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Advanced Vibration

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Text book:

William J. Palm, Mechanical Vibration, John Wiley & Sons Inc., New York, 2007

References:

1. W. T. Thomson, Theory of vibration with applications (4th ed.), prentice Hall, Englewood Cliffs, N.J.,1993.
2. A. Dimarogonas, Vibration for Engineers, (2th ed.), Prentice Hall, Upper Saddle River, N.J.,1996.
3. L. Meirovitch, Fundamental of Vibration, Mc Graw-Hill,New York, 2001.
4. D. J. Inman, Engineering vibration, (2th ed). Prentice Hall, Upper Saddle River, N.J.,2001.
5. R. F. Steidel, An Introduction to Mechanical Vibrations, 3th ed., Jr., Wiley, New York, 1989.
6. Singiresu S. Rao, Mechanical Vibrations S.I. Version, 4E, Prentice Hall, Person Education South Asia Pte Ltd, Singapore, 2005.

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Journals

1. Journal of Sound and Vibration, Academic Press, New York.
2. Journal of Vibration and Acoustics, American Society of Mechanical Engineering(ASME), New York.
3. Journal of Mechanical Systems and Signals, Academic Press, New York.
4. Journal of Multi-body Dynamics, Proceedings Part Institution of Mechanical Engineers(IMechE), PEP Limited, London, UK.

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Topics

1- Introduction to Mechanical Vibration

2- Self-excited Vibration

3- Modal Analysis

4- Continuous Systems

5- Approximate and Numerical Methods



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Grading Policy

Homework	10% of the final grade
Projects	30% of the final grade
Final Exam	60% of the final grade

Note: Homework will be assigned every other session.

No late homework will be accepted.

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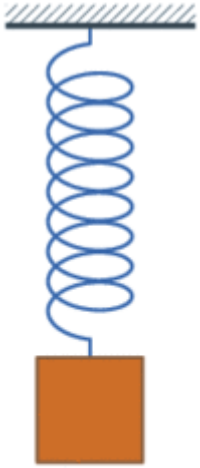
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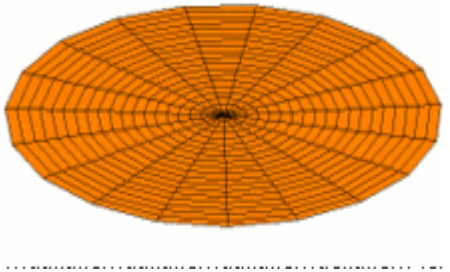
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Chapter1: Introduction

- **Vibration** is a mechanical phenomenon whereby **oscillation** occurs about an equilibrium point. (Showtime)
- The oscillations may be periodic such as the motion of a pendulum or random such as the movement of a tire on a gravel road.



1.Simple harmonic motion of the mass-spring system



2.Vibration of a circular drum

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1.1. Classification of Vibration

1-Linear or Nonlinear

2-Free or Force

3-Damped or Non-damped

4-Discrete or Continuous

5-Deterministic or Random



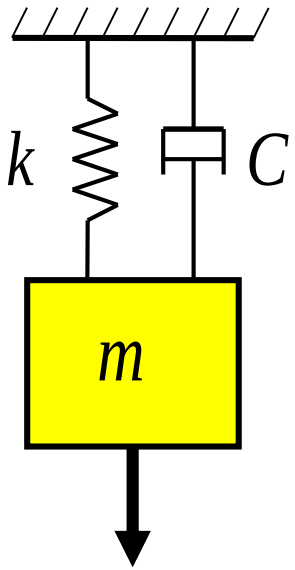
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1.2. Review

- Free Vibration
  - Natural Frequency
  - Mode shape
- Forced Vibration
  - Frequency Response
  - Time response

Damped forced vibration system:



$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$x(t) = x_h(t) + x_p(t)$$

$$\text{If} \quad F(t) = 0 \Rightarrow \begin{cases} m\ddot{x} + c\dot{x} + kx = 0 \\ x(t) = x_h(t) \end{cases}$$

$$\text{If } x(t) = e^{st} \Rightarrow (ms^2 + cs + k)e^{st} = 0$$

$$s_1, s_2 = -\frac{c}{2m} \pm \sqrt{(\frac{c}{2m})^2 - (\frac{k}{m})^2}$$

$$x_h(t) = Ae^{s_1 t} + Be^{s_2 t}$$

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damping ratio: $\zeta = \frac{c}{c_{cr}}$, $\omega_n = \sqrt{\frac{k}{m}}$

If $\zeta < 1$ undamped vibration(oscillatory motion)

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ $x(t) = e^{-\zeta\omega_n t}(c_1 \sin\omega_d t + c_2 \cos\omega_d t)$

If $F(t) = F_0 \sin\omega t$ $x_p = X \sin(\omega t - \phi)$

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad r = \frac{\omega}{\omega_n}$$

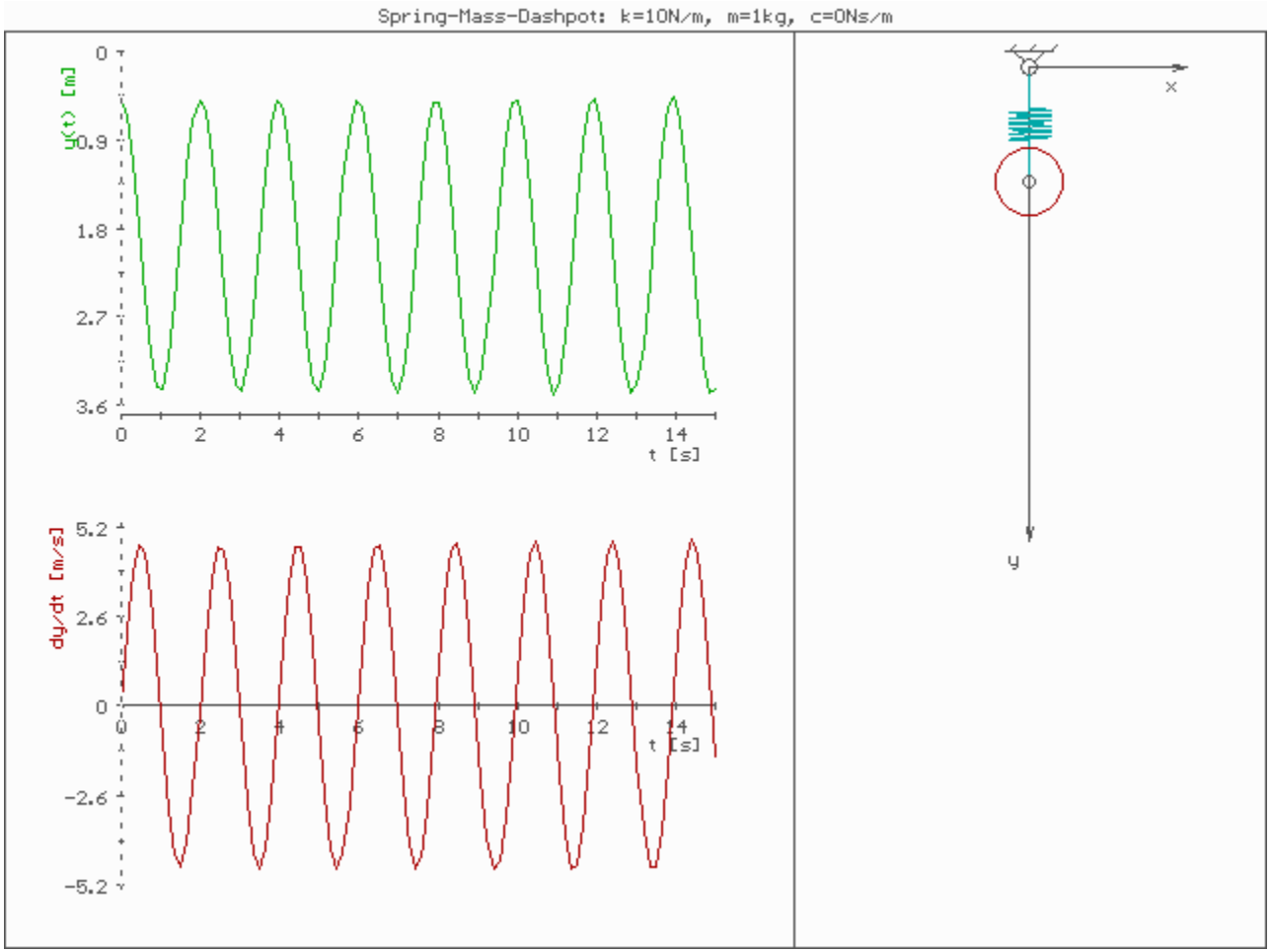
$$\phi = \tan^{-1}\left(2\zeta r / (1-r^2)\right) \qquad \tan\phi = c/D = \frac{2\zeta\frac{\omega}{\omega_n}}{(1-\frac{\omega^2}{\omega_n^2})}$$

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EXAMPLE

1. Spring-mass undamped:

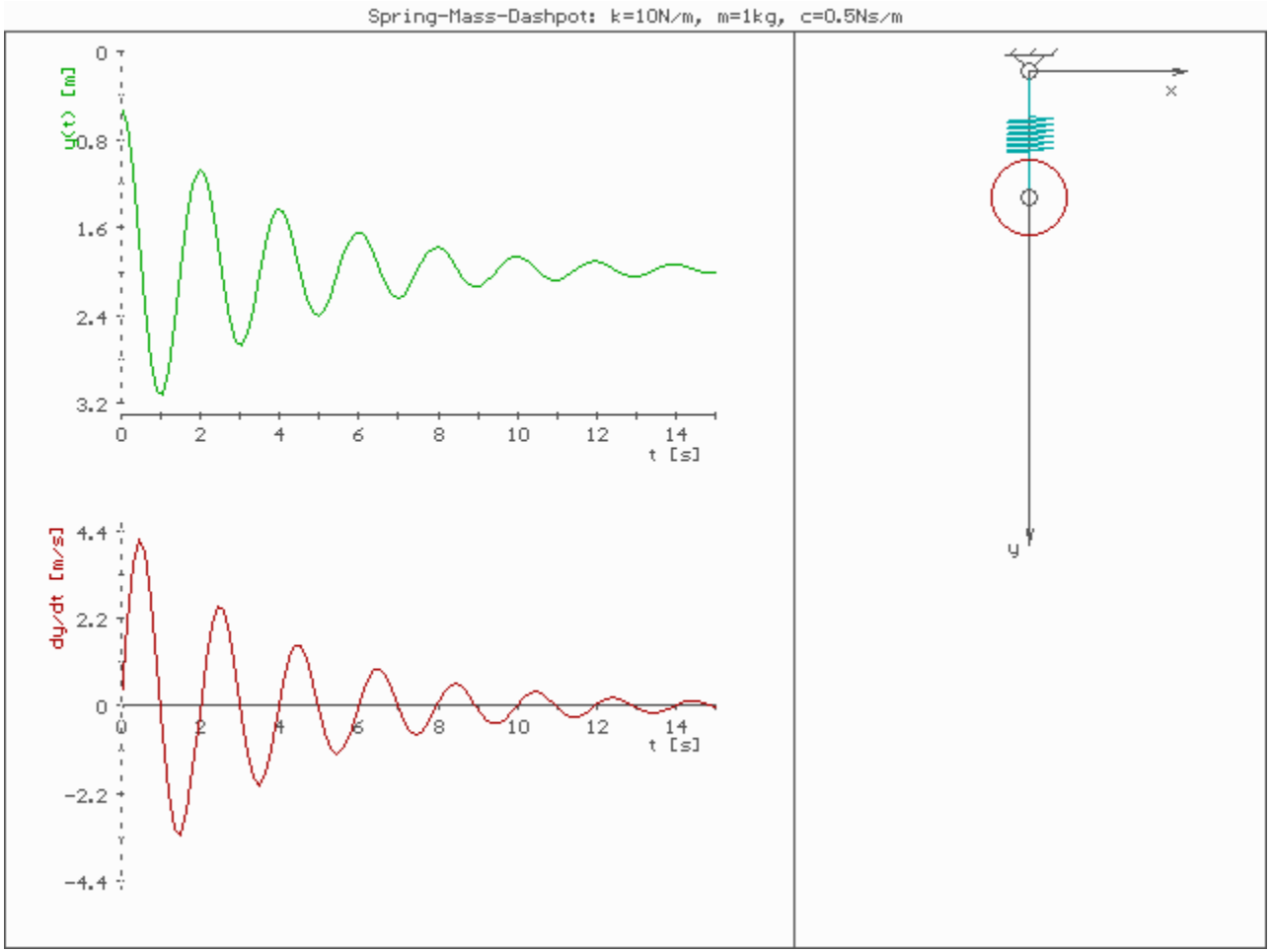


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EXAMPLE

2. Spring-mass underdamped:

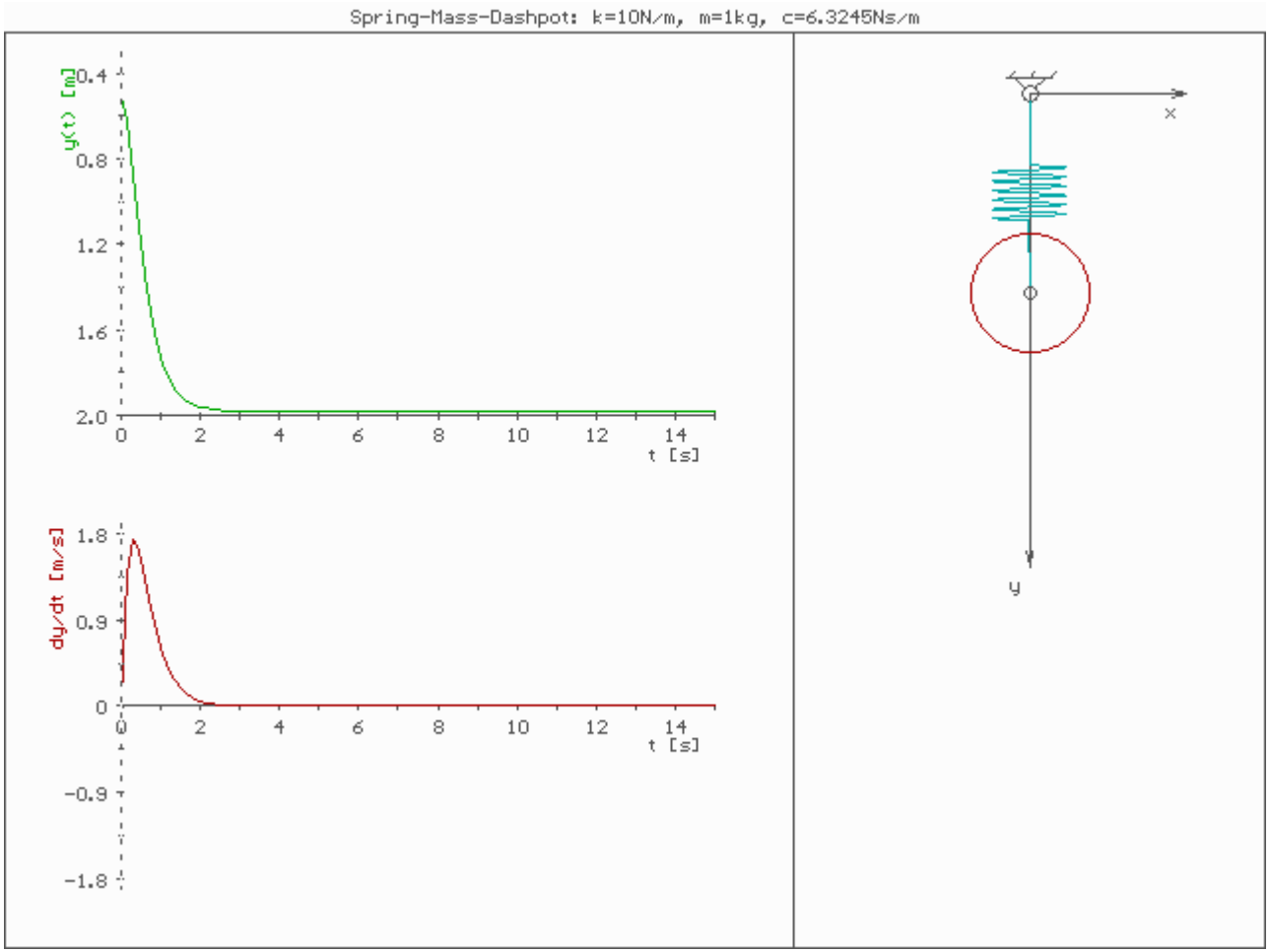


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EXAMPLE

3. Spring-mass critically-damped:

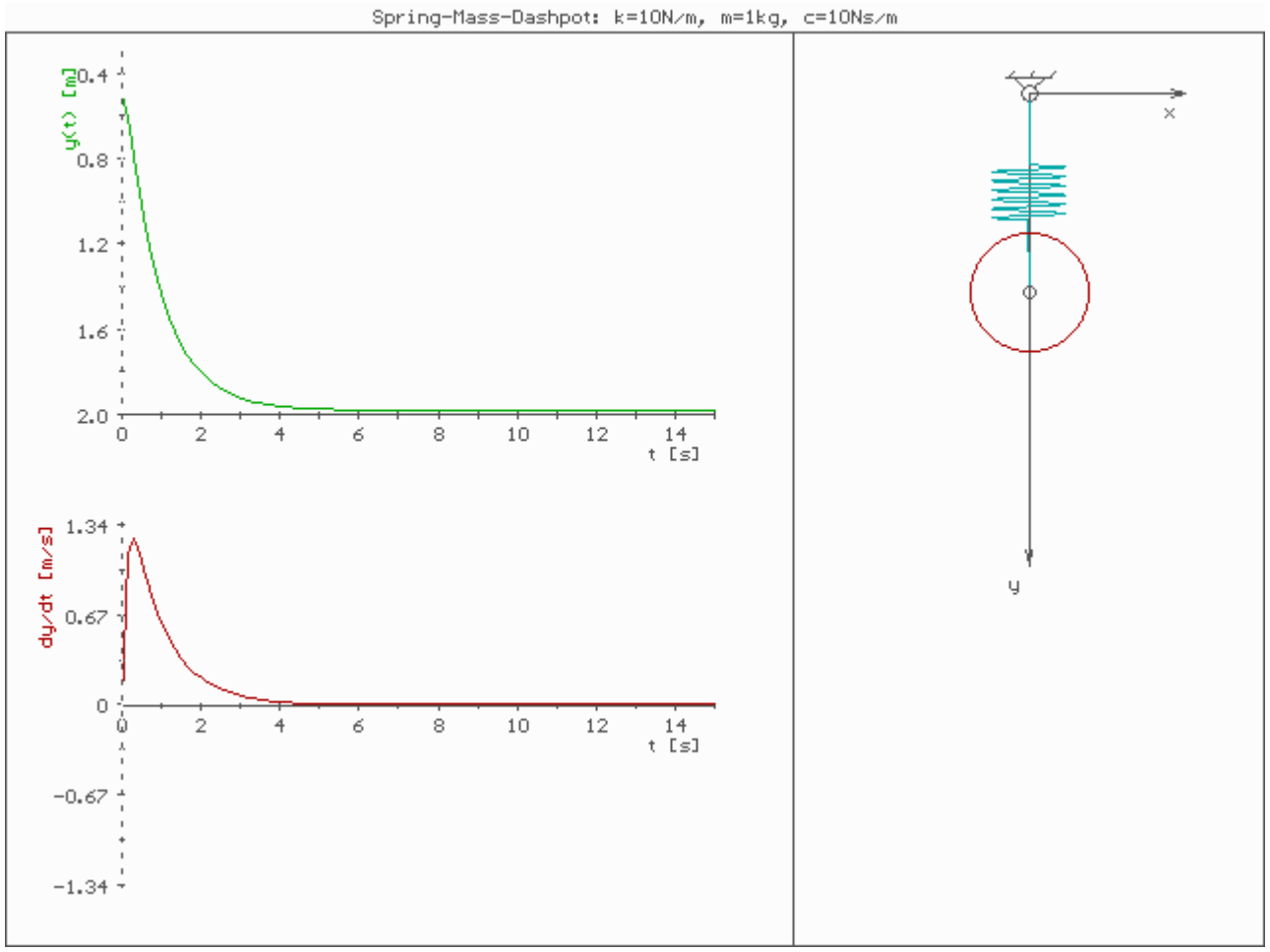


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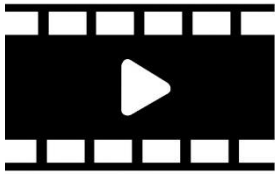
EXAMPLE

4. Spring-mass overdamped:



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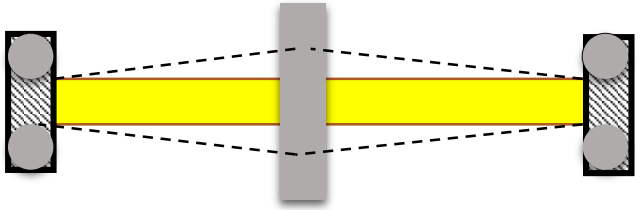
- Vibrations problems in Millenium Bridge opening day Play
- Two degree of freedom spring-mass system- Vibration absorber effect Play
- Sand Vibration Patterns Play
- Amazing Water & Sound Experiment Play
- Resonance Play Play

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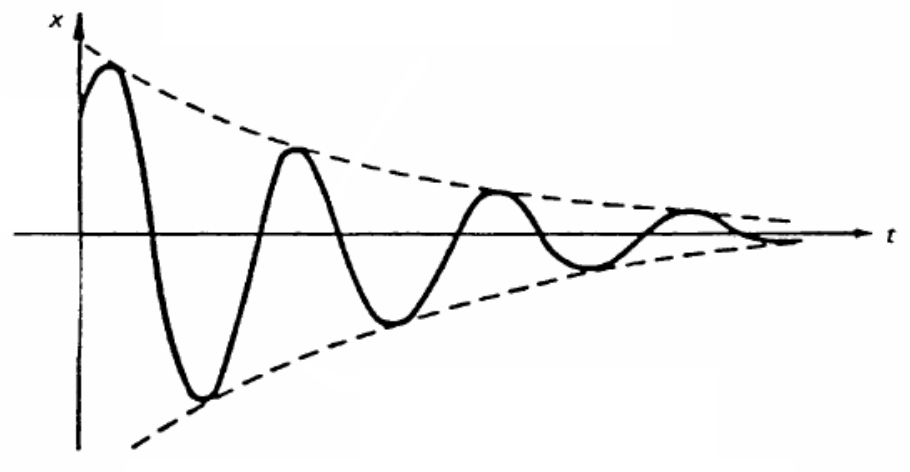
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Chapter 2: Self Excited Vibration

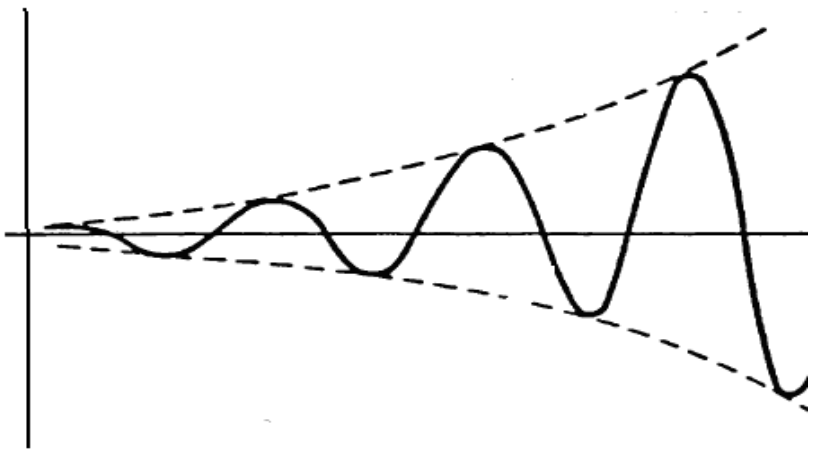
Self Excited Vibration is due to excitation that is a function of the motion itself such as displacement or velocity.



Whirling



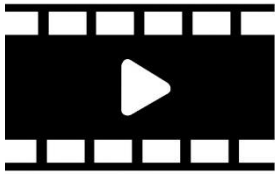
Stable system



unstable system

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Turbulence encountered off the coast of Japan on a United Boeing 777-200. ✈ Boeing 777

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2.1. Dynamic Instability

V: Potential Eneregy

① stable system has min. potential energy

$$\frac{dV}{dx}=0\ ,\ \frac{d^2V}{dx^2}>0$$

② unstable system has max. potential energy

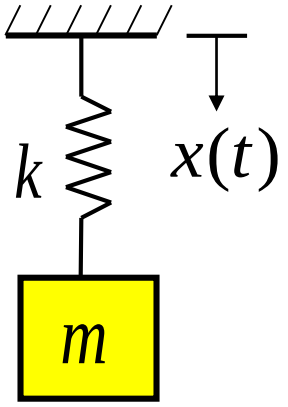
$$\frac{dV}{dx}=0\ ,\frac{d^2V}{dx^2}<0$$

③ Threshold of Instability

$$\boxed{\frac{d^2V}{dx^2}=0}$$

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$$m\ddot{x} + kx = 0$$

$$V(x) = \frac{1}{2}kx^2$$

$$\frac{dV}{dx} = kx$$

$$\frac{d^2V}{dx^2} = k$$

If $k>0$	stable region
If $k<0$	unstable region

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EXAMPLE

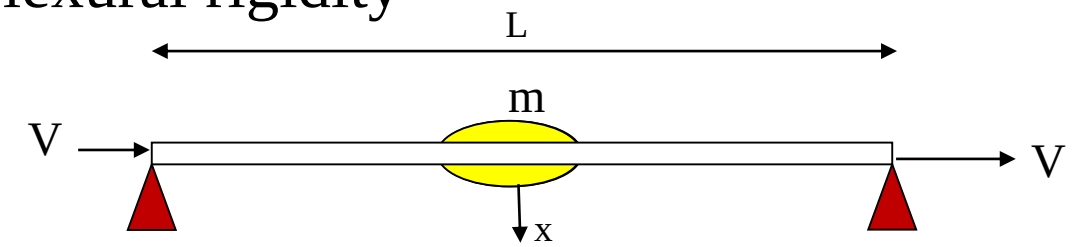
Instability of fluid carrying pipes at high flow velocity

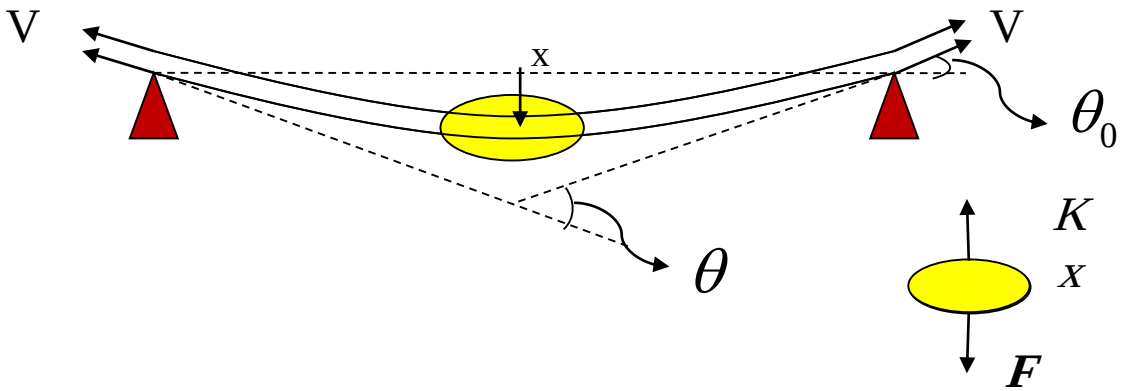
F: force due to change in direction of the flow

ρ: fluid density

A: The cross-sectional area of the pipe

EI: flexural rigidity





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EXAMPLE

$$\theta = 2\theta_0 = \frac{6x}{l}$$

$$k = \frac{48EI}{l^3} \qquad F = 2A\rho V^2 \sin\theta_0$$

$$\text{for small } \theta \rightarrow \quad F = A\rho V^2\theta = \frac{6A\rho V^2}{l}x$$

E. of M. :

$$m\ddot{x} + kx = F \qquad m\ddot{x} + \left(\frac{48EI}{l^3} - \frac{6A\rho V^2}{l}\right)x = 0$$

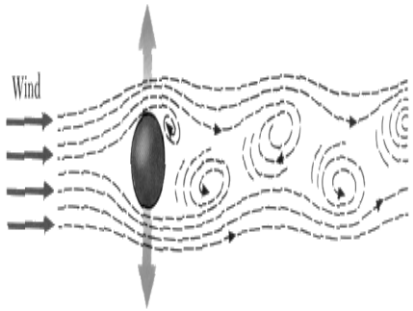
$$\text{for instability } \; k < 0 \qquad \therefore \quad V > \left(\frac{8EI}{A\rho l^2}\right)^{1/2}$$

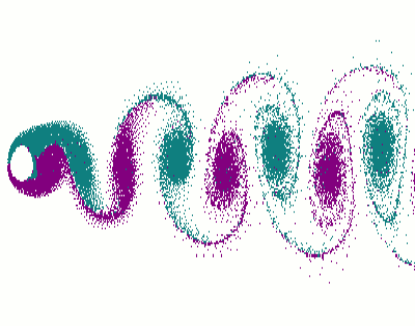
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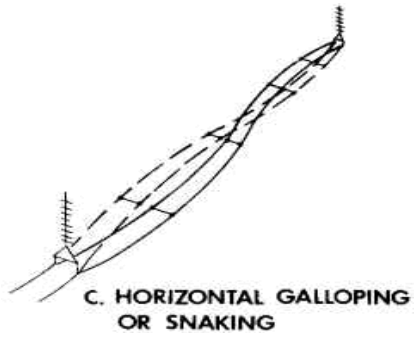
2.2. Fluid-flow-induced Vibration

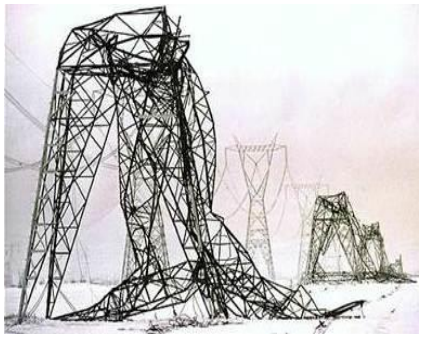
2.2.1. Von Karman Vortex Trails





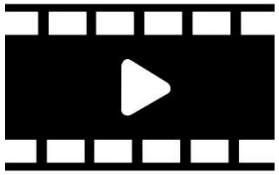
2.2.2. Galloping of Transmission lines





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1.Vortex induced vibration Play Play

2. Stop sign Play

3. Traffic lights Play

4. Galloping (I) Play

5. Galloping (II) Play

6. Vortex (Before) Play

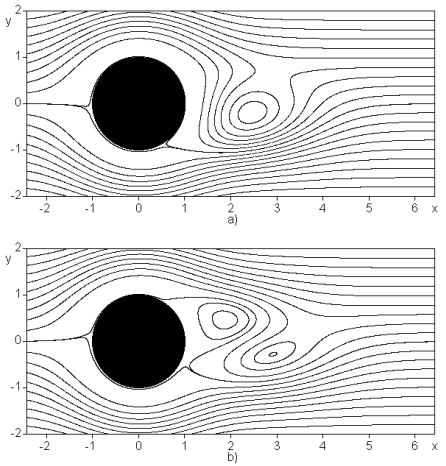
7. Vortex (After) Play

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2.2.1. Von Karman Vortex Trails:

$$F(t)=\frac{C_k \rho V^2 A}{2}\sin(2\pi f t)$$

$$f=\frac{SV}{D} \quad S=\frac{f\,D}{V} \quad \text{Strouhal number}$$



S =0.2 for cylinder
S =0.12 to 0.15 for open-channel section
S =0.18 to 0.22 for flat-thin section
S =0.12 for rectangular section

For a great range of Raynolds numbers

$$R=\frac{VD\rho}{\mu}\rightarrow C_k=1.0$$

$$F_k=0.5\rho V^2 A C_k$$

ρ = Fluid density

C_k = The Von karman coefficient

$F(t)$ = Lateral alternating force

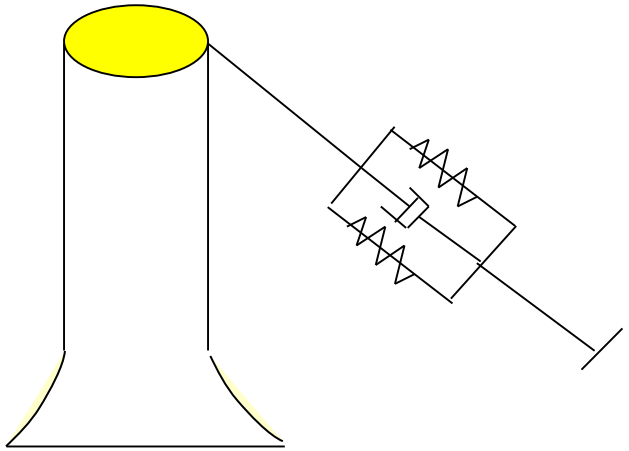
A = Cross-sectional area

D = Diameter for a circular section

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EXAMPLE



Smokestack in 1953, 1 Hz

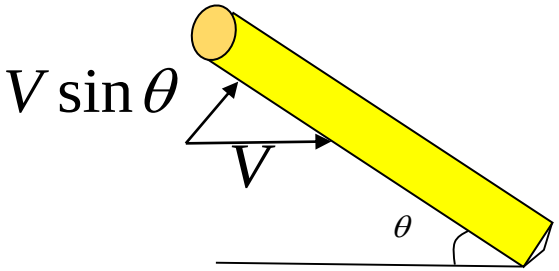
$h = 300$ ft

$D = 16$ ft

$V = 50$ mph

$V > 30$ mph

Tall chimney



Petroleum tower

$$f = \frac{SV}{D}\sin\theta$$

R. D. Blevins, Flow-Induced vibration, Van Nostrand Reinhold Co., New York, 1977

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EXAMPLE

Helical spoilers to stabilize tall chimneys



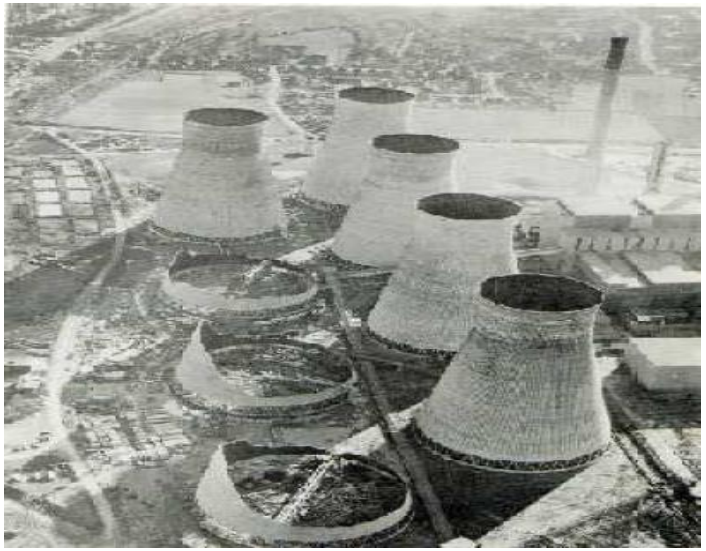


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EXAMPLE

Ferrybridge Power Station , UK (1960)



- 3 of 8 cooling tower collapsed in a wind storm
- Tower height – 375 ft
- Reason: Serious understimation of wind loads in design

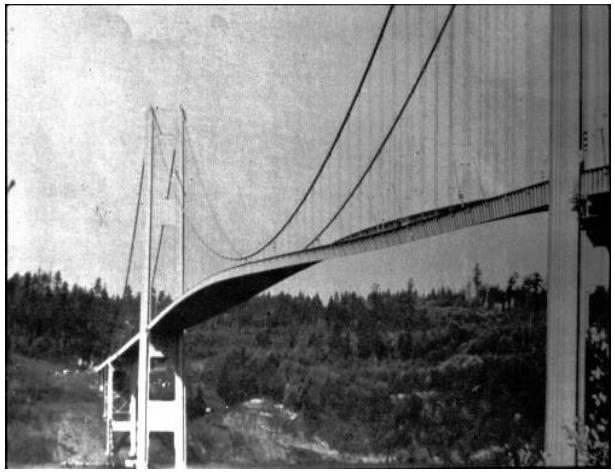


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EXAMPLE

Tacoma Narrows Bridge



November 7, 1940



Nowadays

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EXAMPLE



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EXAMPLE

Tacoma Narrows Bridge

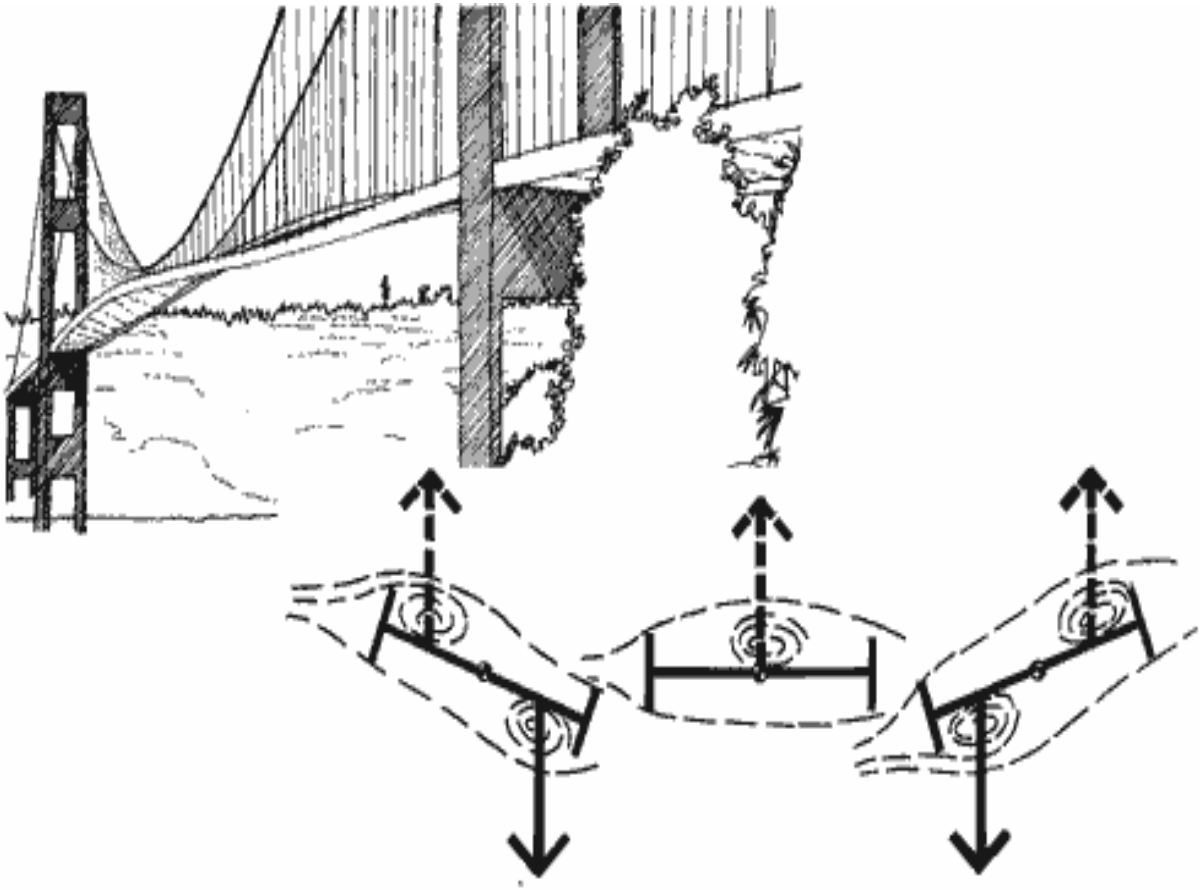


Amplitude of oscillation reached 14 feet!

