIONAL FINS

To find the temperature distribution in a 1D fin (details are given in Section 14.2), a program called heat1.m was developed. The program requires the following quantities:

NN = number of nodes (input)
NE = number of elements (input)
NB = semibandwidth of the overall matrix GK (input)
IEND = 0: means no heat convection from the free end
IEND = any nonzero integer: means that heat convection occurs from the free end (input)
CC = thermal conductivity of the material, k (input)
H = convection heat transfer coefficient, h (input)
TINF = atmospheric temperature, T_∞ (input)
QD = strength of heat source, q (input)
Q = boundary heat flux, q (input)
NODE = array of size NE × 2; NODE (I, J) = global node number corresponding to the J-th (right hand−side) end of element I (input)

XC = array of size NN; XC(I) = x coordinate of node I (input)
A = array of size NE; A(I) = area of cross-section of element I (input)
PERI = array of size NE; PERI (I) = perimeter of element I (input)
TS = array of size NN; TS (I) = prescribed value of temperature of node I (input). If the temperature of node I is not specified, the value of TS (I) is given as 0.0

The program heat1.m requires the following subprograms: adjust.m, decomp.m, and solve.m.
The following example illustrates the use of the program heat1.f.

EXAMPLE 23.6
Find the temperature distribution in the 1D fin considered in Example 14.4 and shown in Fig. 14.1 with two finite elements.

Solution
A main program called main_heat1.m was created to solve the problem. The listing of the program main_heat1.m and the nodal temperatures given by the program are shown below. The results can be seen to agree with the results obtained in Example 14.4 with hand computations.

```
clc; clear all;
% -----------------------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% -----------------------------------------------------
NN=3;
NE=2;
NB=2;
IEND=1;
CC=70;
H=5;
TINF=40;
QD=0;
Q=0;
NODE=[1 2; 2 3] ;
XC= [0; 2.5; 5];
A=[3.1416; 3.1416];
PERI=[6.2832; 6.2832] ;
TS=[140; 0; 0];
PLOAD = HEAT1 (NN, NE, NB, IEND, CC, H, TINF, QD, Q, NODE, XC, A, PERI, TS);

fprintf ('%s\n', 'Node Temperature')
for i = 1:NN
fprintf ('%2.0f %15.4f\n', i, PLOAD (i))
end
```

Node	Temperature
1	140.0000
2	80.4475
3	63.3226

23.7 TEMPERATURE DISTRIBUTION IN ONE-DIMENSIONAL FINS INCLUDING RADIATION HEAT TRANSFER

To find the temperature distribution in a 1D fin including radiation heat transfer (details given in Section 14.6), a program called radiat.m is developed. The program requires the following quantities:

EPSIL = emissivity of the surface (input)
EPS = a small number of the order of 10^{-6} for testing the convergence of the method (input)

SIG = Stefan–Boltzmann constant = 5.7×10^{-8} W/m^2 K^4 (input)
ITER = number of iterations used for obtaining convergence of the solution (output)

The other quantities NN, NE, NB, IEND, CC, H, TINF, QD, NODE, P, PLOAD, XC, A, PERI, and TS have the same meaning as in the case of the subprogram heat1.m. The program radiat.m requires the following subprograms: adjust.m, decomp.m, and solve.m.

The following example illustrates the use of the program radiat.f.

EXAMPLE 23.7

Find the temperature distribution in the 1D fin considered in Example 14.13 using one finite element.

Solution

A main program called main_radiat.m is created to solve the problem. The program main_radiat.m and the nodal temperatures given by the program are given subsequently. The results agree with those obtained in Example 14.13 using hand computations.

```
clear all; clc; close all;
% ----------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ----------------------------------------
fprintf('%s\n', 'Iteration Error Nodal Temperature')
P = zeros (2, 1);
PLOAD = zeros (2, 1);
GK = zeros (2, 2);
EL = zeros (1, 1);
PERI = zeros (1, 1);
NN = 2; NE = 1; NB = 2; IEND = 0; CC = 70; H = 5; TINF = 40; QD = 0; Q=0;EPSIL = 0.1; EPS = 0.0001;
SIG = 5.7e−8;
NODE = [1, 2];
XC = [0;5];
A = 3.1416;
PERI = 6.2832;
TS = [140;0];
PLOAD = radiat
(NN, NE, NB, IEND, CC, H, TINF, QD, Q, EPSIL, EPS, SIG, NODE, XC, A, PERI, TS, P, PLOAD, GK,EL);
```

Iteration	Error	Nodal	Temperature
1	1.0000	140.0000	58.4783
2	0.0145	140.0000	52.2292
3	0.0002	140.0000	52.3106
4	0.0000	140.0000	52.3095

23.8 TWO-DIMENSIONAL HEAT TRANSFER ANALYSIS

To find the solution of a 2D heat transfer problem such as the temperature distribution in a plate (details are given in Section 15.2), a program called heat2.m was developed using linear triangular elements. The program requires the following quantities:

NN = number of nodes (input)
NE = number of triangular elements (input)
NB = semibandwidth of the overall matrix (input)
NODE = array of size NE × 3; NODE (I, J) = global node number corresponding to the J-th corner of element I (input)
XC, YC = array of size NN, XC(I), YC(I) = x and y coordinates of node I (input)
CC = thermal conductivity of the material, k (input)

QD = array of size NE; QD (I) = value of q for element I (input)
ICON = array of size NE; ICON = 1 if element I lies on the convection boundary and = 0 otherwise (input)
NCON = array of size NE × 2; NCON (I, J) = the J-th node of element I that lies on the convection boundary (input); it does not need to be given if ICON (I) = 0 for all I
Q = array of size NE; Q(I) = magnitude of heat flux for element I (input)
TS = array of size NN; TS(I) = specified temperature for node I (input). If the temperature of node I is not specified, the value of TS (I) is to be set equal to 0.0
H = array of size NE; H(I) = convective heat transfer coefficient for element I (input)
TINF = array of size NE; TINF (I) = ambient temperature for element I (input).

The program heat2.m requires the following subprograms: adjust.m, decomp.m, and solve.m. The following example illustrates the use of the program heat2.f.

EXAMPLE 23.8

Find the temperature distribution in the square plate with uniform heat generation considered in Example 15.2 using triangular finite elements.

Solution

A main program called main_heat2.m was created to solve the problem. The program main_ heat2.m and the nodal temperatures given by the program are shown subsequently:

```
clear all; clc;
% -------------------------------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% -------------------------------------------------------------
NN = 9;
NE = 8;
NB = 4;
CC = 30;
Node = [1, 4, 2, 5,4,7,5,8;2,2,3,3,5,5,6,6;4,5,5,6,7,8,8,9]';
XC = [0.0, 5.0, 10.0, 0.0, 5.0, 10.0, 0.0, 5.0, 10.0]';
YC = [0.0, 0.0, 0.0, 5.0, 5.0, 5.0, 10.0, 10.0, 10.0]';
QD = [100.0, 100.0, 100.0, 100.0, 100.0, 100.0, 100.0, 100.0]';
ICON = [0, 0, 0, 0, 0, 0, 0, 0]';
Q = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]';
TS = [0.0, 0.0, 50.0, 0.0, 0.0, 50.0, 50.0, 50.0, 50.0]';
H = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]';
TINF = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]';
PLOAD = HEAT2(NN, NE, NB, NODE, XC, YC, CC, QD, ICON, Q, TS, H, TINF);
fprintf('     Nodal No.    Nodal temperature\n')
for i=1:length (PLOAD)
fprintf('%8.0f %20.4f\n', i, PLOAD(i))
end
```

Nodal No.	Nodal temperature
1	133.3333
2	119.4444
3	50.0000
4	119.4444
5	105.5556
6	50.0000
7	50.0000
8	50.0000
9	50.0000

The results are identical to those obtained in Example 15.2 using hand computations.