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EXAMPLE

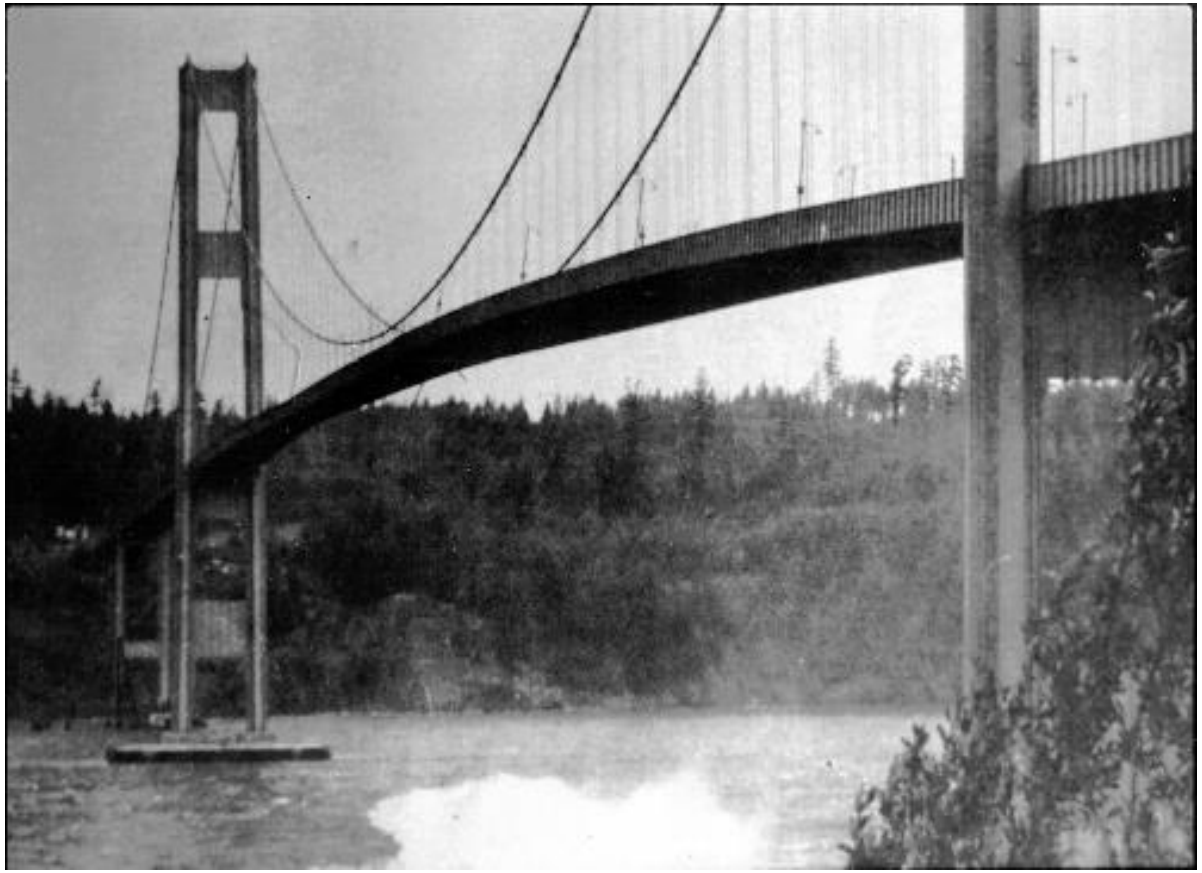


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EXAMPLE

Tacoma Narrows Bridge



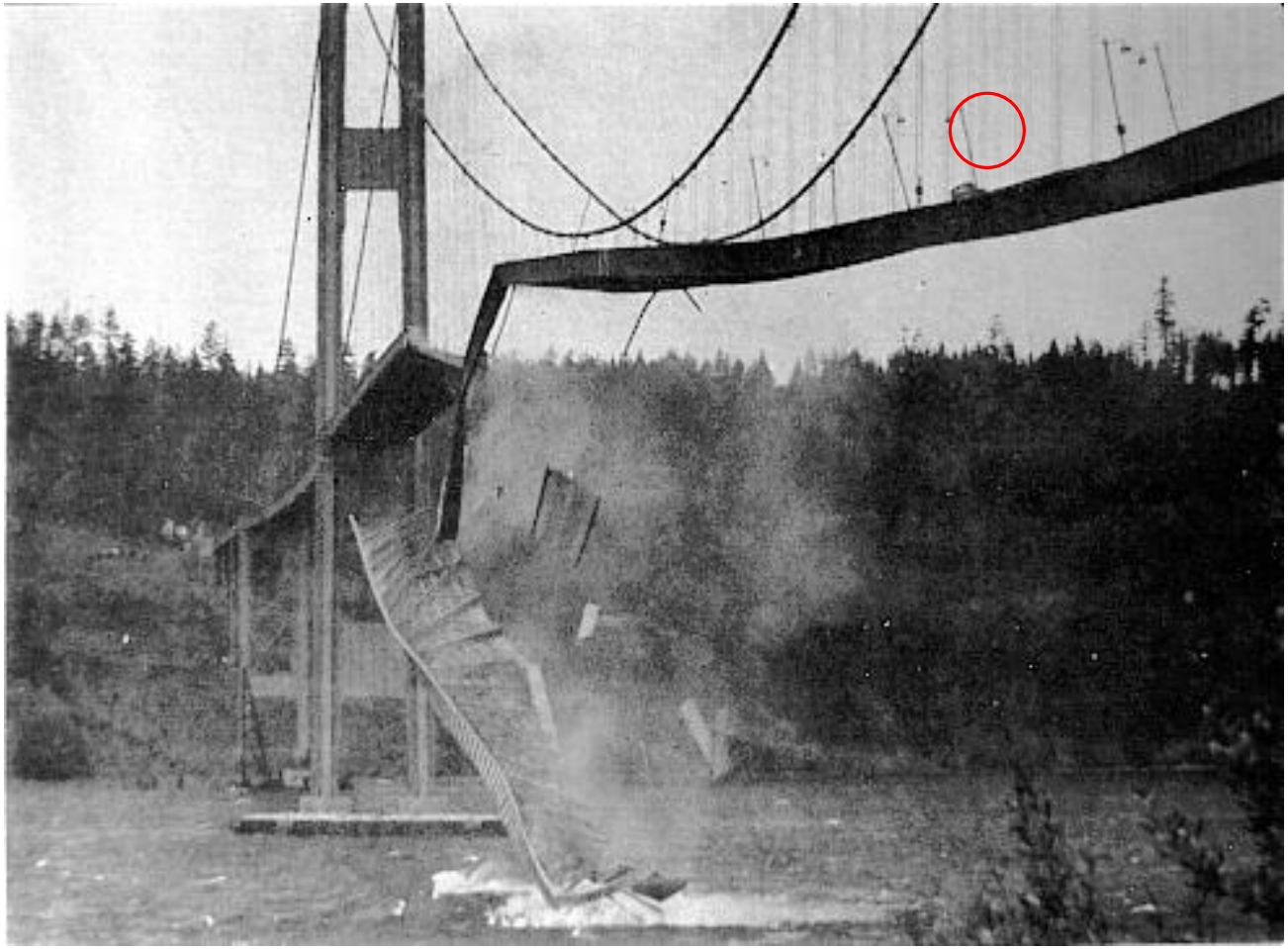
~10:30 A.M. First concrete section breaks away…

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EXAMPLE

Tacoma Narrows Bridge



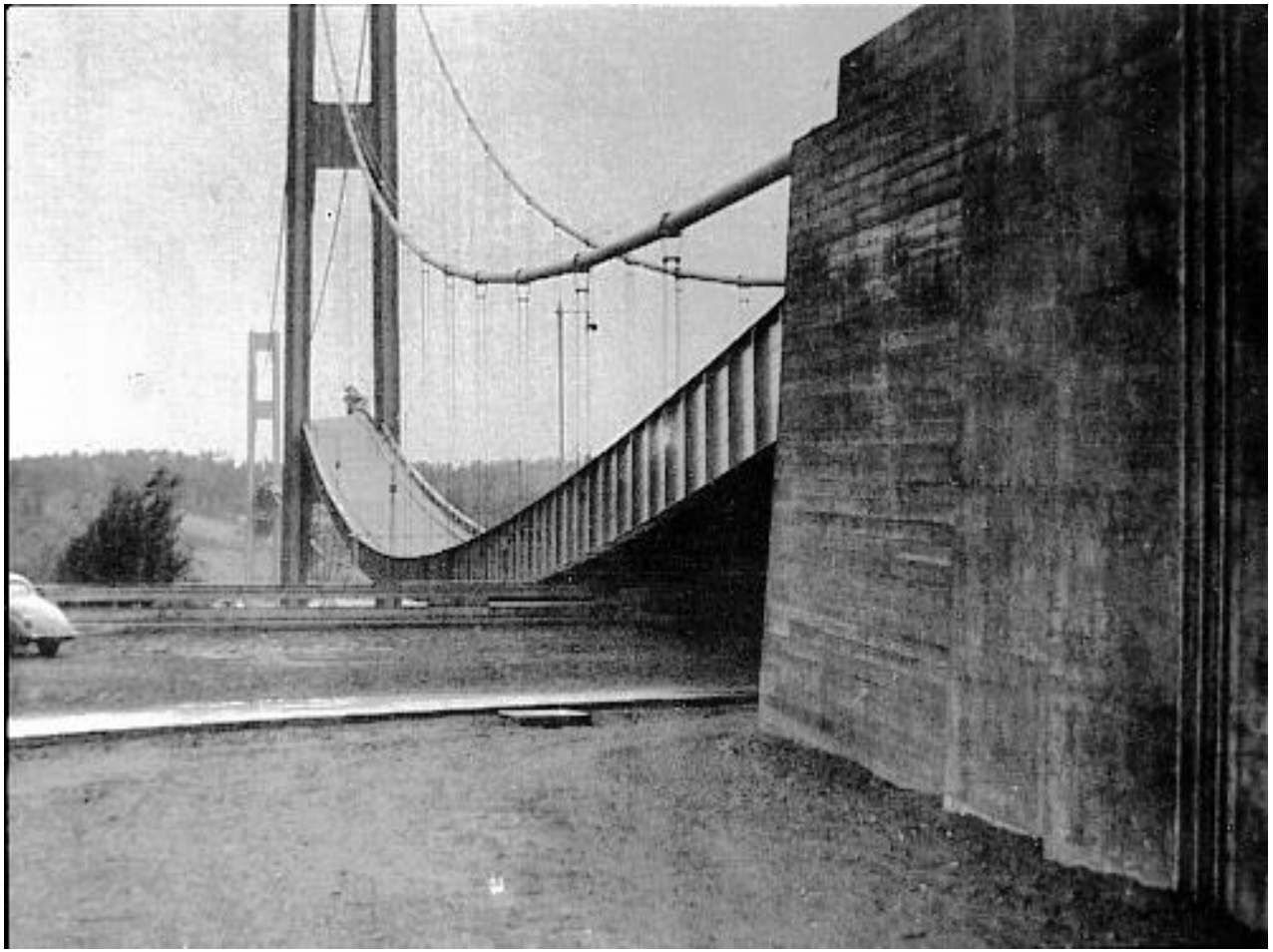
11:02 A.M. 600-ft span breaks away…

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EXAMPLE

Tacoma Narrows Bridge



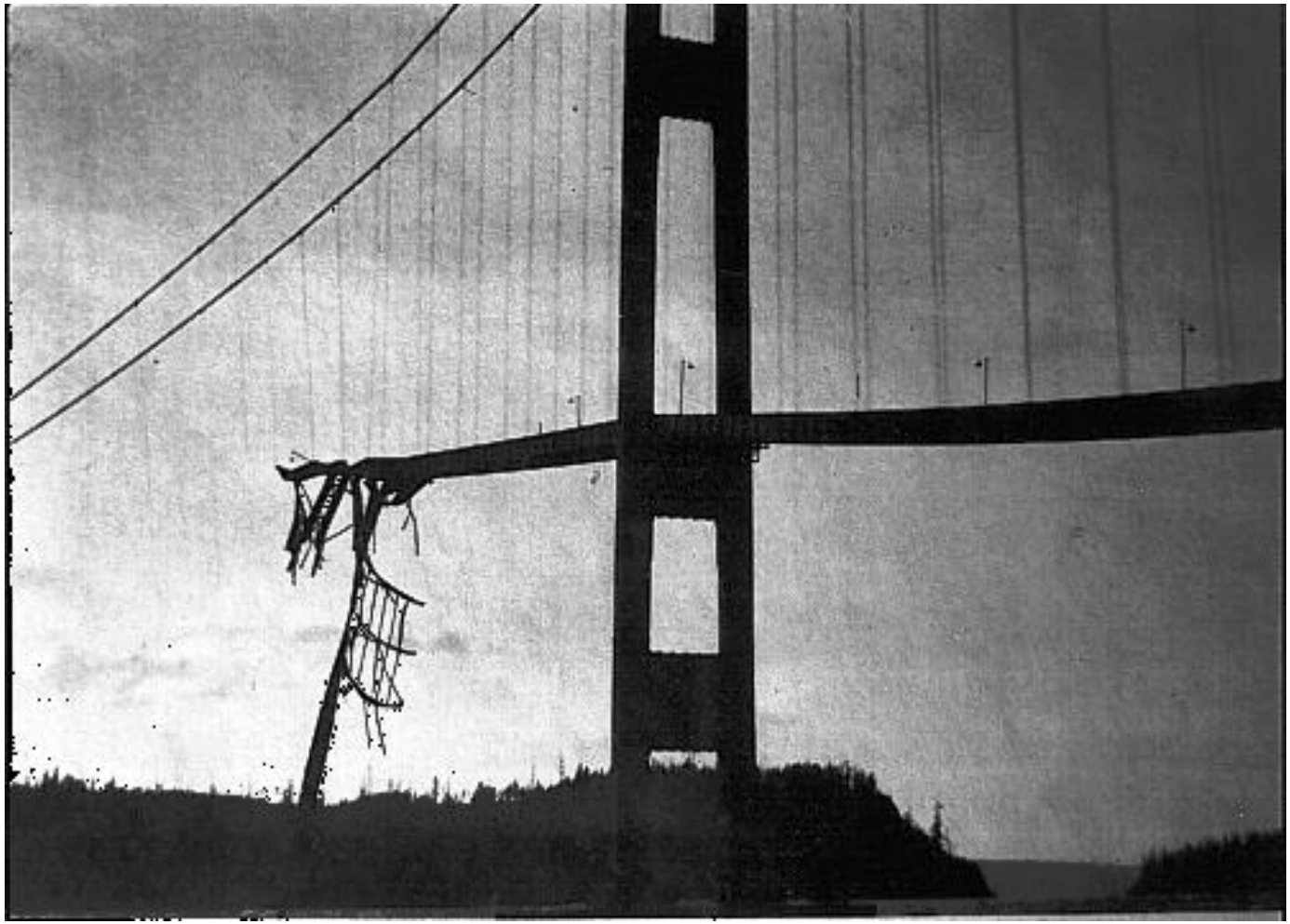
The aftermath…

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EXAMPLE

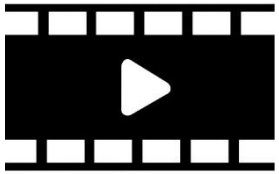
Tacoma Narrows Bridge



…an engineering failure

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Flutter is a dangerous phenomenon encountered in flexible structures subjected to aerodynamic forces.
This includes aircraft, buildings, telegraph wires, stop signs, and bridges.

Aero Elastic Flutter

Flutter at a Glance

Airbus A380 Flutter Test

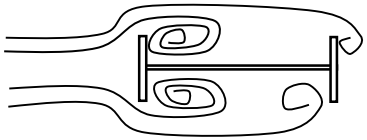
Aircraft Wind Tunnel

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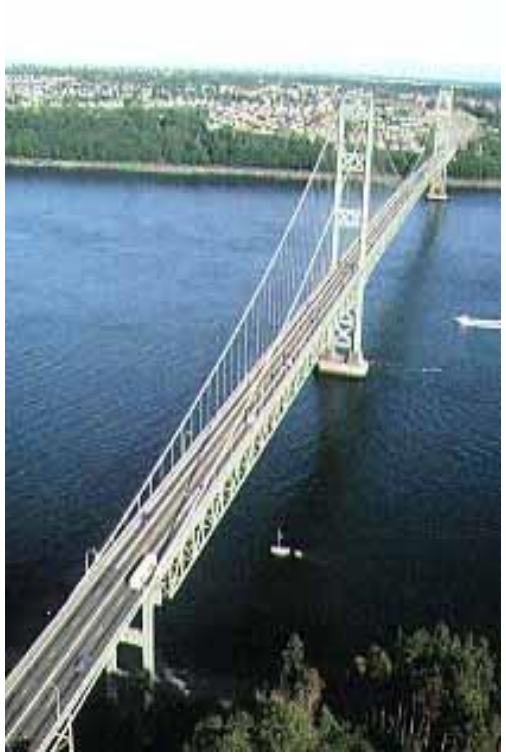
EXAMPLE

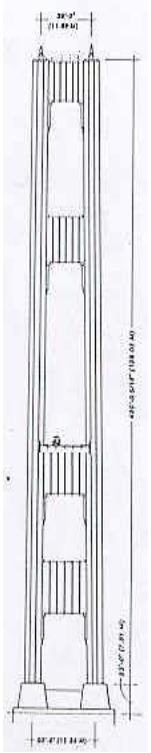
Today's Tacoma Narrows Bridge

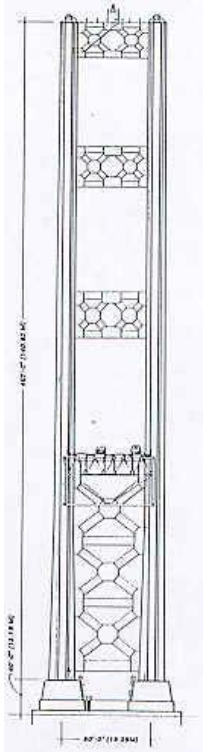


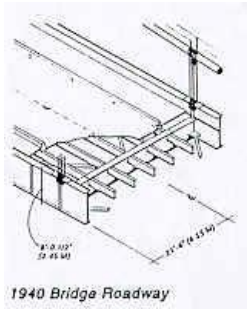
Old section

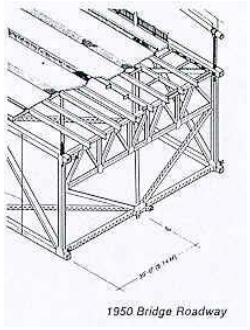
Open side truss →  New section











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EXAMPLE

Stock bridge damper

When vibration has Low amplitude and high frequency



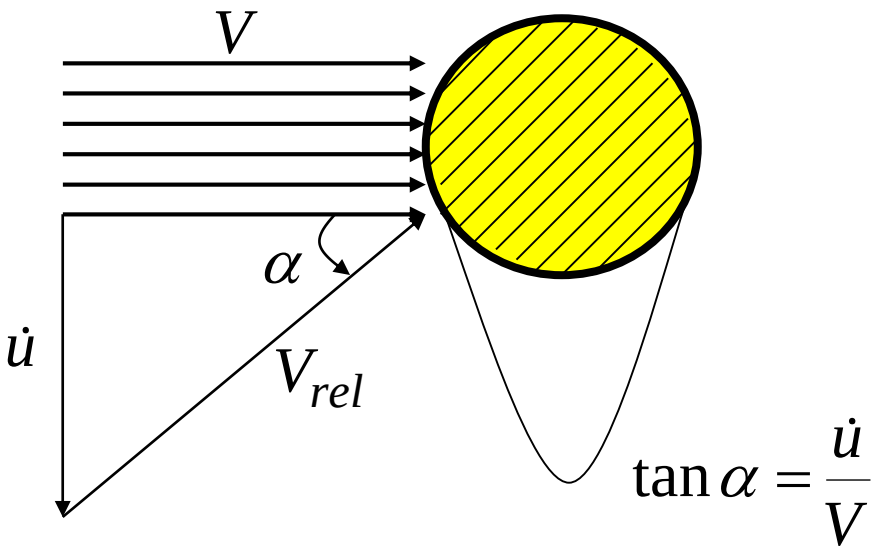


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2.2.2. Galloping of Transmission lines

When vibration has high amplitude and low frequency



$$\tan\alpha = \frac{\dot{u}}{V}$$

α = angle of attack

V_{rel} = relative wind speed



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L, D wind Tunnel test

F = The airodynamic force

D = Drag , L = left

b = width

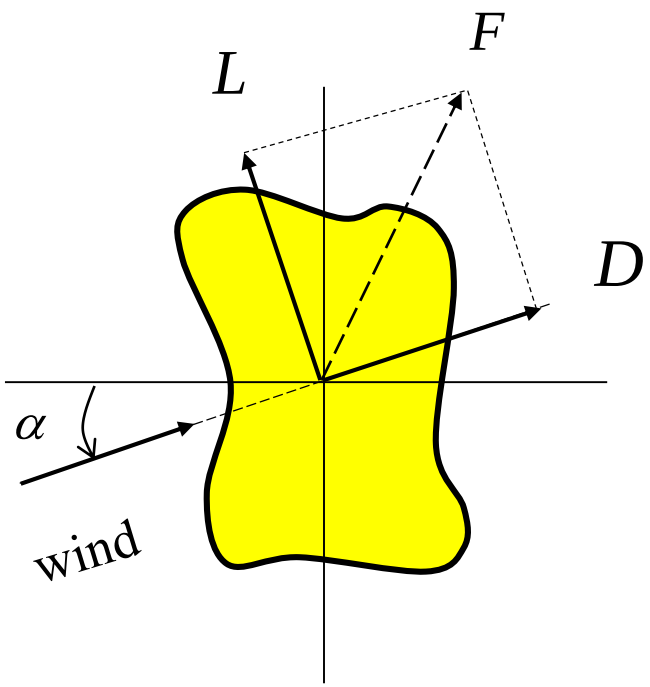
D^* = The hydrolic diameter = $\dfrac{4 \times \text{area of cross section}}{\text{perimeter}}$

C_D, C_L = Aerodynamic parameters

C_v = Linear viscous damping coefficient

$F = L\cos\alpha + D\sin\alpha$

$F = 0.5\rho V^2 b D^* (C_L \cos\alpha + C_D \sin\alpha)$



For a building of rectangular cross-section $a \times b$ $\longrightarrow D = \dfrac{4ab}{2(a+b)} = \dfrac{2ab}{a+b}$

$$D = \frac{4(\pi/4)d^2}{\pi d} = d$$

For a chimney of circular cross-section and diameter d

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for small $\alpha \rightarrow \alpha = \frac{-\dot{x}}{V}$

$$D = C_D \frac{1}{2} \rho h L V^2 \quad \text{Drag}$$

$$L = C_L \frac{1}{2} \rho h L V^2 \quad \text{Lift}$$

C_D, C_L = Drag and lift coefficient

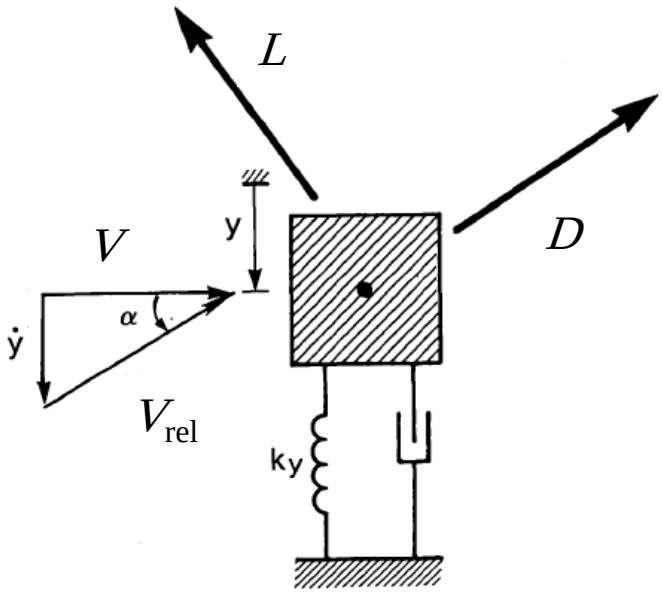
h = Depth of the cross section

L = Length of the body

$$\bar{V} = \frac{V_{rel}}{V} \qquad V_{rel}^2 = \dot{y}^2 + V^2$$

$$F_y = -(C_D \sin\alpha + C_L \cos\alpha) \frac{1}{2} \rho h L \bar{V}^2 \sec^2\alpha = C_{F_y} \frac{1}{2} \rho h L \bar{V}^2$$

Force along y-axis



$$C_{F_y} = -(C_L + C_D \tan\alpha)\sec\alpha$$

The lateral force excites the vibration if the first derivative of C_{fy} at $\alpha = 0$ is > 0.

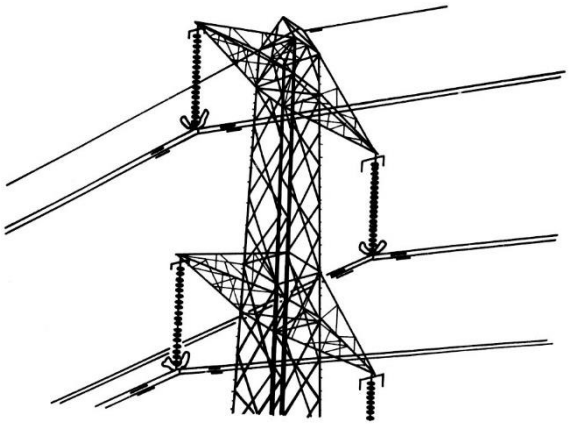
Can be found using wind tunnel tests for particular sections

$$A_1 = \frac{\partial C_{F_y}}{\partial \alpha}\Big|_{\alpha=0} = -(\frac{\partial C_L}{\partial \alpha} + C_D) \succ 0$$

$A_1 \leq 0$ Stable condition, Den Hartog's Criterion

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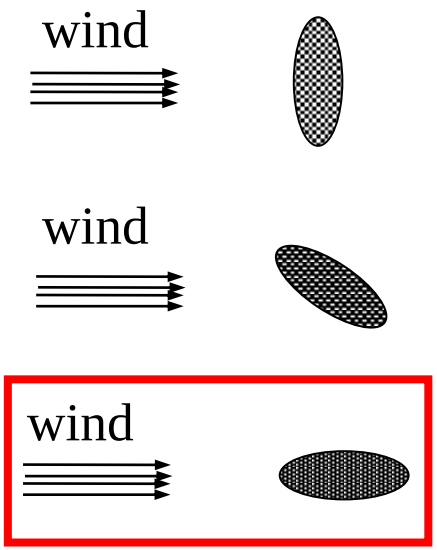


$$\bar{V}_0 = \zeta \frac{2\pi f_1 h}{nA_1}$$ minimum wind velocity for galloping

f_i = natural velocity

m = mass of body per unit length

$$n = \frac{\rho h^2}{4m} = \text{mass parameter}$$



Best condition

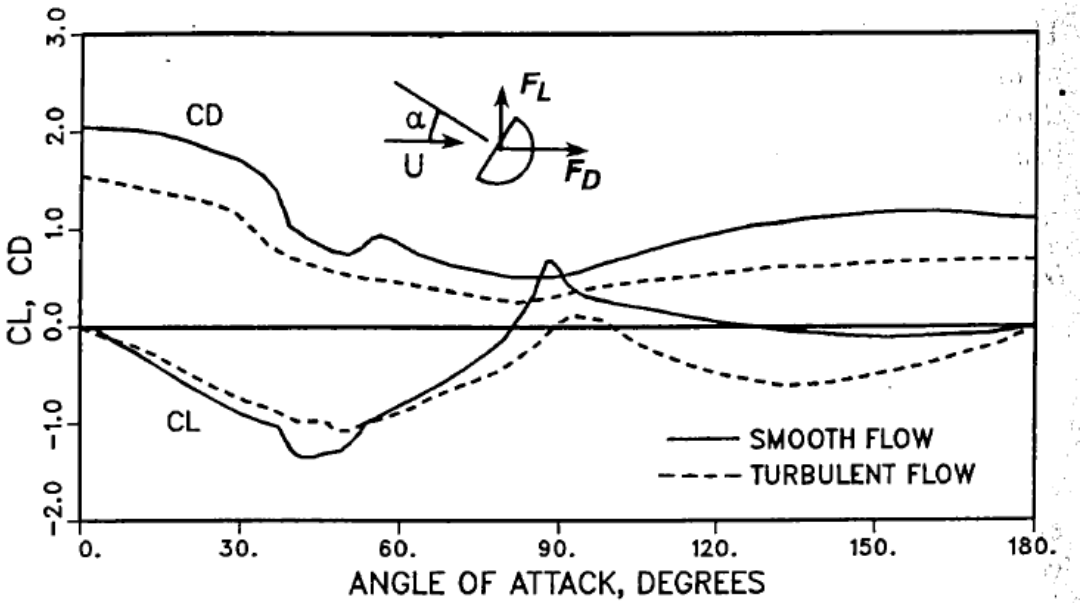
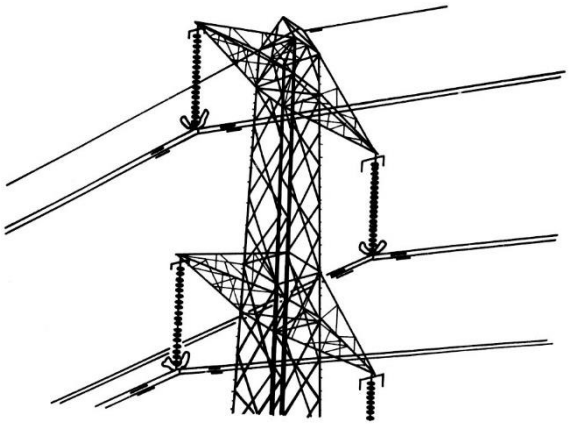
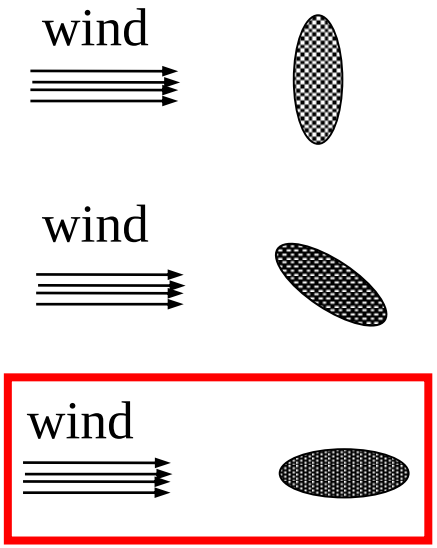


Fig. 4-22 Lift and drag coefficients for D-section in smooth and 11% turbulent flow; $Re = 9 \times 10^4$ (Novak and Tanaka, 1974).

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Best condition

$$\text{for small } \alpha \rightarrow \alpha = \frac{-\dot{x}}{V}$$

$$F = -0.5\rho V^2 bD^* \left(\frac{\partial C_L}{\partial \alpha} + C_D \right) \frac{\dot{x}}{V}$$

If $\left(\frac{dF}{d\alpha} \right) < 0$ Dynamic Instability

If $\left(\frac{dF}{d\alpha} \right) > 0$ Dynamic stability

$$C_V \; -0.5\rho V^2 bD^* \left(\frac{\partial C_L}{\partial \alpha} + C_D \right) < 0$$

$$\frac{dF}{d\alpha} = \left(\frac{dL}{d\alpha} \right) \cos\alpha - L \sin\alpha + \left(\frac{dD}{d\alpha} \right) \sin\alpha + D \cos\alpha =$$

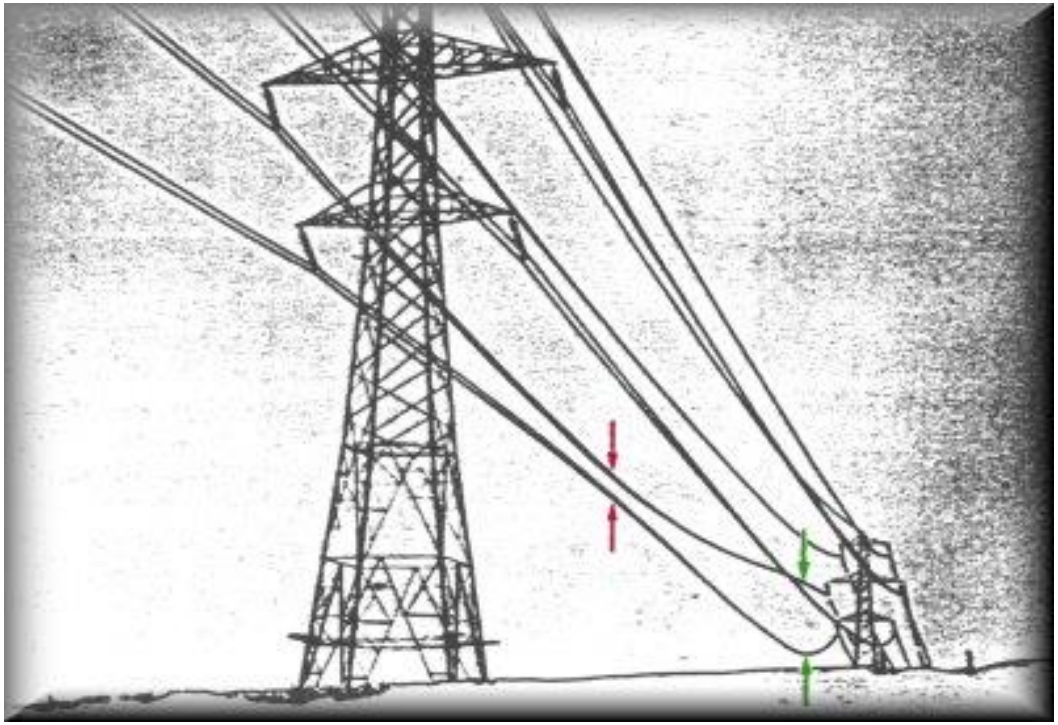
$$\left(\frac{dD}{d\alpha} - L \right) \sin\alpha + \left(D + \frac{dL}{d\alpha} \right) \cos\alpha \cong D + \frac{dL}{d\alpha} < 0$$

$$\rightarrow \frac{dL}{d\alpha} < -D$$

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On the following figure, you can see a two loops galloping occuring on a 220 kV line. The red arrows indicate the point where two of the three phases are too close (risk of short-circuit), while the green arrows show the point where they are the most far away one from each other. Of course, these two points move all over the span as the conductors move.



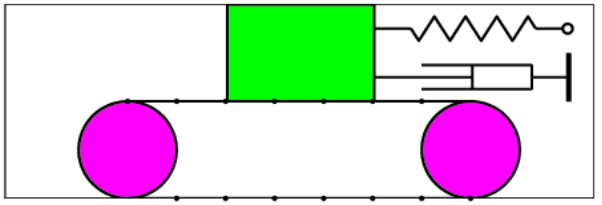
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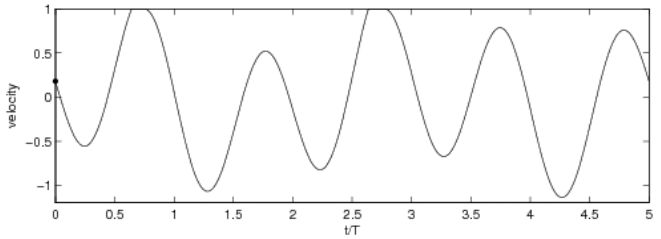
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2.3. Friction-induced Vibration (stick-slip)

The stick-slip phenomenon, is the spontaneous jerking motion that can occur while two objects are sliding over each other.





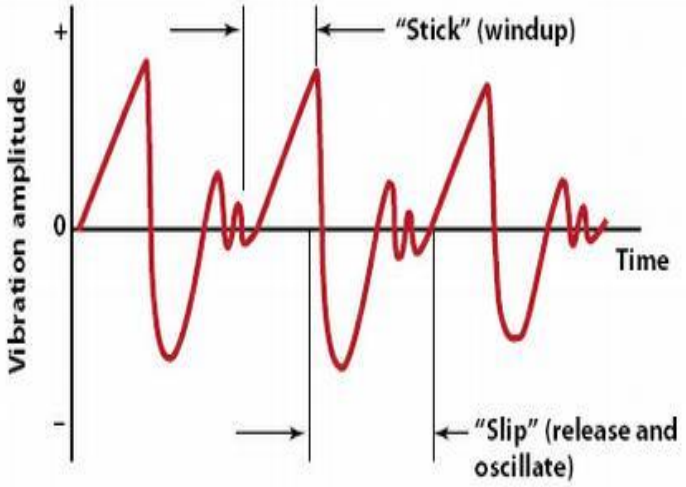


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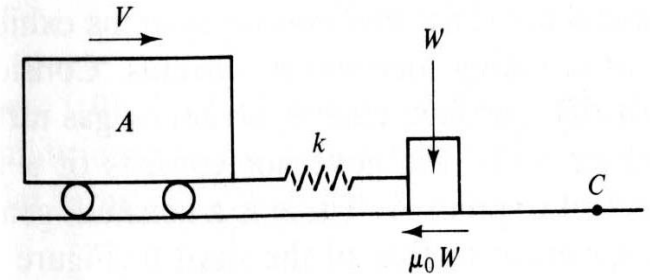
EXAMPLE

Stick-slip is responsible for the squeaking sound of chalk on a Black board one can take a piece of paper, put the paper on a solid Surface , and place a heavy object on it. Put a mark on the paper to Determine the initial position of the object. Now slowly move the paper to the left and suddenly slide it to the right. If you do this a few times then you should observe that the object will be to the left side of its initial position.



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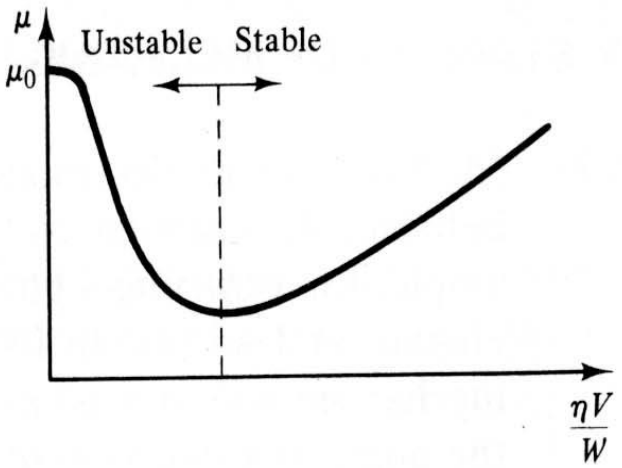


μ_0= static coefficient of friction

μ = kinetic coefficient of friction

If $k\,x < \mu_0\, w$ static friction

If $k\,x > \mu_0\, w$ $F_0 = k\,x - \mu_0\, w$



When motion begins $\mu << \mu_0$

$F = kx - \mu\, w >> F_0$

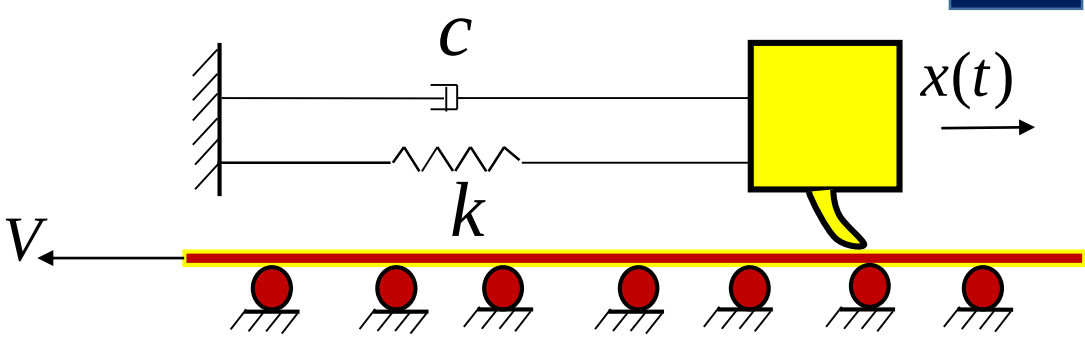
accelerating Force

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EXAMPLE

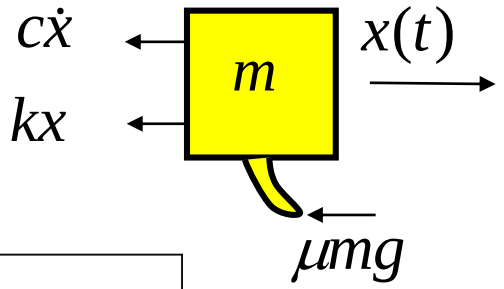
Metal Cutting Stability (plane mill)



U = Cutting Velocity

$\mu = a - bU$

$C = ?$ Critical Value



$$F = -mg\,[a - b(V + \dot{x})]$$

$$m\ddot{x} + c\dot{x} + kx = -mga + mgb\,(V + \dot{x})$$

$$m\ddot{x} + (c - mgb)\dot{x} + kx = -mga + mgbV$$

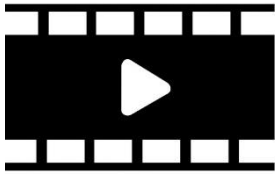
1) If $b < 0 \rightarrow$ stable system

2) If $b > 0 \rightarrow$ (as usually happens)

$$C_v < mgb \quad \{\text{stick-slip chatter}\}$$

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- Wall slip is a unique character of polymeric liquids.

The movie shows Stick-slip oscillation as the sheared polymer was unable to settle down to a stable boundary condition. Play

- Stick-slip device illustrating periodic slip and stress relief Play

- There is Stick-slip in industrial valves Play

- Stick-slip and vibration mitigated while drilling by the use of AST from Tomax AS Play

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Chapter 3. Modal Analysis

Classification of modelling

1- Lumped parameter modelling

2- Distributed parameter modelling

$$[M]_{n\times n}\{\ddot{x}\}_{n\times 1}+[C]_{n\times n}\{\dot{x}\}_{n\times 1}+[K]_{n\times n}\{x\}_{n\times 1}=\{F(t)\}_{n\times 1}$$

Classification of methods

1-Lagrange's method

2-Newton's method

3-Stiffness Influence Coefficients Method

4-Flexibility Influence Coefficient method

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A degree of freedom system

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\Rightarrow x(t) = x_h(t) + x_p(t)$$

$$x_h(t) = X_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi_1)$$

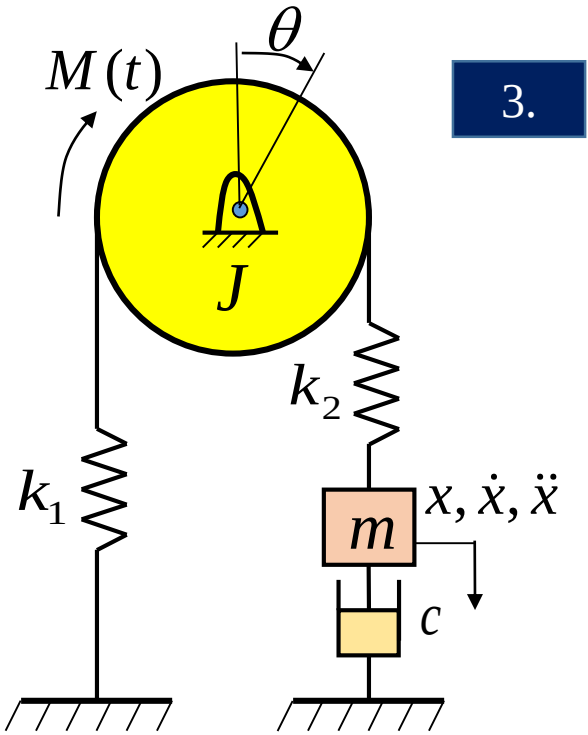
$$x_p(t)\begin{cases} \text{If } \ \mathrm{F(t)} = \mathrm{F_0}\sin\omega t \Rightarrow x_p(t) = X\sin(\omega t - \varphi) \qquad X = \dfrac{\mathrm{F_0}/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \\ \text{If } \ \mathrm{F(t)} = \text{arbitrary function} \quad \Rightarrow \ x_p(t) = \displaystyle\int_0^t F(\tau)h(t-\tau)d\tau \end{cases}$$

$$h(t) = \frac{1}{\omega_d m} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

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EXAMPLE

3.