23.9 CONFINED FLUID FLOW AROUND A CYLINDER USING THE POTENTIAL FUNCTION APPROACH

To find the potential function distribution for confined inviscid and incompressible fluid flow around a cylinder (details are given in Section 18.3), a program called phiflo.m was developed. The potential function approach with linear triangular elements is used. The program requires the following quantities:

NN = number of nodes (input)
NE = number of elements (input)
NB = semibandwidth of the overall matrix GK (input)
XC, YC = array of size NN; XC(I), YC(I) = x and y coordinates of node I (input)
NODE = array of size NE × 3; NODE (I, J) = the global node number corresponding to the J-th corner of element I (input)
GK = array of size NN × NB used to store the matrix $\left[\underset{\sim}{K}\right]$
P = array of size NN used to store vector $\underset{\sim}{\vec{P}}$
Q = array of size NE; Q(I) = velocity of the fluid leaving element I through one of its edges (input)
A = array of size NE; A(I) = area of element I
PS = array of size NN; PS(I) = specified value of ϕ at node I. If the value of ϕ is not specified at node I, the value of PS(I) is to be set equal to −1000.0 (input)
ICON = array of size NE; ICON(I) = 1 if element lies along the boundary on which the velocity is specified, and = 0 otherwise (input)
NCON = array of size NE × 2; NCON(I, J) = the J-th node of element I that lies on the boundary on which the velocity is specified; it does not need to be given if ICON(I) = 0 for all I (input)
PLOAD = array of size NN × 1 used to store the final right hand−side vector. It represents the solution vector (nodal values of ϕ) upon return from the subroutine PHIFLO

The program phiflo.m requires the following subprograms: adjust.m, decomp.m, and solve.m. The following example illustrates the use of the program phiflo.f.

EXAMPLE 23.9

Find the potential function distribution in the confined flow around a cylinder considered in Example 18.4 using triangular finite elements.

Solution

A main program called main_phiflo.m was created to solve the problem. The program main_phiflo.m and the nodal values of potential function given by the program are shown subsequently:

```
clear all; clc; close all;
% ------------------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ------------------------------------------------
A = zeros (13, 1);
PLOAD = zeros (13, 1);
P = zeros (13, 1);
NN = 13; NE = 13; NB = 7;
XC = [0.0, 5.0, 9.17, 12.0, 0.0, 5.0, 9.17, 12.0, 0.0, 5.0, 8.0, 9.17, 12.0];
YC = [8.0, 8.0, 8.0, 8.0, 4.0, 4.0, 5.5, 5.5, 0.0, 0.0, 0.0, 2.83, 4.0];
ICON = [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0];
NODE = [5, 6, 6, 7, 7, 7, 5, 9, 10, 11, 6, 7, 7; . . .
1, 1, 2, 2, 3, 4, 6, 6, 6, 6, 7, 12, 8; . . .
6, 2, 7, 3, 4, 8, 9, 10, 11, 12, 12, 13, 13]';
NCON = [5, 0, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0; . . .
1, 0, 0, 0, 0, 0, 5, 0, 0, 0, 0, 0, 0; . . .
0 0 0 0 0 0 0 0 0 0 0 0 0]';
GK = zeros (NN, NB);
for I = 1:NN
```

Continued

EXAMPLE 23.9 —cont'd

```
PS(I) = -1000.0;
end

PS(4) =0.0;
PS(8) =0.0;
PS(13) =0.0;
Q = [-1.0, 0.0, 0.0, 0.0, 0.0, 0.0, -1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0];
PLOAD = Phiflo(XC, YC, NODE, ICON, NCON, GK, A, PS, PLOAD, Q, P, NN, NE, NB);

fprintf('%s\n', 'Node Potential function')
for i = 1:NN
fprintf('%4.0f %15.4f\n', i, PLOAD(i))
end
```

Node	Potential Function
1	14.9004
2	9.6754
3	4.4818
4	0.0000
5	15.0443
6	10.0107
7	4.7838
8	0.0000
9	15.2314
10	10.5237
11	8.4687
12	6.2288
13	0.0000

23.10 TORSION ANALYSIS OF SHAFTS

To find the stress function distribution in a solid prismatic shaft subjected to a twisting moment (details are given in Section 20.3), a program called torson.m was developed. Linear triangular elements are used to model the cross-section of the shaft. The program requires the following quantities:

NN = number of nodes
NE = number of elements
NB = semibandwidth of the overall matrix GK
XC, YC = array of size NN; XC(I), YC(I) = x and y coordinates of node I
NFIX = number of nodes lying on the outer boundary (number of nodes at which $\phi = 0$)
NODE = array of size NE $\times$ 3; NODE(I, J) = global node number corresponding to the J-th corner of element I
G = shear modulus of the material
THETA = angle of twist in degrees per 100-cm length
IFIX = array of size NFIX; IFIX(I) = the I-th node number at which $\phi = 0$
A = array of size NE denoting the areas of elements; A(I) = area of the I-th triangular element

The program torson.m requires the following subprograms: adjust.m, decomp.m, and solve.m. The following program illustrates the use of the program torson.f.

EXAMPLE 23.10

Find the stresses developed in a prismatic shaft with a 4 $\times$ 4-cm cross-section that is subjected to a twist of 2 degrees per meter length using triangular finite elements. This example is the same as Example 20.1.

EXAMPLE 23.10 —cont'd

Solution

A main program called main_torson.m was created to solve the problem. The program main_torson.m, the nodal values of stress function, and the shear stresses developed in the elements given by the program are shown subsequently:

```
clear all; clc; close all;
% ----------------------------------------------
% Written by Singiresu S. Rao
% The Finite Element Method in Engineering
% ----------------------------------------------
NN = 6; NE = 4; NB = 5; NFIX = 3; G = 0.8e6; THETA = 2.0;
NODE = [2 4 5 6;3 5 6 3;1 2 2 2]' ;
XC = [2 1 2 0 1 2] ;
YC = [2 1 1 0 0 0] ;
IFIX = [1 3 6] ;
A = zeros (4, 1) ;
PLOAD = Torson (NN, NE, NB, NFIX, G, THETA, NODE, XC, YC, IFIX, A) ;

fprintf ('%s\n', 'Node Value of stress function')
for i = 1:NN
fprintf ('%4.0f %15.4f\n', i, PLOAD (i))
end
```

Node	Value of Stress Function
1	0.0000
2	418.8482
3	0.0000
4	651.5416
5	465.3869
6	0.0000

PROBLEMS

23.1 Solve Problem 7.11 using the MATLAB program choleski.m.

23.2 Modify the matrix and the right hand−side vector considered in Problem 6.22 to incorporate the boundary conditions $T_5 = T_6 = 50°C$. Use the MATLAB program adjust.m.