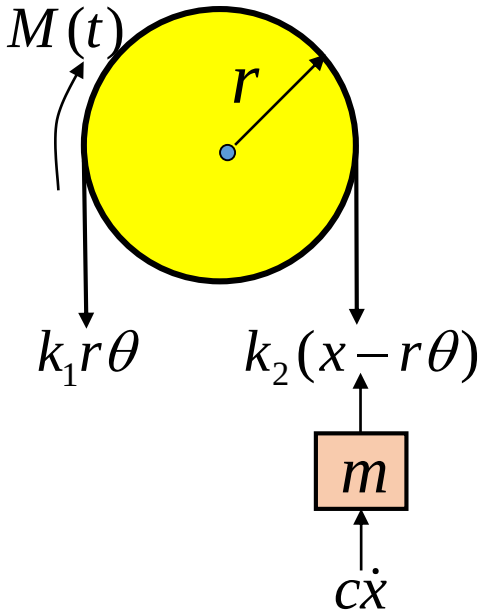
$$\sum M = J\ddot{\theta} \qquad -k_1 r^2\theta + k_2 r(x - r\theta) + M(t) = J\ddot{\theta}$$

$$J\ddot{\theta} + (k_1 + k_2)r^2\theta - k_2 rx = M(t)$$

$$\sum F = m\ddot{x} \qquad -k_2(x - r\theta) - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + k_2 x - k_2 r\theta = 0$$

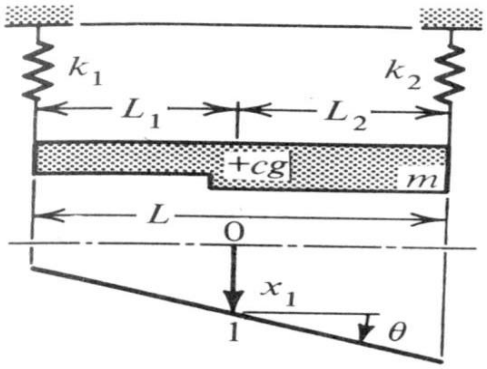
$$\begin{bmatrix} J & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2)r^2 & -k_2 r \\ -k_2 r & k_2 \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} M(t) \\ 0 \end{Bmatrix}$$



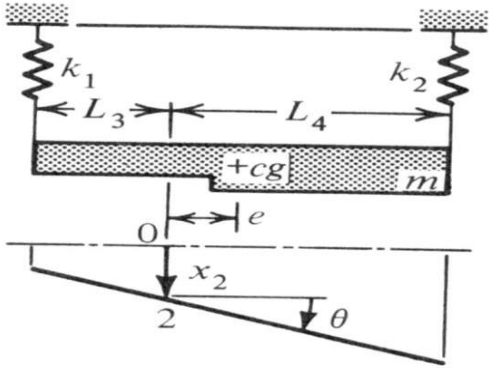
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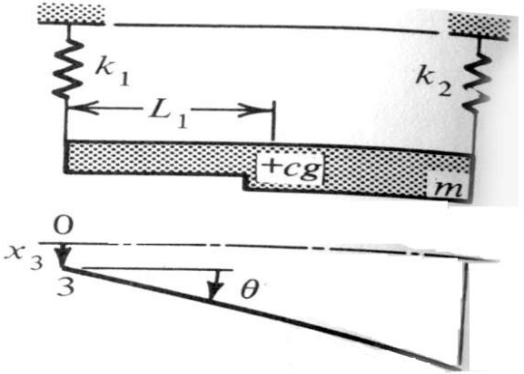
Generalized coordinates and coordinate coupling



1. Static coupling



2. Dynamic coupling



3. Static and dynamic coupling

$$1 \Rightarrow \begin{bmatrix} m & 0 \\ 0 & J_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 L_1 - k_2 L_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 L_1^{\,2} - k_2 L_2^{\,2}) \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

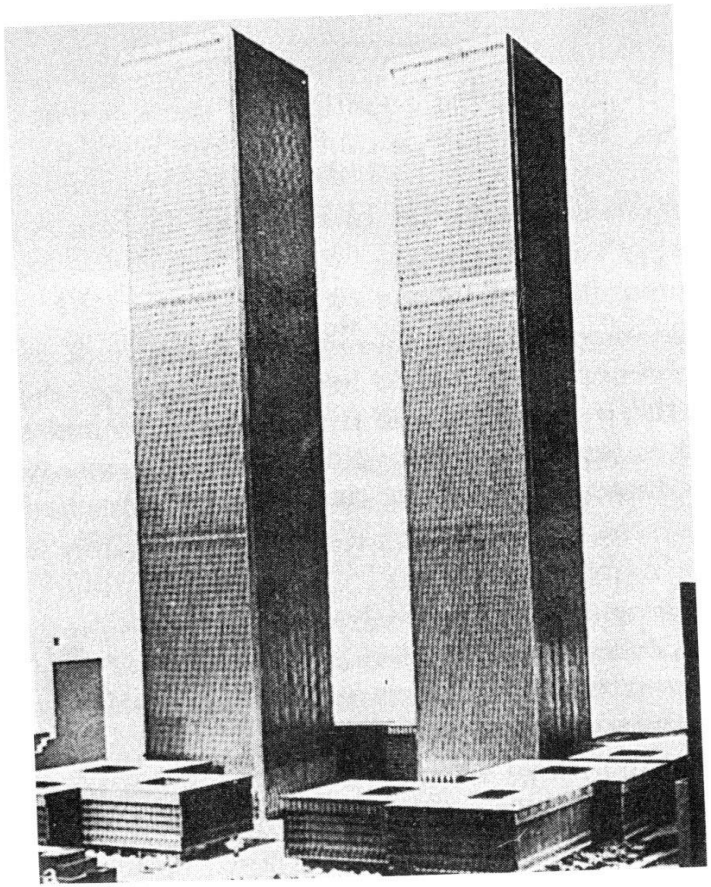
$$2 \Rightarrow \begin{bmatrix} m & m \times e \\ m \times e & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & 0 \\ 0 & (k_1 L_3^{\,2} - k_2 L_4^{\,2}) \end{bmatrix} \begin{Bmatrix} x_2 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$3 \Rightarrow \begin{bmatrix} m & mL_1 \\ mL_1 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} \begin{Bmatrix} x_3 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

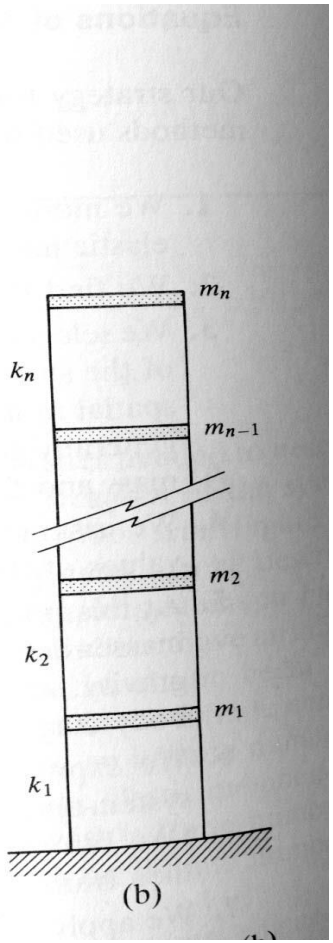
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Stiffness Influence Coefficients Method



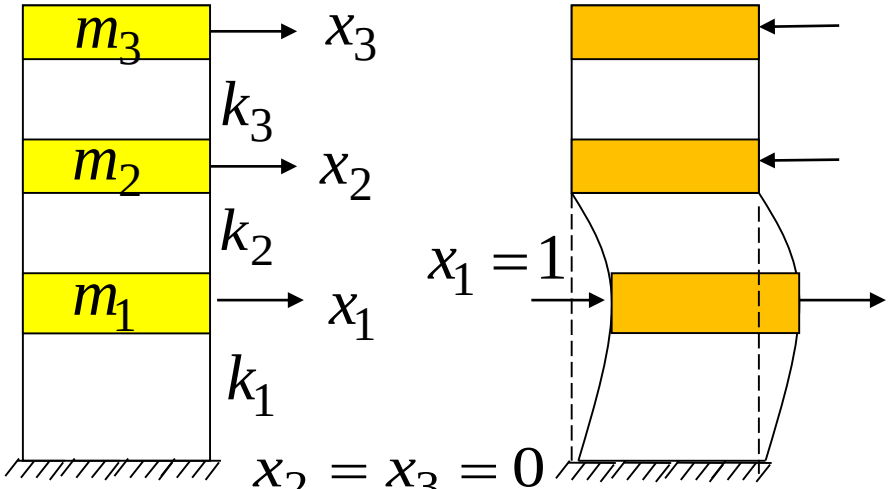
(a)



(b)

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$$k_{31} = 0$$

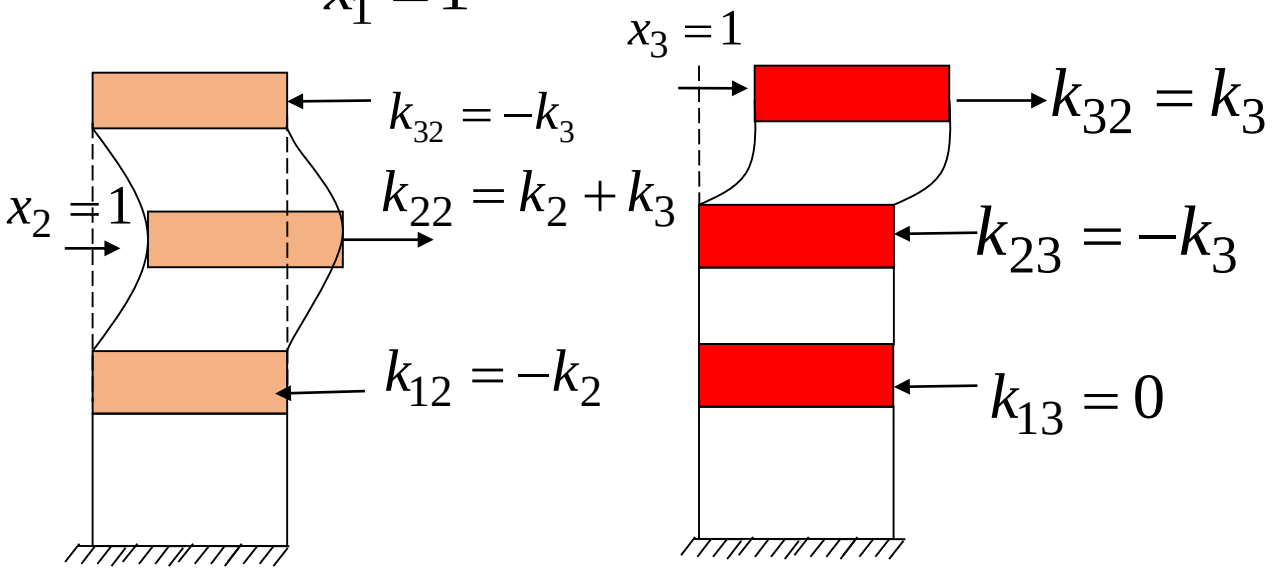
$$k_{21} = -k_2 x_1 = -k_2$$

$$k_{11} = k_1 x_1 + k_2 x_1$$

$$= k_2 + k_1$$

$$x_2 = x_3 = 0$$

$$x_1 = 1$$

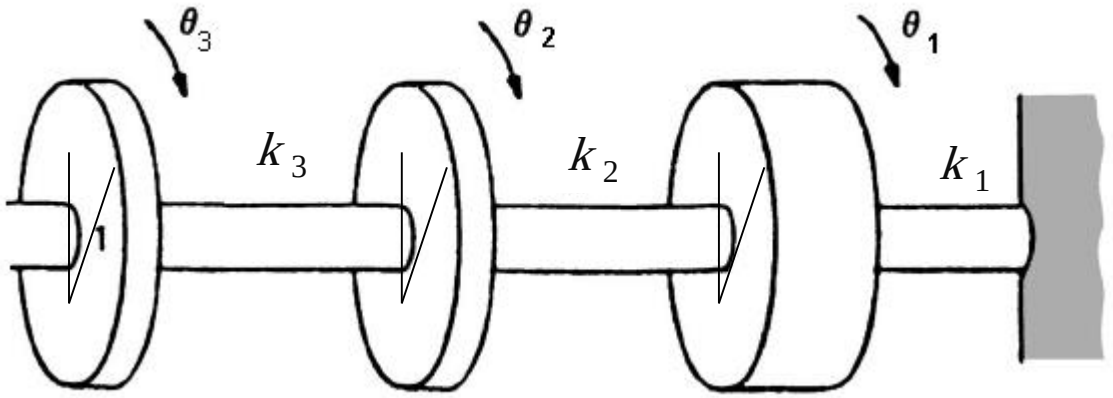


$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

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J_3 J_2 J_1

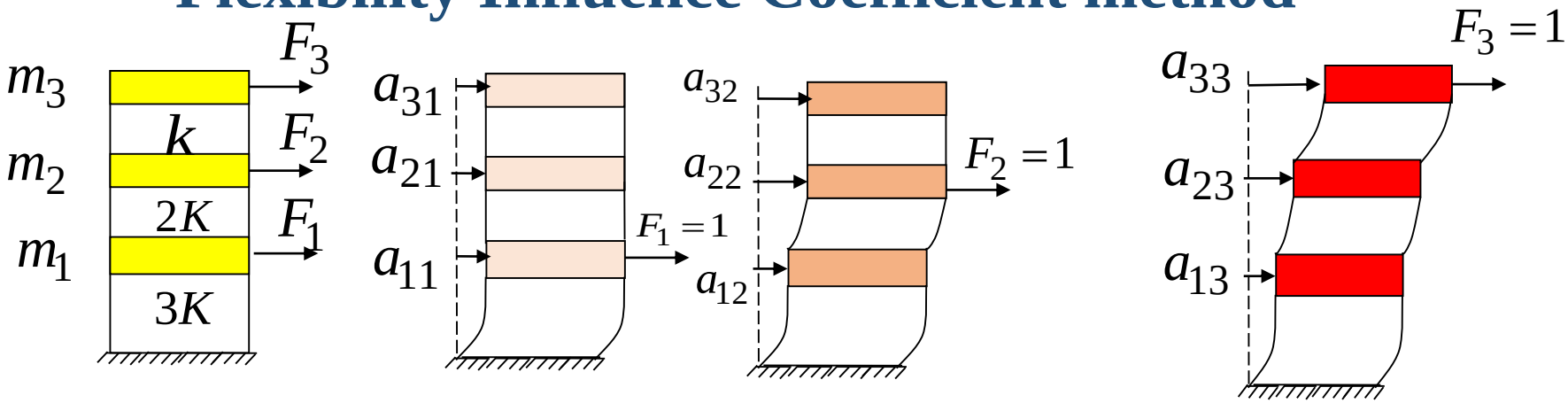
$k_{32} = -k_3$ $k_{22} = k_2 + k_3$ $k_{12} = -k_1$

$$[M] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

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Flexibility Influence Coefficient method



$$[k]^{-1}[M]\{\ddot{x}\}+[I]\{x\}=0 \qquad [k]^{-1}=[A] \qquad \text{flexibility matrix}$$

$$a_{11}=\frac{1}{3k}\ ,\ a_{21}=\frac{1}{3k}\ ,\ a_{31}=\frac{1}{3k}$$

$$a_{12}=\frac{1}{3k}\ ,\ a_{22}=\frac{1}{3k}+\frac{1}{2k}=\frac{5}{6k}\ ,\ a_{32}=\frac{1}{3k}+\frac{1}{2k}=\frac{5}{6k}$$

$$a_{13}=\frac{1}{3k}\ ,\ a_{23}=\frac{1}{3k}+\frac{1}{2k}=\frac{5}{6k}\ ,\ a_{33}=\frac{1}{3k}+\frac{1}{2k}+\frac{1}{k}=\frac{11}{6k}$$

$$[A]=\frac{1}{6k}\begin{bmatrix}2 & 2 & 2\\ 2 & 5 & 5\\ 2 & 5 & 11\end{bmatrix}$$

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3.1. Natural frequencies

$$[M]\{\ddot{x}\}+[K]\{x\}=0$$

$$\{x\}=\{X\,e^{i\omega t}\}\Rightarrow\{\ddot{x}\}=-\omega^2\{X\,e^{i\omega t}\}=-\omega^2 x$$

$$-\omega^2[M]\{x\}+[K]\{x\}=0$$

mode shapes $\left([K]_{n\times n}-\omega^2[M]_{n\times n}\right)\{X\}_{n\times 1}=0$

$\det\left([K]-\omega^2[M]\right)=0 \quad \omega^2=\lambda$ Natural frequencies

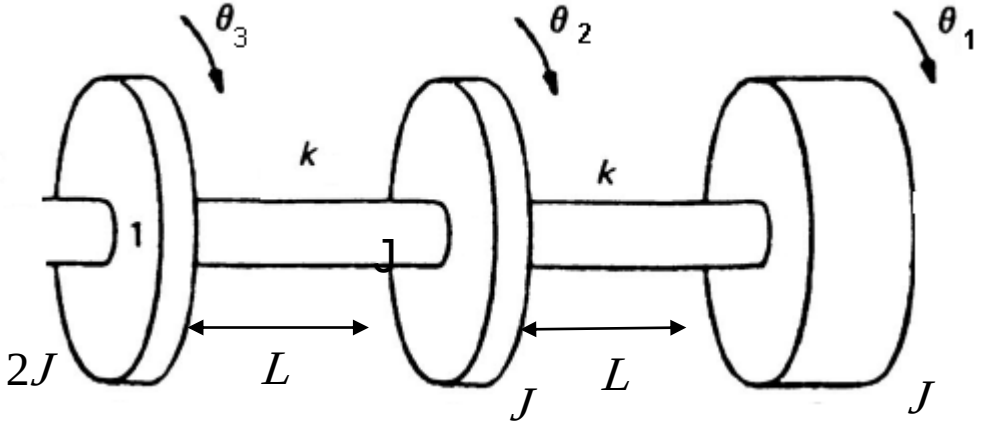
Characteristic eq.

$$\lambda^{n}+a_1\lambda^{n-1}+a_2\lambda^{n-2}+.....+a_n\lambda+a_{n+1}=0$$

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EXAMPLE



$$[M]=\begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 2J \end{bmatrix} \qquad [\mathrm{k}]=\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$\lambda^3-\frac{7}{2}(\frac{k}{J})\lambda^2+2(\frac{k}{J})^2\lambda=0 \quad \rightarrow \quad \begin{cases} \lambda_1=0 & \rightarrow & \mathrm{f}_1=0 \\ \lambda_2=0.72\dfrac{k}{J} & \rightarrow & \mathrm{f}_2=\dfrac{1}{2\pi}\sqrt{0.72\dfrac{k}{J}} \\ \lambda_3=2.78\dfrac{k}{J} & \rightarrow & \mathrm{f}_3=\dfrac{1}{2\pi}\sqrt{2.78\dfrac{k}{J}} \end{cases}$$

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EXAMPLE

$$\begin{bmatrix} k-\lambda J & -k & 0 \\ -k & 2k-\lambda J & -k \\ 0 & -k & k-2\lambda J \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = 0$$

$$\text{If } \theta_1 = 1, \lambda = 0.72\frac{k}{J} \rightarrow \theta_2 = 0.28, \quad \theta_3 = -0.64$$

$$\{\theta_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \{\theta_2\} = \begin{Bmatrix} 1 \\ 0.28 \\ -0.64 \end{Bmatrix} \qquad \{\theta_3\} = \begin{Bmatrix} 1 \\ -1.78 \\ 0.39 \end{Bmatrix}$$

$$[U] = [\theta_1 \quad \theta_2 \quad \theta_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.28 & -1.78 \\ 1 & -0.64 & 0.39 \end{bmatrix}$$

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$$\begin{cases} (\omega_r^2[M]\{x\}_r)^T = ([K]\{x\}_r)^T \rightarrow \omega_r^2\{x\}_r{}^T[M] = \{x\}_r{}^T[K] \\ \qquad \times\{x\}_s \rightarrow \omega_r^2\{x\}_r^T[M]\{x\}_s = \{x\}_r^T[K]\{x\}_s \\ \omega_s^2[M]\{x\}_s = ([K]\{x\}_s \rightarrow \omega_s^2\{x\}_s{}^T[M] = \{x\}_s{}^T[K] \\ \qquad \times\{x\}_r^T \rightarrow \omega_s^2\{x\}_r^T[M]\{x\}_s = \{x\}_r^T[K]\{x\}_s \\ (\omega_r^2 - \omega_s^2)\{x\}_r^T[M]\{x\}_s = 0 \qquad \omega_r \neq \omega_s \end{cases}$$

$$\{x\}_r^T[M]\{x\}_s = 0 \quad , \quad \{x\}_r^T[K]\{x\}_s = 0 \quad \text{Orthogonality}$$

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$$\sum_{i=1}^{n} M_{\mathrm{i}} (x_i)_r (x_i)_s = 0 \qquad \text{If } \mathrm{r} \neq \mathrm{s}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} M_{\mathrm{ij}} (x_i)_r (x_j)_s = 0$$

$$K_r = \omega_r^2 M_r$$

$$M_r = \underset{1\times n}{\{x\}_r^T} \underset{n\times n}{[M]} \underset{n\times 1}{\{x\}_r} = \sum_{i=1}^{n} M_{\mathrm{i}} (x_i{}^2)_r \quad \text{General mass}$$

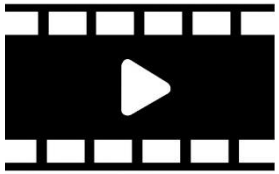
$$K_{\mathrm{r}} = \{x\}_r^T [K] \{x\}_r \qquad \text{General stiffness}$$

$$\Rightarrow \omega_r^2 \underbrace{\{x\}_r^T [M] \{x\}_r}_{M_r} = \underbrace{\{x\}_r^T [K] \{x\}_r}_{\mathrm{K_r}}$$

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3.2. Modal Matrix



1.Mode Shapes for Multiple Degree-of-Freedom Oscillators Play

2. Multiple Degree-of-Freedom system forced vibration Play

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3.2. Modal Matrix

$$[M]\{\ddot{X}\}+[K]\{X\}=\{F(t)\}$$ Coupled Eqs.

$$[U]=[\{X\}_1 \quad \{X\}_2 \quad \{X\}_3 \ldots \{X\}_n]_{n\times n}$$ Modal Matrix

$$\{X\}=[U]\{\delta\} \qquad \{\ddot{X}\}=[U]\{\ddot{\delta}\}$$

$$[U]^T[M][U]\{\ddot{\delta}\}+[U]^T[K][U]\{\delta\}=0 \quad \langle \mathrm{A} \rangle$$

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$$[U]=\left[\begin{Bmatrix}X_1\\X_2\\\vdots\\X_n\end{Bmatrix}_1 \quad \begin{Bmatrix}X_1\\X_2\\\vdots\\X_n\end{Bmatrix}_2 \quad \begin{Bmatrix}X_1\\X_2\\\vdots\\X_n\end{Bmatrix}_3 \quad \cdots \quad \begin{Bmatrix}X_1\\X_2\\\vdots\\X_n\end{Bmatrix}_n\right]$$

$$[U]^T=\begin{bmatrix}[X_1 \quad X_2 \ldots X_n]_1\\[X_1 \quad X_2 \ldots X_n]_2\\\vdots\\[X_1 \quad X_2 \ldots X_n]_n\end{bmatrix}$$

$$[X_1 \quad X_2 \ldots X_n]_r[\mathrm{M}]\begin{Bmatrix}X_1\\X_2\\\vdots\\X_n\end{Bmatrix}=\{X\}_r^T[M]\{X\}_s=0 \qquad \text{If} \quad r\neq s$$

$$\{X\}_r^T[M]\{X\}_r=M_r$$

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$$[U]^T[M][U]=\begin{bmatrix} M_1 & & 0 \\ & M_2 & \\ & & \ddots \\ 0 & & M_n \end{bmatrix}=\begin{bmatrix} \ddots & & 0 \\ & M_r & \\ 0 & & \ddots \end{bmatrix}$$

$\langle \mathrm{A} \rangle \Rightarrow$

$$[U]^T[K][U]=\begin{bmatrix} K_1 & & 0 \\ & K_2 & \\ & & \ddots \\ 0 & & K_n \end{bmatrix}=\begin{bmatrix} \ddots & & 0 \\ & \omega_{\mathrm{r}}^2 M_r & \\ 0 & & \ddots \end{bmatrix}$$

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$$\begin{Bmatrix} \ddot{\delta}_1 + \omega_1^2 \delta_1 = 0 \\ \ddot{\delta}_2 + \omega_2^2 \delta_2 = 0 \\ \vdots \\ \ddot{\delta}_r + \omega_r^2 \delta_r = 0 \\ \vdots \\ \ddot{\delta}_n + \omega_n^2 \delta_n = 0 \end{Bmatrix} \qquad \delta_r = A_r \cos\omega_r t + B_r \sin\omega_r t$$

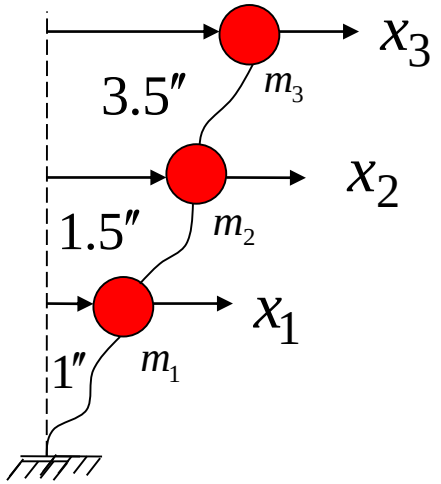
$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = [\mathrm{U}] \begin{Bmatrix} A_1 \cos\omega_1 t + B_1 \sin\omega_1 t \\ A_2 \cos\omega_2 t + B_2 \sin\omega_2 t \\ \vdots \\ A_n \cos\omega_n t + B_n \sin\omega_n t \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \delta_1 \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}_1 + \delta_2 \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}_2 + \cdots + \delta_n \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}_n$$

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EXAMPLE



I.C.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 1.0 \\ 1.5 \\ 3.5 \end{Bmatrix} \qquad \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}_{t=0} = 0$$

$$\omega_1 = 12.23 \,\text{rad/sec} \qquad \omega_2 = 25.55 \,\text{rad/sec} \qquad \omega_3 = 35.79 \,\text{rad/sec}$$

$$[U] = \begin{bmatrix} 1 & 1 & 1 \\ 2.05 & 0.54 & -1.34 \\ 2.93 & -1.78 & 0.86 \end{bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{t=0} = \begin{Bmatrix} 1.0 \\ 1.5 \\ 3.5 \end{Bmatrix} = [U] \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \quad , \quad \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = [U] \begin{Bmatrix} B_1\omega_1 \\ B_2\omega_2 \\ B_3\omega_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \mathrm{B}_1 = \mathrm{B}_2 = \mathrm{B}_3 = 0$$

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EXAMPLE

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = [U]^{-1} \begin{Bmatrix} 1.0 \\ 1.5 \\ 3.5 \end{Bmatrix} = \begin{Bmatrix} 0.896 \\ -0.461 \\ 0.065 \end{Bmatrix}$$

$$\delta_1 = 0.896 \cos 12.23t$$

$$\delta_2 = -0.461 \cos 25.55t$$

$$\delta_3 = 0.065 \cos 35.79t$$

$$x_1(t) = 0.896 \cos(12.23t) - 0.461 \cos(25.55t) + 0.065 \cos(35.79t)$$

$$x_2(t) = 1.837 \cos(12.23t) - 0.249 \cos(25.55t) + 0.087 \cos(35.79t)$$

$$x_3(t) = 2.625 \cos(12.23t) + 0.821 \cos(25.55t) + 0.056 \cos(35.79t)$$

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3.3. Proportional Damping

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=0$$

$$(1)\ [C]=\alpha[M]\rightarrow[U]^T[C][U]=\alpha[U]^T[M][U]=\begin{bmatrix}\ddots & & \mathbf{0}\\ & \alpha M_r & \\ \mathbf{0} & & \ddots\end{bmatrix}$$

$$C_r=\zeta_r C_{cr}=2\zeta_r M_r\omega_r=\alpha M_r$$

$$(2)\ [C]=\beta[K]\rightarrow[U]^T[C][U]=\beta[U]^T[K][U]=\begin{bmatrix}\ddots & & \mathbf{0}\\ & \beta\omega_r^2 M_r & \\ \mathbf{0} & & \ddots\end{bmatrix}$$

$$2\zeta_r M_r\omega_r=\beta\omega_r^2 M_r$$

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$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \vdots \\ \ddot{\delta}_n \end{Bmatrix} + \begin{bmatrix} \ddots & & \\ & 2\zeta\omega_r M_r & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_n \end{Bmatrix} + \begin{bmatrix} \ddots & & \\ & \omega_1^2 M_r & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{Bmatrix} = 0$$

$$\ddot{\delta}_r + 2\zeta\omega_r\dot{\delta}_r + \omega_n^2\delta_r = 0 \qquad r = 1,2,...,\text{n}$$

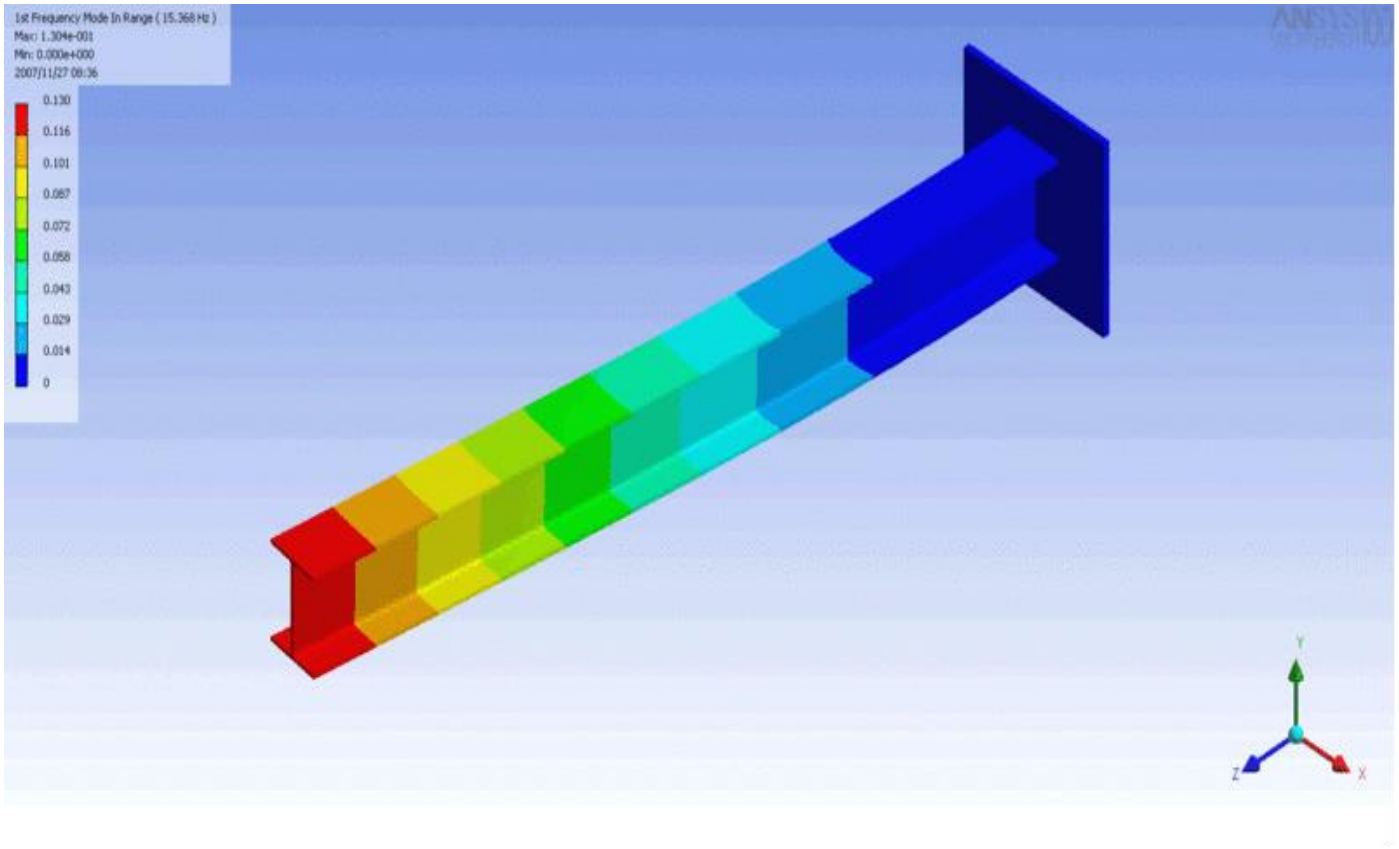
$$\delta_r = e^{-\zeta\omega_n t}(A_r \cos\omega_d t + B_r \sin\omega_d t)$$

(3) $[C] = \alpha[M] + \beta[k]$ $\qquad 2\zeta\omega_r M_r = (\alpha + \beta\omega_n^2)M_r$

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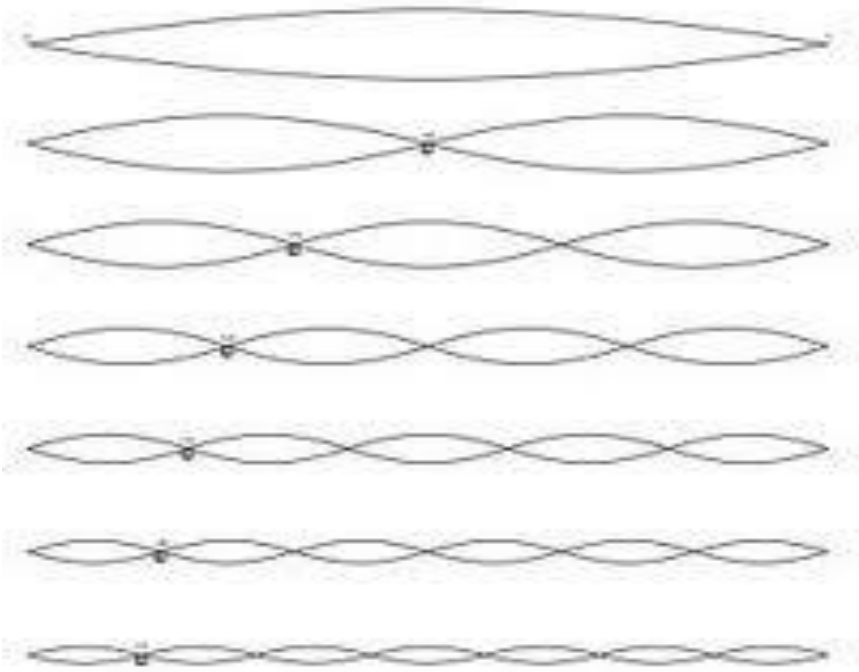
Chapter4. Continuous Vibration



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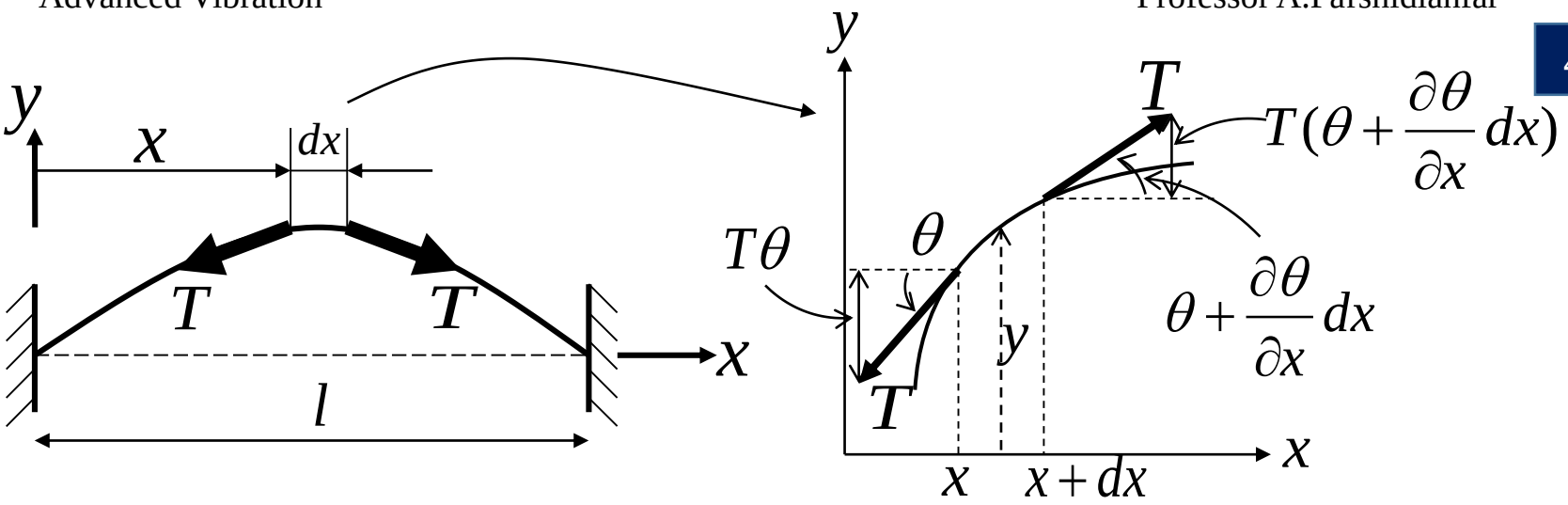
4.1. Lateral vibration of taut string



The piano
The guitar

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 y}{\partial t^2} \quad , \quad C^2 = \frac{T}{m}$$

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B.V.P. (Boundary Value Problem) Small Deflection

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$$ one dimensional wave eq.

$$C^2 = \frac{T}{m}$$ wave speed , m = mass per unit lenght

$$y = y(x,t) \quad , \quad \theta = \frac{\partial y}{\partial x} \quad , \quad M = \text{total mass}$$

$$\sum F = M\ddot{y} \rightarrow T(\theta + \frac{\partial \theta}{\partial x} dx) - T\theta = m dx \frac{\partial^2 y}{\partial t^2} \rightarrow \frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$$

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B.C. $y(0,t)=0$ & $y(l,t)=0$

I.C. $y(x,0)=f(x)$ & $\frac{\partial y(x,0)}{\partial t}=g(x)$, $t=0$

$$y(x,t)=\varphi(x)q(t) \qquad C^2\frac{1}{\varphi(x)}\frac{d^2\varphi(x)}{dx^2}=\frac{1}{q(t)}\frac{d^2q(t)}{dt^2}=-\omega^2$$

$\varphi(x)=$ eigenfunction(mode shapes)

$\omega=$ natural frequency

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$$\left\{\begin{array}{l}\dfrac{d^2\varphi(x)}{d\,x^2}+\dfrac{\omega^2}{c^2}\varphi(x)=0\\[2ex]\dfrac{d^2q(t)}{d\,t^2}+\omega^2 q(t)=0\end{array}\right\}\rightarrow\left\{\begin{array}{l}\varphi(x)=A\sin\dfrac{\omega\,x}{c}+B\cos\dfrac{\omega\,x}{c}\\[2ex]q(t)=D\sin(\omega t+\psi)\end{array}\right\}$$

$$\text{Formal Solution}: \; y(x,t)=(A\sin\frac{\omega\,x}{c}+B\cos\frac{\omega\,x}{c}).\sin(\omega t+\psi)$$

$$\underline{\text{B.C.}}\quad y(0,t)=0\;\Rightarrow\;B=0\;,\;\sin\frac{\omega\,l}{c}=0\rightarrow\;\frac{\omega\,l}{c}=n\pi\quad\text{for}\;\;n=0,1,2,...$$

$$\therefore\;\;\omega_n=\frac{n\pi\,c}{l}\,(\text{rad/sec})\;\;\;n=1,2,3,...$$

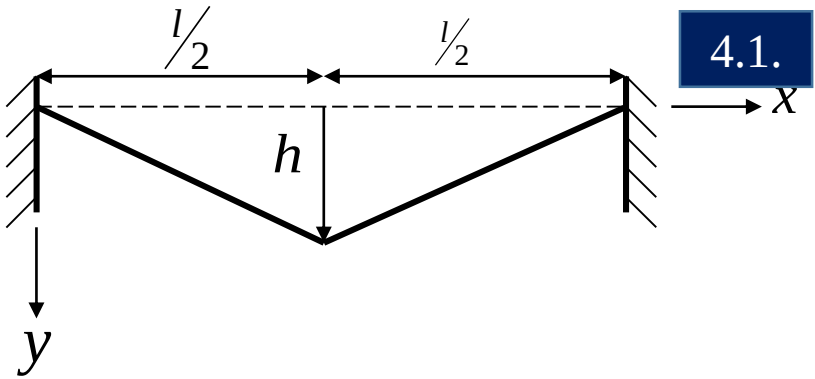
The General Solution :

$$y(x,t)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi\,x}{l}\sin(\omega_n t+\psi_n)=\sum_{n=1}^{\infty}\sin\frac{n\pi\,x}{l}(C_n\sin\omega_n t+D_n\cos\omega_n t)$$

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4.1.

EXAMPLE



Vibrating String

$$y(x,0)=\begin{cases}2hx/l & \text{for } 0\le x\le l/2\\ 2h(1-x/l) & \text{for } l/2\le x\le l\end{cases}$$

$$\dot{y}(x,0)=0\ ,\quad \dot{y}(x,0)=\sum_{n=1}^{\infty}C_n\omega_n\sin\frac{n\pi x}{l}\rightarrow\ C_n=0\ ,\ y(x,0)=\sum_{n=1}^{\infty}D_n\sin\frac{n\pi x}{l}$$

$$\int_0^l D_n\sin\frac{n\pi x}{l}\sin\frac{n\pi x}{l}dx=\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}dx+\int_{l/2}^{l}2h\left(1-\frac{x}{l}\right)\sin\frac{n\pi x}{l}dx$$

$$D_n=\begin{cases}0 & \text{for n even}\\ \dfrac{8h}{n^2\pi^2}(-1)^{(n-1)/2} & \text{for n odd}\end{cases}$$

$$y(x,t)=\frac{8h}{\pi^2}\left(\sin(\pi x/l)\cos(\omega_1 t)-\frac{1}{9}\sin(3\pi x/l)\cos(\omega_3 t)+\frac{1}{25}\sin(5\pi x/l)\cos(\omega_5 t)+\cdots\right)$$

$$y(x,t)=\frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left((2n+1)\frac{\pi x}{l}\right)\cos\left((2n+1)\frac{\pi c}{l}t\right)$$

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Forced vibration of taut string:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{1}{m} f(x,t) \ \text{(I)} \ \text{or} \ \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{1}{T} f(x,t) \ , \ c^2 = \frac{T}{m}$$

Example:

$$\begin{cases} y(x,t) = Y(x)\cos\omega t \\ f(x,t) = F_0 \cos\omega t \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 y}{\partial x^2} = \dfrac{d^2 Y}{d x^2}\cos\omega t \\ \dfrac{\partial^2 y}{\partial t^2} = -\omega^2 Y(x)\cos\omega t \end{cases}$$

$$\text{(I)} \Rightarrow \quad \frac{d^2 Y}{d x^2} + \frac{\omega^2}{c^2} Y(x) = -\frac{F_0}{mc^2}$$

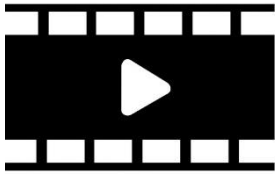
$$Y(x) = Y_h(x) + Y_p(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c} - \frac{F_0}{m\omega^2}$$

B.C. $\quad y(0) = y(l) = 0 \rightarrow A = \dfrac{F_0}{m\omega^2}, B = \dfrac{F_0(1-\cos\frac{\omega l}{c})}{m\omega^2 \sin\frac{\omega l}{c}} = \dfrac{F_0}{m\omega^2}\tan\dfrac{\omega l}{2c}$

$$y(x,t) = \frac{F_0}{m\omega^2}\left[\left(\cos\frac{\omega x}{c} + \tan\frac{\omega l}{2c}\sin\frac{\omega x}{c} - 1\right)\cos\omega t - 1\right]$$

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1. Standing Waves on String Play

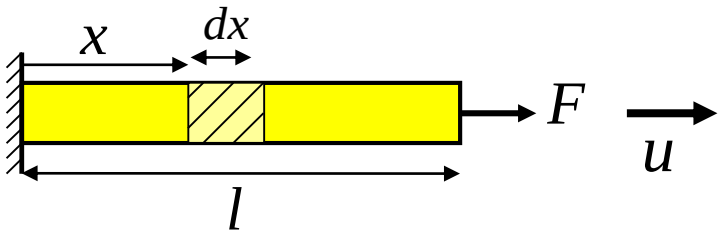
2. String vibration lab Play

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4.2. Longitudinal vibration of rods

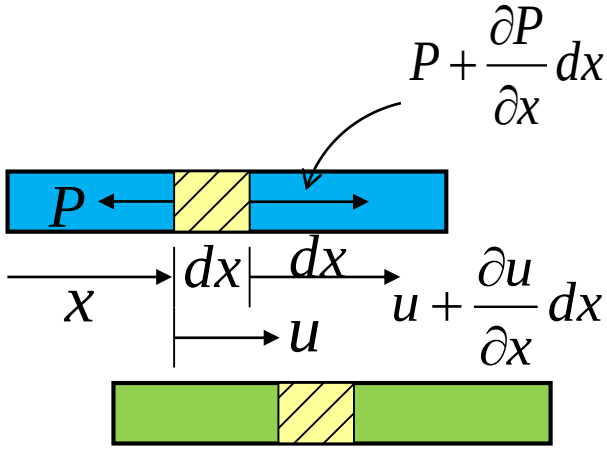
$u(x,t) =$ The axial displacement

$$\sigma = E\varepsilon \ , \quad \frac{P}{A} = E\frac{\partial u}{\partial x}$$



$$\sum F = m\ddot{u} \rightarrow (P + \frac{\partial P}{\partial x}dx) - P = dm\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \frac{\partial P}{\partial x} = \rho A\frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\partial}{\partial x}(EA\frac{\partial u}{\partial x}) = \rho A\frac{\partial^2 u}{\partial t^2}$$



If $EA = \text{const.}$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ The wave eq.

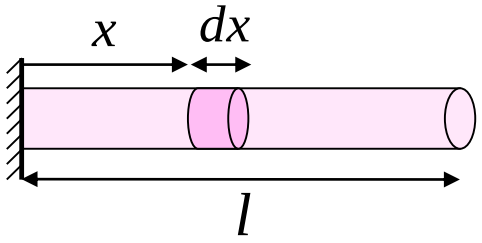
$$c^2 = \frac{E}{\rho}$$ The velocity of wave propagation for L.V.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{EA}p(x,t) \quad \text{Forced vibration}$$

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4.3. Torsional vibration of shafts

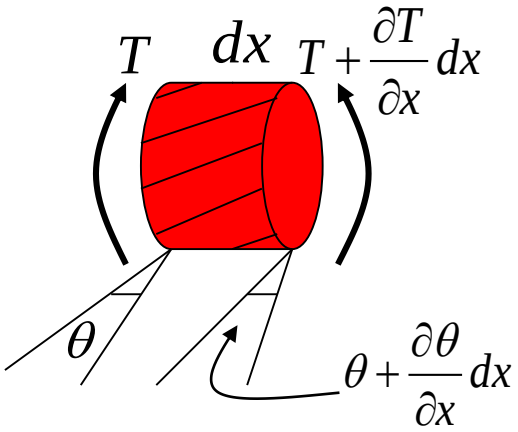


Circular Shaft

$$\sum M = J\ddot{\theta}$$

$\theta(x,t)$ = angular displacement

$$T = I_P G \frac{\partial \theta}{\partial x} \quad \text{Hook's law}$$



I_P = The polar moment of inertia of the cross section

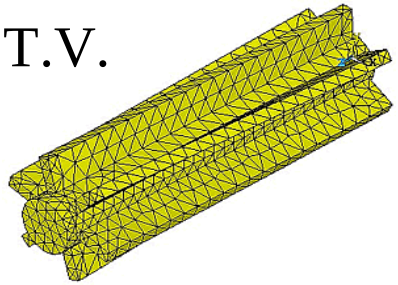
$I_P G$ = The torsional stiffness

$$(T+\frac{\partial T}{\partial x}dx)-T = \rho I_P dx \frac{\partial^2 \theta}{\partial t^2} \qquad \therefore \frac{\partial T}{\partial x} = \rho I_P \frac{\partial^2 \theta}{\partial t^2}$$

$$\frac{\partial}{\partial x}(I_P G \frac{\partial \theta}{\partial x}) = \rho I_P \frac{\partial^2 \theta}{\partial t^2} \qquad \text{If } I_P G = \text{const.} \rightarrow \quad \frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}$$

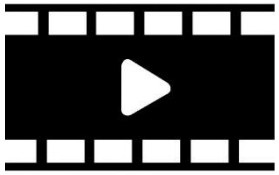
$$c^2 = \frac{G}{\rho} \quad \text{The velocity of wave propagation for T.V.}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} + \frac{T(x,t)}{I_P G}$$



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1. Torsional vibration introduction Play

2. Torsional vibration in engine Play

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EXAMPLE

Some worker have been Imprisonment in Mine. They impact on compressor and Simultaneously they Shout . Peoples out of mine hear the voices with difference 0.2 seconds. Calculate the distance.

Δt= 0.2 sec

ρ_{air} = 1.29 kg/m^3

ρ_{steel} = 7800 kg/m^3

E_{air} = 1.37×10^5 N/m^2

E_{steel} = 210 Gpa

C_{air} =$(E_a/\rho_a)^{1/2}$ =326 m/s

C_{pipe} =$(E_p/\rho)^{1/2}$ =5189 m/s

$C_{pipe} \times t_p = l$, $C_a \times t_a = l$

$t_a = t_p + \Delta t$

$l = C_{pipe}\, C_a .\Delta t / \Delta C$

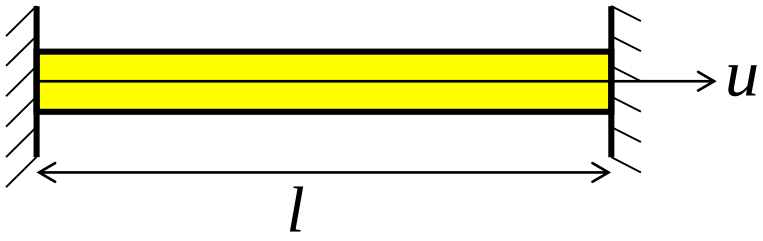
l= 69.5 m

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EXAMPLE

A long uniform rod with both end fixed



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad c^2 = \frac{E}{\rho}$$

$$u(x,t) = (A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c})(c_1\sin\omega t + D\cos\omega t)$$

$\underline{B.C.}$ $\quad u(0,t) = 0 \rightarrow \quad B = 0$

$$u(l,t) = 0 \rightarrow \quad A\sin\frac{\omega l}{c} = 0 = \sin n\pi$$

$$\frac{\omega l}{c} = n\pi \quad \rightarrow \quad \omega_n = \frac{n\pi c}{l} = \frac{n\pi}{l}\sqrt{E/\rho}$$ Natural frequency

$$u_n(x,t) = \sin\frac{n\pi x}{l}(c_1\sin\omega_n t + D\cos\omega_n t)$$

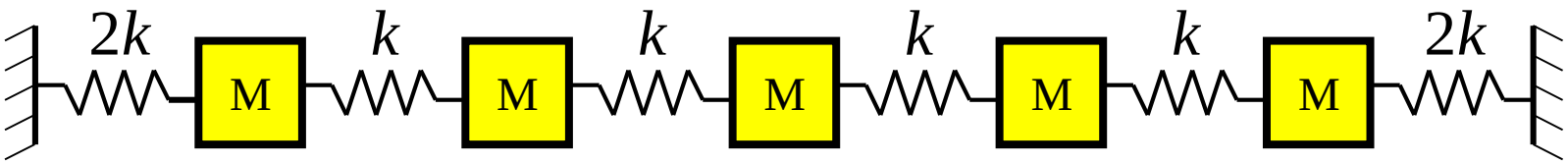
The general solution: $u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}(c_1\sin\omega_n t + D\cos\omega_n t)$

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EXAMPLE

Lumped model



$$M = \frac{ml}{5}, k = \frac{5EA}{l} \qquad [k]\{u\} = \omega^2 [M]\{u\}$$

$$[M] = \frac{ml}{5}[I] \ , \ [I] = \text{unit matrix}$$

$$[k] = \frac{5EA}{l} \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

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EXAMPLE

$$\begin{Bmatrix} (\omega_1)_a = 3.0902\sqrt{EA/ml^2} \\ (\omega_1)_e = 3.1416\sqrt{EA/ml^2} \end{Bmatrix} \rightarrow \quad \{u^{(1)}\} = \begin{Bmatrix} 0.3090 \\ 0.8090 \\ 1.000 \\ 0.8090 \\ 0.3090 \end{Bmatrix}$$

$$\begin{Bmatrix} (\omega_2)_a = 5.8779\sqrt{EA/ml^2} \\ (\omega_2)_e = 6.2832\sqrt{EA/ml^2} \end{Bmatrix}$$

$$\begin{Bmatrix} (\omega_3)_a = 8.0902\sqrt{EA/ml^2} \\ (\omega_3)_e = 9.4248\sqrt{EA/ml^2} \end{Bmatrix}$$

$$\begin{Bmatrix} (\omega_4)_a = 9.5106\sqrt{EA/ml^2} \\ (\omega_4)_e = 12.5664\sqrt{EA/ml^2} \end{Bmatrix}$$

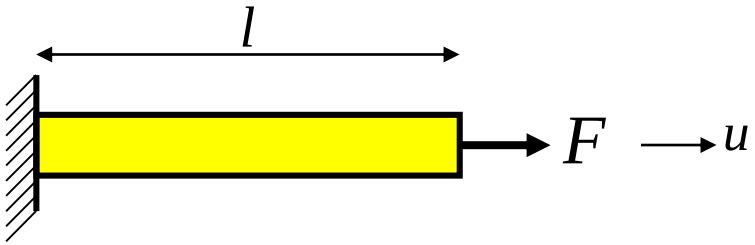
$$\begin{Bmatrix} (\omega_5)_a = 10.000\sqrt{EA/ml^2} \\ (\omega_5)_e = 15.708\sqrt{EA/ml^2} \end{Bmatrix}$$

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4.3.

EXAMPLE



(I.V.P.)

$$\omega_n = \frac{n\pi c}{2l} \quad n = 1,3,5,... \qquad u(x,t) = \sum_{n=1,3,...}^{\infty} \sin\frac{n\pi x}{2l}(C_n \sin\omega_n t + D_n \cos\omega_n t)$$

$$u(x,0) = {Fx}/{AE} \quad (\text{I}) \quad , \quad {\partial u(x,0)}/{\partial t} = 0 \quad (\text{II})$$

$$\dot{u} = \frac{\partial u(x,0)}{\partial t} = \sum_{n=1,3,...}^{\infty} \sin\frac{n\pi x}{2l}(C_n\omega_n \cos\omega_n t - D_n\omega_n \sin\omega_n t)$$

$$(\text{II}) \Rightarrow C_n = 0 \qquad (\text{I}) \Rightarrow \sum_{n=1,3,...}^{\infty} D_n \sin\frac{n\pi x}{2l} = \frac{Fx}{AE}$$

$$D_n \int_0^l \sin^2\frac{n\pi x}{2l}dx = \int_0^l \frac{Fx}{AE}\sin\frac{n\pi x}{2l}dx \rightarrow D_n = \frac{8Fl}{n^2\pi^2 AE}(-1)^{(n-1)/2} \quad n = 1,3,...$$

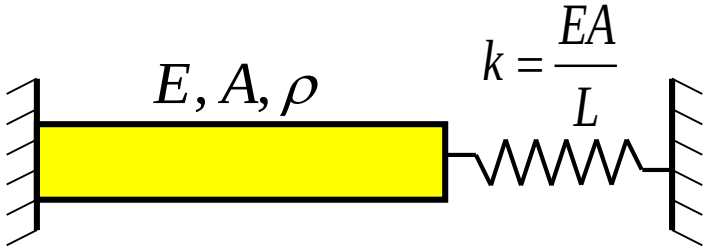
$$u(x,t) = \frac{8Fl}{\pi^2 AE}\sum_{n=1,3,...}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi c}{2l}t \qquad c^2 = \frac{E}{\rho}$$

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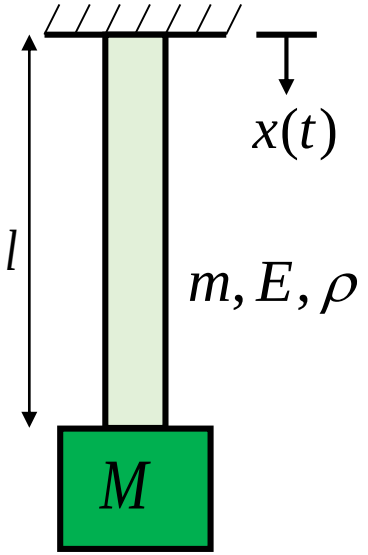
H.W1:

Find first & second frequency & mode shape of Longitudinal vibration



$$\omega_1 = \frac{2.029}{L}\sqrt{\frac{E}{\rho}} \qquad \phi^{(1)}(x) = \sin 2.029\frac{x}{L}$$

$$\omega_2 = \frac{4.913}{L}\sqrt{\frac{E}{\rho}} \qquad \phi^{(2)}(x) = \sin 4.913\frac{x}{L}$$



H.W2: Longitudinal vibration equation of the rod

$$\frac{\omega L}{c}\tan\frac{\omega L}{c} = \frac{mL}{M}$$

$$\underline{B.C.}\left\{\begin{array}{l} u(0,t) = 0 \\ M\dfrac{\partial^2 u(L,t)}{\partial t^2} = -AE\dfrac{\partial u(L,t)}{\partial x} \end{array}\right\}$$

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(1) If $\frac{mL}{M} \to \infty \Rightarrow \tan\frac{\omega L}{c} \to \infty \Rightarrow \cos\frac{\omega L}{c} = 0$

(2) If $M >> mL$ or $\frac{mL}{M} \to 0 \Rightarrow \tan\frac{\omega L}{c} = 0 \Rightarrow$

$$\sin\frac{\omega L}{c} = 0 = \sin n\pi \Rightarrow \omega_1 = \sqrt{\frac{EA}{ML}} = \sqrt{\frac{k}{M}}$$

(3) $\frac{mL}{M} = 0.1 \Rightarrow \frac{\omega_1 L}{c} = 0.32$, $c = \sqrt{\frac{E}{\rho}} \Rightarrow$

$$(\omega_1)_{\text{exact}} = 0.32\frac{c}{L} = 0.32\sqrt{\frac{E}{\rho L^2}} = 0.32\sqrt{\frac{EA}{mL^2}} = 0.32\sqrt{\frac{\text{stiffness}}{\text{mass}}}$$

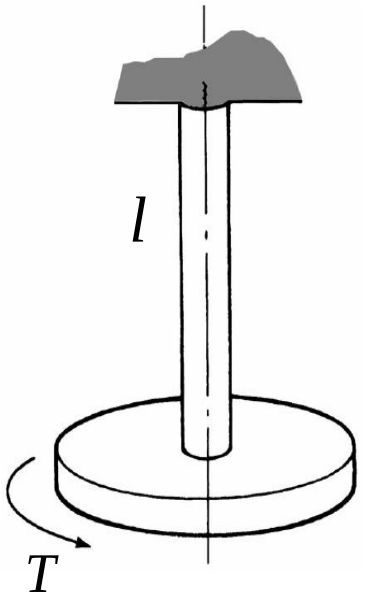
$$(\omega_1)_{\text{app.}} = \sqrt{\frac{k}{M_{eff.}}} = \sqrt{\frac{k}{M + \frac{mL}{3}}} = 0.311\sqrt{\frac{\text{stiffness}}{\text{mass}}}$$

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EXAMPLE

A drill pipe of an oil well



$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} \qquad c^2=\frac{G}{\rho}$$

$$\theta(x,t)=(A\sin\frac{\omega x}{c}+B\cos\frac{\omega x}{c})(C\sin\omega t+D\cos\omega t)$$

B.C. $\begin{cases} x=0: \quad \theta(0,t)=0 \quad \Rightarrow \quad B=0 \\ x=L: \sum M=J\ddot{\theta} \rightarrow J_0\dfrac{\partial^2\theta}{\partial t^2}(L,t)=-I_pG\dfrac{\partial\theta}{\partial x}(L,t) \end{cases}$

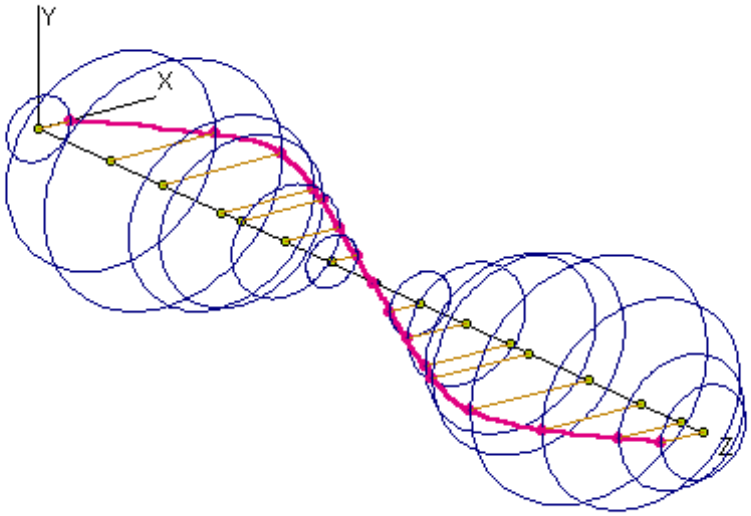
$$-J_0\omega^2\sin\frac{\omega L}{c}=-I_pG\frac{\omega}{c}\cos\frac{\omega L}{c}$$

$$\tan\frac{\omega L}{c}=\frac{I_pG}{J_0\omega c} \qquad \frac{\omega L}{c}\tan\frac{\omega L}{c}=\frac{J_{\text{shaft}}}{J_0}$$

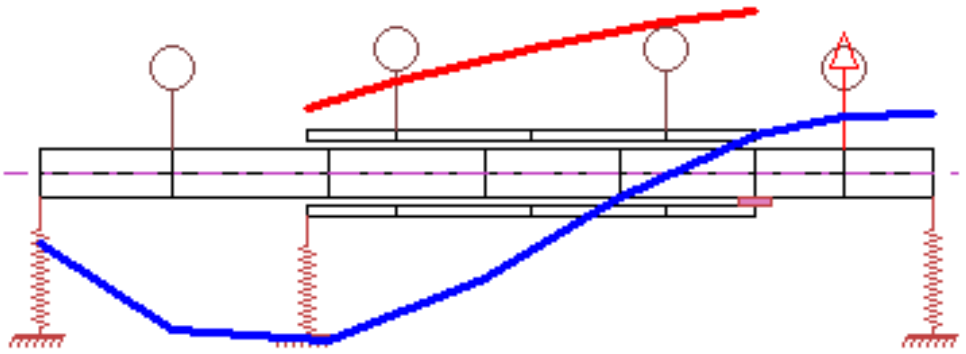
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4.4. Lateral vibration of beams:



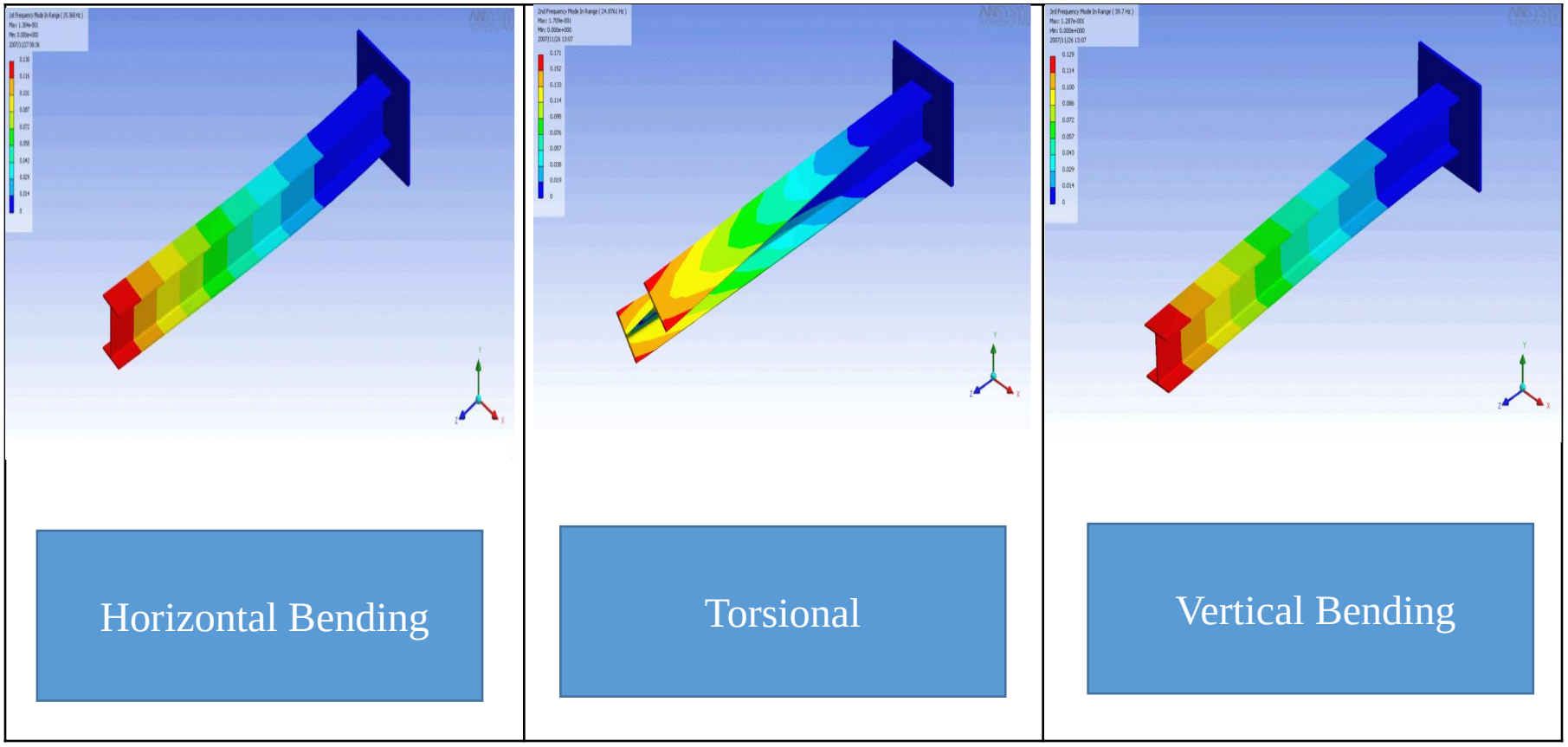
Critical Speed Mode Shape, Mode No.= 2
Spin/Whirl Ratio = 1, Stiffness: Kxx
Critical Speed = 15292 rpm = 254.87 Hz



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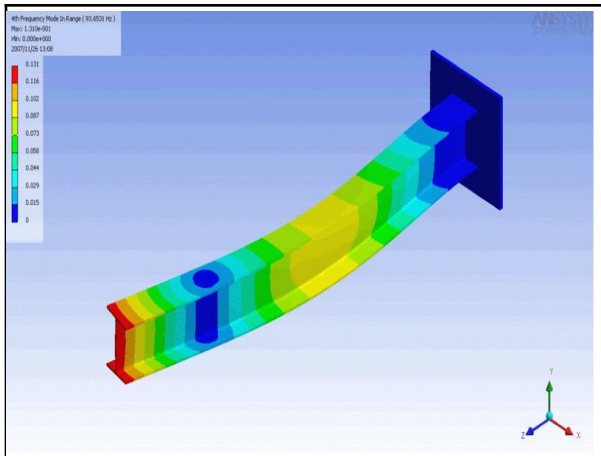
First mode



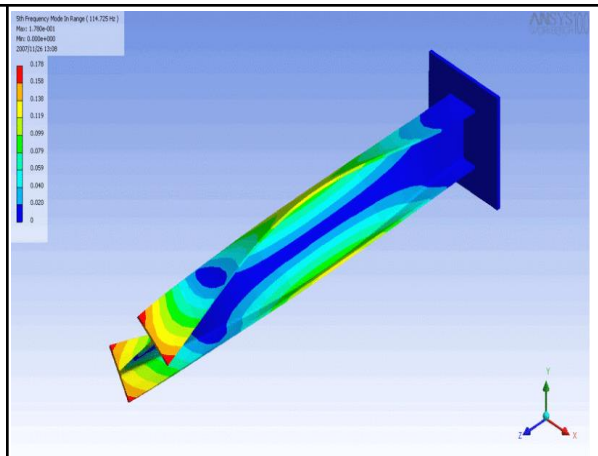
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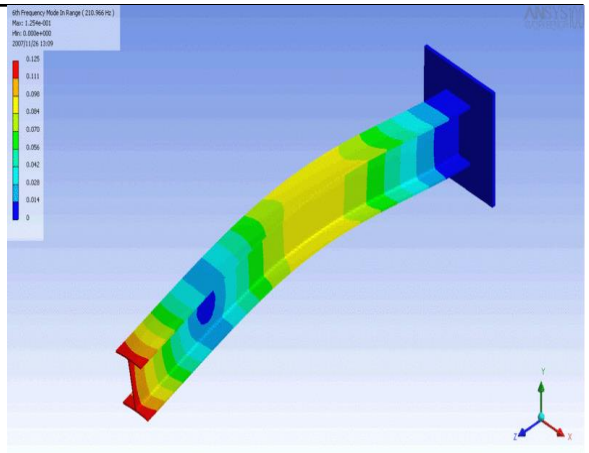
Second mode



Horizontal Bending



Torsional

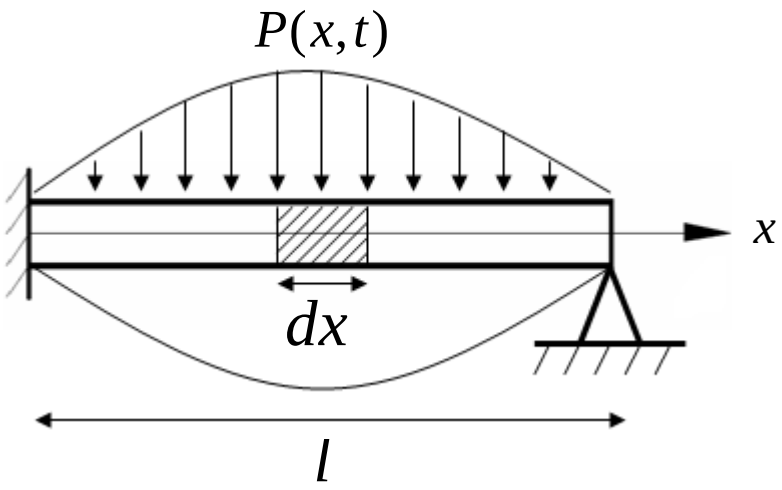


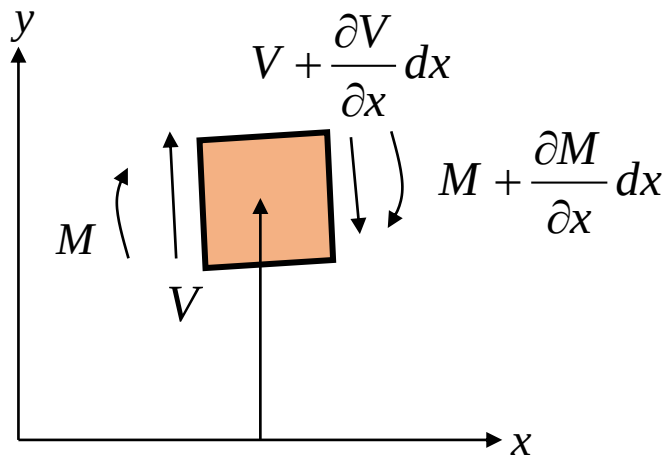
Vertical Bending

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4.4.1. The Euler-Bernoulli beams





$$\begin{cases} M = EI\dfrac{\partial^2 y}{\partial x^2} \\ V = -EI\dfrac{\partial^3 y}{\partial x^3} \end{cases} , \qquad M = \frac{EI}{\rho} \quad \rightarrow \quad \rho = \frac{\left[1+\left(\dfrac{\partial y}{\partial x}\right)^2\right]^{3/2}}{\dfrac{\partial^2 y}{\partial x^2}} \cong \left(\frac{\partial^2 y}{\partial x^2}\right)^{-1}$$

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$$\sum M = 0 \rightarrow \frac{\partial M}{\partial x}dx - Vdx = 0 \rightarrow \frac{\partial M}{\partial x} = V$$

$$\sum F = m\ddot{y} \rightarrow -(V + \frac{\partial V}{\partial x}dx) + V = m\frac{\partial^2 y}{\partial t^2}dx \rightarrow -\frac{\partial V}{\partial x} = m\frac{\partial^2 y}{\partial t^2}$$

$$-\frac{\partial^2 M}{\partial x^2} = m\frac{\partial^2 y}{\partial t^2} \quad , \quad EI\frac{\partial^2 y}{\partial x^2} = M \rightarrow -\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 y}{\partial x^2}) = m\frac{\partial^2 y}{\partial t^2}$$

$$(c^2 = \frac{EI}{m} = \frac{EI}{\rho A}) \qquad \text{If} \quad EI = \text{const.} \qquad -\frac{\partial^4 y}{\partial x^4} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$\text{Forced vibration}: \qquad -\frac{\partial^4 y}{\partial x^4} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} + \frac{1}{EI}p(x,t)$$

$$\text{Or}: \qquad \rho A\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = p(x,t)$$

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Boundary Conditions:

1) Free – Free



$$\text{Free} \quad \text{at} \quad x=0 \rightarrow \begin{cases} \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \\ \dfrac{\partial^3 y}{\partial x^3}=0 \quad (V) \end{cases}$$

$$\text{Free} \quad \text{at} \quad x=l \rightarrow \begin{cases} \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \\ \dfrac{\partial^3 y}{\partial x^3}=0 \quad (V) \end{cases}$$

2) Pinned – Free



$$\text{Pinned} \quad \text{at} \quad x=0 \rightarrow \begin{cases} y=0 \\ \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \end{cases}$$

$$\text{Free} \quad \text{at} \quad x=l \rightarrow \begin{cases} \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \\ \dfrac{\partial^3 y}{\partial x^3}=0 \quad (V) \end{cases}$$

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3) Clamped – Free



$$\text{Clamped at } x=0 \rightarrow \begin{cases} y=0 \\ \dfrac{\partial y}{\partial x}=0 \quad (\theta) \end{cases}$$

$$\text{Free at } x=l \rightarrow \begin{cases} \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \\ \dfrac{\partial^3 y}{\partial x^3}=0 \quad (V) \end{cases}$$

4) Pinned – Pinned



$$\text{Pinned at } x=0 \rightarrow \begin{cases} y=0 \\ \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \end{cases}$$

$$\text{Pinned at } x=l \rightarrow \begin{cases} y=0 \\ \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \end{cases}$$

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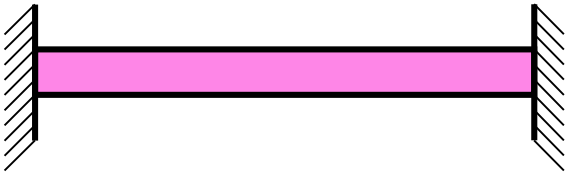
5) Clamped – Pinned



$$\text{Clamped} \quad \text{at} \quad x=0 \rightarrow \begin{cases} y=0 \\ \dfrac{\partial y}{\partial x}=0 \quad (\theta) \end{cases}$$

$$\text{Pinned} \quad \text{at} \quad x=l \rightarrow \begin{cases} y=0 \\ \dfrac{\partial^2 y}{\partial x^2}=0 \quad (M) \end{cases}$$

6) Clamped – Clamped



$$\text{Clamped} \quad \text{at} \quad x=0 \rightarrow \begin{cases} y=0 \\ \dfrac{\partial y}{\partial x}=0 \quad (\theta) \end{cases}$$

$$\text{Clamped} \quad \text{at} \quad x=l \rightarrow \begin{cases} y=0 \\ \dfrac{\partial y}{\partial x}=0 \quad (\theta) \end{cases}$$

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Free vibration of beams:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0$$

$$y(x,t) = \varphi(x)\sin(\omega t) \quad \rightarrow \quad -\rho A \omega^2 \varphi(x)\sin(\omega t) + EI \frac{d^4\varphi(x)}{dx^4}\sin(\omega t) = 0$$

$$,\sin(\omega t) \neq 0 \quad \rightarrow \quad -\rho A \omega^2 \varphi(x) + EI \frac{d^4\varphi(x)}{dx^4} = 0 \quad \Rightarrow \quad \rho A \omega^2 \varphi(x) = EI \frac{d^4\varphi(x)}{dx^4}$$

$$\Rightarrow \qquad \frac{d^4\varphi(x)}{dx^4} = \beta^4 \varphi(x) \qquad (\beta^4 = \frac{\rho A \omega^2}{EI})$$

1) If $\omega = 0$: $\frac{d^4\varphi(x)}{dx^4} = 0 \rightarrow \varphi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ (Rigid Body Motion)

2) If $\omega \neq 0$: $\varphi(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$

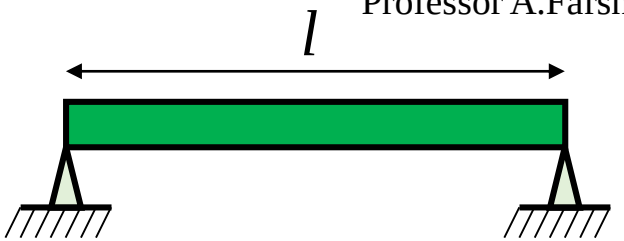
$$\cosh\beta x = \frac{e^{\beta x} + e^{-\beta x}}{2} \quad , \quad \sinh\beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}$$

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EXAMPLE

S.S-B.



$$-\frac{\partial^4 y}{\partial x^4}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t)=\phi(x)q(t) \quad \rightarrow \quad -c^2\frac{1}{\phi(x)}\frac{d^4\phi(x)}{dx^4}=\frac{1}{q(t)}\frac{d^2q(t)}{dt^2}=-\omega^2$$

$$\text{If} \quad \beta^4=\frac{\omega^2}{c^2}=\frac{m\omega^2}{EI}$$

$$\begin{cases} \dfrac{d^4\phi(x)}{dx^4}-\beta^4\phi(x)=0 & (1) \\ \dfrac{d^2q(t)}{dt^2}+\omega^2 q(t)=0 & (2) \end{cases}$$

$$(1) \quad \phi(x)=A\cos\beta x+B\sin\beta x+C\cosh\beta x+D\sinh\beta x$$

$$(2) \quad q(t)=c_1\sin\omega t+c_2\cos\omega t$$

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EXAMPLE

1) $y(0,t)=0 \rightarrow \phi(0)=0 \rightarrow A+C=0$

2) $\dfrac{\partial^2 y(0,t)}{\partial x^2}=0 \rightarrow \dfrac{d^2\phi(0)}{dx^2}=0 \rightarrow \beta^2(-A+C)=0$

$$\Rightarrow A=C=0$$

3) $y(l,t)=0 \rightarrow \phi(l)=0 \rightarrow B\sin\beta l+D\sinh\beta l=0 \quad (1)$

4) $\dfrac{\partial^2 y(l,t)}{\partial x^2}=0 \rightarrow \dfrac{d^2\phi(l)}{dx^2}=0 \rightarrow \beta^2(-B\sin\beta l+D\sinh\beta l)=0 \quad (2)$

$$(1),(2)\Rightarrow \begin{bmatrix}\sin\beta l & \sinh\beta l\\ -\sin\beta l & \sinh\beta l\end{bmatrix}\begin{Bmatrix}B\\D\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \rightarrow \begin{vmatrix}\sin\beta l & \sinh\beta l\\ -\sin\beta l & \sinh\beta l\end{vmatrix}=0$$

$$(\sinh\beta l\neq 0) \rightarrow \sin\beta l=0=\sin n\pi \rightarrow \beta l=n\pi \rightarrow \beta=\frac{n\pi}{l}$$

$$\omega_n=\beta^2 c=\frac{n^2\pi^2}{l^2}c \qquad \therefore \quad \omega_n=n^2\pi^2\sqrt{\frac{EI}{ml^4}} \qquad n=1,2,3,\ldots$$

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EXAMPLE

$$(1)+(2) \rightarrow \quad 2D\sinh\beta l = 0 \quad \rightarrow \quad D = 0$$

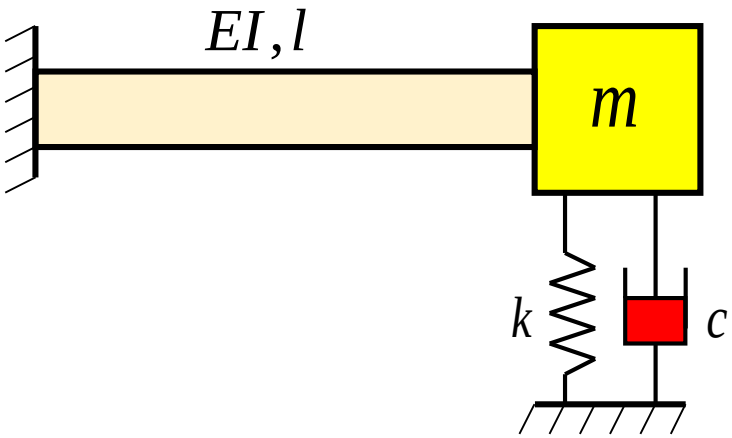
$$\phi(x) = B\sin\beta\, x = B\sin\frac{n\pi\, x}{l} \qquad \text{The natural mode}$$

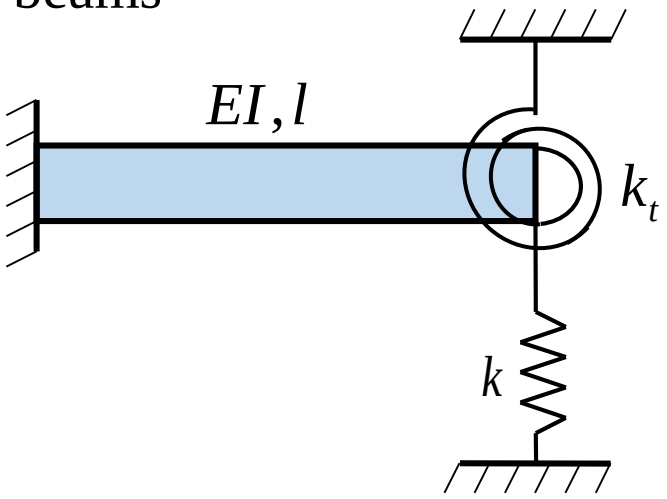
$$y(x,t) = \sum_{n=1}^{\infty} B\sin\frac{n\pi\, x}{l}\left[(c_1)\sin\omega_n t + (c_2)\cos\omega_n t\right] =$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi\, x}{l}\left[(c_1)\sin\omega_n t + (c_2)\cos\omega_n t\right]$$

H.W.3:

Write boundary condition of follow beams



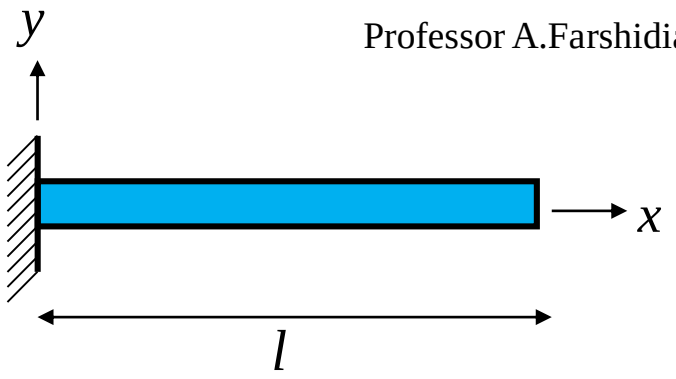


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EXAMPLE

C.F-B.



$$-\frac{\partial^4 y}{\partial x^4}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t)=\phi(x)q(t) \quad \rightarrow \quad -c^2\frac{1}{\phi(x)}\frac{d^4\phi(x)}{dx^4}=\frac{1}{q(t)}\frac{d^2q(t)}{dt^2}=-\omega^2$$

$$\text{if} \quad \beta^4=\frac{\omega^2}{c^2}=\frac{m\omega^2}{EI}$$

$$\begin{cases} \dfrac{d^4\phi(x)}{dx^4}-\beta^4\phi(x)=0 & (1) \\ \dfrac{d^2q(t)}{dt^2}+\omega^2 q(t)=0 & (2) \end{cases}$$

$$(1) \quad \phi(x)=A\cos\beta x+B\sin\beta x+C\cosh\beta x+D\sinh\beta x$$

$$(2) \quad q(t)=c_1\sin\omega t+c_2\cos\omega t$$

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EXAMPLE

B.C.

$$\text{At } x=0 \rightarrow \begin{cases} \varphi(x)=0 \rightarrow A+C=0 \rightarrow A=-C \\ \dfrac{d\varphi(0)}{dx}=0 \rightarrow \beta(D+B)=0 \xrightarrow{\beta\neq 0} D=-B \end{cases}$$

$$\varphi(x)=A(\cos\beta x-\cosh\beta x)+B(\sin\beta x-\sinh\beta x)$$

$$\text{At } x=l \rightarrow \begin{cases} \dfrac{d^2\varphi(l)}{dx^2}=0 \rightarrow -\beta^2 A(\cos\beta l+\cosh\beta l)-\beta^2 B(\sin\beta l+\sinh\beta l) \\ \dfrac{d^3\varphi(l)}{dx^3}=0 \rightarrow \beta^3 A(\sin\beta l-\sinh\beta l)-\beta^3 B(\cos\beta l+\cosh\beta l) \end{cases}$$

$$\begin{bmatrix} -(\cos\beta l+\cosh\beta l) & -(\sin\beta l+\sinh\beta l) \\ (\sin\beta l-\sinh\beta l) & -(\cos\beta l+\cosh\beta l) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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EXAMPLE

$$\begin{vmatrix} -(\cos\beta l+\cosh\beta l) & -(\sin\beta l+\sinh\beta l) \\ (\sin\beta l-\sinh\beta l) & -(\cos\beta l+\cosh\beta l) \end{vmatrix}=0$$

$$\rightarrow \quad (\cos\beta l+\cosh\beta l)^2+(\sin^2\beta l-\sinh^2\beta l)=0$$

$$\cos^2\beta l+\cosh^2\beta l+2\cos\beta l\cosh\beta l+\sin^2\beta l-\sinh^2\beta l=0$$

$$(\sin^2\beta l+\cos^2\beta l=1\ ,\ \ \cosh^2\beta l-\sinh^2\beta l=1)\ \rightarrow\ 2+2\cos\beta l\cosh\beta l=0$$

$$\cos\beta l\cosh\beta l+1=0 \qquad \text{Frequency Equation}$$

$$\omega_n=(\beta_n l)^2\sqrt{\frac{EI}{ml^4}}\ ,\ \ \cos\beta\ l\cosh\beta\ l=-1$$

$$\rightarrow \beta_1 l=1.875\ ,\ \beta_2 l=4.694\ ,\ \beta_3 l=7.855\ ,\ \beta_4 l=10.946$$

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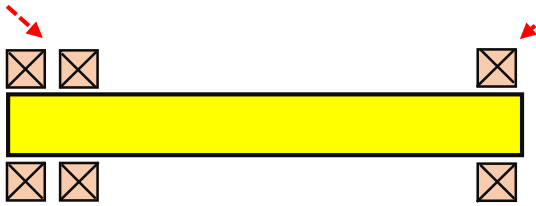
Boundary Conditions	Frequency Equations
Free-Free	$\cos \beta l \cosh \beta l - 1 = 0$
Pinned-Free	$\tan \beta l - \tanh \beta l = 0$
Clamped-Free	$\cos \beta l \cosh \beta l + 1 = 0$
Pinned-Pinned	$\sin \beta l = 0$
Clamped-Pinned	$\tan \beta l - \tanh \beta l = 0$
Clamped-Clamped	$\cos \beta l \cosh \beta l - 1 = 0$

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EXAMPLE

Self Align Bearing Bearing



$$\begin{cases} E = 206\ \text{GN/m}^3 \\ \rho = 7860\,\text{kg/m}^3 \\ L = 1.5\ \text{m} \\ d = 20\ \text{mm} \\ \text{Speed} = 1000\text{rpm} \end{cases}$$

What are three first critical speeds?

$$\tan \beta L - \tanh \beta L = 0$$

$$\omega_n = (\beta_n L)^2 \sqrt{\frac{EI}{\rho A L}}$$



Clamped-Pinned

$$x = 0 \Rightarrow \begin{cases} y = 0 \\ \dfrac{\partial y}{\partial x} = 0 \end{cases} \qquad x = L \Rightarrow \begin{cases} y = 0 \\ \dfrac{\partial^2 y}{\partial x^2} = 0 \end{cases}$$

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EXAMPLE

$$\omega_n = (\beta_n L)^2 \sqrt{\frac{EI}{\rho AL}} \rightarrow \begin{cases} \beta_1 L = 3.927 \\ \beta_2 L = 7.064 \\ \beta_3 L = 10.210 \\ \beta_4 L = 13.352 \\ \beta_5 L = 16.443 \end{cases}$$

$$A = \pi r^2 \qquad I = \frac{\pi}{4} r^4 \qquad \frac{EI}{\rho AL} = 129.63 \frac{1}{s^2}$$

$$\begin{cases} \omega_1 = 175.4 \, \text{rad/sec} \\ \omega_2 = 568.5 \, \text{rad/sec} \\ \omega_3 = 1186.0 \, \text{rad/sec} \end{cases}$$