23.3 Find the nodal displacements and element stresses in the truss considered in Problem 9.7 and Fig. 9.19 using the MATLAB program truss3D.m.

23.4 Find the stresses developed in the plate shown in Fig. 10.16 using at least 10 CST elements. Use the MATLAB program cst.m.

23.5 Find the nodal displacements and element stresses developed in the box beam described in Section 10.3.3 and Fig. 10.7 using the finite element model shown in Fig. 10.8. Use the MATLAB program CST3D.m.

23.6 Solve Problem 14.8 using the MATLAB program heat1.m.

23.7 Solve Problem 14.8 by including radiation heat transfer from the lateral and end surfaces of the fin using the MATLAB program radiat.m. Assume $\varepsilon = 0.1$.

23.8 Find the temperature distribution in the plate considered in Example 15.2 using 16 triangular finite elements. Use the MATLAB program heat2.m.

23.9 Determine the velocity distribution in the region ABCDEA shown in Fig. 18.1 using at least 25 triangular elements. Use the MATLAB program phiflo.m.

23.10 Find the stress distribution in the elliptic shaft subjected to a torsional moment described in Problem 20.7 and Fig. 20.3 using the MATLAB program torson.m. Use at least 10 triangular elements in a quarter of the cross-section.

Appendix A

Comparison of the Finite Element Method With Other Methods of Analysis

The analysis problems, particularly those in the broad area of engineering mechanics, can be solved using several methods. The problems, also known as field problems, are governed by ordinary or partial differential equations. The common methods available for the solution of a general field problem are shown in Fig. A.1. The finite element method of solution is compared to some of the other methods of analysis by considering the problem of free vibration (natural frequencies) of a uniform beam. This is known as an eigenvalue problem.

A.1 DERIVATION OF THE EQUATION OF MOTION FOR THE VIBRATION OF A BEAM [A.1]

By considering the dynamic equilibrium of an element of the beam, shown in Fig. A.2, the following equations can be derived.

Vertical force equilibrium equation: $F - (F + \mathrm{d}F) - p\mathrm{d}x = 0$

or

$$-\frac{\mathrm{d}F}{\mathrm{d}x} = p \tag{A.1}$$

where F denotes the shear force at a distance x, $\mathrm{d}F$ indicates its increment over an elemental distance $\mathrm{d}x$, and p represents the external force acting on the beam per unit length.

Moment equilibrium about the point A:

$$M - (M + \mathrm{d}M) + F\,\mathrm{d}x - p\,\mathrm{d}x\frac{\mathrm{d}x}{2} = 0$$

which can be written, by neglecting the term involving $(\mathrm{d}x)^2$, as

$$\frac{\mathrm{d}M}{\mathrm{d}x} = F \tag{A.2}$$

where M is the bending moment at x and $\mathrm{d}M$ is its increment over $\mathrm{d}x$.

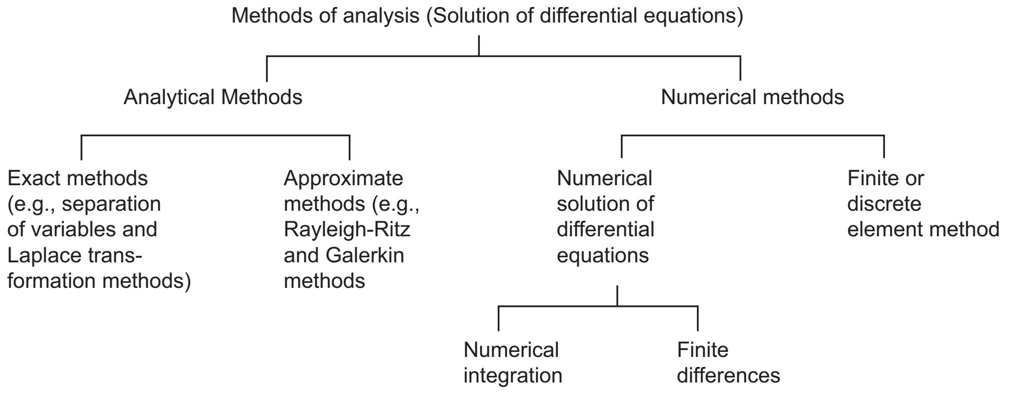


FIGURE A.1 Different methods of analysis.

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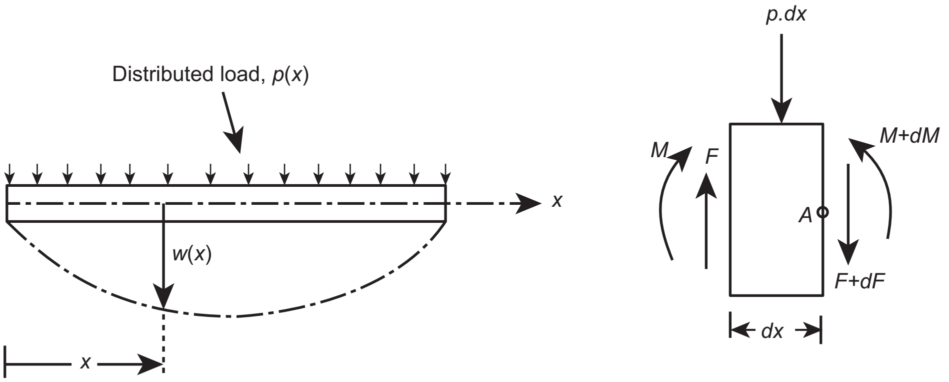

FIGURE A.2 Free body diagram of an element of beam.

Combining Eqs. (A.1) and (A.2), we obtain

$$-\frac{d^2M}{dx^2} = -\frac{dF}{dx} = p(x) \tag{A.3}$$

The curvature of the deflected center line of the beam, $w(x)$, is given by [A.2]

$$\frac{1}{R} = -\frac{\dfrac{d^2w}{dx^2}}{\left(1+\left(\dfrac{dw}{dx}\right)^2\right)^{\frac{3}{2}}} \tag{A.4}$$

For small deflections, Eq. (A.4) can be approximated as

$$\frac{1}{R} = -\frac{d^2w}{dx^2} \tag{A.5}$$

From strength of materials, the following relation is available [A.2]:

$$\frac{1}{R} = -\frac{M}{E\,I(x)} \tag{A.6}$$

where R is the radius of curvature, I is the second area moment of the cross section of the beam, and E is the Young's modulus of the material of the beam. By combining Eqs. (A.3), (A.5), and (A.6), we obtain

$$M(x) = -E\,I(x)\frac{d^2w}{dx^2} \tag{A.7}$$

and

$$p(x) = \frac{d^2}{dx^2}\left[E\,I(x)\frac{d^2w}{dx^2}\right] \tag{A.8}$$

According to D'Alembert's principle, for free vibration, we have

$$p(x) = \text{inertia force} = -m\frac{d^2w}{dt^2} \tag{A.9}$$

where m is the mass of the beam per unit length. If the cross section of the beam is constant throughout its length, the final beam vibration equation takes the form

$$E\,I\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = 0 \tag{A.10}$$

Eq. (A.10) is solved by satisfying the specified boundary conditions. For example, for a fixed-fixed beam, the boundary conditions are given by

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\} \quad \text{at } x = 0 \text{ and } x = L \tag{A.11}$$

where L is the length of the beam.