Gyration radius

$$I = Ak^2 \rightarrow \begin{cases} k = \frac{R}{2} \, \text{Circle} \\ k = \frac{b}{\sqrt{2}} \, \text{Rectangle} \end{cases} \rightarrow \omega_n = (\beta_n L)^2 \frac{k}{L^2} \sqrt{\frac{EI}{\rho}}$$

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EXAMPLE

$$\omega_n = (\beta_n L)^2 \frac{k}{L^2}\sqrt{\frac{EI}{\rho}} \rightarrow \omega_n \propto \frac{\text{Cross-Section}}{(\text{Length})^2} \rightarrow \frac{\omega_n'}{\omega_n} = \frac{L^2}{(L')^2}$$

$$\omega_n' = \frac{L^2}{(L')^2} \times \omega_n = \frac{1.5^2}{(L')^2} \times 1675 \geq 1000 \rightarrow L' \leq 0.614\text{m} \rightarrow L' = 600\text{mm}$$

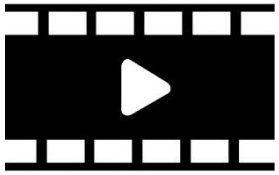
$$\omega_n = (\beta_n L)^2 \frac{k}{L^2}\sqrt{\frac{EI}{\rho}} \rightarrow \frac{\omega_n'}{\omega_n} = \left(\frac{\beta L}{(\beta L)'}\right)^2$$



$$\omega_n' = \left(\frac{4.73}{3.927}\right)^2 \times 1675 = 2430\text{rpm}$$

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- Vibration of a cantilever beam & wing Play

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4.4.1.1. Principal Modes

$$\frac{B}{A}=\frac{\sin\beta_n l-\sinh\beta_n l}{\cos\beta_n l+\cosh\beta_n l}=-\sigma_n \qquad \text{(Clamped-Free)}$$

$$\varphi(x)=A\left[\cos\beta_n x-\cosh\beta_n x-\sigma_n(\sin\beta_n x-\sinh\beta_n x)\right]$$

If $A=-1.0$

$$\rightarrow \varphi_r(x)=\cosh\beta_r x-\cos\beta_r x-\sigma_r(\sinh\beta_r x-\sin\beta_r x)$$

(Clamped-Free & Clamped-pinned & Clamped-Clamped)

$$\varphi_r(x)=\cosh\beta_r x+\cos\beta_r x-\sigma_r(\sinh\beta_r x+\sin\beta_r x)$$

(Free-Free & pinned -Free)

$$\sigma_r=\frac{\sinh\beta_r l-\sin\beta_r l}{\cosh\beta_r l+\cos\beta_r l} \qquad \text{for (Clamped-Free)}$$

$$\sigma_r=\frac{\cosh\beta_r l-\cos\beta_r l}{\sinh\beta_r l-\sin\beta_r l} \qquad \text{for (others)}$$

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Orthogonality of principal modes:

$$\omega_r \rightarrow \quad \rho A\omega_r^2\varphi_r(x) = EI\frac{d^4\varphi_r}{dx^4} \qquad \times\ \varphi_s(x) \quad (1)$$

$$\omega_s \rightarrow \quad \rho A\omega_s^2\varphi_s(x) = EI\frac{d^4\varphi_s}{dx^4} \qquad \times\ \varphi_r(x) \quad (2)$$

$$(1)-(2) \xrightarrow{\omega_r \neq \omega_s} (\omega_r^2-\omega_s^2)\rho A\varphi_r(x)\varphi_s(x) = EI(\varphi_s\frac{d^4\varphi_r}{dx^4} - \varphi_r\frac{d^4\varphi_s}{dx^4})$$

$$(\omega_r^2-\omega_s^2)\rho A\int_0^l \varphi_r(x)\varphi_s(x)dx = EI\int_0^l (\underbrace{\varphi_s\frac{d^4\varphi_r}{dx^4} - \varphi_r\frac{d^4\varphi_s}{dx^4}}_{I})dx$$

$$\begin{cases} \varphi_s = u \rightarrow \dfrac{d\varphi_s}{dx} = du \\ \dfrac{d^4\varphi_r}{dx^4} = dv \rightarrow \dfrac{d^3\varphi_r}{dx^3} = v \end{cases} \Rightarrow \quad I = uv - \int vdu$$

$$I = \left[\varphi_s\frac{d^3\varphi_r}{dx^3} - \varphi_r\frac{d^3\varphi_s}{dx^3}\right]_0^l - \int_0^l (\underbrace{\frac{d\varphi_s}{dx}\frac{d^3\varphi_r}{dx^3}}_{I'} - \frac{d\varphi_r}{dx}\frac{d^3\varphi_s}{dx^3})dx$$

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$$\begin{cases} \dfrac{d\varphi_s}{dx} = u \quad \rightarrow \dfrac{d^2\varphi_s}{dx^2} = du \\ \dfrac{d^3\varphi_r}{dx^3} = dv \quad \rightarrow \quad \dfrac{d^2\varphi_r}{dx^2} = v \end{cases} \Rightarrow \quad I' = uv - \int v du$$

$$I' = \left[\varphi_s \frac{d^3\varphi_r}{dx^3} - \varphi_r \frac{d^3\varphi_s}{dx^3} - \frac{d\varphi_s}{dx}\frac{d^2\varphi_r}{dx^2} + \frac{d\varphi_r}{dx}\frac{d^2\varphi_s}{dx^2} \right]_0^l +$$

$$\int_0^l \left(\frac{d^2\varphi_s}{dx^2}\frac{d^2\varphi_r}{dx^2} - \frac{d^2\varphi_r}{dx^2}\frac{d^2\varphi_s}{dx^2} \right) dx = 0$$

$$\Rightarrow \quad (\omega_r^2 - \omega_s^2)\rho A \int_0^l \varphi_r(x)\varphi_s(x)dx = 0$$

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for $\omega_r \neq \omega_s$

$$\rightarrow \int_0^l \varphi_r(x)\varphi_s(x)dx = 0$$

The first orthogonality condition for a beam in flexture

$$\int_0^l \left(\frac{d^2\varphi_r}{dx^2}\frac{d^2\varphi_s}{dx^2}\right)dx = 0$$

The second orthogonality condition

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$$\text{for } \omega_r = \omega_s \quad \rightarrow \quad \int_0^l [\varphi_r(x)]^2 dx = Z \qquad \text{Scaling}(*)$$

$$y_r(x,t) = \varphi_r(x)\sin\omega_r t \quad , \quad y(x,t) = \sum_{r=1}^{\infty} k_r \varphi_r(x)\sin(\omega_r t + \varepsilon_r) \quad (A) \quad r = 1,2,3,\ldots$$

$$T = \frac{1}{2}\int_0^l \rho A (\frac{\partial y}{\partial t})^2 dx = \frac{1}{2}\rho A \int_0^l (\frac{\partial y}{\partial t})^2 dx$$

$$\frac{\partial y}{\partial t} = \sum_{r=1}^{\infty} \omega_r k_r \varphi_r(x)\cos(\omega_r t + \varepsilon_r) \xrightarrow{(*)}$$

$$T = \frac{1}{2}\rho A Z \sum_{r=1}^{\infty} \omega_r^2 k_r^2 \cos^2(\omega_r t + \varepsilon_r) \quad \text{Kinetic Energy}$$

$$V = \frac{1}{2} EI \int_0^l (\frac{\partial^2 y}{\partial x^2})^2 dx = \frac{1}{2} EIZ \sum_{r=1}^{\infty} \omega_r^2 k_r^2 \sin^2(\omega_r t + \varepsilon_r) \quad \text{Potential Energy}$$

$$\int_0^l \rho A \varphi_r(x)\varphi_s(x) dx = \delta_{rs} = \begin{cases} 0 & if \quad r \neq s \\ 1 & if \quad r = s \end{cases}$$

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I.V.P.

$$At\ \ t=0: \begin{cases} y(x,t)=y(x,0)=g(x) \\ \dfrac{\partial y(x,0)}{\partial t}=0 \end{cases}$$

$$(A)\rightarrow \begin{cases} g(x)=\sum_{r=1}^{\infty} k_r\varphi_r(x)\sin(\varepsilon_r) \qquad (1) \qquad \times\varphi_s \\ 0=\sum_{r=1}^{\infty} \omega_r k_r\varphi_r(x)\cos(\varepsilon_r) \qquad (2) \qquad \times\varphi_s \end{cases}$$

$$(1)\rightarrow \quad \int_0^l \varphi_s(x)g(x)dx=\int_0^l \varphi_s(x)\sum_{r=1}^{\infty}[k_r\sin(\varepsilon_r)\varphi_r(x)]dx=$$

$$\rightarrow \quad \int_0^l \varphi_s(x)g(x)dx=Zk_s\sin(\varepsilon_s) \quad s=1,2,3,\ldots$$

$$(2)\rightarrow \quad 0=\int_0^l \varphi_s(x)\sum_{r=1}^{\infty}[\omega_r k_r\varphi_r(x)\cos(\varepsilon_r)]dx=Z\omega_s k_s\cos(\varepsilon_s)$$

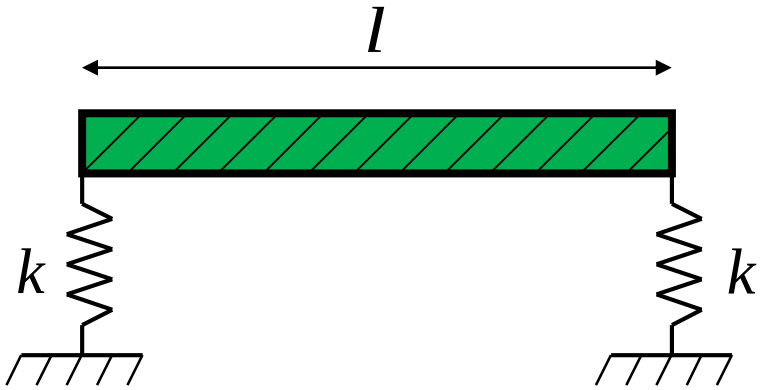
$$\therefore\ \cos(\varepsilon_s)=0 \rightarrow \varepsilon_s=\frac{\pi}{2} \rightarrow k_s=\frac{1}{Z}\int_0^l g(x)\varphi_s(x)dx$$

$$(A)\rightarrow \quad y(x,t)=\sum_{r=1}^{\infty} k_r\varphi_r(x)\cos(\omega_r t)$$

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H.W.4: Find the frequency equation of this system.



$$\cos(\beta L)\cosh(\beta L)-1+2\frac{k}{EI\,\beta^3}\left(\sin(\beta L)\cosh(\beta L)-\cos(\beta L)\sinh(\beta L)\right)$$
$$-2\left(\frac{k}{EI\,\beta^3}\right)^3\sin(\beta L)\sinh(\beta L)=0$$

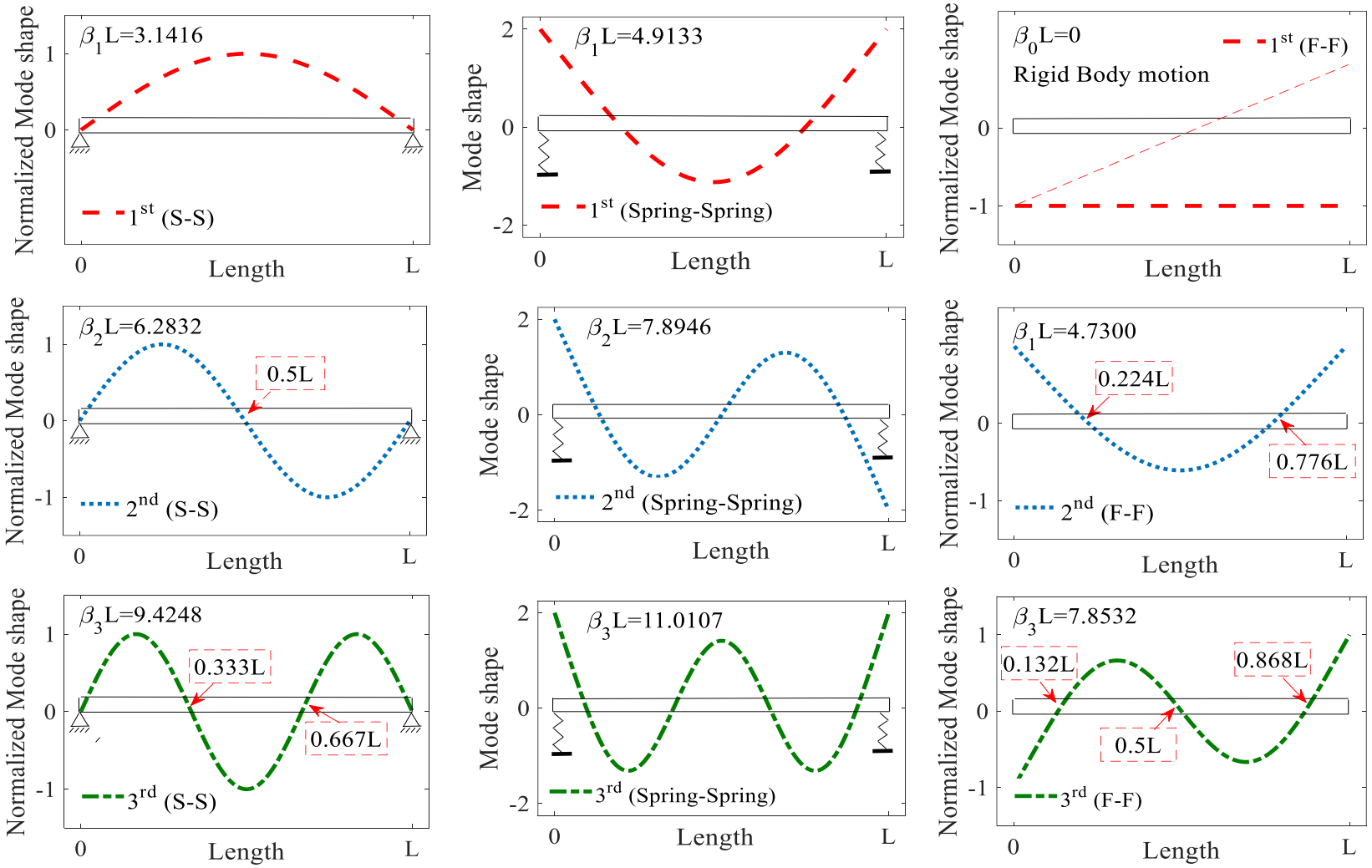
1- In free-free case: $k=0 \rightarrow \quad \cos(\beta L)\cosh(\beta L)=-1$

2- In pinned-pinned case: $k=\infty \rightarrow -2\left(\frac{k}{EI\,\beta^3}\right)^3\sin(\beta L)\sinh(\beta L)=0 \rightarrow \sin(\beta L)=0$

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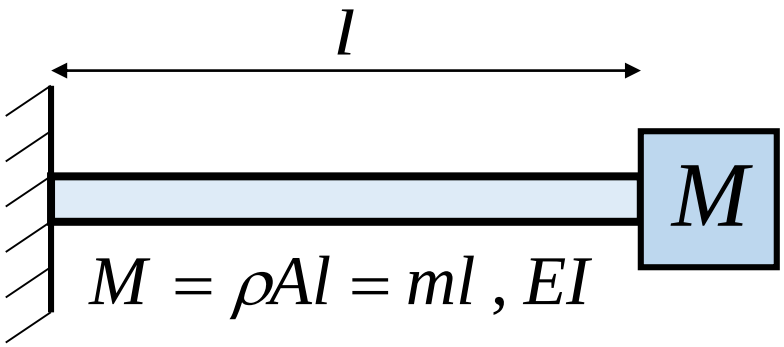
H.W.4: Find the frequency equation of this system.

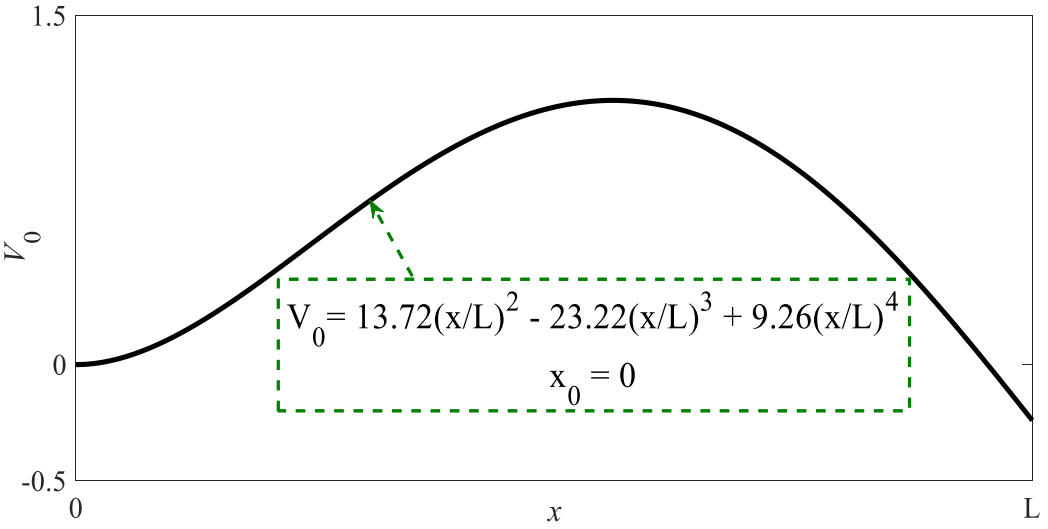


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H.W.5: Find the lateral frequency of the beam.





Solution: $$y(x,t) = \sum_{r=1}^{\infty} C_r \left\{ \sin(\beta_r x) - \sinh(\beta_r x) - \frac{\sin(\beta_r L) + \sinh(\beta_r L)}{\cos(\beta_r L) + \cosh(\beta_r L)} \left(\cos(\beta_r x) - \cosh(\beta_r x) \right) \right\} \sin(\omega_r t)$$

$$C_1 = -0.0404, C_2 = 0.7761, C_3 = -0.0003$$

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General Homework (Palm)

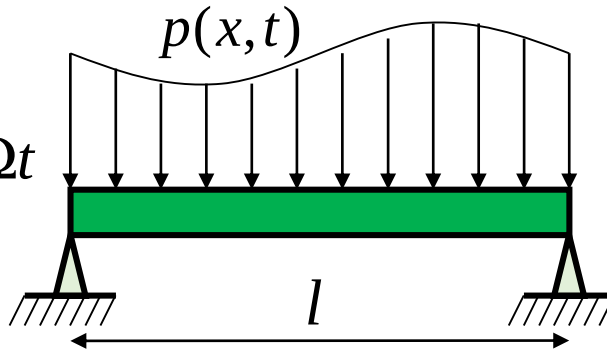
- 10.2
- 10.5
- 10.10
- 10.18
- 10.19
- 10.21
- 10.24



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4.4.1.2. Forced Harmonic Response



$$\rho A\frac{\partial^2 y(x,t)}{\partial t^2}+EI\frac{\partial^4 y(x,t)}{\partial x^4}=p(x,t)=\psi(x)\sin\Omega t$$

$$y(x,t)=\sum_{r=1}^{\infty}Q_r\varphi_r(x)\sin\Omega t$$

$$-\rho A\Omega^2\sum_{r=1}^{\infty}Q_r\varphi_r(x)\sin\Omega t+EI\sin\Omega t\sum_{r=1}^{\infty}Q_r\frac{d^4\varphi_r}{dx^4}=\psi(x)\sin\Omega t$$

$$\rightarrow\quad -\rho A\Omega^2\sum_{r=1}^{\infty}Q_r\varphi_r(x)+EI\sum_{r=1}^{\infty}Q_r\frac{d^4\varphi_r}{dx^4}=\psi(x)\quad(1)$$

$$\rho A\omega_r^2\varphi_r(x)=EI\frac{d^4\varphi_r}{dx^4}\quad(2)$$

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$$(2) in (1) \rightarrow \quad -\rho A \Omega^2 \sum_{r=1}^{\infty} Q_r \varphi_r(x) + \rho A \sum_{r=1}^{\infty} \omega_r^2 Q_r \varphi_r(x) = \psi(x) \qquad \times \varphi_s$$

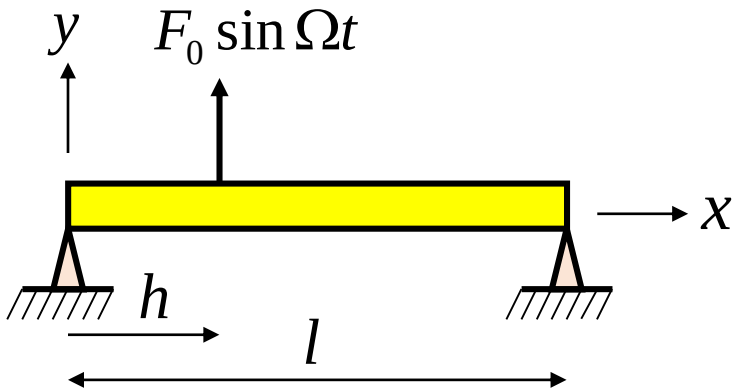
$$\rho A (\omega_s^2 - \Omega^2) Q_s \underbrace{\int_0^l [\varphi_s(x)]^2 dx}_{Z} = \int_0^l \psi(x) \varphi_s(x) dx$$

$$Q_s = \frac{\Psi_s}{\rho A Z (\omega_s^2 - \Omega^2)} \quad , \quad \Psi_s = \int_0^l \psi(x) \varphi_s(x) dx$$

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H.W.6: Find frequency response of the beam.



Answer:

$$y(x,t) = \frac{F_0 \sin \Omega t}{\rho A Z} \sum_{r=1}^{\infty} \frac{\varphi_r(h)\varphi_r(x)}{\omega_r^2 - \Omega^2} = \alpha_{xh} F_0 \sin \Omega t$$

$$\alpha_{xh} = \frac{1}{\rho A Z} \sum_{r=1}^{\infty} \frac{\varphi_r(h)\varphi_r(x)}{\omega_r^2 - \Omega^2}$$

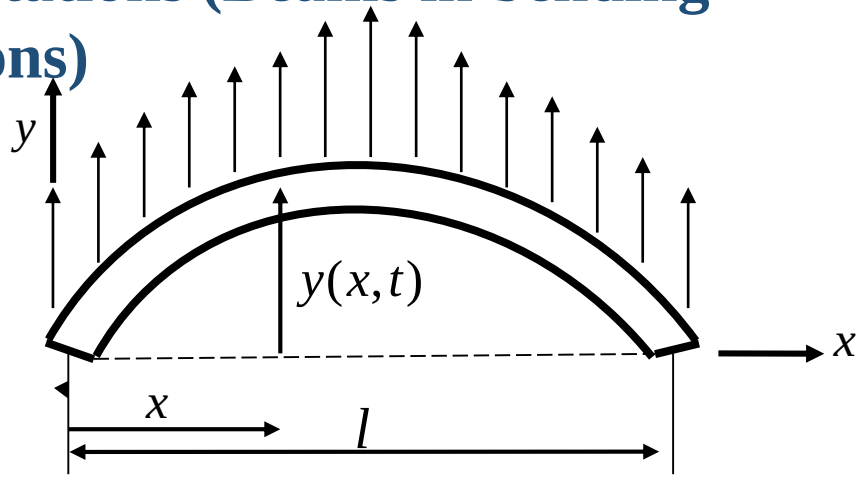
$$\text{If} \quad \omega_r = \Omega \quad \rightarrow \quad \text{Resonance}$$

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4.4.1.3. Response to initial excitations (Beams in bending vibrations)

$$\rho A\frac{\partial^2 y(x,t)}{\partial t^2}+EI\frac{\partial^4 y(x,t)}{\partial x^4}=f(x,t)$$



Initial Excitation :

$$y(x,0)=y_0(x),\frac{\partial y(x,0)}{\partial t}=V_0(x)$$

$$if\ \ f(x,t)=0 \quad \text{Using the expansion theorem :} \quad y(x,t)=\sum_{r=1}^{\infty}\phi_r(x)\eta_r(t)$$

$\phi_r(x)=$ The normal modes of the system

$\eta_r(t)=$ Time – dependent function

$$\rho A\sum_{r=1}^{\infty}\phi_r(x)\frac{d^2\eta_r(t)}{dt^2}+EI\sum_{r=1}^{\infty}\frac{d^4\phi_r(x)}{dx^4}\eta_r(t)=0 \qquad \times\phi_s(x)$$

$$\rho A\sum_{r=1}^{\infty}\left[\int_0^l\phi_r(x)\phi_s(x)dx\right]\frac{d^2\eta_r(t)}{dt^2}+EI\sum_{r=1}^{\infty}\left[\int_0^l\phi_s(x)\frac{d^4\phi_r(x)}{dx^4}dx\right]\eta_r(t)=0 \quad (1)$$

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$$\rho A\int_0^l \varphi_r(x)\varphi_s(x)dx = 0 \quad for \quad \omega_r \neq \omega_s \qquad r,s = 1,2,\ldots$$ The orthogonality relations

$$\rho A\,\omega_r^2\varphi_r(x) = EI\,\frac{d^4\varphi_r(x)}{dx^4} \quad \rightarrow \quad \omega_r^2\rho A\int_0^l \varphi_r(x)\varphi_s(x)dx = \mathrm{EI}\int_0^l \varphi_s(x)\frac{d^4\varphi_r(x)}{dx^4}dx \qquad (2)$$

$$\int_0^l \rho A\,\varphi_r^2(x)dx = 1.0 \qquad r = 1,2,\ldots$$ Normalized the natural modes

$$\rho A\int_0^l \varphi_r(x)\varphi_s(x)dx = \delta_{rs} \qquad r,s = 1,2,\ldots$$

$$(2) \rightarrow \quad EI\int_0^l \varphi_s(x)\frac{d^4\varphi_r(x)}{dx^4}dx = \omega_r^2\delta_{rs} \qquad r,s = 1,2,\ldots$$

$$(1) \rightarrow \quad \frac{d^2\eta_r(t)}{dt^2} + \omega_r^2\eta_r(t) = 0 \qquad r = 1,2,\ldots$$

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The Modal Equation : $\ddot{\eta}_r(t)+\omega_r^2\eta_r(t)=0$

$\eta_r(t)=$ The modal coordinates

$$\eta_r(t)=c_r\cos(\omega_r t-\varphi_r)=\eta_r(0)\cos\omega_r t+\frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t \qquad r=1,2,\ldots$$

$$\begin{cases} c_r=\sqrt{\eta_r^2(0)+[\dfrac{\dot{\eta}_r(0)}{\omega_r}]^2} & \text{The amplitudes} \\ \varphi_r=\tan^{-1}\dfrac{\dot{\eta}_r(0)}{\omega_r\eta_r(0)} & \text{The phase angle} \end{cases}$$

$$y(x,0)=\sum_{r=1}^{\infty}\varphi_r(x)\eta_r(0)=y_0(x) \qquad \times\rho A\varphi_s(x)$$

$$\begin{cases} \eta_r(0)=\displaystyle\int_0^l \rho A\varphi_r(x)y_0(x)dx \\ \dot{\eta}_r(0)=\displaystyle\int_0^l \rho A\varphi_r(x)V_0(x)dx \end{cases} \qquad r=1,2,\ldots$$

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4.4.1.4. Response to external excitations

$$f(x,t) \neq 0 \quad , \quad y(x,t) = \sum_{r=1}^{\infty} \varphi_r(x)\eta_r(t)$$

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r(t) \qquad r = 1,2,... \qquad \text{The Modal Equation}$$

$$N_r(t) = \int_0^l \varphi_r(x) f(x,t) dx \qquad \text{The Modal Forces}$$

For harmonic excitation: $f(x,t) = F(x)\cos\Omega t$

$$N_r(t) = \left[\int_0^l \varphi_r(x) F(x) dx \right] \cos\Omega t = F_r \cos\Omega t \qquad r = 1,2,...$$

$$F_r = \int_0^l \varphi_r(x) F(x) dx \qquad r = 1,2,... \qquad \text{The Modal Force Amplitude}$$

$$\eta_r(t) = \frac{F_r}{\omega_r^2 - \Omega^2} \cos\Omega t \qquad r = 1,2,...$$

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The steady state harmonic response:

$$y(x,t)=\left[\sum_{r=1}^{\infty}\frac{F_r}{\omega_r^2-\Omega^2}\varphi_r(x)\right]\cos\Omega t$$

If $F(x)=m(x)\varphi_k(x) \rightarrow F_r=0$

for $r\neq k \rightarrow y(x,t)=\dfrac{F_k}{\omega_k^2-\Omega^2}\varphi_k(x)\cos\Omega t$ If $\omega_r=\Omega \rightarrow$ Resonance

If $f(x,t)=$ arbitrary $\rightarrow$ $N_r(t)=$ arbitrary

The convolution integrals:

$$\eta_r(t)=\frac{1}{\omega_r}\int_0^t N_r(t-\tau)\sin\omega_r\tau d\tau \qquad r=1,2,3,...$$

The response to arbitrary:

$$y(x,t)=\sum_{r=1}^{\infty}\frac{\varphi_r(x)}{\omega_r}\int_0^t N_r(t-\tau)\sin\omega_r\tau d\tau$$

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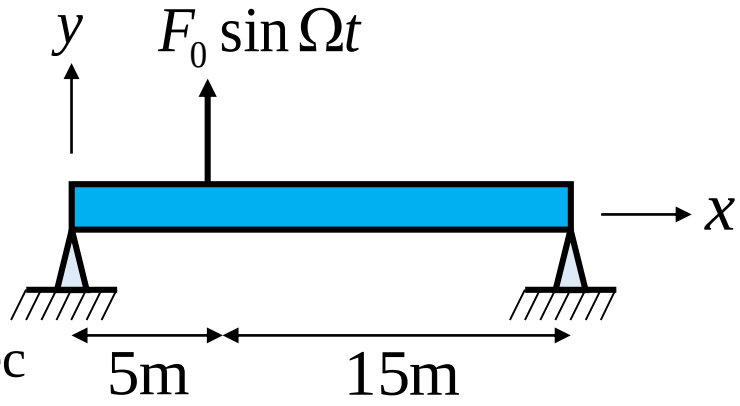
EXAMPLE

Find system response to excitation.

$E = 200$ Gpa

$\rho = 7800 \ {}^{\text{kg}}/_{\text{m}^3}$

$A = 1^{\text{m}} \times 1^{\text{m}}$



a) Natural frequencies $\langle 10000 \text{ rpm} = 1047 \text{ rad/sec}$

b) $y(x,t) = ?$

$$\omega_r = (\beta_r l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

Simply-Supported Beam: $\sin \beta l = 0 = \sin n\pi \rightarrow \beta_r l = r\pi$

$\omega_1 = 36.07$ rad/sec , $\omega_2 = 144.3$ rad/sec , $\omega_3 = 324.6$ rad/sec

$\omega_4 = 577.1$ rad/sec , $\omega_5 = 901.7$ rad/sec , $\omega_6 = 1298 \rangle 1047$ rad/sec

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EXAMPLE

$$\text{Mode shape}: \phi_r(x) = \sin\frac{r\pi x}{l} \quad , \quad Z = \int_0^l [\phi(x)]^2 dx = \frac{l}{2}$$

$$\rho AZ = 7800 \times 1^2 \times \frac{20}{2} = 78 \times 10^3 \quad , \quad I = \frac{1}{3}(1\times 1)\times(\frac{1}{2})^2 \quad , \quad h = 5m = \frac{l}{4}$$

$$\phi_1(h) = \sin(\frac{\pi}{4}) = 0.7071 \; , \; \phi_2(h) = \sin(\frac{\pi}{2}) = 1 \; , \; \phi_3(h) = \sin(\frac{3\pi}{4}) = 0.7071$$

$$\phi_4(h) = \sin\pi = 0 \quad , \quad \phi_5(h) = \sin(\frac{5\pi}{4}) = 0.7071$$

$$y(x,t) = \frac{50\sin\Omega t}{78\times 10^3}[\frac{0.7071\Omega^2 \sin(\frac{\pi x}{l})}{(36.07)^2 - \Omega^2} + \frac{\Omega^2 \sin(\frac{2\pi x}{l})}{(144.3)^2 - \Omega^2} + \frac{0.7071\Omega^2 \sin(\frac{3\pi x}{l})}{(324.6)^2 - \Omega^2} + \frac{0.7071\Omega^2 \sin(\frac{5\pi x}{l})}{(901.7)^2 - \Omega^2}]$$

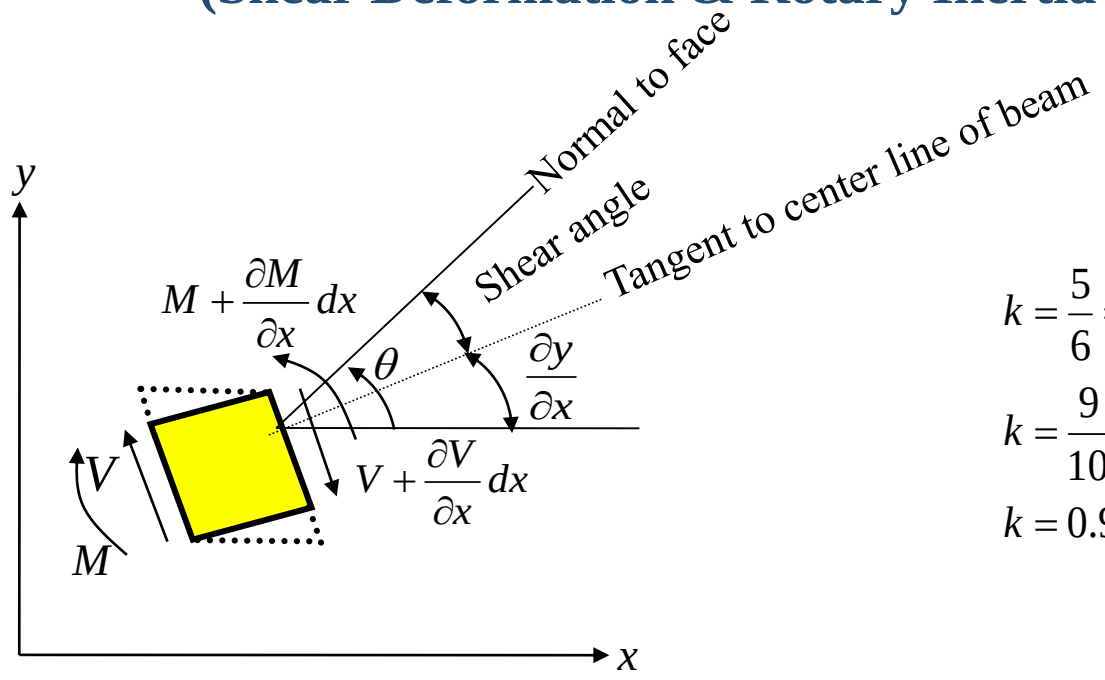
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4.4.2. The Timoshenko beam

Rotating Shafts

(Shear Deformation & Rotary Inertia effects)



$k = \frac{5}{6} = 0.83$ for section

$k = \frac{9}{10} = 0.85$ for section

$k = 0.92$ for section

Lateral vibration of beam with rotary inertia and shear deformation

θ = The slope due to the moment M

m = Mass per unit lenght of the beam

J = The rotary inertia per unit lenght of the beam

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$$\begin{cases} EI\,\dfrac{\partial\theta(x,t)}{\partial x}=M\,(x,t) \\ KAG\left(\theta(x,t)-\dfrac{\partial y\,(x,t)}{\partial x}\right)=V\,(x,t) \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial x}\left(EI\,\dfrac{\partial\theta}{\partial x}\right)+KAG\left(\dfrac{\partial y}{\partial x}-\theta\right)=\rho I\,\dfrac{\partial^2\theta}{\partial t^2} \\ \dfrac{\partial}{\partial x}\left(KAG\left\{\dfrac{\partial y}{\partial x}-\theta\right\}\right)+f\,(x,t)=\rho A\,\dfrac{\partial^2 y}{\partial t^2} \end{cases}$$

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$$\sum M = J\ddot{\theta} \rightarrow J\frac{\partial^2\theta(x,t)}{\partial t^2} = \frac{\partial M}{\partial x} - V$$

$$\sum F = m\ddot{x} \rightarrow -(V + \frac{\partial V}{\partial x}dx) + V = m\frac{\partial^2 y}{\partial t^2}dx$$

$$-\frac{\partial V}{\partial x} = m\frac{\partial^2 y}{\partial t^2} \qquad EI\frac{\partial \theta}{\partial x} = M$$

$$\frac{\partial y}{\partial x} = \theta - (\text{shear angle}) \ , \ \text{shear angle} = \frac{V}{kAG}$$

$$k = \text{a shape factor of the cross-section}$$

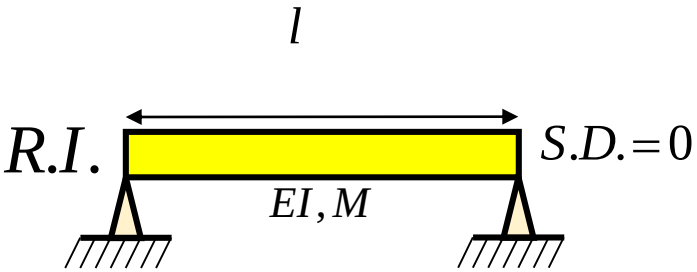
for a uniform beam: $EI = const.$

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} - (J + \frac{EIm}{kAG})\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{Jm}{kAG}\frac{\partial^4 y}{\partial t^4} = 0$$

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EXAMPLE



$m = \rho.A \quad , \quad J = \rho.I$ without R.I. $\rightarrow \omega_n = \beta^2 c = (n\pi/l)^2 \sqrt{EI/\rho A}$

R.I. $\rightarrow \quad EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \rho I\frac{\partial^4 y}{\partial x^2 \partial t^2} = 0$

$$y(x,t) = c\sin\frac{n\pi\, x}{l}.\sin(\omega_c t + \psi)$$

ω_c = Natural frequency corrected for rotary Inertia effect

$$\omega_c^2\left[1 + \frac{I}{A}\left(\frac{n\pi}{l}\right)^2\right] = \frac{EI}{\rho A}\left(\frac{n\pi}{l}\right)^4$$

$$\omega_c^2 \cong \omega_n^2\left[1 - \frac{1}{2}\frac{I}{A}\left(\frac{n\pi}{l}\right)^2\right]$$

1) decrease the N.F.

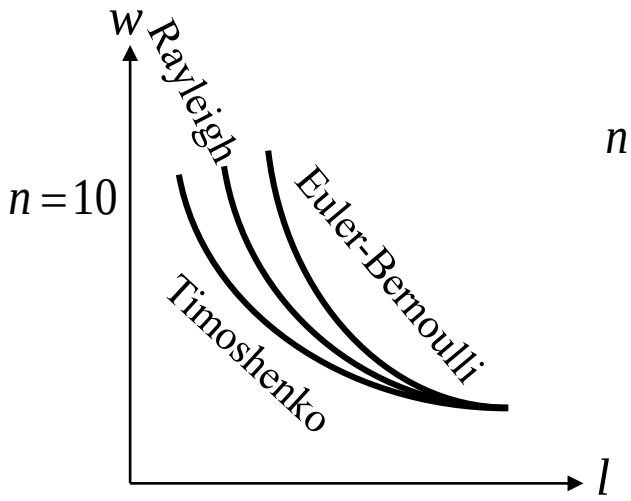
2) more pronounced for higher modes

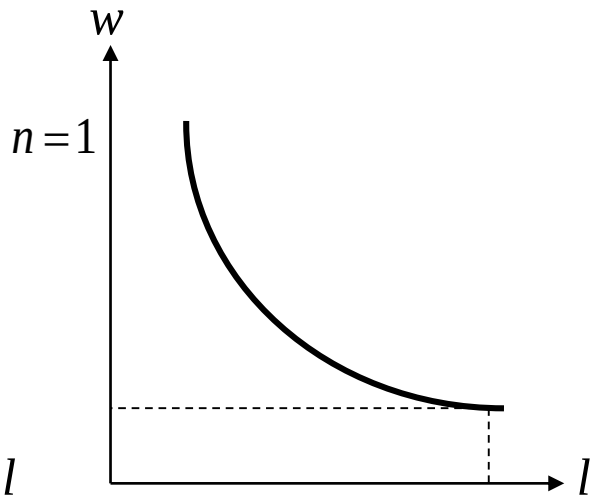
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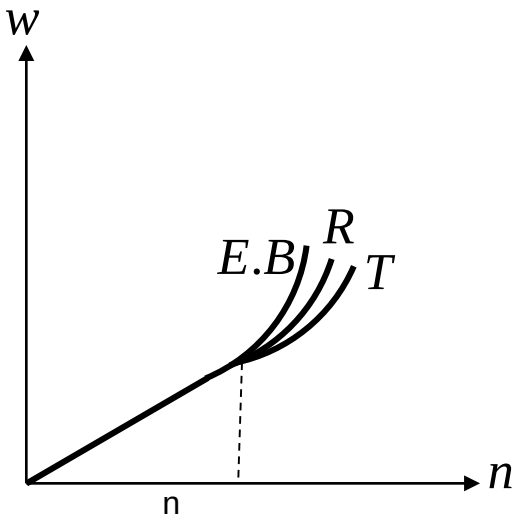
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$$\omega_c^2 = \frac{E\,I\,n^4\pi^4}{l^4 - J\,n^2\pi^2} \qquad R.I. \neq 0\ ,\ S.D. = 0 \qquad \text{(Rayleigh beam)}$$

$$\omega_c^2 = \frac{E\,I\,n^4\pi^4}{l^4 - J\,n^2\pi^2\left(1+\frac{A}{k\,G}\right)} \qquad R.I. \neq 0\ ,\ S.D. \neq 0 \qquad \text{(Timoshenko beam)}$$



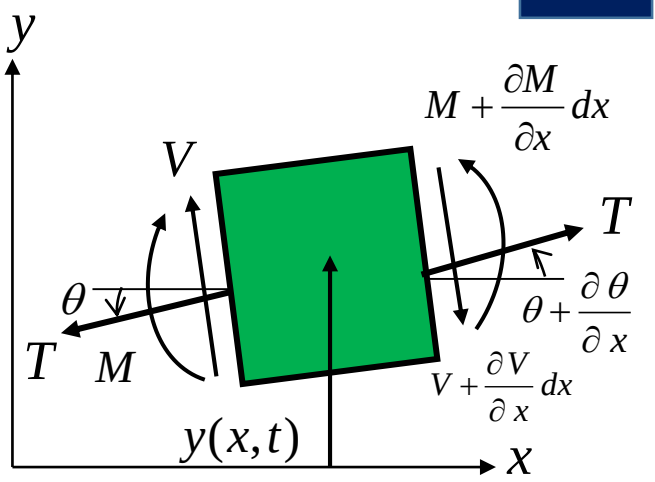




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4.4.3. Effect of axial loading



T = constant for small deflection

$m = \dfrac{\text{mass}}{\text{lenght}}$, $\theta = \dfrac{\partial y}{\partial x}$ = slope, V = shear force

$$-(V+\frac{\partial V}{\partial x}dx)+V+T(\theta+\frac{\partial \theta}{\partial x}dx)-T\theta = m\frac{\partial^2 y}{\partial t^2}dx$$

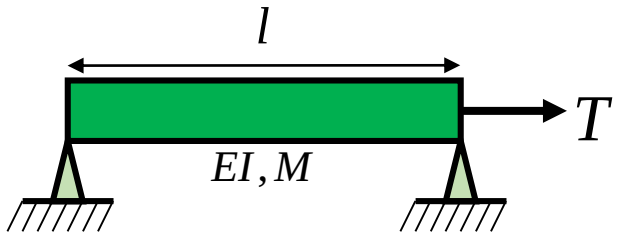
$$\therefore \left\{\begin{array}{l} -\dfrac{\partial V}{\partial x}+T\dfrac{\partial^2 y}{\partial x^2}=m\dfrac{\partial^2 y}{\partial t^2} \\ \dfrac{\partial M}{\partial x}=V \quad , \quad EI\dfrac{\partial^2 y}{\partial x^2}=M \end{array}\right\} \Rightarrow \quad -\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 y}{\partial x^2})+T\frac{\partial^2 y}{\partial x^2}=m\frac{\partial^2 y}{\partial t^2}$$

If EI = const. $\quad -EI\dfrac{\partial^4 y}{\partial x^4}+T\dfrac{\partial^2 y}{\partial x^2}=m\dfrac{\partial^2 y}{\partial t^2}$

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EXAMPLE



$$-EI\frac{\partial^4 y}{\partial x^4}+T\frac{\partial^2 y}{\partial x^2}=m\frac{\partial^2 y}{\partial t^2}$$

$$y(x,t)=\phi(x).q(t) \qquad \begin{cases} EI\dfrac{d^4\phi(x)}{dx^4}-T\dfrac{d^2\phi(x)}{dx^2}-m\omega^2\phi(x)=0 \\ \dfrac{d^2q(t)}{dt^2}+\omega^2 q(t)=0 \end{cases}$$

$$q(t)=A\sin\omega t+B\cos\omega t \qquad \phi(x)=Ce^{Sx},\quad \mathrm{C,S=const.}$$

$$EIS^4-TS^2-m\omega^2=0 \quad \text{characteristic eq.}$$

$$S_1^2,\ S_2^2=\frac{-T}{2EI}[1\pm(1+4m\omega^2\frac{EI}{T^2})^{0.5}]$$

if $S_1^2=a^2,\quad S_2^2=-b^2$

$$\phi(x)=c_1\sinh ax+c_2\cosh ax+c_3\sin bx+c_4\cos bx$$

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EXAMPLE

$$\underline{B.C.} \quad \left\{ \varphi(0) = \frac{d^2\varphi(x)}{dx^2} = \varphi(l) = \frac{d^2\varphi(l)}{dx^2} = 0 \right\} \rightarrow$$

$$\therefore c_1 = c_2 = c_4 = 0$$

$$c_3 \sin bl = 0 \rightarrow bl = n\pi \;, n = 1, 2, \ldots$$

$$b^2 = \left(\frac{n\pi}{l}\right)^2 = -\frac{T}{2EI}\left[1 - \left(1 + 4m\omega^2 \frac{EI}{T^2}\right)^{0.5}\right]$$

$$\omega_n^2 = \left(\frac{n\pi}{l}\right)^2 \frac{T}{m} + \left(\frac{n\pi}{l}\right)^4 \frac{EI}{m}$$

$$if \quad T = 0 \rightarrow \quad \omega_n = (\omega_n)_{\text{a S.S. beam}}$$

$$if \quad EI = 0 \rightarrow \quad \omega_n = (\omega_n)_{\text{a flexible taut string}}$$

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4.4.4. The Eigenvalue problem:

$$M\left(\ddot{w}\right)+L\left(w\right)=0$$

$$-\frac{\partial^4 y}{\partial x^4}=-\frac{1}{c^2}\frac{\partial^4 y}{\partial t^4},c=\left(\frac{EI}{m}\right)^{0.5}$$

$$M=\frac{1}{c^2},w=y,L=\frac{\partial^4}{\partial x^4},w\left(x,t\right)=\phi\left(x\right)q\left(t\right),\frac{L\left(\phi\right)}{M\left(\phi\right)}=-\frac{\ddot{q}}{q}=\lambda$$

$$\left\{\begin{array}{c}\ddot{q}+\lambda q=0\\ L\left(\phi\right)=\lambda M\left(\phi\right)\end{array}\right\},\lambda=\text{EigneValue},\phi=\text{EigenFunction}$$

Orthogonality:

$$\left\{\begin{array}{l}L\left(\phi_i\right)=\lambda_i M\left(\phi_i\right)\\ L\left(\phi_j\right)=\lambda_j M\left(\phi_j\right)\end{array}\right.\text{or}\left\{\begin{array}{l}\phi_j L\left(\phi_i\right)=\lambda_i\phi_j M\left(\phi_i\right)\\ \phi_i L\left(\phi_j\right)=\lambda_j\phi_i M\left(\phi_j\right)\end{array}\right.$$

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4.4.4. The Eigenvalue problem:

$$\int_{\sum}\left[\phi_j L\left(\phi_i\right)-\phi_i L\left(\phi_j\right)\right]d\sigma=\int_{\sum}\left[\lambda_i\phi_j M\left(\phi_i\right)-\lambda_j\phi_i M\left(\phi_j\right)\right]d\sigma$$

$$\text{Self adjoint}\begin{cases}\int_{\sum}\phi_j L\left(\phi_i\right)d\sigma=\int_{\sum}\phi_i L\left(\phi_j\right)d\sigma\\ \int_{\sum}\lambda_i\phi_j M\left(\phi_i\right)d\sigma=\int_{\sum}\lambda_j\phi_i M\left(\phi_j\right)d\sigma\end{cases}$$

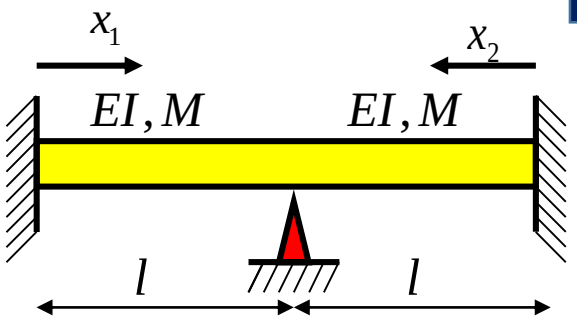
$$\begin{cases}\int\left[\phi_j L\left(\phi_i\right)-\phi_i L\left(\phi_j\right)\right]d\sigma=0\\ \left(\lambda_i-\lambda_j\right)\int_{\sum}\phi_i M\left(\phi_j\right)d\sigma=0\end{cases}$$

$$\text{since}\ \lambda_i\neq\lambda_j\rightarrow\begin{cases}\int_{\sum}\phi_i M\left(\phi_j\right)d\sigma=0\ \ \text{for}\ \ \mathrm{i}\neq\mathrm{j}\\ \int_{\sum}\phi_i L\left(\phi_j\right)d\sigma=0\ \ \text{for}\ \ \mathrm{i}\neq\mathrm{j}\end{cases}$$

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Two span beam



$$\begin{cases} \phi_1(x_1) = A_1\sin(\beta x_1) + B_1\cos(\beta x_1) + C_1\sinh(\beta x_1) + D_1\cosh(\beta x_1) \\ \phi_2(x_2) = A_2\sin(\beta x_2) + B_2\cos(\beta x_2) + C_2\sinh(\beta x_2) + D_2\cosh(\beta x_2) \end{cases}$$

$$\begin{cases} -\dfrac{\partial^4 y_1}{\partial x_1^4} = \dfrac{1}{c^2}\dfrac{\partial^2 y_1}{\partial t^2} \\ -\dfrac{\partial^4 y_2}{\partial x_2^4} = \dfrac{1}{c^2}\dfrac{\partial^2 y_2}{\partial t^2} \end{cases}, \quad \begin{cases} y_1(x_1,t) = \phi_1(x_1)q(t) \\ y_2(x_2,t) = \phi_2(x_2)q(t) \end{cases}$$

$$B.C.\ \text{at } x_1 = 0 \quad \begin{cases} \phi_1(0) = 0 \rightarrow B_1 = -D_1 \\ -\dfrac{d\phi_1(0)}{dx_1} = 0 \rightarrow A_1 = -C_1 \end{cases}$$

$$\text{at } x_2 = 0 \quad \begin{cases} \phi_2(0) = 0 \rightarrow B_2 = -D_2 \\ -\dfrac{d\phi_2(0)}{dx_2} = 0 \rightarrow A_2 = -C_2 \end{cases}$$

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Two span beam

$$\text{at } x_1 = L,\ x_2 = L \quad \begin{cases} \phi_1(L) = 0 \rightarrow B_1(\cos\beta L - \cosh\beta L) + A_1(\sin\beta L - \sinh\beta L) = 0 \\ \dfrac{d\phi_1(L)}{dx_1} = -\dfrac{d\phi_2(L)}{dx_2} \\ \phi_2(L) = 0 \rightarrow B_2(\cos\beta L - \cosh\beta L) + A_2(\sin\beta L - \sinh\beta L) = 0 \\ \dfrac{d^2\phi_2(L)}{dx_2^2} = -\dfrac{d^2\phi_1(L)}{dx_1^2} \end{cases}$$

$$\Rightarrow B_1 = -\frac{A_1(\sin\beta L - \sinh\beta L)}{(\cos\beta L - \cosh\beta L)}$$

$$B_2 = -\frac{A_2(\sin\beta L - \sinh\beta L)}{(\cos\beta L - \cosh\beta L)}$$

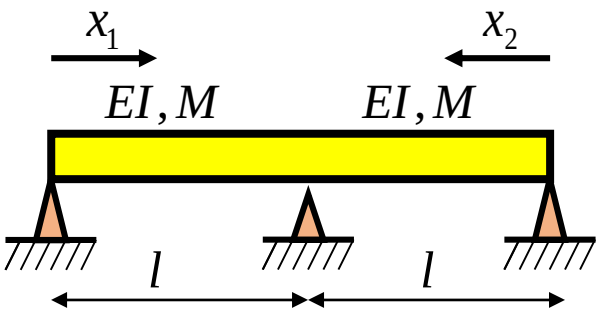
$$\text{Frequency Equation:} \begin{cases} \cos\beta L \cosh\beta L = 1 \\ \tan\beta L = \tanh\beta L \end{cases}$$

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H.W.7:

Find frequency equation of this *two span beam.*



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4.5. Forced Vibration

1) Distributed harmonic excitation

2) Harmonic motion of the boundaries

3) Harmonic force on the boundary

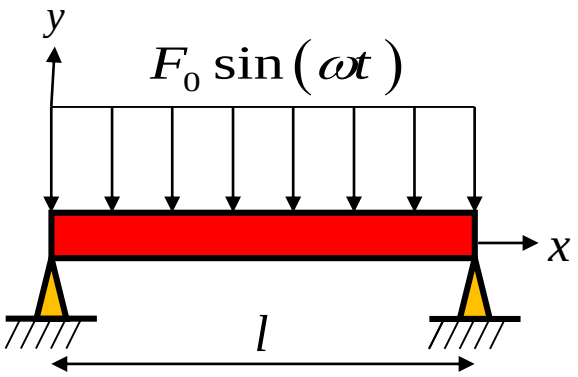
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EXAMPLE

4.5.1. Distributed harmonic excitation

$$\text{Beam} \rightarrow \quad -\frac{\partial^4 y}{\partial x^4} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} + \frac{1}{EI}p(x,t)\ , \quad c^2 = \frac{EI}{m}$$



$$c^2\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = \frac{F_0}{\rho.A}\sin\omega t \qquad \rho.A = m$$

$$y(x,t) = Y(x).\sin\omega t\,, \quad \frac{d^4Y(x)}{dx^4} - (\frac{\omega}{c})^2 Y(x) = \frac{F_0}{\rho A c^2}\ , \ \beta^2 = \frac{\omega}{c}$$

$$Y(x) = c_1\sin\beta x + c_2\cos\beta x + c_3\sinh\beta x + c_4\cosh\beta x - \frac{F_0}{\rho A \omega^2}$$

$$Y(0) = Y(l) = 0 \qquad \frac{d^2Y(0)}{dx^2} = \frac{d^2Y(l)}{dx^2} = 0$$

$$c_2 = c_4 = \frac{F_0}{2\rho A \omega^2}\ , \quad c_1 = \frac{F_0}{2\rho A \omega^2}.\tan\frac{\beta l}{2}\ , \quad c_3 = -\frac{F_0}{2\rho A \omega^2}.\tanh\frac{\beta l}{2}$$

$$Y(x,t) = \frac{F_0\sin\omega t}{2\rho A \omega^2}\left\{\frac{\cos\beta(\frac{l}{2}-x)}{\cos\frac{\beta l}{2}} + \frac{\cosh\beta(\frac{l}{2}-x)}{\cosh\frac{\beta l}{2}} - 2\right\}$$

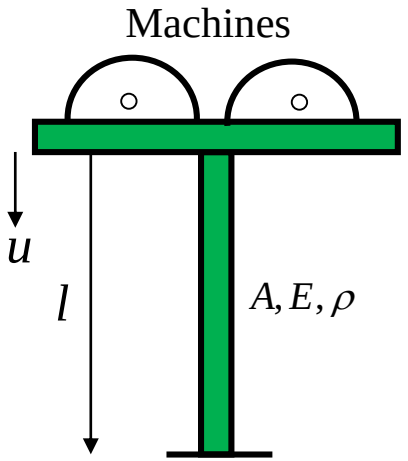
$$\Rightarrow \text{If} \quad \frac{\beta l}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \qquad \omega = \frac{n^2\pi^2 c}{l^2} \quad n = 1,3,5,\ldots$$

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4.5.2. Harmonic motion of the boundaries

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{p(x,t)}{EA}\ ,\ c = \sqrt{E/\rho}$$



If $p(x,t) = 0$ $u(x,t) = U(x).\cos\omega t$

$$\frac{d^2U(x)}{dx^2} + (\frac{\omega}{c})^2 U(x) = 0 \qquad U(x) = A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}$$

$$u(x,t) = (a\sin\frac{\omega x}{c} + b\cos\frac{\omega x}{c}).\cos\omega t$$

$$U(l) = 0 \quad U(0) = Y_0\cos\omega t \quad b = Y_0 \quad a = -Y_0\frac{\cos(\omega l/c)}{\sin(\omega l/c)}$$

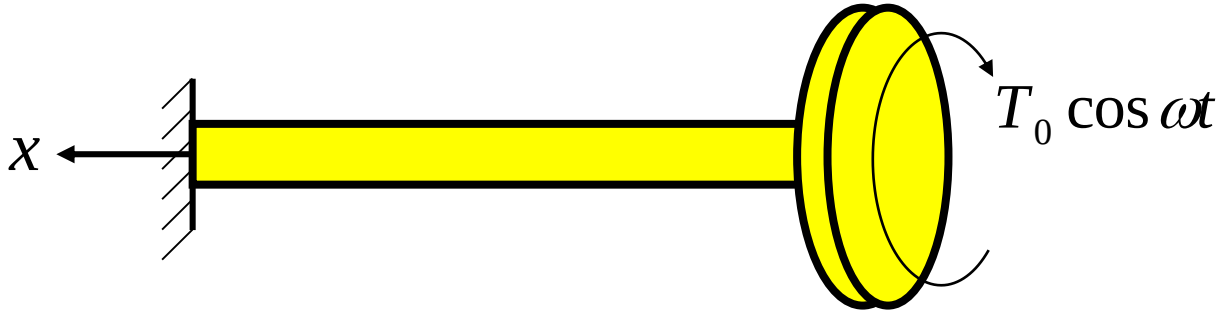
$$u(x,t) = Y_0(\cos\frac{\omega x}{c} - \frac{\cos(\omega l/c).\sin(\omega x/c)}{\sin(\omega l/c)}).\cos\omega t$$

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4.5.3. Harmonic force on the boundary

یک نیروی مرزی همانند یک شرط مرزی غیرهمگن می‌تواند عمل کند. به عنوان مثال یک ترک که در انتهای سمت راست یک شافت برابر $T_0\cos\omega t$ اعمال شود را می‌توان به صورت شرط مرزی زیر در نظر گرفت.



$$I_P G \frac{\partial \theta(0,t)}{\partial x} = T_0 \cos \omega t \qquad \theta(x,t) = \theta(x) \cos \omega t \quad \ldots$$

$$I_P G \frac{d\,\theta(0,t)}{dx} = T_0 \qquad \theta(x,t) = \theta(x) \cos \omega t \quad \ldots$$

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EXAMPLE

4.5.3. Harmonic force on the boundary

Forced longitudinal vibration of a column

در مثال قبل اگر یک نیروی هارمونیک نامیزان $F\cos\omega t$ توسط ماشین M ایجاد شود، ارتعاشات طولی ستون را بیابید.

$$\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \qquad c=\sqrt{E/\rho}$$

$$u(x,t)=U(x)\cos\omega t \;\rightarrow\; \frac{d^2U(x)}{dx^2}+\frac{\omega^2}{c^2}U(x)=0\Rightarrow U(x)=a\sin\left(\frac{\omega x}{c}\right)+b\cos\left(\frac{\omega x}{c}\right)$$

$$\text{B.C.:}\quad \begin{cases} x=0: & M\dfrac{\partial^2 u(0,t)}{\partial t^2}+EA\dfrac{\partial u(0,t)}{\partial x}=F\cos\omega t \\ x=L: & u(L,t)=0 \end{cases}$$

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EXAMPLE

4.5.3. Harmonic force on the boundary

$$\begin{cases} M\dfrac{\omega^2}{c^2}\left(a\sin\left(\dfrac{\omega x}{c}\right)+b\cos\left(\dfrac{\omega x}{c}\right)\right)_{x=0}\cos\omega t+EA\dfrac{\omega}{c}\left(a\cos\left(\dfrac{\omega x}{c}\right)-b\sin\left(\dfrac{\omega x}{c}\right)\right)_{x=0}\cos\omega t=F\cos\omega t \\ \left(a\sin\left(\dfrac{\omega x}{c}\right)+b\cos\left(\dfrac{\omega x}{c}\right)\right)\cos\omega t=0 \end{cases}$$

$$\begin{cases} M\dfrac{\omega^2}{c^2}b+EA\dfrac{\omega}{c}a\cos\omega t=F \\ a\sin\left(\dfrac{\omega L}{c}\right)+b\cos\left(\dfrac{\omega L}{c}\right)=0 \end{cases} \Rightarrow$$

$$b=-\frac{F}{M\dfrac{\omega^2}{c^2}+EA\dfrac{\omega}{c}\left(1\Big/\tan\left(\dfrac{\omega L}{c}\right)\right)}$$

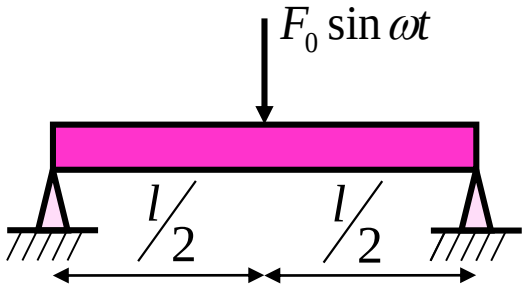
$$a=-\frac{b}{\tan\left(\dfrac{\omega L}{c}\right)}=\frac{F}{M\dfrac{\omega^2}{c^2}\tan\left(\dfrac{\omega L}{c}\right)+EA\dfrac{\omega}{c}}$$

$$u(x,t)=F\left(\frac{\sin\left(\dfrac{\omega x}{c}\right)}{M\dfrac{\omega^2}{c^2}\tan\left(\dfrac{\omega L}{c}\right)+EA\dfrac{\omega}{c}}-\frac{\cos\left(\dfrac{\omega x}{c}\right)}{M\dfrac{\omega^2}{c^2}+EA\dfrac{\omega}{c}\left(1\Big/\tan\left(\dfrac{\omega L}{c}\right)\right)}\right)\cos\omega t$$

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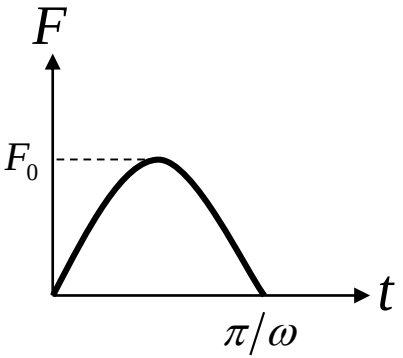
EXAMPLE



$$y(x,t) = \frac{2F_0}{AL\rho} \sum_{n=1,2}^{\infty} \frac{\sin(n\pi/2)}{(\omega_n^2 - \omega^2)} \sin\left(\frac{n\pi x}{L}\right) \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t\right)$$

$$\omega_n = \frac{n^2 \pi^2 c}{L^2}$$

$$y(x,T) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) (A_n \sin \omega_n T - B_n \sin \omega_n T) \quad T = t - \frac{\pi}{\omega}$$



$$y\left(x, \frac{\pi}{\omega}\right) = -\frac{2F_0\omega}{AL\rho} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{(\omega_n^2 - \omega^2)\omega_n} \sin\left(\frac{\omega_n \pi}{\omega}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\dot{y}\left(x, \frac{\pi}{\omega}\right) = -\frac{2F_0\omega}{AL\rho} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{(\omega_n^2 - \omega^2)\omega_n} \left\{1 + \cos\left(\frac{\omega_n \pi}{\omega}\right)\right\} \sin\left(\frac{n\pi x}{L}\right)$$

$$y(x,T)\Big|_{T=0} = y\left(x, \frac{\pi}{\omega}\right), \quad \dot{y}(x,T)\Big|_{T=0} = \dot{y}\left(x, \frac{\pi}{\omega}\right)$$

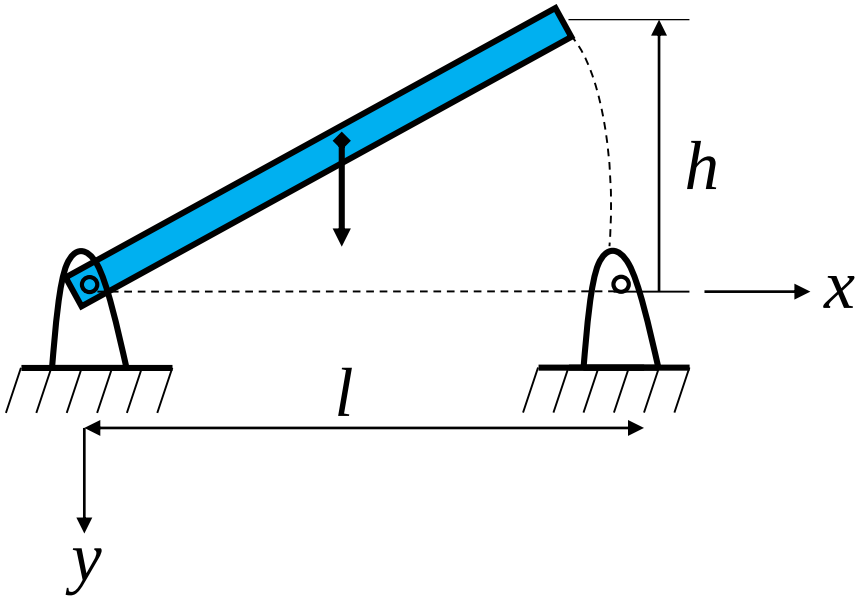
$$A_n = -\frac{2F_0\omega}{AL\rho} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{(\omega_n^2 - \omega^2)\omega_n} \left\{1 + \cos\left(\frac{\omega_n \pi}{\omega}\right)\right\}, \quad B_n = -\frac{2F_0\omega}{AL\rho} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{(\omega_n^2 - \omega^2)\omega_n} \sin\left(\frac{\omega_n \pi}{\omega}\right)$$

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H.W.8:

One tip of follow beam drop from inertia state .if there is no energy losses & jumping in strike time , find free frequency equation?



Answer :

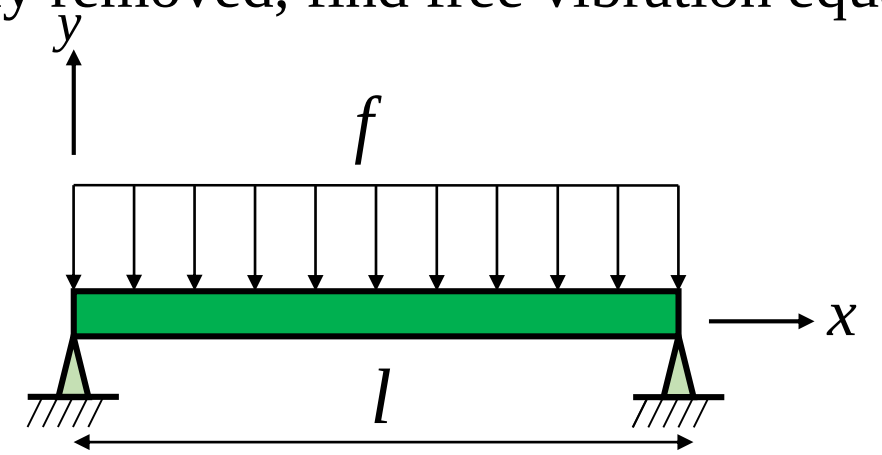
$$y(x,t)=\frac{2\sqrt{3gh}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\omega_n}(-\cos n\pi)\sin\frac{n\pi x}{l}\sin\omega_n t$$

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H.W.9:

The load is suddenly removed, find free vibration equation?



Answer :

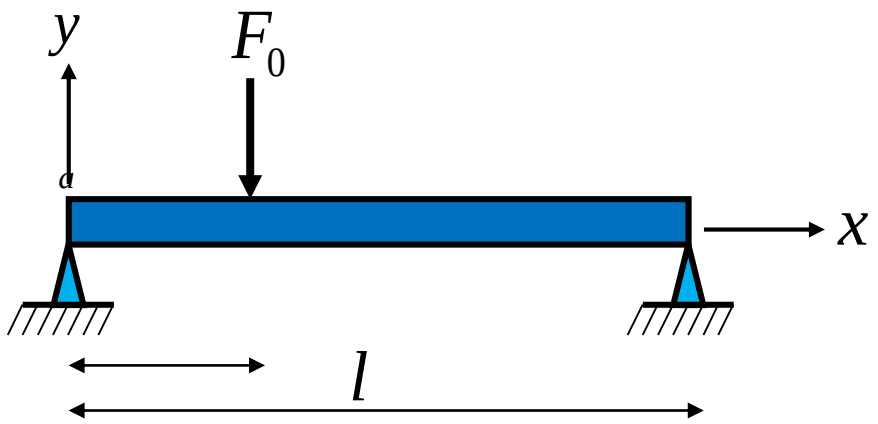
$$y(x,t)=\frac{4\,f\,l^4}{\pi^5 EI}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^5}\cos\omega_n t.\sin\frac{n\,\pi\,x}{l}$$

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H.W.10:

The load is suddenly removed, find free vibration equation?



Answer :

$$y(x,t)=\frac{2F_0 l^3}{\pi^4 EI}\sum_{n=0}^{\infty}\frac{1}{n^4}\sin\frac{n\ \pi\ x}{l}.\sin\frac{n\ \pi\ a}{l}.\cos\omega_n t$$

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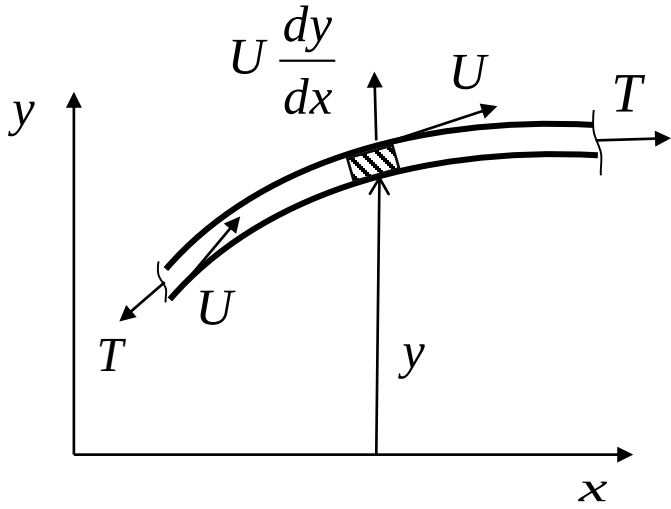
4.5.4. Special cases

1) The moving taut string

$$\left\{\begin{array}{l}\text{a- wire production}\\ \text{b- belt transmission}\end{array}\right\}$$

$$u=\frac{\partial y}{\partial t}+U\frac{\partial y}{\partial x} \qquad F_y=m dx\frac{du}{dt} \qquad F_y=m dx\frac{d}{dt}\left(\frac{\partial y}{\partial t}+U\frac{\partial y}{\partial x}\right)$$

$$u=u(x,t)\rightarrow \quad \frac{du}{dt}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}U+\frac{\partial u}{\partial t}$$



$$F_y=m dx\ \left[\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}+U\frac{\partial y}{\partial x}\right)U+\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}+U\frac{\partial y}{\partial x}\right)\right]$$

$$=m\left(U^2\frac{\partial^2 y}{\partial x^2}+2U\frac{\partial^2 y}{\partial x\,\partial t}+\frac{\partial^2 y}{\partial t^2}\right)dx$$

$$F_y=T\frac{\partial^2 y}{\partial x^2}dx\ ,\quad F_y=T\frac{\partial\theta}{\partial x}dx \qquad \therefore (T-mU^2)\frac{\partial^2 y}{\partial x^2}-2mU\frac{\partial^2 y}{\partial x\,\partial t}=m\frac{\partial^2 y}{\partial t^2}$$

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$$\text{If } T = mU^2 \rightarrow -2mU\frac{\partial^2 y}{\partial x\,\partial t} = m\frac{\partial^2 y}{\partial t^2}$$

$$\therefore \quad \frac{\partial}{\partial t}\left[2mU\frac{\partial y}{\partial x} + m\frac{\partial y}{\partial t}\right] = 0 \quad \therefore \quad 2mU\frac{\partial y}{\partial x} + m\frac{\partial y}{\partial t} = \mathrm{f}(x)$$

$$\frac{\partial y}{\partial t} = Y_1(x) \rightarrow \quad y_{(t)} = Y_1(x)t + Y_2(x) \quad \Rightarrow \quad \text{instability}$$

$$U = (T/m)^{1/2} \quad \text{The threshold of instability}$$

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2) Vibration of pipe at high flow velocity



EI = pipe flextural rigidity

m_p = mass per unit lenght of pipe

m_f = mass per unit lenght of carring fluid

U = constant fluid velocity

V = shear force

$u(x,t) = \frac{\partial u}{\partial t} + U\left(\frac{\partial u}{\partial x}\right)$ fluid velocity in y-axis $\frac{\partial y}{\partial t}$ = pipe velocity in y-axis

$$F = m_f\, dx \left(U^2 \frac{\partial^2 y}{\partial x^2} + 2U \frac{\partial^2 y}{\partial x\, \partial t} + \frac{\partial^2 y}{\partial t^2}\right) + m_p\, dx \frac{\partial^2 y}{\partial t^2}$$

$$F_y = -\frac{\partial V}{\partial x} dx = -\frac{\partial^2 M}{\partial x^2} dx = -\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) dx$$

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2) Vibration of pipe at high flow velocity



$$\therefore \quad \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+m_f U^2\frac{\partial^2 y}{\partial x^2}+2m_f U\frac{\partial^2 y}{\partial x\,\partial t}+(m_f+m_p)\frac{\partial^2 y}{\partial t^2}=0$$

بر اساس یک حل تقریبی که کل جرم سیال m را در وسط تیر (لوله) قرار دهیم و از وزن لوله صرف نظر شود

$$U \succ \left(\frac{8EI}{m_f L^2}\right)^{\frac{1}{2}}$$

For instability

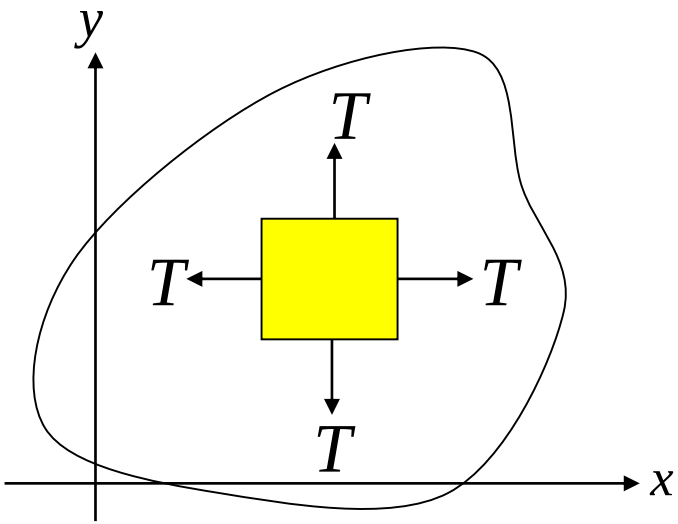
A: سطح مقطع لوله

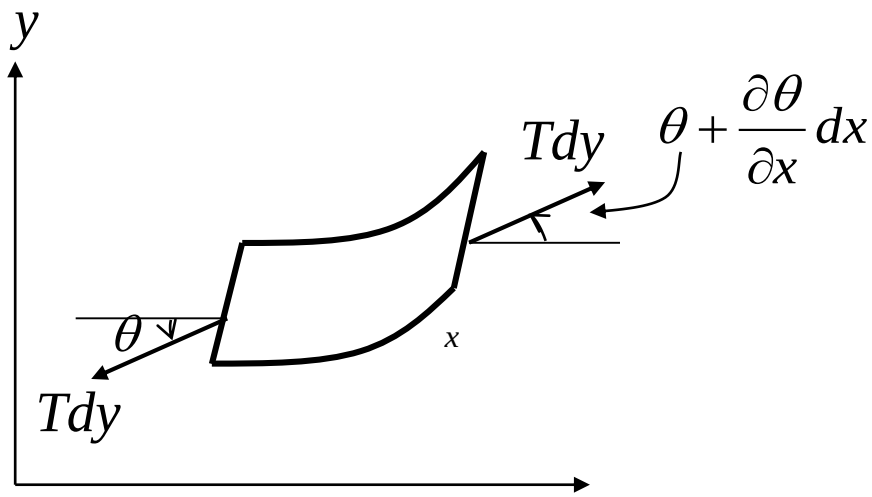
$$m_f = A\rho$$

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4.6. Vibration of Membranes





$$Tdy(\theta+\frac{\partial\theta}{\partial x}dx)-Tdy\theta=T\frac{\partial\theta}{\partial x}dxdy$$

$$Tdx(\varphi+\frac{\partial\varphi}{\partial y}dy)-Tdx\varphi=T\frac{\partial\varphi}{\partial y}dxdy$$

$$\varphi=\frac{\partial w}{\partial y}\qquad\theta=\frac{\partial w}{\partial x}$$

$$F=T(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})dxdy$$

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$$\rho\ dxdy\frac{\partial^2 w}{\partial t^2}=T(\frac{\partial^2 w}{\partial\ x^2}+\frac{\partial^2 w}{\partial\ y^2})dxdy\ +p(x,y)dxdy$$

$$\frac{\partial^2 w}{\partial t^2}=c^2\nabla^2 w+\frac{1}{\rho}\,p(x,y)$$

$$\nabla^2=\frac{\partial^2}{\partial\ x^2}+\frac{\partial^2}{\partial\ y^2}\qquad c=\sqrt{T/\rho}$$

$$w(x,y,t)=W(x,y)Q(t)$$

$$\text{if } p\ (x,y)=0\qquad \frac{\partial^2 w}{\partial\ t^2}=-\omega^2\ W(x,y)Q(t)$$

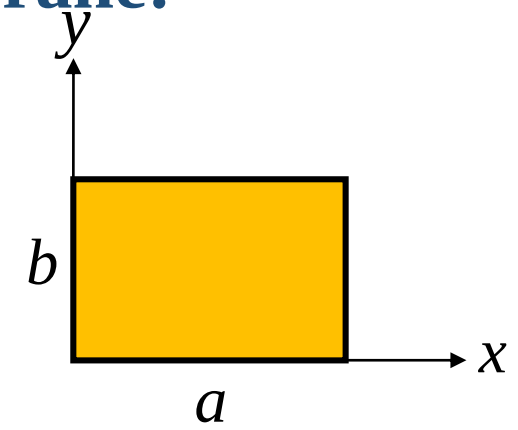
$$\boxed{\nabla^2 W(x,y)+\frac{\omega^2}{c^2}W(x,y)=0}$$

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EXAMPLE

Vibration of rectangular membrane:



$$w(x,y,t)=W(x,y)q(t)=X_{(x)}Y_{(y)}\,q_{(t)}$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \qquad c=\sqrt{T/\rho}$$

$$\frac{d^2X}{dx^2}Y.q+\frac{d^2Y}{dy^2}X.q=\frac{1}{c^2}\frac{d^2q}{dt^2}XY \;\rightarrow\; X''Y\,q+Y''X\,q=c^{-2}\ddot{q}\,X\,Y$$

$$c^2\frac{X''}{X}+c^2\frac{Y''}{Y}=\frac{\ddot{q}}{q} \qquad \left\{\begin{array}{l} c^2\dfrac{X''}{X}=-\lambda_1^2 \\ c^2\dfrac{Y''}{Y}=-\lambda_2^2 \\ \dfrac{\ddot{q}}{q}=-\lambda_3^2=-\omega^2 \end{array}\right\} \qquad \lambda_3=\omega=\text{natural frequency}$$

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EXAMPLE

$$\begin{Bmatrix} X(x) = c_1 \sin\frac{\lambda_1}{c}x + c_2 \cos\frac{\lambda_1}{c}x \\ Y(y) = c_3 \sin\frac{\lambda_2}{c}y + c_4 \cos\frac{\lambda_2}{c}y \\ q(t) = c_5 \sin\omega t + c_6 \cos\omega t \end{Bmatrix}$$

$$\underline{B.C.} \quad x = y = 0 \ \& \ x = a, y = b \ \rightarrow \ W(x,y,t) = 0 \rightarrow$$

$$\begin{Bmatrix} c_2 = 0 \\ c_1 \sin\frac{\lambda_1}{c}a = 0 \rightarrow \ \lambda_1 = \frac{n\pi c}{a} \\ c_4 = 0 \\ c_3 \sin\frac{\lambda_2}{c}b = 0 \rightarrow \ \lambda_2 = \frac{m\pi c}{b} \end{Bmatrix} \quad \omega^2 = \left(\frac{n\pi c}{a}\right)^2 + \left(\frac{m\pi c}{b}\right)^2$$

$$X(x)Y(y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(m\pi x/a)\sin(n\pi y/b)\{A\sin(\omega t) + B\cos(\omega t)\}$$

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Circular Membrane :

$$\nabla^2 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} \right) = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \qquad c = \sqrt{T_0/\rho}$$

Seperation of variable $\rightarrow \mathrm{w}(r,\theta,t) = R(r)T(t)\Theta(\theta)$

$$\frac{\ddot{T}}{T} = c^2\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right] = -\omega^2$$

$$\begin{cases} \underbrace{\ddot{T} + \omega^2 T = 0}_{(I)} \\ \underbrace{r^2\frac{R''}{R} + r\frac{R'}{R} + (\omega/c)^2 r^2}_{(II)} = -\frac{\Theta''}{\Theta} = m^2 \quad (III) \end{cases}$$

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$$(I)\rightarrow T(t)=A_1\cos(\omega t)+B_1\sin(\omega t)$$

$$(II)\rightarrow R''+\frac{1}{r}R'+(\beta^2-\frac{m^2}{r^2})R=0,\beta^2=\frac{m^2}{c^2}$$

Bessel's equation

$$R_m(r)=A_{3m}J_m(\beta r)+B_{3m}Y_m(\beta r)$$

The second kind

The first kind

$$(III)\rightarrow \Theta(\theta)=A_2\cos(m\theta)+B_2\sin(m\theta)$$

$$w(r,\theta)=c_{1m}J_m(\beta r)\sin(m\theta)+c_{2m}J_m(\beta r)\cos(m\theta)+c_{3m}Y_m(\beta r)\sin(m\theta)+c_{4m}Y_m(\beta r)\cos(m\theta)$$

$$w(l,\theta)=0 \qquad c_{3m}=c_{4m}=0$$

$$\rightarrow w(l,\theta)=c_{1m}J_m(\beta l)\sin(m\theta)+c_{2m}J_m(\beta l)\cos(m\theta)$$

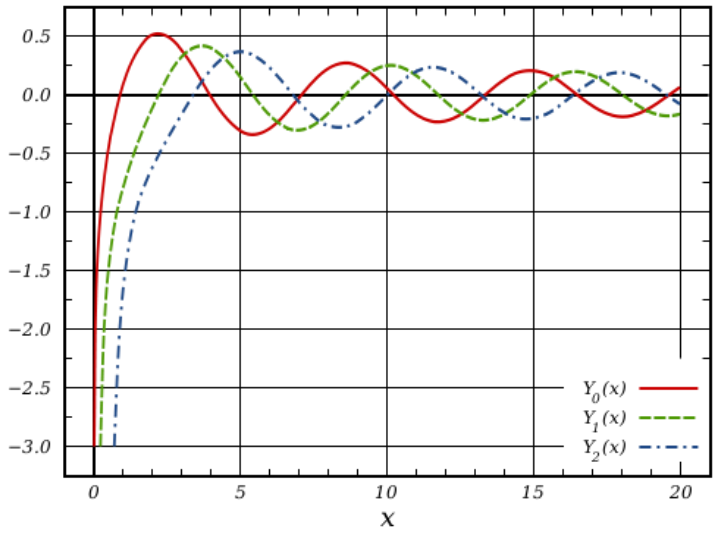
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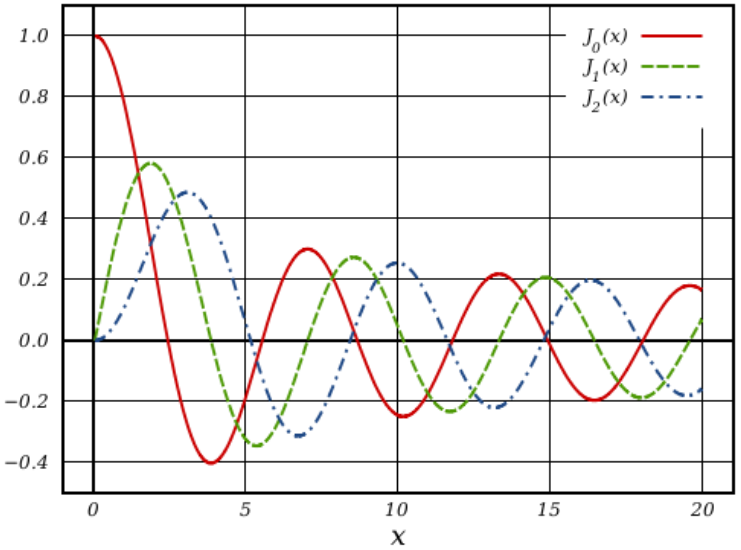
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$$\text{Bessel's equation}: x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$$

$$\text{The first kind: } J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}$$

$$\text{The second kind: } Y_\alpha(x)=\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}$$





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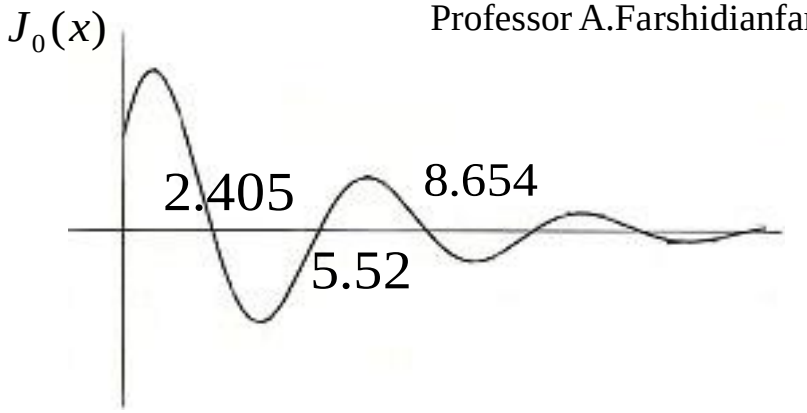
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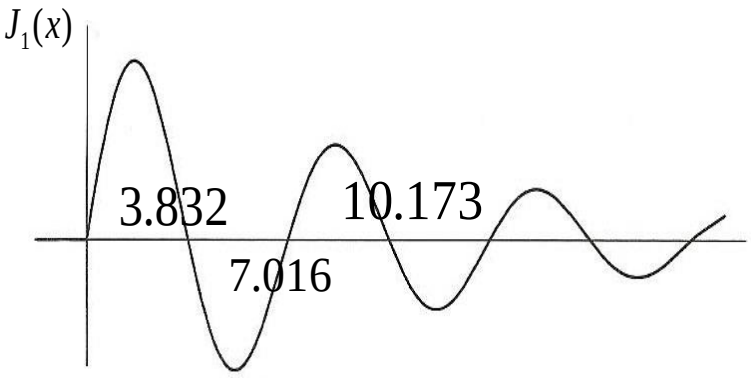
$$\omega_{mn} = C\,\beta_{mn}$$