A.2 EXACT ANALYTICAL SOLUTION (USING SEPARATION OF VARIABLES TECHNIQUE)

For free vibration, we assume harmonic motion and hence

$$w(x,t) = W(x)e^{i\,\omega\,t} \tag{A.12}$$

where $W(x)$ is purely a function of x and ω is the circular natural frequency of vibration. Substituting Eq. (A.12) into Eq. (A.10) we obtain

$$\frac{d^4 W}{dx^4} - \lambda W = 0 \tag{A.13}$$

where

$$\lambda = \frac{m\,\omega^2}{E\,I} \equiv \beta^4(\text{assumed}) \tag{A.14}$$

The general solution of Eq. (A.13) can be expressed as

$$W(x) = C_1 \sin\beta\,x + C_2 \cos\beta\,x + C_3 \sinh\beta\,x + C_4 \cosh\beta\,x \tag{A.15}$$

where C_1, C_2, C_3, and C_4 are constants to be determined from the boundary conditions. In view of Eq. (A.12), the boundary conditions for a fixed-fixed beam can be written as

$$\left.\begin{aligned} W(x=0) &= W(x=L) = 0 \\ \frac{dW}{dx}(x=0) &= \frac{dW}{dx}(x=L) = 0 \end{aligned}\right\} \tag{A.16}$$

If we substitute Eq. (A.15) into Eq. (A.16), we get four linear homogeneous equations in the unknowns C_1, C_2, C_3, and C_4. For a nontrivial solution, the determinant of the coefficient matrix must be zero. This leads to condition [A.1]

$$\cos\beta\,L \cosh\beta\,L = 1 \tag{A.17}$$

Eq. (A.17) is called the frequency equation, and there will be an infinite number of solutions to it. Let the nth root of Eq. (A.17) be $\beta_n L$. By using this solution in Eqs. (A.14) and (A.15), we obtain the natural frequencies as

$$\omega_n^2 = \frac{\beta_n^4 E\,I}{m} \tag{A.18}$$

where $\beta_1 L = 4.73$, $\beta_2 L = 7.85$, $\beta_3 L = 11.00$, and so on, and mode shapes as

$$W_n(x) = B_n\left\{\cosh\beta_n x - \cos\beta_n x - \left(\frac{\cos\beta_n L - \cosh\beta_n L}{\sin\beta_n L - \sinh\beta_n L}\right)(\sinh\beta_n x - \sin\beta_n x)\right\} \tag{A.19}$$

where B_n is a constant. Finally, the solution of w is given by

$$w_n(x) = W_n(x)e^{i\,\omega_n t},\, n = 1,2,3,\ldots \tag{A.20}$$

A.3 APPROXIMATE ANALYTICAL SOLUTION (RAYLEIGH METHOD)

To find an approximate value of the first (or fundamental) natural frequency and the corresponding mode shape, the Rayleigh energy method can be used. In this method, the maximum kinetic energy of the system during motion is equated to the maximum potential energy. For a uniform beam, the potential (or strain) energy, π, is given by [A.1]

$$\pi = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x \tag{A.21}$$

and the kinetic energy, T, by

$$T = \frac{1}{2}\int_0^L m(x)\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{A.22}$$

By assuming harmonic variation of $w(x, t)$ as

$$w(x,t) = W(x)\cos \omega t \tag{A.23}$$

the maximum values of π and T can be found as

$$(\pi)_{\max} = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \tag{A.24}$$

and

$$(T)_{\max} = \frac{\omega^2}{2}\int_0^L m(x)W^2(x)\,\mathrm{d}x \tag{A.25}$$

By equating $(\pi)_{\max}$ and $(T)_{\max}$, we obtain

$$\omega^2 = \frac{\displaystyle\int_0^L EI\left(\frac{\mathrm{d}^2 W}{\mathrm{d}x^2}\right)^2 \mathrm{d}x}{\displaystyle\int_0^L m(x)W^2 \mathrm{d}x} \tag{A.26}$$

To find the value of ω^2 using Eq. (A.26), we assume an approximate deflection or mode shape $W(x)$, known as admissible deflection, which satisfies the geometric boundary conditions but not necessarily the governing differential equation, Eq. (A.13), of the beam and substitute it in Eq. (A.26). For example, we can choose

$$W(x) = 1 - \cos\frac{2\pi x}{L} \tag{A.27}$$

to satisfy the boundary conditions stated in Eq. (A.16) and not the equation of motion, Eq. (A.13). By substituting Eq. (A.27) into Eq. (A.26), we can find the approximate value of the first natural frequency $(\widetilde{\omega}_1)$ as

$$(\widetilde{\omega}_1) = \frac{22.792}{L^2}\sqrt{\frac{EI}{m}} \tag{A.28}$$

which can be seen to be 1.87% larger than the exact value of

$$\omega_1 = \frac{22.3729}{L^2}\sqrt{\frac{EI}{m}} \tag{A.29}$$

A.4 APPROXIMATE ANALYTICAL SOLUTION (RAYLEIGH–RITZ METHOD)

If we want to find the approximate values of many natural frequencies, we need to substitute an assumed solution made up of a series of admissible functions that satisfy the forced (or geometric) boundary conditions of the beam in Eq. (A.16). This method is an extension of the Rayleigh method known as the Rayleigh–Ritz method. In the Rayleigh–Ritz method, if we want to find n natural frequencies, we use the deflection function

$$W(x) = C_1 f_1(x) + C_2 f_2(x) + \ldots + C_n f_n(x) \tag{A.30}$$

where C_1, C_2, …, C_n are constants and $f_1, f_2, \ldots, f_n$ are admissible functions. By substituting Eq. (A.30) in Eq. (A.26), we obtain ω^2 as a function of C_1, C_2, …, C_n. Since the actual frequency ω will be smaller than $\widetilde{\omega}$ [A.1], we want to choose C_1, C_2, …, C_n such that they make $\widetilde{\omega}$ a minimum. For this, we set the derivative of $\widetilde{\omega}^2$, given by Eq. (A.26) after substitution of Eq. (A.30), with respect to C_1, C_2, …, C_n, to zero:

$$\frac{\partial(\widetilde{\omega}^2)}{\partial C_1} = \frac{\partial(\widetilde{\omega}^2)}{\partial C_2} = \cdots = \frac{\partial(\widetilde{\omega}^2)}{\partial C_n} = 0 \tag{A.31}$$

A typical jth term in Eq. (A.31) gives the following equation:

$$\frac{\partial}{\partial C_j}\int_0^L EI\left(\frac{d^2W}{dx^2}\right)^2 dx - \omega_j^2\frac{\partial}{\partial C_j}\int_0^L m(x)W^2(x)\,dx = 0 \tag{A.32}$$

Eq. (A.31) represents a set of n homogeneous algebraic equations in terms of the n unknown constants C_1, C_2, …, C_n (as given by Eq. A.32 for $j = 1, 2, \ldots, n$). As such, for a nontrivial solution of the constants C_1, C_2, …, C_n, the determinant of their coefficient matrix must be set equal to zero. The process enables us to find the desired n approximate natural frequencies of the beam (or system). The process is illustrated by using a two-term solution for the vibration of the uniform fixed-fixed beam considered in Section A.3.