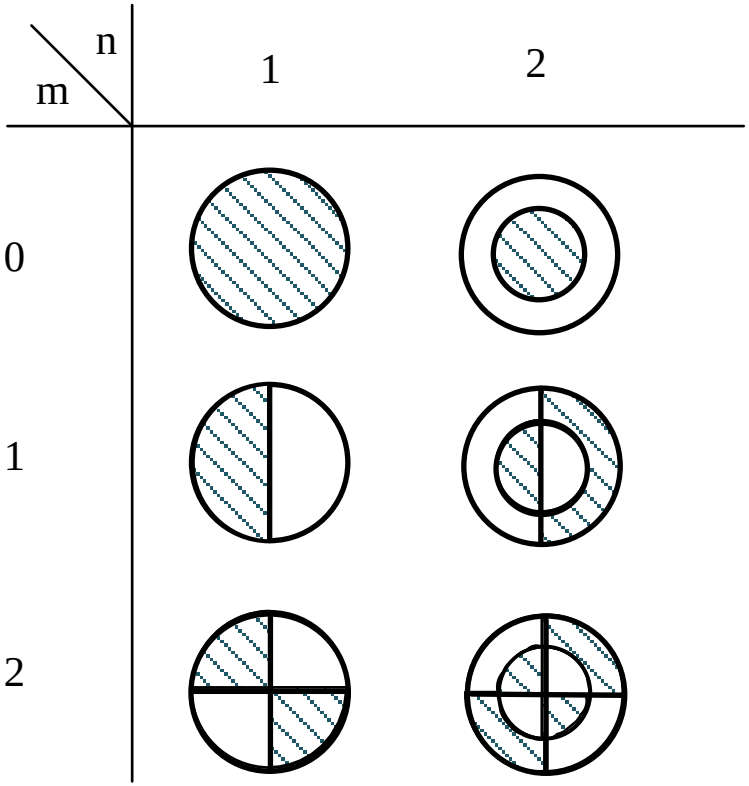
$$\text{If} \begin{cases} m=1 \\ n=1 \end{cases} \Rightarrow \omega_{11} = 3.832\frac{c}{l}$$

$$\text{If} \begin{cases} m=1 \\ n=2 \end{cases} \Rightarrow \omega_{12} = 7.016\frac{c}{l}$$

$$\text{If} \begin{cases} m=0 \\ n=1 \end{cases} \Rightarrow \omega_{01} = 2.405\frac{c}{l}$$

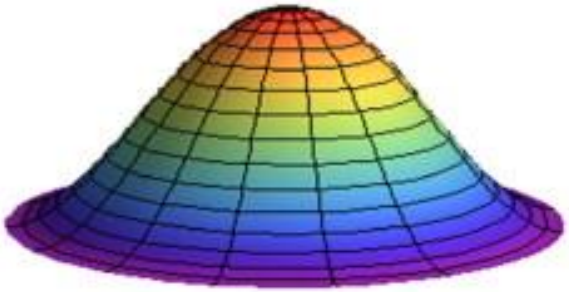




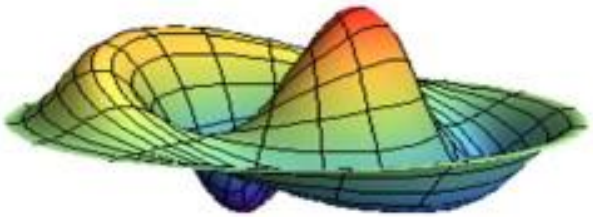


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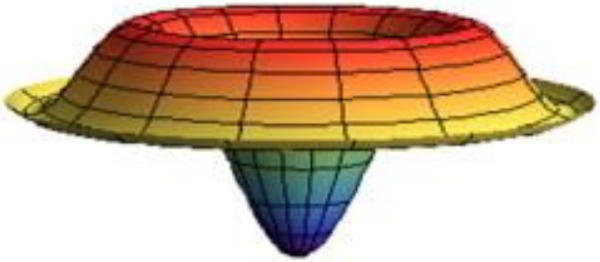
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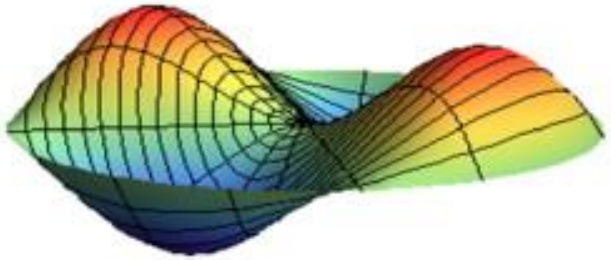
$\omega(0,0) = 930.506$



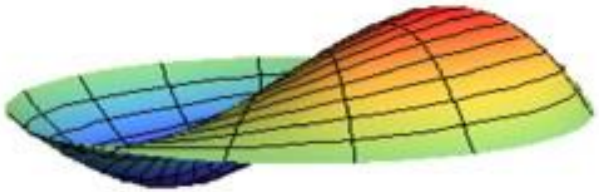
$\omega(1,1) = 5540.17$



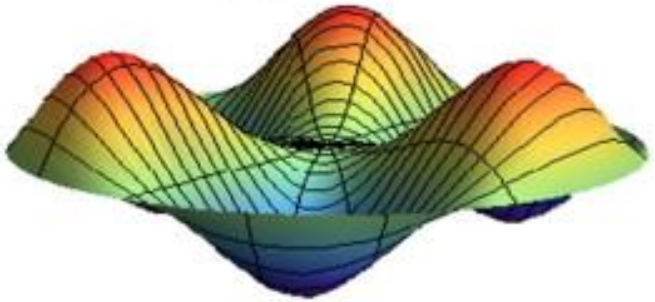
$\omega(0,1) = 3622.343$



$\omega(2,0) = 3177.072$



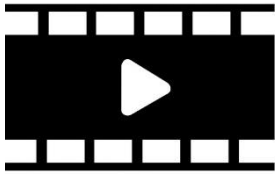
$\omega(1,0) = 1936.41$



$\omega(3,0) = 4648.28$

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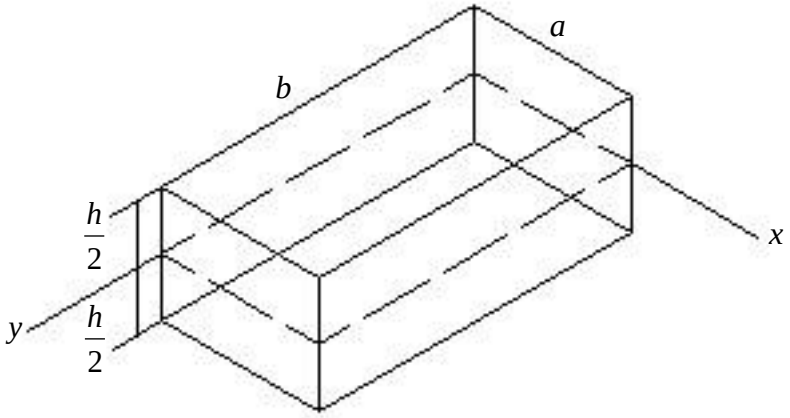
1. Mode Shapes for a Circular Membrane Play

2. Two modes of vibration of a square membrane Play

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4.7. Vibration of plates



$$T = \frac{1}{2}\int_0^a\int_0^b \rho b \left(\frac{\partial w}{\partial t}\right)^2 dx\,dy$$

$$V = \frac{D}{2}\int_0^a\int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\upsilon\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\upsilon)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right] dx\,dy$$

$$D = \frac{E h^3}{12(1-\upsilon^2)}$$ The plate constant

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Hamilton's principle: $\delta \int_{t_1}^{t_2} L\,dt = 0$

$$L = T - V$$

$$D\nabla^4 w + \rho h \ddot{w} = 0$$

$$\nabla^4 w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial y^2 \partial x^2} + \frac{\partial^4 w}{\partial y^4}$$

ρ = mass density of plate

h =Thickness of plate

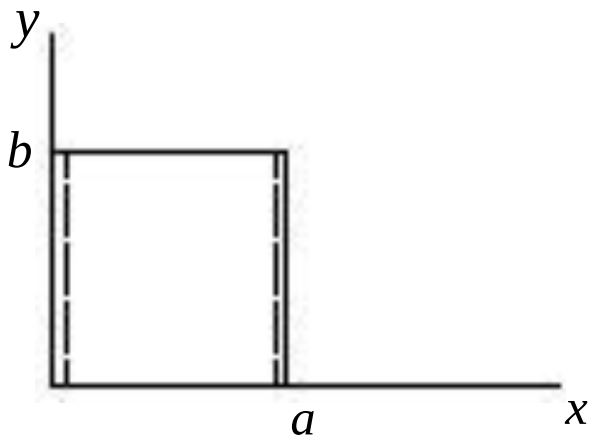
$$w(x, y, t) = W(x, y) q(t) \quad \rightarrow \quad \nabla^4 W(x, y) = \gamma^2 W(x, y)$$

$$\gamma^2 = \frac{\rho h}{D} \omega^2$$

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EXAMPLE



$$W(x,y)=Y(y)\sin\left(\frac{m\pi x}{a}\right)$$

$$\text{B.C.}\begin{cases} W(0,y)=W(a,y)=0 \\ \dfrac{\partial^2 W(0,y)}{\partial x^2}=\dfrac{\partial^2 W(a,y)}{\partial x^2}=0 \end{cases}$$

$$\frac{\partial^4 Y}{\partial y^4}+Y\left[\left(\frac{m\pi}{a}\right)^4-\gamma^2\right]-2\frac{\partial^2 Y}{\partial y^2}\left(\frac{m\pi}{a}\right)^2=0$$

$$Y(y)=ae^{\phi y} \quad \rightarrow \quad \phi^4-2\phi^2\left(\frac{m\pi}{a}\right)^2+\left\{\left(\frac{m\pi}{a}\right)^4-\gamma^2\right\}=0$$

$$\phi_{1,2}^2=\left(\frac{m\pi}{a}\right)^2\pm\sqrt{\gamma^2}$$

$$\phi_{1,2}^2=\alpha^2,-\beta^2, \qquad \alpha=\sqrt{\gamma+\left(\frac{m\pi}{a}\right)^2}, \qquad \beta=\sqrt{\gamma-\left(\frac{m\pi}{a}\right)^2}$$

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$$\varphi_1 = \alpha \quad , \quad \varphi_2 = -\alpha \quad , \quad \varphi_3 = +i\beta \quad , \quad \varphi_4 = -i\beta$$

$$Y = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 \cos \beta y + C_4 \sin \beta y$$

$$\omega_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \sqrt{\frac{D}{\rho h}} \qquad m, n = 1, 2, \ldots.$$

$$w(x, y) = \sin \frac{m\pi}{a} x \ \sin \frac{n\pi}{b} y$$

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$$\varphi_1 = \alpha, \quad \varphi_2 = -\alpha, \quad \varphi_3 = +i\beta, \quad \varphi_4 = -i\beta$$

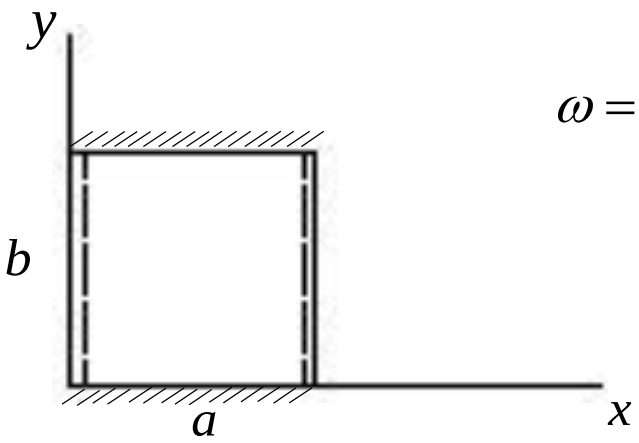
$$Y = C_1 \cosh(\alpha y) + C_2 \sinh(\alpha y) + C_3 \cos(\beta y) + C_4 \sinh(\beta y)$$

For a plate clamped at $y = 0$, $y = b$

$$Y(0) = Y(b) = Y'(0) = Y'(b) = 0$$

Characteristic equation:

$$2 - 2\cos(\beta b)\cosh(\alpha\beta) + \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\sin(\beta b)\sinh(\alpha\beta) = 0$$



$$\omega = \frac{\lambda}{b^2}\sqrt{\frac{D}{\rho h}}$$

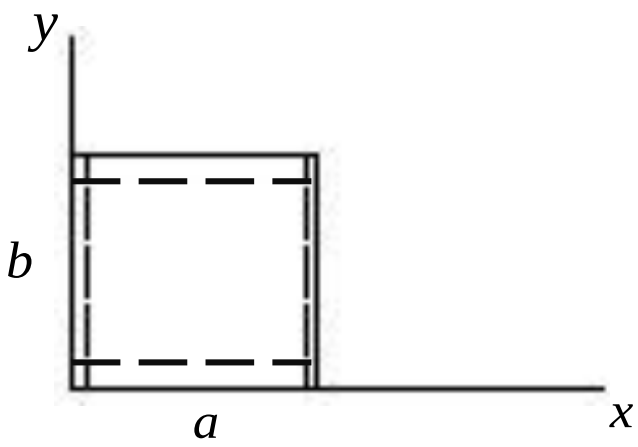
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For a plate supported on all the four edges:

$$w_0 = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right); \quad \text{m, n=1,2,3,...}$$

$$\omega_{m,n} = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho h}}; \quad \text{m, n=1,2,3,...}$$



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Chapter 5. Approximate and Numerical Methods

5.1. Dunkerley's Equation

The fundemental frequency of multi – rotor system.

$\{q\}=[d_{ij}]\{-m\ddot{q}\}$ The free vibration of undanped system

$\{q\}=$ The displacement vector

$[d_{ij}]=$ The flexibility matrix $=[K_{ij}]^{-1}$

$[-m\ddot{q}]=$ The vector of inertia forces

Assumeing harmonic modes, $\{\ddot{q}\}=\{-\omega^2 q\}$

$\{q\}=[d_{ij}]\{\omega^2 mq\}$ The ferquency equation

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{Bmatrix} \omega^2 m_1 q_1 \\ \omega^2 m_2 q_2 \end{Bmatrix}\rightarrow\begin{bmatrix} \frac{1}{\omega^2}-d_{11}m_1 & -d_{12}m_2 \\ -d_{21}m_1 & \frac{1}{\omega^2}-d_{22}m_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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$$\Delta(\omega)=\begin{bmatrix} \frac{1}{\omega^2}-d_{11}m_1 & -d_{12}m_2 \\ -d_{12}m_2 & \frac{1}{\omega^2}-d_{22}m_2 \end{bmatrix}=0$$

$$(\frac{1}{\omega^2})^2-(d_{11}m_1+d_{22}m_2)(\frac{1}{\omega^2})+m_1m_2\,(d_{11}d_{22}-d_{12}d_{21})=0$$

$$(\frac{1}{\omega^2}-\frac{1}{\omega_1^2})(\frac{1}{\omega^2}-\frac{1}{\omega_2^2})=0$$

$$(\frac{1}{\omega^2})^2-(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2})(\frac{1}{\omega^2})+\frac{1}{\omega_1^2\,\omega_2^2}=0$$

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}=d_{11}m_1+d_{22}m_2 \qquad \text{if } \omega_1<<\omega_2 \rightarrow \frac{1}{\omega_2^2}<<\frac{1}{\omega_1^2}$$

$$\rightarrow \frac{1}{\omega_1^2}\approx d_{11}m_1+d_{22}m_2$$

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$$\frac{1}{\omega_1^2} \approx d_{11}m_1 + d_{22}m_2 + \ldots\ldots + d_{nn}m_n = \sum_{i=1}^{n} d_{ii}m_i$$

d_{ii} :The influence coefficient

$$d_{ii}m_i = \frac{m_i}{k_{ii}} = \frac{1}{\omega_{ii}^2}$$

$$\rightarrow \quad \frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2} + \ldots\ldots + \frac{1}{\omega_{nn}^2} = \sum_{i=1}^{n} \frac{1}{\omega_{ii}^2}$$

The estimated fundamental frequency < The exact value

$$\omega_n^2 \approx \frac{1}{d_{11}m_1} + \frac{1}{d_{22}m_2} + \frac{1}{d_{33}m_3}$$ (Southwell equation)

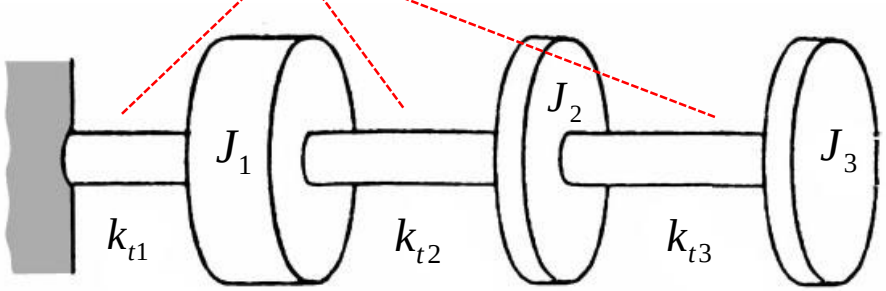
معادلهٔ دانکرلی روش تقریبی برای یافتن کوچکترین فرکانس طبیعی و معادلهٔ ساوت ول روش تقریبی برای یافتن بزرگترین فرکانس است. دقت روش بستگی به جدائی فرکانس‌های طبیعی از هم دارد. این معادلات کاربرد محدودی دارند، اما ساده هستند و به ما امکان انجام یک تخمین سریع از پایین‌ترین و بالاترین فرکانس طبیعی با استفاده از همان داده‌ها را می‌دهند.

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EXAMPLE

$$k_{t1} = k_{t2} = k_{t3} = k_t$$



$$J_1 = J_2 = J_3 = J$$

$$d_{11} = \frac{1}{k_t} \qquad \omega_{11}^2 = \frac{k_t}{J}$$

$$d_{22} = \frac{2}{k_t} \qquad \omega_{22}^2 = \frac{k_t}{2J}$$

$$d_{33} = \frac{3}{k_t} \qquad \omega_{33}^2 = \frac{k_t}{3J}$$

$$\frac{1}{\omega_1^2} = \frac{J}{k_t} + \frac{2J}{k_t} + \frac{3J}{k_t} = \frac{6J}{k_t}$$

$$\begin{cases} (\omega_1)_{app.} = 0.408\sqrt{k_t/J} \\ (\omega_1)_{ex.} = 0.445\sqrt{k_t/J} \end{cases} \Rightarrow \text{error} = 8\ \%$$

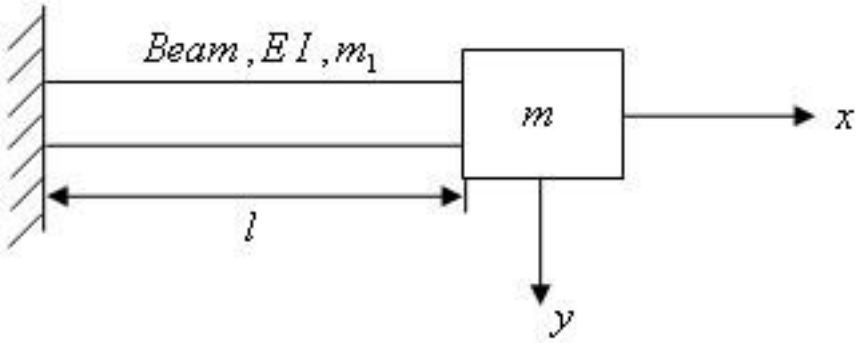
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EXAMPLE

$$\omega_{11}^2 = \frac{3EI}{ml^3}$$

$$\omega_{22}^2 = (3.52)^2 \frac{EI}{m_1 l^3} \cong 12.7 \frac{EI}{m_1 l^3}$$



if $m_1 = m$

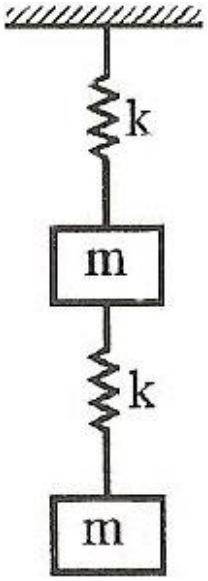
$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = 0.412 \frac{ml^3}{EI}$$

$$\omega_1 = 1.56 \sqrt{\frac{EI}{ml^3}}$$

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EXAMPLE



$$\frac{1}{\omega_1^2}=\frac{1}{\omega_{11}^2}+\frac{1}{\omega_{22}^2}=\frac{m}{k}+\frac{2m}{k}=\frac{3m}{k}$$

$$(\omega_1)^2_{\text{app.}}=0.333\,k/m$$

$$(\omega_1)^2_{\text{ex.}}=0.36\,k/m$$

$$\text{The error}=7.5\,\%$$

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5.2. Rayleigh Method

The fundemental frequency of multi-rotor systems.

$[M]\{\ddot{x}\}+[k]\{x\}=0$ free vibration of undamped system.

$\{\varphi\}^{(i)^T}\left([k]\{\varphi\}^{(i)}=\omega_i^2[M]\{\varphi\}^{(i)}\right)$ $i=1,2,3,....n$

$$\begin{cases}\omega_i=\text{eigenvalue}\\ \varphi^i=\text{eigenvector}\end{cases}$$

$$\omega_i^2=\frac{\{\varphi\}^{(i)^T}[k]\{\varphi\}^{(i)}}{\{\varphi\}^{(i)^T}[M]\{\varphi\}^{(i)}}$$

$$R_r(z)=\frac{\omega_r^2}{1+\varepsilon^2}$$

$\{z\}=\text{orbitrary vector}$

$$\omega^2=\frac{\{z\}^T[k]\{z\}}{\{z\}^T[M]\{z\}}=R(z)$$ Rayleigh's quotient

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P.S. of Rayleigh Method

A 10% error in the form of the eigenvector (natural mode) will result in only 1% error in the eigenvalue (natural frequency)

$$\omega_1^2 \le R_r(z) \le \omega_n^2$$

The Rayliegh's quotient is never lower than the smallest eigenvalue or higher than the greatest eigenvalue.

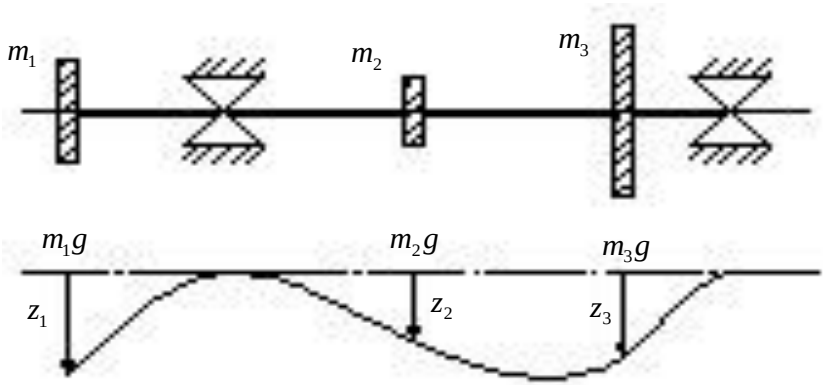
$$R_r(z) = \frac{\omega_r^2}{1+\varepsilon^2}$$

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Rayleigh's principle.

In a conservative system.



$$E = T + V = \text{Constant.}$$

$$T_{\max} + 0 = 0 + V_{\max} \rightarrow T_{\max} = V_{\max}$$

usually the dynamic deflections are estimeted from the static deflections.

$$\{z\} = \{z_1, z_2, ..., z_n\} \quad \text{for a discrete system.}$$

$$T_{\max} = \frac{1}{2}\{z\}^T [M]\{z\}\omega^2 \quad \text{for a diagonal mass matrix.}$$

$$T_{\max} = \frac{1}{2}\left[m_1(\omega z_1)^2 + m_2(\omega z_2)^2 + ... + m_n(\omega z_n)^2\right] = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i z_i^2$$

$$V_{\max} = \omega_{1\rightarrow 2} = \frac{1}{2}(m_1 g z_1 + m_2 g z_2 + ... + m_n g z_n) = \frac{g}{2}\sum_{i=1}^{n} m_i z_i$$

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$$T_{\max} = V_{\max} \quad \Rightarrow \quad \omega^2 = \frac{g\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i z_i^2}$$

For continuous system: $y(x) = \text{mode shape}$

$$M\left(\ddot{W}\right) + L\left(W\right) = 0, \qquad W(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t)$$

$$T = \frac{1}{2}\int_0^l \dot{W} M\left(\dot{W}\right) dx, \qquad V = \frac{1}{2}\int_0^l W L\left(W\right) dx$$

$$W\left(x,t\right) = \psi(x) q(t) \qquad \psi(x) = \text{Assumed mode shape}$$

$$\lambda_R = \frac{\int_0^l \psi L\left(\psi\right) dx}{\int_0^l \psi M\left(\psi\right) dx}$$ Rayleigh's quotient for continuous system

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Beam

$$\omega^2 = R(y) = \frac{\int_0^l E\ I(x) \left[\frac{d^2 y(x)}{dx^2}\right]^2 dx}{\int_0^l \rho(x)\, y^2(x)\, dx}$$

Shaft

$$\omega^2 = R(\theta) = \frac{\int_0^l G\ J(x) \left[\frac{d\theta}{dx}\right]^2 dx}{\int_0^l I(x)\ \theta^2(x)\, dx}$$

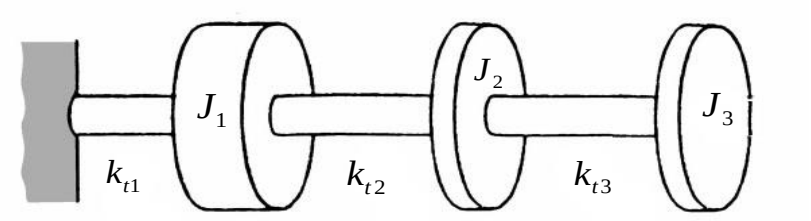
Bar

$$\omega^2 = R(u) = \frac{\int_0^l E\ A(x) \left[\frac{du(x)}{dx}\right]^2 dx}{\int_0^l \rho(x) u^2(x)\, dx}$$

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EXAMPLE



$$k_{t1} = k_{t2} = k_{t3} = K_t$$

$$J_1 = J_2 = J_3 = J$$

$$\begin{cases} A_1 = 1 \\ A_2 = 2 \\ A_3 = 3 \end{cases}$$

$$\begin{cases} T = \frac{1}{2} J\dot{\theta}_1^2 + \frac{1}{2} J\dot{\theta}_2^2 + \frac{1}{2} J\dot{\theta}_3^2 \\ V = \frac{1}{2} K\theta_1^2 + \frac{1}{2} K(\theta_2 - \theta_1)^2 + \frac{1}{2} K(\theta_3 - \theta_2)^2 \end{cases}$$

$$\begin{cases} \theta_1 = A_1 \cos\omega t \\ \theta_2 = A_2 \cos\omega t \\ \theta_3 = A_3 \cos\omega t \end{cases}$$

$$T_{\max} = \frac{1}{2} J\omega^2 \left(A_1^2 + A_2^2 + A_3^2\right)$$

$$V_{\max} = \frac{1}{2} K\left[A_1^2 + \left(A_2 - A_1\right)^2 + \left(A_3 - A_2\right)^2\right]$$

$$\Rightarrow T_{\max} = V_{\max}$$

$$\omega^2 = \frac{K}{J} = \frac{1+1+1}{1+4+9} \Rightarrow \omega = 0.463\sqrt{K_t/J}$$

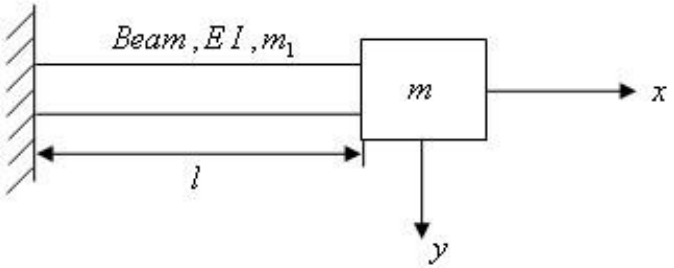
$$\begin{cases} \omega_{\text{exact}} = 0.445\sqrt{K_t/J} \\ \omega = 0.463\sqrt{K_t/J} \end{cases} \Rightarrow 4\% \text{ error}$$

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EXAMPLE

Find the natural frequency.



$$y(x)=\frac{1}{2}\left[3(\frac{x}{l})^2-(\frac{x}{l})^3\right]\frac{Fl^3}{3EI}$$

$$y(l)=y_l=\frac{Fl^3}{3EI} \quad \rightarrow \quad y(x)=\frac{1}{2}\left[3(\frac{x}{l})^2-(\frac{x}{l})^3\right]y_l$$

y_l = as the coordinate

$$\dot{y}(x)=\frac{1}{2}\left[3(\frac{x}{l})^2-(\frac{x}{l})^3\right]\dot{y}_l$$

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$$T=\frac{1}{2}\int_0^l \dot{y}^2\,dm_1+\frac{1}{2}m\dot{y}_l^2=\frac{1}{2}\left(\frac{33m_1}{140}+m\right)\dot{y}_l^2$$ The kinetic energy

$$v=\frac{1}{2}k\,y_l^2=\frac{1}{2}\frac{3EI}{l^3}y_l^2$$ The potential energy

$$y_l=A\cos\omega t\ \rightarrow\ \dot{y}_l=-A\,\omega\sin\omega t\ \rightarrow\ \begin{cases}(y_l)_{\max}=A\\(\dot{y}_l)_{\max}=\omega A\end{cases}$$

$$T_{\max}=V_{\max}$$

$$\frac{1}{2}\left(\frac{33}{140}m_1+m\right)\omega^2A^2=\frac{1}{2}\left(\frac{3EI}{l^3}\right)A^2$$

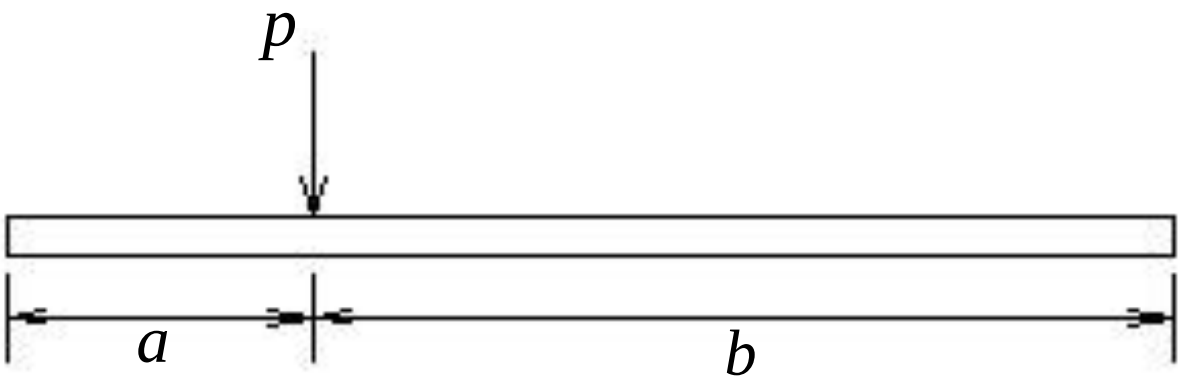
$$\omega^2=\left(\frac{\frac{3EI}{l^3}}{0.235m_1+m}\right)=\frac{k_{eq}}{m_{eq}}$$ If $m_1=m \Rightarrow \omega^2=2.429\frac{EI}{ml^3}$

$$(\omega_1)_R=1.559\sqrt{\frac{EI}{ml^3}},\qquad (\omega_1)_D=1.554\sqrt{\frac{EI}{ml^3}}$$

$$(\omega_1)_D\le\omega_{\text{exact}}\le\omega_{Ray}$$

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$$y(x)=\begin{cases}\dfrac{Pbx}{6EIl}(l^2-b^2-x^2) & 0\le x\le a\\ \dfrac{-Pa(l-x)}{6EIl}(a^2+x^2-2lx) & a\le x\le l\end{cases}$$

$$y(x=a)=\frac{Pa^2b^2}{3EIl}$$

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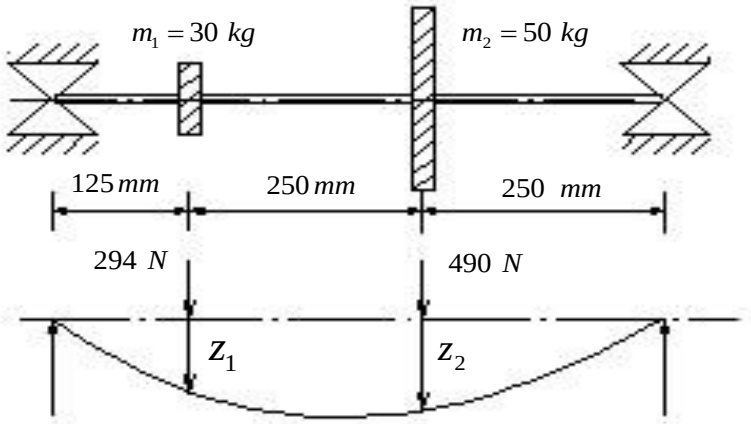
EXAMPLE

$E = 200 \text{ Gpa}$

$d = 25 \text{ mm}$

$$I = \frac{\pi}{64} d^4$$

$$I = 1.1917 \times 10^{-8} \quad \text{m}^4$$



$$\omega^2 = \frac{g\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i z_i^2} = \frac{g(m_1 z_1 + m_2 z_2)}{m_1 z_1^2 + m_2 z_2^2} = 15640$$

$$\omega = 125 \text{ rad/sec} = 1194 \text{ rpm}$$

$$z_1 = y_1' + y_1'' = \frac{0.6125}{EI} + \frac{1.2760}{EI} = \frac{1.8885}{EI} = 0.49 \text{ mm}$$

$$z_2 = y_2' + y_2'' = \frac{0.7656}{EI} + \frac{2.2986}{EI} = \frac{3.0625}{EI} = 0.799 \text{ mm}$$

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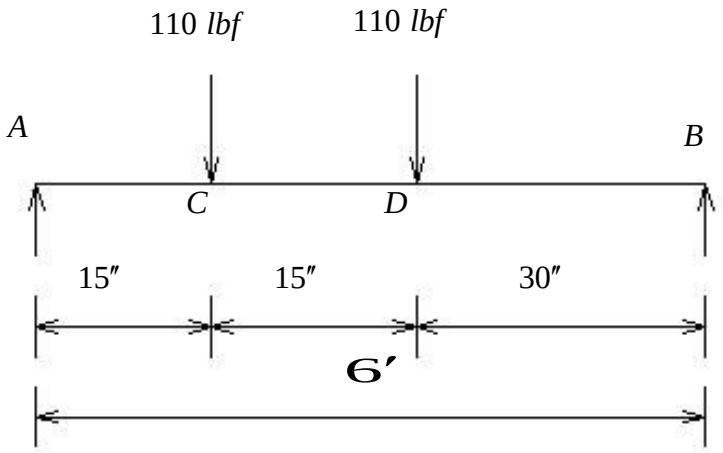
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EXAMPLE

$$d_o = 3\text{in} \quad , \quad d_i = 1\frac{1}{2}\text{in} \quad , \quad \rho_{\text{shaft}} = 0.28\,\text{lbf}/\text{in}^3 \quad , \quad E = 29\times10^6\ \text{psi}$$

$a)$ Dankerly's Method

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_s^2} \qquad \text{(I)}$$



$$\omega_{11}^2 = \frac{k}{m} = \frac{g}{\Delta} = \frac{g}{y_c} \quad , \quad \omega_{22}^2 = \frac{k}{m} = \frac{g}{y_D}$$

$$y_c = \frac{15^2\times110\times45^2}{3\times29\times10^{26}\times3.727\times60} = 0.002577\ \text{in}$$

$$y_D = \frac{110\times30^4}{3\times29\times10^{26}\times3.727\times60} = 0.004581\ \text{in}$$

$$\text{g} = 12\times32.2\,\text{ft}/\text{sec}^2 = 386\ \text{in}/\text{sec}^2$$

$$\omega_s^2 = \frac{\pi^4 EI}{\rho A l^4} = \frac{\pi^4\times386\times29\times10^6\times3.727}{0.28\times A\times60^4} = 211200\ \text{rad}^2/\text{sec}^2$$

$$\text{(I)} \rightarrow \frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_s^2}$$

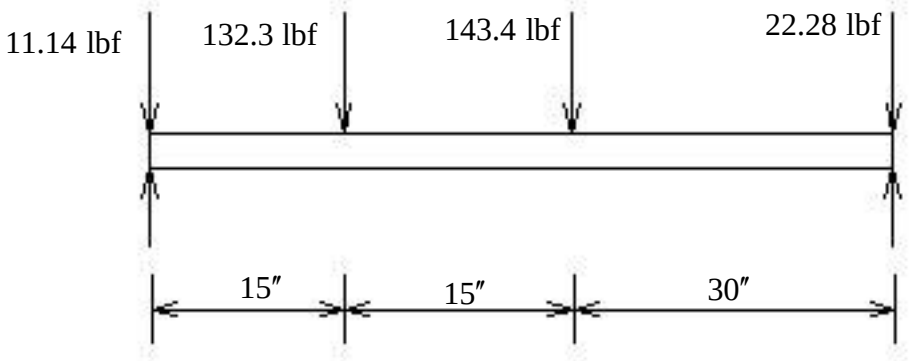
$$(\omega_1)_D \approx 207.34 \ \text{rad/s}$$

$$(\omega_1)_D \times \frac{60}{2\pi} = (n_1)_D = 1980\ \text{rpm}$$

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b) Rayleigh's Method:



$$\begin{cases} 7.5\times 1.482 = 11.14\ lb \\ 15\times 1.482 = 22.28\ lb \end{cases}$$

$$z_1 = y_1' + y_1'' = \ldots\ldots$$

$$z_2 = y_2' + y_2'' = \ldots\ldots$$

$$\omega^2 = \frac{g\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i z_i^2} = 44240$$

$$(\omega_1)_R = 210.33 \text{ rad/sec}$$

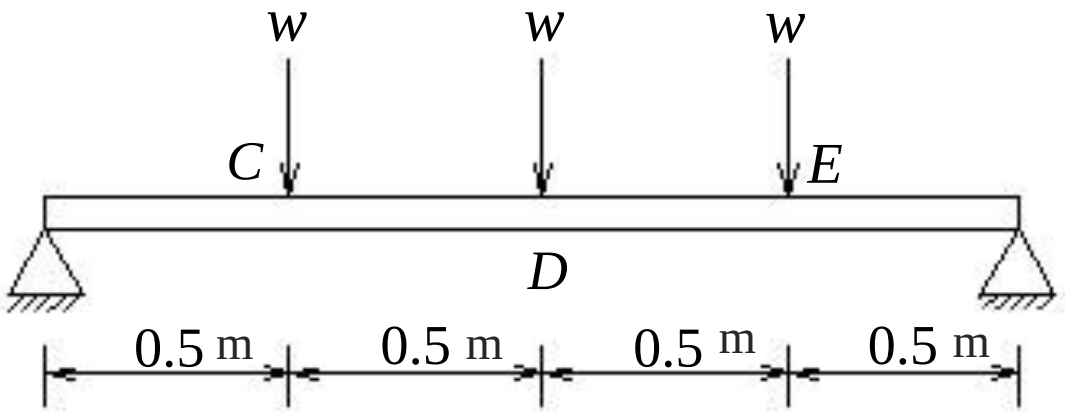
$$(n_1)_R = 2008 \text{ rpm}$$

$$(n_1)_D < (n_1)_{\text{exact}} < (n_1)_R$$

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H.W.11:



$\rho = 7840 \ \ \mathrm{kg/m^3}$

$E_{\text{steel}} = 210 \ \ \mathrm{Gpa}$

d=4 cm

$m = 21 \ \ \mathrm{kg}$

Ans:

$(n_1) = 516 \ \ \mathrm{rpm}$

$(n_2) = 2114 \ \ \mathrm{rpm}$

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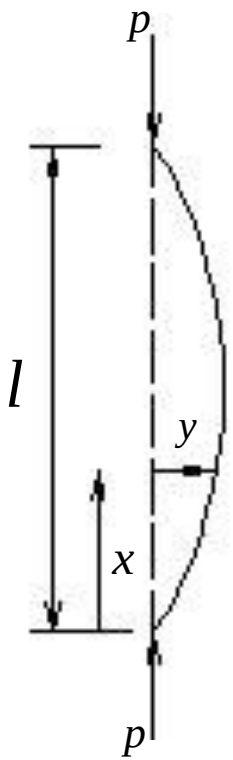
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EXAMPLE

$$\begin{cases} y_0 = \text{total deflection mid span of strut} \\ y = \text{total deflection any point of strut} \end{cases}$$

$\delta_0 = \max$ initial eccentriciting of strut at mid span when $P = 0$

$\delta =$ deflection at any point on strut when $P = 0$



$$\frac{y}{\delta} = \frac{y_0}{\delta_0} = \frac{P_{cr}}{P_{cr} - P} \qquad P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$y = y_0 \sin\frac{\pi x}{l}$$ deflection from of the strut

$$y = \left(\frac{P_{cr}}{P_{cr} - P}\right)\delta_0 \sin\frac{\pi x}{l}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g\int_0^l \delta\, dx}{\int_0^l \delta^2\, dx}}$$ the frequency of transverse vibration of a beam