Let the two-term solution of the beam be given by

$$W(x) = C_1\left(1 - \cos\frac{2\pi x}{L}\right) + C_2\left(1 - \cos\frac{4\pi x}{L}\right) \tag{A.33}$$

Substitution of Eq. (A.33) into Eq. (A.26) gives

$$\widetilde{\omega}^2 = \frac{\dfrac{8\pi^4}{L^3}EI\left(C_1^2 + 16C_2^2\right)}{\dfrac{mL}{2}\left(2C_1^2 + 3C_2^2 + 4C_1C_2\right)} \tag{A.34}$$

The conditions for the minimum of $\widetilde{\omega}^2$, namely,

$$\frac{\partial(\widetilde{\omega}^2)}{\partial C_1} = \frac{\partial(\widetilde{\omega}^2)}{\partial C_2} = 0 \tag{A.35}$$

lead to the following algebraic eigenvalue problem:

$$\frac{16\pi^4 EI}{L^3}\begin{bmatrix}1 & 0\\0 & 10\end{bmatrix}\begin{Bmatrix}C_1\\C_2\end{Bmatrix} = \widetilde{\omega}^2 mL\begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{Bmatrix}C_1\\C_2\end{Bmatrix} \tag{A.36}$$

The solution of Eq. (A.36) gives two natural frequencies as

$$\widetilde{\omega}_1 = \frac{22.35}{L^2}\sqrt{\frac{EI}{m}} \quad \text{with} \quad \begin{Bmatrix}C_1\\C_2\end{Bmatrix} = \begin{Bmatrix}1.0\\0.5750\end{Bmatrix} \tag{A.37}$$

and

$$\widetilde{\omega}_2 = \frac{124.0}{L^2}\sqrt{\frac{EI}{m}} \quad \text{with} \quad \begin{Bmatrix}C_1\\C_2\end{Bmatrix} = \begin{Bmatrix}1.0\\-1.4488\end{Bmatrix} \tag{A.38}$$

The first and second natural frequencies shown in Eqs. (A.37) and (A.38) can be compared with the correct values (given by the exact solution) for which the constants in the numerators on the right-hand side of the expressions of the natural frequencies are 22.3729 and 61.6727, respectively.

A.5 APPROXIMATE ANALYTICAL SOLUTION (GALERKIN METHOD)

To find an approximate solution of the free vibration equation of the beam, Eq. (A.13), with the boundary conditions stated in Eq. (A.16), we can use the Galerlin method. In this method, we assume the solution to be of the form

$$W(x) = C_1 f_1(x) + C_2 f_2(x) + \ldots + C_n f_n(x) \tag{A.39}$$

where C_1, C_2, ..., C_n are constants and $f_1, f_2, ..., f_n$ are functions that satisfy all the specified boundary conditions (not just the geometric boundary conditions, but free boundary conditions also, if applicable). Since the solution assumed, Eq. (A.39), is not the exact solution, it will not satisfy the governing differential equation, Eq. (A.13). Thus when the assumed solution, Eq. (A.39), is substituted into the left-hand side expression of Eq. (A.13), we obtain a quantity different from zero (known as the residual, R). The values of the unknown constants C_1, C_2, ..., C_n are obtained by setting the integral of the residual multiplied by each of the functions $f_i(x)$, also termed the weighting functions, over the length of the beam equal to zero; that is,

$$\int_0^L f_i(x)R(x)\,\mathrm{d}x = 0, \quad i = 1, 2, \ldots, n \tag{A.40}$$

Because the method uses the integrals of residual multiplied by the weighting functions, the Galerkin method is also known as a weighted residual method. Eq. (A.40) represents a system of linear homogeneous equations (since the problem is an eigenvalue problem) in the unknowns C_1, C_2, ..., C_n. These equations can be solved to find the natural frequencies and mode shapes of the problem.

For illustration of the procedure, consider a two-term solution as

$$W(x) = C_1 f_1(x) + C_2 f_2(x) \tag{A.41}$$

where

$$f_1(x) = \cos\frac{2\pi x}{L} - 1 \tag{A.42}$$

$$f_2(x) = \cos\frac{4\pi x}{L} - 1 \tag{A.43}$$

Substitution of Eq. (A.41) into Eq. (A.13) gives the residual R as

$$R = C_1\left\{\left(\frac{2\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{2\pi x}{L} + C_1\beta^4 + C_2\left\{\left(\frac{4\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{4\pi x}{L} + C_2\beta^4 \tag{A.44}$$

Thus, the application of Eq. (A.40) leads to

$$\int_{x=0}^{L}\left(\cos\frac{2\pi x}{L} - 1\right)\left[C_1\left\{\left(\frac{2\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{2\pi x}{L} + C_1\beta^4 + C_2\left\{\left(\frac{4\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{4\pi x}{L} + C_2\beta^4\right]\mathrm{d}x = 0 \tag{A.45}$$

$$\int_{x=0}^{L}\left(\cos\frac{4\pi x}{L} - 1\right)\left[C_1\left\{\left(\frac{2\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{2\pi x}{L} + C_1\beta^4 + C_2\left\{\left(\frac{4\pi}{L}\right)^4 - \beta^4\right\}\cos\frac{4\pi x}{L} + C_2\beta^4\right]\mathrm{d}x = 0 \tag{A.46}$$

Upon carrying out the indicated integrations, Eqs. (A.45) and (A.46) yield

$$C_1\left[\frac{1}{2}\left\{\left(\frac{2\pi}{L}\right)^4 - \beta^4\right\} - \beta^4\right] - C_2\beta^4 = 0 \tag{A.47}$$

$$-C_1\beta^4 + C_2\left[\frac{1}{2}\left\{\left(\frac{4\pi}{L}\right)^4 - \beta^4\right\} - \beta^4\right] = 0 \tag{A.48}$$

For a nontrivial solution of the homogeneous equations in C_1 and C_2, Eqs. (A.47) and (A.48), the determinant of the coefficient matrix of C_1 and C_2 must be zero. This gives

$$\begin{vmatrix} \frac{1}{2}\left\{\left(\frac{2\pi}{L}\right)^4 - \beta^4\right\} - \beta^4 & -\beta^4 \\ -\beta^4 & \frac{1}{2}\left\{\left(\frac{4\pi}{L}\right)^4 - \beta^4\right\} - \beta^4 \end{vmatrix} = 0 \tag{A.49}$$

which can be simplified as

$$(\beta L)^8 - 15900(\beta L)^4 + 7771000 = 0 \tag{A.50}$$

The solution of Eq. (A.50) gives $\beta L = 4.741$ and 11.140. Thus, the first two natural frequencies of the beam are given by

$$\omega_1 = \frac{22.48}{L^2}\sqrt{\frac{E\,I}{m}} \quad \text{with} \quad \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 21.0 \\ 1 \end{Bmatrix} \tag{A.51}$$

$$\omega_2 = \frac{124.1}{L^2}\sqrt{\frac{E\,I}{m}} \quad \text{and with} \quad \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} -0.69 \\ 1.00 \end{Bmatrix} \tag{A.52}$$

The first and second natural frequencies shown in Eqs. (A.51) and (A.52) can be compared with the correct values (given by the exact solution) for which the constants in the numerators on the right-hand side of the expressions of the natural frequencies are 22.3729 and 61.6727, respectively.

A.6 FINITE DIFFERENCE METHOD OF NUMERICAL SOLUTION

The main idea in the finite difference method is to use approximations to derivatives. We can derive finite difference approximations to various order derivatives (similar to central difference method). Let $f = f(x)$ be any given function of x. The Taylor's series expansion of f around any point x gives

$$f(x+\Delta x) \approx f(x) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x \frac{(\Delta x)^2}{2!} + \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_x \frac{(\Delta x)^3}{3!} + \left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_x \frac{(\Delta x)^4}{4!} \tag{A.53}$$

$$f(x-\Delta x) \approx f(x) - \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x \frac{(\Delta x)^2}{2!} - \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_x \frac{(\Delta x)^3}{3!} + \left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_x \frac{(\Delta x)^4}{4!} \tag{A.54}$$

By taking the first two terms only on the right-hand side and subtracting Eq. (A.54) from Eq. (A.53), we obtain [A.3]

$$f(x+\Delta x) - f(x-\Delta x) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x$$

or

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{A.55}$$

Considering the first three terms only on the right-hand side and adding Eqs. (A.54) and (A.53) gives

$$f(x+\Delta x) + f(x-\Delta x) = f(x) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x \frac{(\Delta x)^2}{2!} + f(x) - \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_x \Delta x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x \frac{(\Delta x)^2}{2!}$$

or

$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \tag{A.56}$$

Using $\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x$ in place of $f(x)$, Eq. (A.56) gives [A.3]

$$\left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_x = \frac{\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x+\Delta x} - 2\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x-\Delta x}}{(\Delta x)^2} \tag{A.57}$$

By substituting Eq. (A.56) on the right-hand side of Eq. (A.57), we obtain [A.3]

$$\left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_x = \left[\left\{\frac{f(x+2\Delta x) - 2f(x+\Delta x) + f(x)}{(\Delta x)^2}\right\} - 2\left\{\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}\right\} + \left\{\frac{f(x) - 2f(x-\Delta x) + f(x-2\Delta x)}{(\Delta x)^2}\right\}\right] \Big/ (\Delta x)^2 \tag{A.58}$$

Eq. (A.58) can be simplified to obtain the central difference formula for the fourth derivative as

$$\left.\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}\right|_x = \frac{1}{(\Delta x)^4}\{f(x+2\Delta x) - 4f(x+\Delta x) + 6f(x) - 4f(x-\Delta x) + f(x-2\Delta x)\} \tag{A.59}$$

To find the approximate solution of

$$\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} - \beta^4 W = 0 \tag{A.60}$$

we first divide the length of the beam (or solution region) into a suitable number of segments and establish the grid or node points as indicated in Fig. A.3 and then approximate the governing differential equation around points 1 and 2 in Fig. A.3 by using a central finite difference formula for the fourth derivative. For this, we need two fictitious or hypothetical node points, 1 and 4, as shown in Fig. A.3.

By approximating Eq. (A.60) at points 1 and 2, we obtain

$$(W_{-1} - 4W_0 + 6W_1 - 4W_2 + W_3) - \beta_1^4 W_1 = 0 \tag{A.61}$$

$$(W_0 - 4W_1 + 6W_2 - 4W_3 + W_4) - \beta_2^4 W_2 = 0 \tag{A.62}$$

where

$$\beta_i^4 = \left(\frac{L}{3}\right)^2 \beta^4, \quad i = 1, 2 \tag{A.63}$$

The boundary conditions of the beam can be expressed as

$$\begin{aligned} &W_0 = W_3 = 0 \\ &\left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_0 = \left.\frac{\mathrm{d}W}{\mathrm{d}x}\right|_3 = 0 \quad \text{or} \quad W_{-1} = W_1, W_3 = W_4 \end{aligned} \tag{A.64}$$

By substituting the conditions of Eq. (A.64), Eqs. (A.61) and (A.62) reduce to

$$7W_1 - 4W_2 = \beta_1^4 W_1 \tag{A.65}$$

$$-4W_1 + 7W_2 = \beta_2^4 W_2 \tag{A.66}$$

which can be expressed in matrix form as

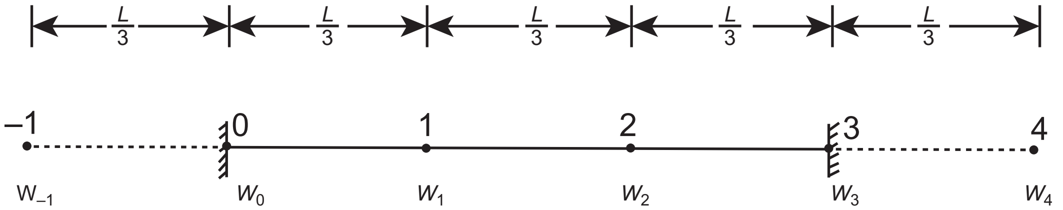

FIGURE A.3 Introduction of hypothetical nodes in finite differences method of solution.

$$\begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \beta_1^4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} \tag{A.67}$$

By solving the standard eigenvalue problem shown in Eq. (A.67), we can obtain the approximate first two natural frequencies and mode shapes of the beam as

$$\omega_1 = \frac{15.59}{L^2}\sqrt{\frac{EI}{m}} \quad , \quad \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{A.68}$$

$$\omega_2 = \frac{29.85}{L^2}\sqrt{\frac{EI}{m}} \quad , \quad \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{A.69}$$

The first and second natural frequencies given by Eqs. (A.68) and (A.69) can be compared with the correct values (given by the exact solution) for which the constants in the numerators on the right-hand side of the expressions of the natural frequencies are 22.3729 and 61.6727, respectively.

A.7 FINITE ELEMENT METHOD OF NUMERICAL SOLUTION (DISPLACEMENT APPROACH)

In the finite element method, we divide the given beam (or structure) into several elements and assume a suitable solution within each of the elements. From this we formulate the necessary equations from which the approximate solution can be obtained easily. Fig. A.4 shows the beam divided into two elements of equal length. Within each element, the solution for the transverse displacement is assumed to be a cubic equation as

$$\begin{aligned} w(x) &= W_1^{(e)} \cdot \frac{1}{l^3}\left(2x^3 - 3lx^2 + l^3\right) + W_3^{(e)} \cdot \frac{1}{l^3}\left(3lx^2 - 2x^3\right) \\ &\quad + W_2^{(e)} \cdot \frac{1}{l^2}\left(x^3 - 2lx^2 + l^2x\right) + W_4^{(e)} \cdot \frac{1}{l^2}\left(x^3 - lx^2\right) \end{aligned} \tag{A.70}$$

where $W_1^{(e)}$ to $W_4^{(e)}$ denote the displacements at the ends of the element e (to be determined), and l indicates the length of the element.

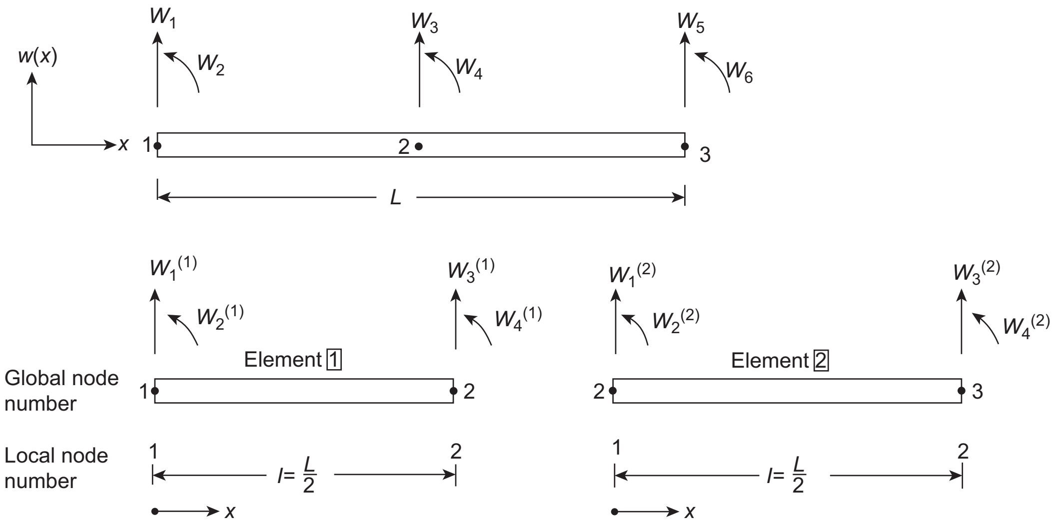

FIGURE A.4 Finite element idealization of the beam.

The strain and kinetic energies of the element can be expressed as

$$\pi^{(e)} = \frac{1}{2}\int_0^l E\,I\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x = \frac{1}{2}\vec{W}^{(e)^T}\left[K^{(e)}\right]\vec{W}^{(e)} \tag{A.71}$$

and

$$T^{(e)} = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x = \frac{1}{2}\dot{\vec{W}}^{(e)^T}\left[M^{(e)}\right]\dot{\vec{W}}^{(e)} \tag{A.72}$$

where ρ is the mass density, A is the cross-sectional area of the element, and a dot over $\vec{W}^{(e)}$ represents the time derivative of the nodal displacement vector of element e, $\vec{W}^{(e)}$. We can obtain, after substituting Eq. (A.70) into Eqs. (A.71) and (A.72), the stiffness matrix $[K^{(e)}]$ and the mass matrix $[M^{(e)}]$ as

$$\left[K^{(e)}\right] = \frac{2E\,I}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \tag{A.73}$$

$$\left[M^{(e)}\right] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{A.74}$$

Fig. A.4 shows that

$$\vec{W}^{(1)} = \text{vector of unknown nodal displacements of element } 1 = \begin{Bmatrix} W_1^{(1)} \\ W_2^{(1)} \\ W_3^{(1)} \\ W_4^{(1)} \end{Bmatrix} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$$

and

$$\vec{W}^{(2)} = \text{vector of unknown nodal displacements of element } 2 = \begin{Bmatrix} W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{Bmatrix} = \begin{Bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix}$$

By assembling the stiffness and mass matrices of the two elements (details are given in Chapter 6), we obtain the assembled stiffness and mass matrices as

$$[\underline{K}] = \frac{2EI}{l^3}\begin{array}{cl} \begin{array}{cccccc} W_1 = W_1^{(1)} & W_2 = W_2^{(1)} & W_3 = W_3^{(1)} = W_1^{(2)} & W_4 = W_4^{(1)} = W_2^{(2)} & W_5 = W_3^{(2)} & W_6 = W_4^{(2)} \end{array} & \\ \begin{bmatrix} 6 & 3l & -6 & 3l & 0 & 0 \\ 3l & 2l^2 & -3l & l^2 & 0 & 0 \\ -6 & -3l & 6+6 & -3l+3l & -6 & 3l \\ 3l & l^2 & -3l+3l & 2l^2+2l^2 & -3l & l^2 \\ 0 & 0 & -6 & -3l & 6 & -3l \\ 0 & 0 & 3l & l^2 & -3l & 2l^2 \end{bmatrix} & \begin{array}{l} W_1 = W_1^{(1)} \\ W_2 = W_2^{(1)} \\ W_3 = W_3^{(1)} = W_1^{(2)} \\ W_4 = W_4^{(1)} = W_2^{(2)} \\ W_5 = W_3^{(2)} \\ W_6 = W_4^{(2)} \end{array} \end{array} \tag{A.75}$$

$$[M]=\frac{\rho A l}{420}\begin{array}{c}
\begin{array}{cccccc} W_1=W_1^{(1)} & W_2=W_2^{(1)} & W_3=W_3^{(1)}=W_1^{(2)} & W_4=W_4^{(1)}=W_2^{(2)} & W_5=W_3^{(2)} & W_6=W_4^{(2)} \end{array}\\
\left[\begin{array}{cccccc}
156 & 22l & 54 & -13l & 0 & 0\\
22l & 4l^2 & 13l & -3l^2 & 0 & 0\\
54 & 13l & 156+156 & -22l+22l & 54 & -13l\\
-13l & -3l^2 & -22l+22l & 4l^2+4l^2 & 13l & -3l^2\\
0 & 0 & 54 & 13l & 156 & -22l\\
0 & 0 & -13l & -3l^2 & -22l & 4l^2
\end{array}\right]
\end{array}
\begin{array}{l}
W_1=W_1^{(1)}\\
W_2=W_2^{(1)}\\
W_3=W_3^{(1)}=W_1^{(2)}\\
W_4=W_4^{(1)}=W_2^{(2)}\\
W_5=W_3^{(2)}\\
W_6=W_4^{(2)}
\end{array} \tag{A.76}$$

After deleting the rows and columns corresponding to the degrees of freedom W_1, W_2, W_5, and W_6 (since $W_1 = W_2 = W_5 = W_6 = 0$ are the boundary conditions), in Eqs. (A.75) and (A.76), we obtain the stiffness and mass matrices of the beam as

$$[K] = \frac{2E\,I}{l^3}\begin{bmatrix} 12 & 0 \\ 0 & 4l^2 \end{bmatrix} = \frac{16E\,I}{L^3}\begin{bmatrix} 12 & 0 \\ 0 & L^2 \end{bmatrix} \tag{A.77}$$

$$[M] = \frac{\rho Al}{420}\begin{bmatrix} 312 & 0 \\ 0 & 8l^2 \end{bmatrix} = \frac{\rho AL}{840}\begin{bmatrix} 312 & 0 \\ 0 & 2L^2 \end{bmatrix} \tag{A.78}$$

Once the stiffness and mass matrices of the complete beam are available, we can formulate the eigenvalue problem as

$$[K]\vec{W} = \lambda[M]\vec{W} \tag{A.79}$$

where $\vec{W} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$ is the eigenvector or mode shape and λ is the eigenvalue, with $\lambda = \omega^2$. The solution of Eq. (A.79) gives two natural frequencies and the corresponding mode shapes of the beam as

$$\omega_1 = \frac{22.7}{L^2}\sqrt{\frac{E\,I}{m}}\quad , \quad \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{A.80}$$

and

$$\omega_2 = \frac{82.0}{L^2}\sqrt{\frac{E\,I}{m}}\quad , \quad \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{A.81}$$

It can be seen that the two natural frequencies are 1.4680% and 32.96% larger than the correct values (of the constants in the numerators on the right-hand side of the expressions of the natural frequencies) 22.3729 and 61.6727 corresponding to the exact solution.

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Appendix B

Green–Gauss Theorem (Integration by Parts in Two and Three Dimensions)

In the derivation of finite element equations for two-dimensional problems, we need to evaluate integrals of the type:

$$\iint_S \psi \frac{\partial \phi}{\partial x} \mathrm{d}x\, \mathrm{d}y \tag{B.1}$$

where S is the area or region of integration and C is its bounding curve. We can integrate Eq. (B.1) by parts, first with respect to x, using the basic relation:

$$\int_{x_l}^{x_r} u\, \mathrm{d}v = -\int_{x_l}^{x_r} v\, \mathrm{d}u + uv\Big|_{x_l}^{x_r} \tag{B.2}$$

to obtain

$$\iint_S \psi \frac{\partial \phi}{\partial x} \mathrm{d}x\, \mathrm{d}y = -\iint_S \frac{\partial \psi}{\partial x} \phi\, \mathrm{d}x\, \mathrm{d}y + \int_{y=y_1}^{y_2} (\psi\phi)\big|_{x_l}^{x_r} \cdot \mathrm{d}y \tag{B.3}$$

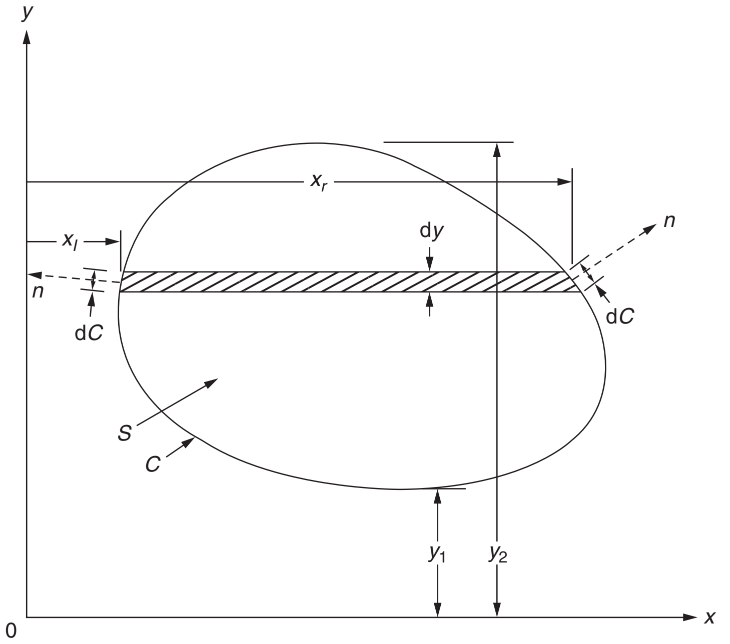

FIGURE B.1 Limits of integration for x and y.

The Finite Element Method in Engineering. http://dx.doi.org/10.1016/B978-0-12-811768-2.15002-3

where (x_l, x_r) and (y_1, y_2) denote the limits of integration for x and y, as shown in Fig. B.1. However, dy can be expressed as:

$$\mathrm{d}y = \pm \mathrm{d}C \cdot l_x \tag{B.4}$$

where dC is an element of the boundary curve, l_x is the cosine of the angle between the normal n and the x direction, and the plus−minus sign is applicable to the right and left hand−side boundary curves (Fig. B.1). Thus, the last term of Eq. (B.3) can be expressed in integral form as:

$$\int_{y_1}^{y_2} (\psi\phi)\big|_{x_l}^{x_r} \cdot \mathrm{d}y = \oint_C \psi\phi \, \mathrm{d}C \, l_x \tag{B.5}$$

Thus, the integral of Eq. (B.1) can be evaluated as:

$$\iint_S \psi \frac{\partial\phi}{\partial x} \mathrm{d}x \, \mathrm{d}y = -\iint_S \frac{\partial\psi}{\partial x} \phi \, \mathrm{d}x \, \mathrm{d}y + \oint_C \psi\phi l_x \, \mathrm{d}C \tag{B.6}$$

Similarly, if integral (B.1) contains the term $(\partial\phi/\partial y)$ instead of $(\partial\phi/\partial x)$, it can be evaluated as:

$$\iint_S \psi \frac{\partial\phi}{\partial y} \mathrm{d}x \, \mathrm{d}y = -\iint_S \frac{\partial\psi}{\partial y} \phi \, \mathrm{d}x \, \mathrm{d}y + \oint_C \psi\phi l_y \, \mathrm{d}C \tag{B.7}$$

where l_y is the cosine of the angle between the normal n and the y direction.

Eqs. (B.6) and (B.7) can be generalized to the case of three dimensions as:

$$\iiint_V \psi \frac{\partial\phi}{\partial x} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = -\iiint_V \frac{\partial\psi}{\partial x} \phi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z + \oint_S \psi\phi l_x \, \mathrm{d}S \tag{B.8}$$

where V is the volume or domain of integration and S is the surface bounding the domain V. Expressions similar to Eq. (B.8) can be written if the quantity $(\partial\phi/\partial y)$ or $(\partial\phi/\partial z)$ appears instead of $(\partial\phi/\partial x)$ in the original integral.

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ABOUT THE AUTHOR

Singiresu S. Rao is a professor in the Department of Mechanical and Aerospace Engineering at the University of Miami, Coral Gables, Florida. He was chairman of the department during 1998–2011. Prior to that, he worked as a professor in the School of Mechanical Engineering at Purdue University, West Lafayette, Indiana; professor of Mechanical Engineering at San Diego State University, San Diego, California; and professor of Mechanical Engineering at the Indian Institute of Technology, Kanpur. He was a visiting research scientist for two years at NASA Langley Research Center.

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