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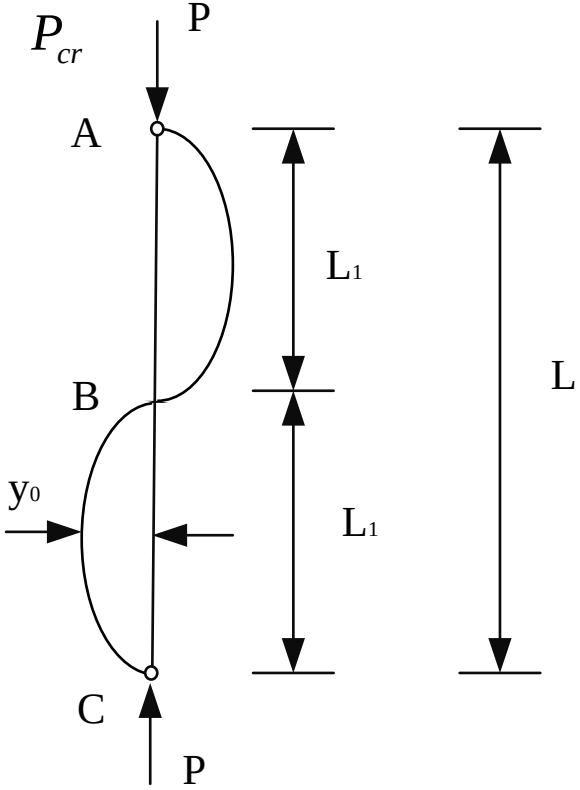
$$n_p = \frac{1}{2\pi}\sqrt{\frac{g\int_0^l y\,dx}{\int_0^l y^2\,dx}}$$ the frequency of transverse vibration of a laterally

$$\left(\frac{n_p}{n}\right)^2 = \frac{\int_0^l y\,dx \int_0^l \delta^2\,dx}{\int_0^l y^2\,dx \int_0^l \delta\,dx} = \frac{\left(\frac{y_0}{\delta_0}\right)\int_0^l \delta\,dx}{\left(\frac{y_0}{\delta_0}\right)^2 \int_0^l \delta^2\,dx} \frac{\int_0^l \delta^2\,dx}{\int_0^l \delta\,dx} = \frac{\delta_0}{y_0} = \frac{P_{cr}-P}{P_{cr}}$$

$$\left(\frac{\omega_{p1}}{\omega}\right)^2 = 1-\frac{P}{P_{cr}} \quad , \quad \omega_1 = \pi^2\frac{EI}{ml^4}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{g\sum \omega_i\,\delta_i}{\sum \omega_i\,\delta_i^2}}$$

$$\left(\frac{\omega_{p2}}{\omega}\right)^2 = 1-\frac{P}{P_{cr2}} \quad , \quad \omega_2^2 = \frac{16\pi^2 EI}{ml^4} \quad \rightarrow \quad P_{cr2} = \frac{4\pi^2 EI}{l^2}$$



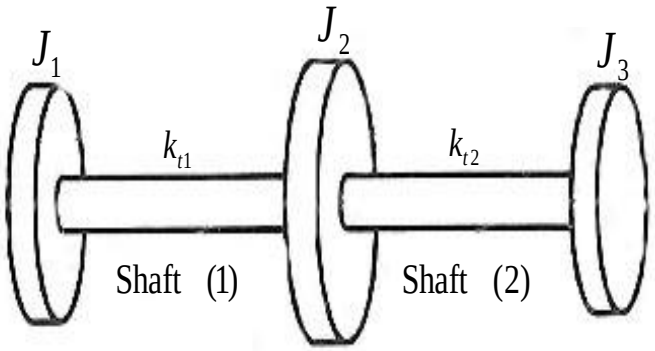
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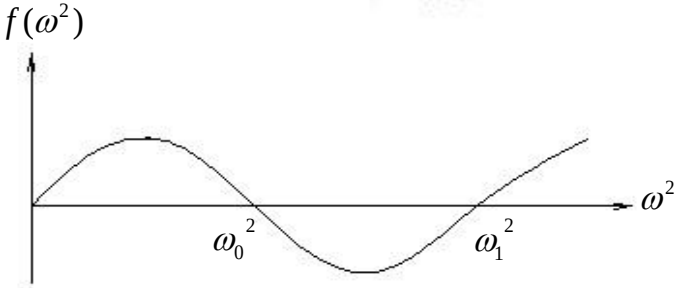
5.3. Holzer Method

$$\begin{cases} J_1\ddot{\theta}_1 = -K_{t1}(\theta_1-\theta_2) \\ J_2\ddot{\theta}_2 = -K_{t1}(\theta_2-\theta_1)-K_{t2}(\theta_2-\theta_3) \\ J_3\ddot{\theta}_3 = -K_{t2}(\theta_3-\theta_2) \end{cases}$$

$$\theta_i = \theta_i \sin\omega t \qquad i = 1,2,3$$



$$\begin{cases} -\omega^2 J_1\theta_1 = -K_{t1}(\theta_1-\theta_2) & (1) \\ -\omega^2 J_2\theta_2 = -K_{t1}(\theta_2-\theta_1)-K_{t2}(\theta_2-\theta_3) & (2) \\ -\omega^2 J_3\theta_3 = -K_{t2}(\theta_3-\theta_2) & (3) \end{cases}$$



$$(1+2+3) \Rightarrow \sum_{i=1}^{3} J_i\theta_i\omega^2 = 0$$

assume a trial ferquency

$$\begin{cases} \theta_1 = 1.0 \\ \theta_2 = \theta_1 - \omega^2 J_1\theta_1 / K_{t1} \\ \theta_3 = \theta_2 - \omega^2(J_1\theta_1 + J_2\theta_2)/K_{t2} \end{cases}$$

$$\begin{cases} f(\omega^2) = \sum j_i\theta_i\omega^2 \\ \theta_j = \theta_{j-1} - \dfrac{\omega^2}{K_{(j-1)}}\sum_{i=1}^{j-1} j_i\theta_i \end{cases} \qquad j = 2,3,\ldots,n$$

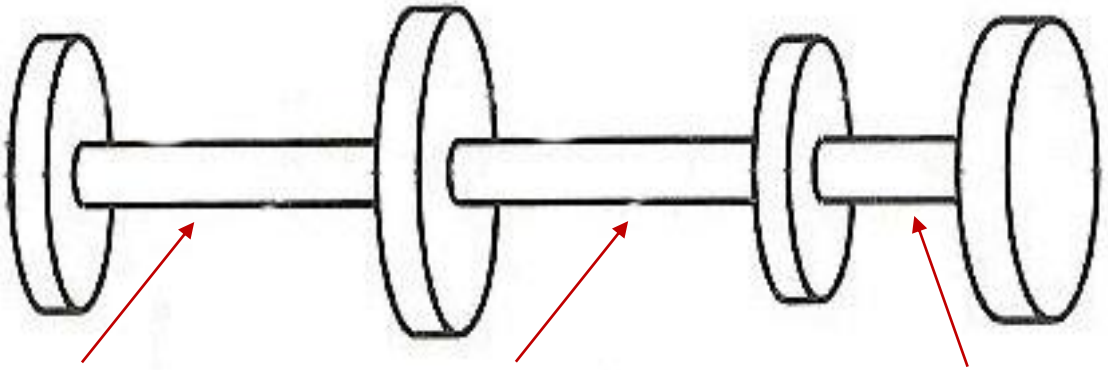
$$T = f(\omega^2) = \sum j_i\theta_i\omega^2 \quad \text{residul Torque}$$

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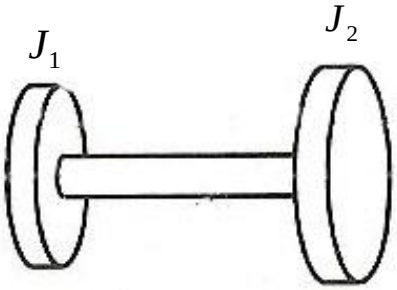
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EXAMPLE

$J_1 = 0.4 \ \text{kg.m}^2$ $J_2 = 0.1 \ \text{kg.m}^2$ $J_3 = 0.4 \ \text{kg.m}^2$ $J_4 = 0.1 \ \text{kg.m}^2$



$k_{t1} = 10 \ \text{kN.m/rad}$ $k_{t2} = 20 \ \text{kN.m/rad}$ $k_{t3} = 10 \ \text{kN.m/rad}$



$$\omega_1 = 0$$

$$\omega_2 = \sqrt{k_t \left(\frac{J_1 + J_2}{J_1 J_2}\right)}$$

$$k_t = \frac{1}{1/k_{t1} + 1/k_{t2}} = \frac{10^3}{1/10 + 1/20} = \frac{20}{3} \ \text{kN.m/rad}$$

$$\omega = \sqrt{k_t / J_1 + k_t / (J_3 + J_4)} =$$

$$\sqrt{(20/3)(10^3)\left[1/0.4 + 1/(0.4 + 0.1)\right]} = 173 \ \text{rad/sec}$$

First trial

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A. First trial: First mode: ω=173 rad/s

Row	J	θ	$J\theta\omega^2$	$\sum J\theta\omega^2$	k_r	$\frac{1}{k_r}\sum J\theta\omega^2$
	(m^2.kg)	(rad)	(kN.m)	(kN.m)	(kN.m/rad)	(rad)
1	0.4	1.000	11.97	11.97	10	1.197
2	0.1	-0.197	-0.59	11.38	20	0.569
3	0.4	-0.766	-9.17	2.21	10	0.218
4	0.1	-1.043	-3.12	-0.92		

Residual torque

B. Second trial: First mode: ω=165 rad/s

Row	J	θ	$J\theta\omega^2$	$\sum J\theta\omega^2$	k_r	$\frac{1}{k_r}\sum J\theta\omega^2$
	(m^2.kg)	(rad)	(kN.m)	(kN.m)	(kN.m/rad)	(rad)
1	0.4	1.000	10.89	10.89	10	1.089
2	0.1	-0.089	-0.24	10.65	20	0.532
3	0.4	-0.621	-6.77	3.88	10	0.388
4	0.1	-1.009	-2.74	1.13		

Residual torque

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EXAMPLE

First trial ω=173 rad/sec

The residual torque is -0.92 kN.m
The negative sign is indicates that the trial frequency is too high for the first modes, then:

Second trial ω=165 rad/sec

The residual torque is 1.3 kN.m.
The linear interpolation between the first and second trial gives:

Third trial ω=170 rad/sec

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C. Third trial: First mode: ω=170 rad/s

Row	J	θ	$J\theta\omega^2$	$\sum J\theta\omega^2$	k_r	$\frac{1}{k_r}\sum J\theta\omega^2$
	(m^2.kg)	(rad)	(kN.m)	(kN.m)	(kN.m/rad)	(rad)
1	0.4	1.000	11.56	11.56	10	1.156
2	0.1	-0.156	-0.45	11.11	20	0.555
3	0.4	-0.711	-8.22	2.88	10	0.288
4	0.1	-1.000	-2.89	-0.01		

Residual torque

D. Second mode: ω=354 rad/s

Row	J	θ	$J\theta\omega^2$	$\sum J\theta\omega^2$	k_r	$\frac{1}{k_r}\sum J\theta\omega^2$
	(m^2.kg)	(rad)	(kN.m)	(kN.m)	(kN.m/rad)	(rad)
1	0.4	1.00	50.1	50.1	10	5.01
2	0.1	-4.01	-50.3	-0.2	20	-0.01
3	0.4	-4.00	-200.7	-200.9	10	-20.09
4	0.1	16.09	-201.6	-0.6		

Residual torque

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E. Third mode: ω=588 rad/s

Row	J (m^2.kg)	θ (rad)	$J\theta\omega^2$ (kN.m)	$\sum J\theta\omega^2$ (kN.m)	k_r (kN.m/rad)	$\frac{1}{k_r}\sum J\theta\omega^2$ (rad)
1	0.4	1.00	138.3	138.3	10	13.83
2	0.1	-12.83	-443.6	-305.3	20	-15.26
3	0.4	-2.43	336.7	31.4	10	3.14
4	0.1	-0.70	-24.4	7.0		

Residual torque

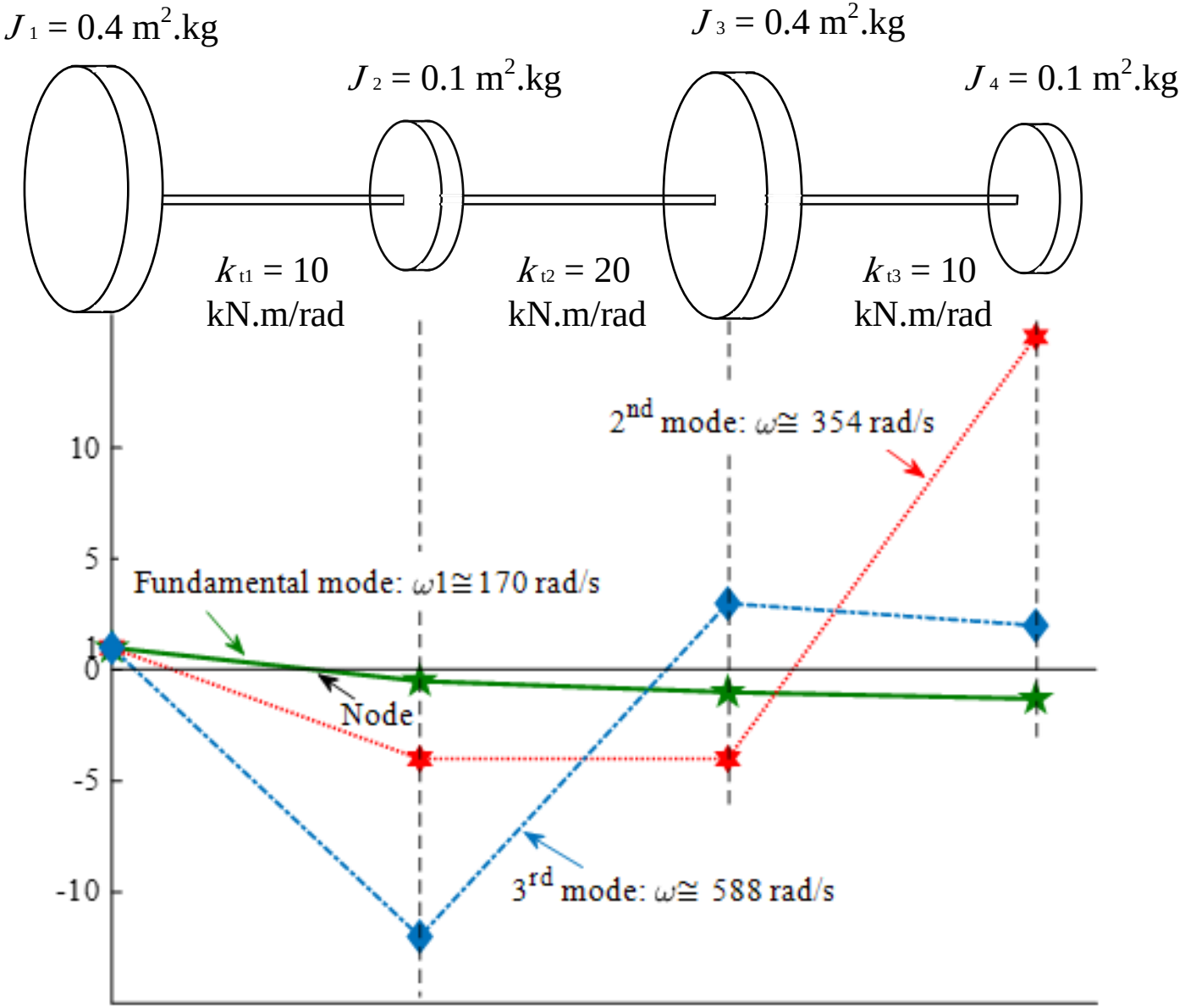
Advantageous of Holzer:

1. Gives exact mode shapes (Modal vectors)
2. Provides torque in elastic axis with natural frequency, which can easily leads to find shear stress

$$\tau = \frac{16T}{\pi d^3}$$

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with damping

$$\begin{cases} \Delta U_{\text{in}} = \Delta U_{\text{out}} \\ \sum \pi \theta_i M_i \sin \phi_i = \sum \pi C_i \theta_i^2 \omega \end{cases}$$

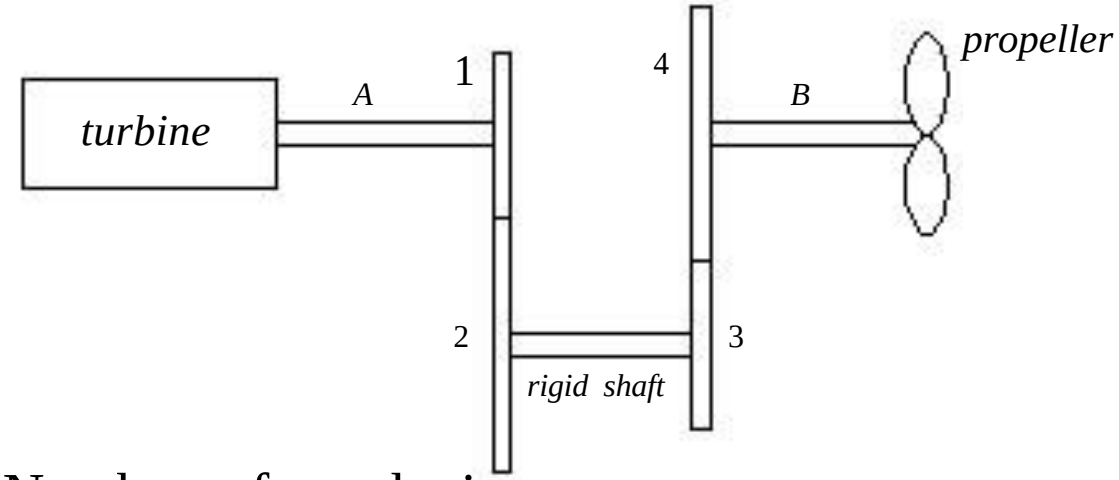
Forced Vibration

$$\begin{cases} \sum (J_i \omega^2 \theta_i - M_0) = 0 \\ M = M_i \sin \omega t \end{cases}$$

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H.W. 12:



$J_T = 2000\ \text{kg.m}^2$

$J_p = 500\ \text{kg.m}^2$

$J_1 = 250\ \text{kg.m}^2$

$J_2 = 1000\ \text{kg.m}^2$

$J_3 = 350\ \text{kg.m}^2$

$J_4 = 1500\ \text{kg.m}^2$

$k_A = 50\ \text{kN.m/rad}$

$k_B = 288\ \text{kN.m/rad}$

Number of teeth in gear:

$z_1 = 40$

$z_2 = 120$

$z_3 = 50$

$z_4 = 200$

Ans:

$f_1 = 1.919\ \text{Hz}$

$f_2 = 3.839\ \text{Hz}$

The kinetic Energy

Help:

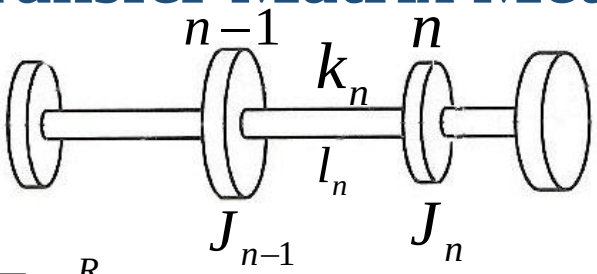
$$\text{Gear ratio} = \frac{\text{Speed of motor shaft}}{\text{Speed of impeller shaf}} = \frac{\Omega_A}{\Omega_B} = G$$

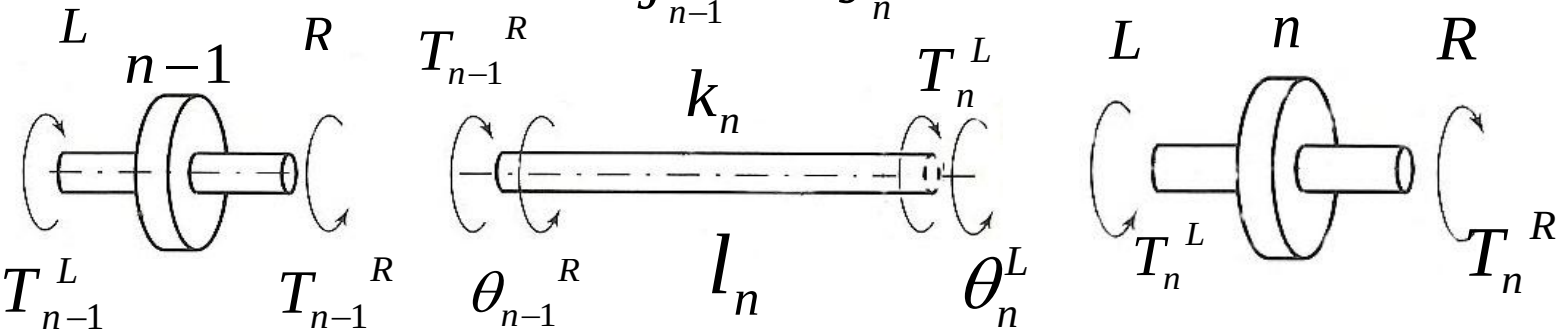
$$(\text{K.E.})_1 = (\text{K.E.})_{2,\text{eq.}} \Rightarrow \frac{1}{2} I_{B'} \Omega_A^2 = \frac{1}{2} I_B \Omega_B^2 \Rightarrow I_{B'} = \frac{I_B}{G^2}$$

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5.4. Transfer Matrix Method (TMM)





Torsional system with subsystems:

$$T_n^R - T_n^L = J_n \ddot{\theta}_n \qquad \text{Eq. of motion}$$

The point inertia matrix

$$\begin{cases} T_n^R = -J_n \omega^2 \theta_n + T_n^L \\ \theta_n = \theta_n^R = \theta_n^L \end{cases} \text{ for harmonic motion} \Rightarrow \begin{Bmatrix} \theta \\ T \end{Bmatrix}_n^R = \begin{bmatrix} 1 & 0 \\ J_n \omega^2 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_n^L$$

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shaft:

$$T_{n-1}^{R} = T_{n}^{L} \quad , \quad \theta_{n}^{L} - \theta_{n-1}^{R} = \frac{T_{n-1}^{R}}{K_{n}}$$

The field matrix

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n}^{L} = \begin{bmatrix} 1 & 1/K \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n}^{R} = \begin{bmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{bmatrix}_{n} \begin{bmatrix} 1 & 1/K \\ 0 & 1 \end{bmatrix}_{n} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

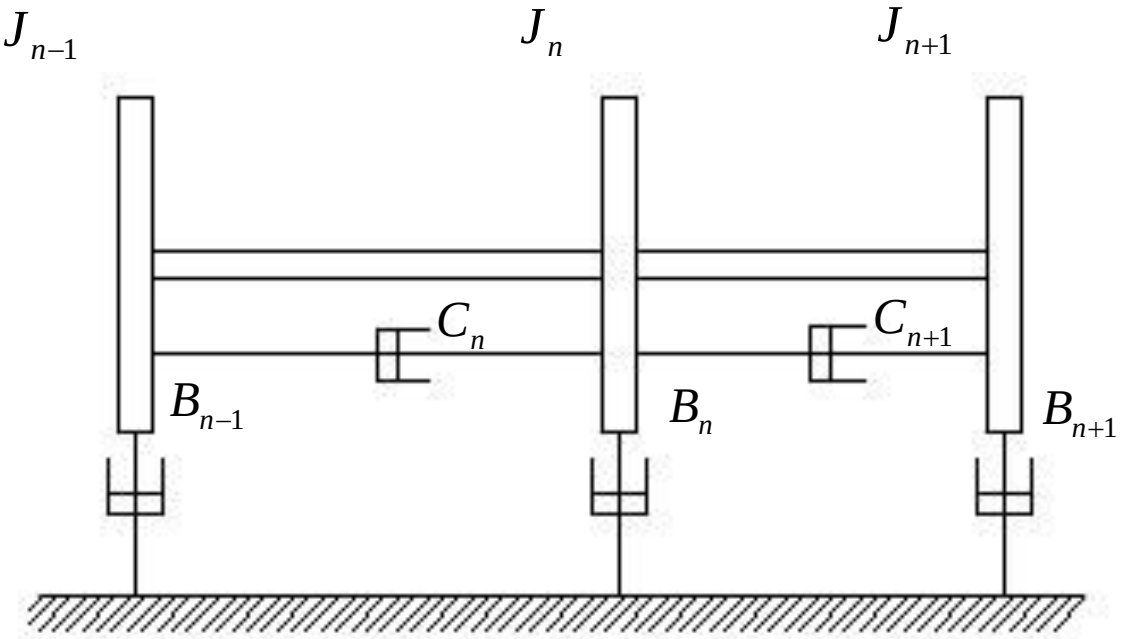
$$= \begin{bmatrix} 1 & 1/K \\ -J\omega^2 & 1 - J\omega^2/K \end{bmatrix}_{n} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

$$\{Z\}_{n}^{R} = [H]_{n} \{Z\}_{n-1}^{R}$$ The recurrence formula

$$\{Z\}_{n}^{R} = [H_n \; H_{n-1} \; H_{n-2} \; \dots \; H_2 \; H_1]_{n} \{Z\}_{0}^{R}$$

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Torsional system with damping

$$\begin{Bmatrix}\theta\\ T\end{Bmatrix}_n^R = \begin{bmatrix}1 & 0\\ i\omega B - J\omega^2 & 1\end{bmatrix}_n \begin{Bmatrix}\theta\\ T\end{Bmatrix}_n^L$$

The point inetia matrix

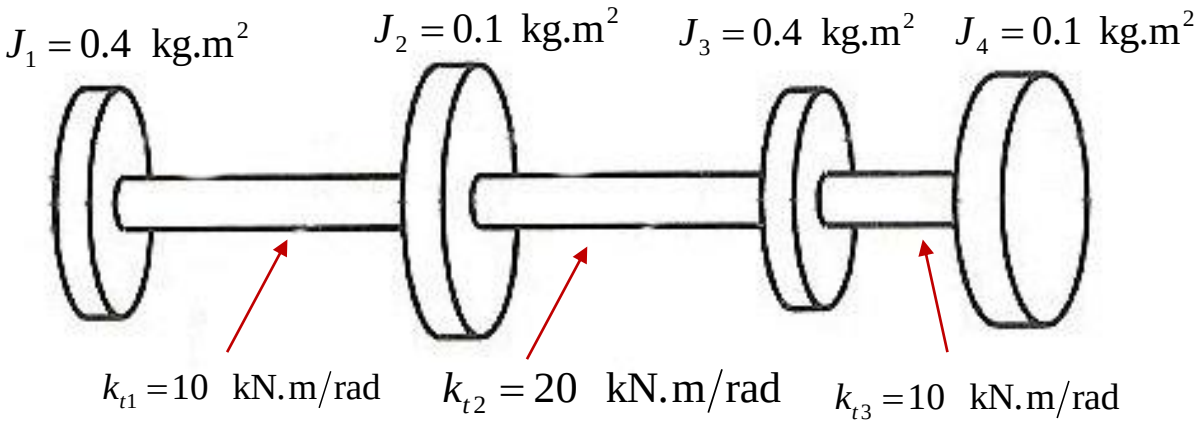
$$\begin{Bmatrix}\theta\\ T\end{Bmatrix}_n^L = \begin{bmatrix}1 & \dfrac{1}{K + i\omega c}\\ 0 & 1\end{bmatrix} \begin{Bmatrix}\theta\\ T\end{Bmatrix}_{n-1}^R$$

The field matrix

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EXAMPLE



assuming $\theta_1 = 1.0$ rad ; $T_1^L = 0$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^R = \begin{bmatrix}1 & 0\\ -J\omega^2 & 1\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^L = \begin{bmatrix}1 & 0\\ 0.4\omega^2 & 1\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} = \begin{Bmatrix}1\\ -0.4\omega^2\end{Bmatrix}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{bmatrix}1 & \dfrac{1}{K_2}\\ -J\omega^2 & 1-\dfrac{J\omega^2}{K_2}\end{bmatrix}_2 \begin{Bmatrix}\theta\\T\end{Bmatrix}_1^R = \begin{bmatrix}1 & 10^{-4}\\ -0.1\omega^2 & 1-\omega^2\left(10^{-5}\right)\end{bmatrix}\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^R = \begin{bmatrix}1 & 10^{-4}\\ -0.1\omega^2 & 1-\omega^2\left(10^{-5}\right)\end{bmatrix}\begin{Bmatrix}1\\0.4\omega^2\end{Bmatrix}$$

$\vdots$

$$T_4^R = 0.8\left(10^{-15}\right)\omega^8 - 4\left(10^{-10}\right)\omega^6 + 4.55\left(10^{-5}\right)\omega^4 - \omega^2 = 0 \quad \Rightarrow \quad \omega = 170 \text{ rad/sec}$$

$$\begin{Bmatrix}\theta\\T\end{Bmatrix}_1^R = \begin{Bmatrix}1\\-11.56\end{Bmatrix}, \quad \begin{Bmatrix}\theta\\T\end{Bmatrix}_2^R = \begin{Bmatrix}-0.156\\-11.11\end{Bmatrix}, \quad \begin{Bmatrix}\theta\\T\end{Bmatrix}_3^R = \begin{Bmatrix}-0.711\\-2.88\end{Bmatrix}, \quad \begin{Bmatrix}\theta\\T\end{Bmatrix}_4^R = \begin{Bmatrix}-1.00\\0.01\end{Bmatrix}$$

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EXAMPLE

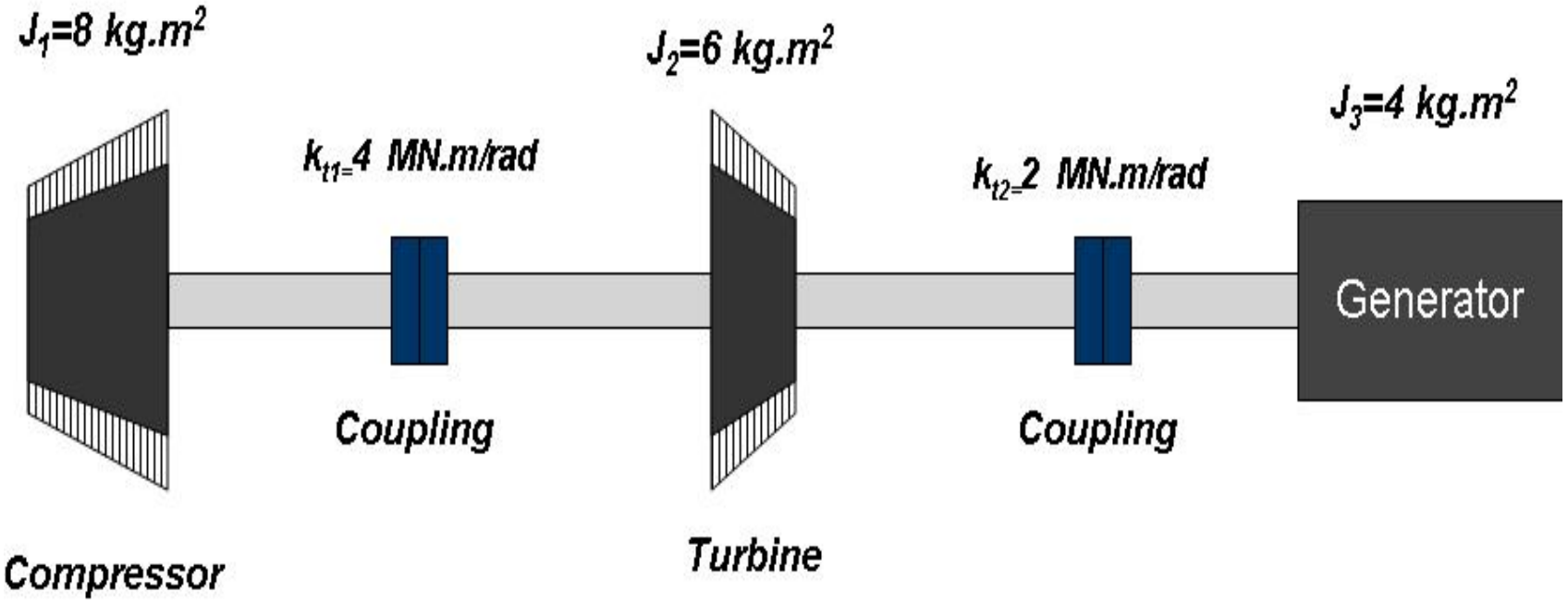
Natural frequencies and state vectors

ω (rad/s)	$\begin{bmatrix}\theta\\T\end{bmatrix}_1^R$	$\begin{bmatrix}\theta\\T\end{bmatrix}_2^R$	$\begin{bmatrix}\theta\\T\end{bmatrix}_3^R$	$\begin{bmatrix}\theta\\T\end{bmatrix}_4^R$
170	1.00 -11.56	-0.156 -11.11	-0.711 -2.88	-1.00 0.01
354	1.00 -50.1	-4.01 0.2	-4.00 200.9	16.09 0.6
588	1.00 -138.3	-12.83 305.3	2.43 -31.4	-0.70 -7.00

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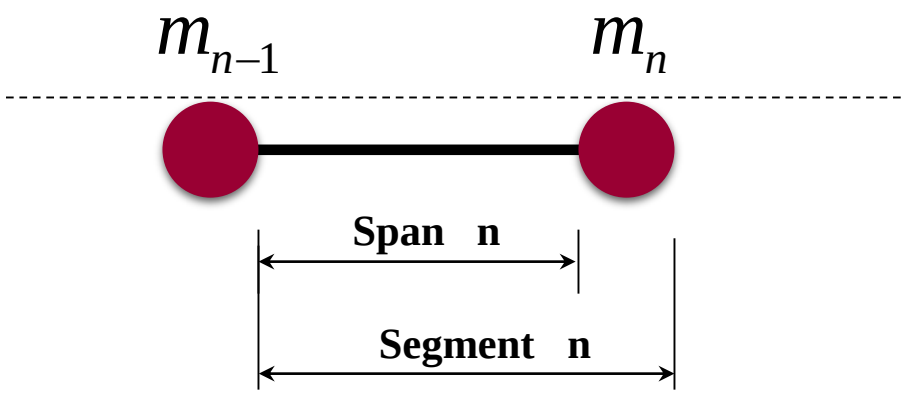
H.W.13:

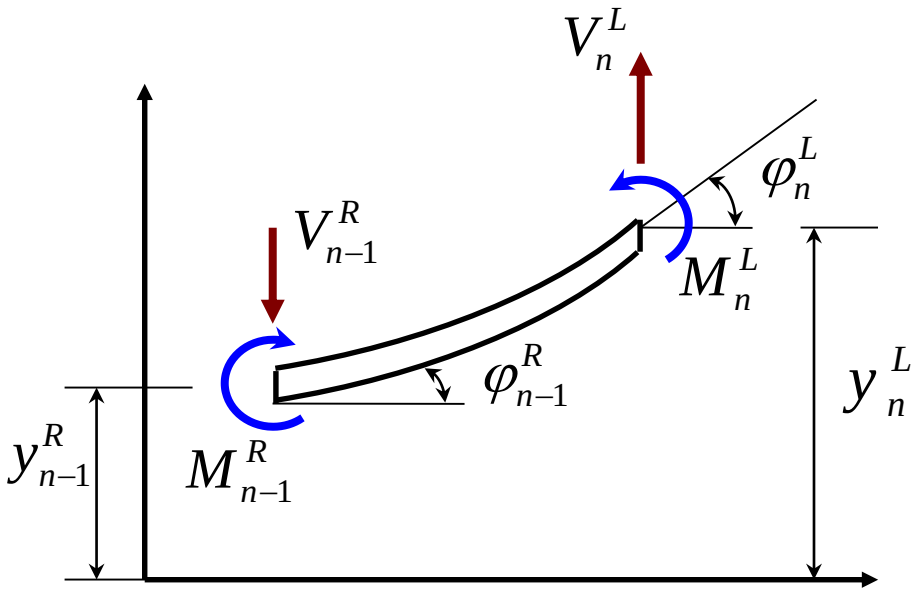


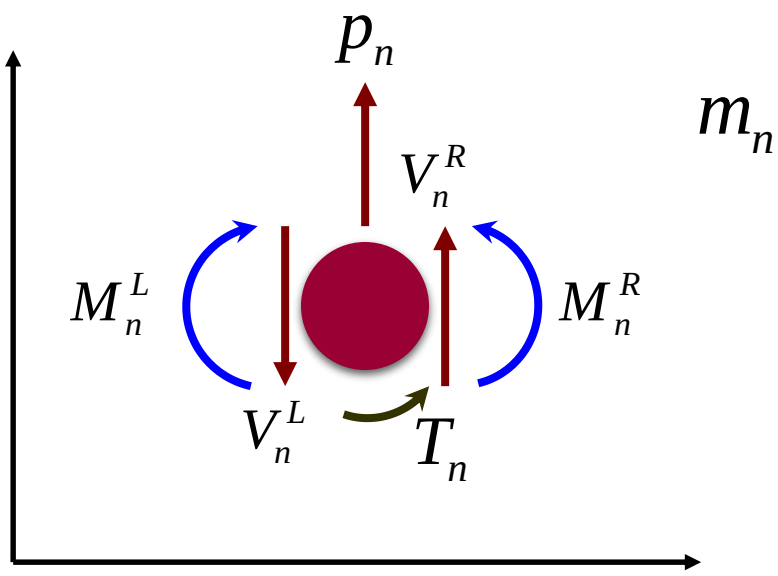
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5.5. Myklestad – Prohl Method

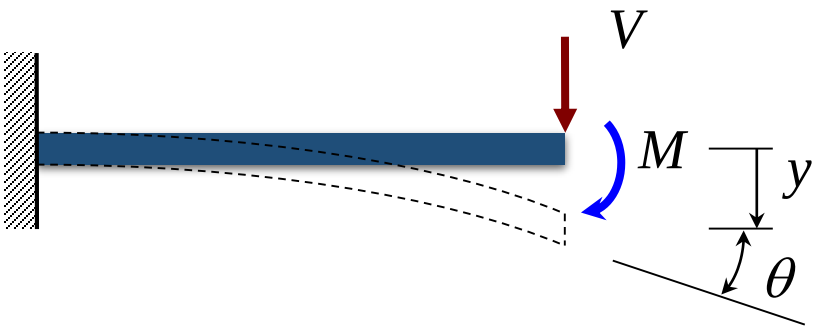






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$$\begin{cases} V = \dfrac{dM}{dx} \\ M = EI\dfrac{d\varphi}{dx} \\ \varphi = \dfrac{dy}{dx} \end{cases}$$

$$\begin{cases} \theta = \dfrac{Vl^2}{2EI} + \dfrac{Ml}{EI} \\ y = \dfrac{Vl^3}{3EI} + \dfrac{Ml^2}{2EI} \end{cases}$$

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$$\begin{cases} V_n^L = V_{n-1}^R \\ M_n^L = M_{n-1}^R - l_n V_{n-1}^R \end{cases}$$

$$\begin{cases} \varphi_n^L - \varphi_{n-1}^R = (\dfrac{l}{EI})_n M_n^L + (\dfrac{l^2}{2EI})_n V_n^L \\ \varphi_n^L = \varphi_{n-1}^R + (\dfrac{l}{EI})_n M_{n-1}^R - (\dfrac{l^2}{2EI})_n V_{n-1}^R \end{cases}$$

$$\begin{cases} y_n^L - y_{n-1}^R = l_n \varphi_{n-1}^R + (\dfrac{l^2}{2EI})_n M_n^L + (\dfrac{l^3}{3EI})_n V_n^L \\ y_n^L = y_{n-1}^R + l_n \varphi_{n-1}^R + (\dfrac{l^2}{2EI})_n M_{n-1}^R - (\dfrac{l^3}{6EI})_n V_{n-1}^R \end{cases}$$

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$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_n^L = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & -\frac{l^3}{6EI} \\ 0 & 1 & \frac{l}{EI} & -\frac{l^2}{2EI} \\ 0 & 0 & 1 & -l \\ 0 & 0 & 0 & 1 \end{bmatrix}_n \begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_{n-1}^R$$

The Field transfer matrix

$$\{z\}_n^L = [F]_n \{z\}_{n-1}^R$$

$$V_n^R = V_n^L - \omega^2 m_n y_n^L \quad , \quad M_n^R = M_n^L - \omega^2 J_n \varphi_n^L$$

$$\varphi_n^L = \varphi_n^R \quad , \quad y_n^L = y_n^R$$

$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_n^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 J & 1 & 0 \\ -\omega^2 m & 0 & 0 & 1 \end{bmatrix}_n \begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_n^L$$

The Point transfer matrix

$$\{z\}_n^R = [P]_n \{z\}_n^L = [P]_n [F]_n \{z\}_{n-1}^R = [H]_n \{z\}_{n-1}^R$$

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$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_n^R = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & -\frac{l^3}{6EI} \\ 0 & 1 & \frac{l}{EI} & -\frac{l^2}{2EI} \\ 0 & (-\omega^2 J) & (1-\omega^2 J\frac{l}{EI}) & (-l+\omega^2 J\frac{l^2}{2EI}) \\ (-\omega^2 m_n) & (-\omega^2 ml) & (-\omega^2 m\frac{l^2}{2EI}) & (1+\omega^2 m\frac{l^3}{6EI}) \end{bmatrix}_n \begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_{n-1}^R$$

The transfer matrix

Common B.C.

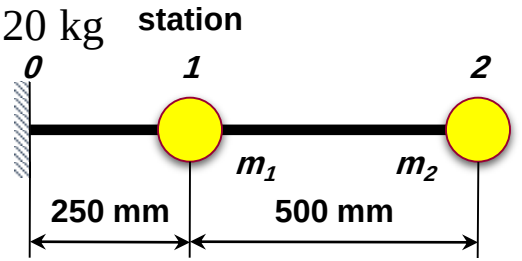
	y	φ	M	V
Simple support	0	φ	0	V
Free	y	φ	0	0
Fixed	0	0	M	V

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EXAMPLE

Find the natural frequencies of the lumped-mass .Asuume $m_1 = m_2 = 20$ kg and the flexural stiffnessEI of the beam is 3 KN.m^2 .



$$\{Z\}_1^R = H_1\{Z\}_0^R \quad \text{and} \quad \{Z\}_2^R = H_2\{Z\}_1^R = H_2H_1\{Z\}_0^R$$

Where $\{Z\}_0^R = \{Y \quad \Phi \quad M \quad V\}_0^R = \{0 \quad 0 \quad M_0 \quad V_0\}$

and M_0 and V_0 are the unknown moment and shear at the fixed end.So we have

$$\begin{bmatrix} Y \\ \Phi \\ M \\ V \end{bmatrix}_1^R = \begin{bmatrix} 1 & 0.25 & 10.4(10^{-6}) & -0.87(10^{-6}) \\ 0 & 1 & 83.3(10^{-6}) & -10.4(10^{-6}) \\ 0 & 0 & 1 & -0.25 \\ -20\omega^2 & -5\omega^2 & -208.3(10^{-6})\omega^2 & 1+17.36(10^{-6})\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M \\ V \end{bmatrix}_0^R \quad \text{and}$$

$$\begin{bmatrix} Y \\ \Phi \\ M \\ V \end{bmatrix}_2^R = \begin{bmatrix} 1 & 0.50 & 41.7(10^{-6}) & -6.9(10^{-6}) \\ 0 & 1 & 166.67(10^{-6}) & -41.7(10^{-6}) \\ 0 & 0 & 1 & -0.50 \\ -20\omega^2 & -10\omega^2 & -833.3(10^{-6})\omega^2 & 1+138.9(10^{-6})\omega^2 \end{bmatrix} \begin{bmatrix} Y \\ \Phi \\ M \\ V \end{bmatrix}_1^R$$

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From $\{Z\}_2^R = H_2\{Z\}_1^R = H_2H_1\{Z\}_0^R$,we get

$$\begin{bmatrix} \mathrm{Y} \\ \Phi \\ \mathrm{M} \\ \mathrm{V} \end{bmatrix}_2^R = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathrm{M}_0 \\ \mathrm{V}_0 \end{bmatrix}$$

M_2^R and V_2^R must be zero at the free end of the beam,that is

$$M_2^R = 0 = H_{33}M_0 + H_{34}V_0$$
$$V_2^R = 0 = H_{43}M_0 + H_{44}V_0$$

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$$V_2^R = 0 = H_{43}M_0 + H_{44}V_0$$

For a nontirival solution:

$$\begin{vmatrix} H_{33} & H_{34} \\ H_{43} & H_{44} \end{vmatrix} = 0$$

$$\begin{vmatrix} 1+104.2(10^{-6})\omega^2 & -0.75-8.68(10^{-6})\omega^2 \\ -2.08(10^{-6})\omega^2 - 28.9(10^{-9})\omega^4 & 1+486(10^{-6})\omega^2 + 2.41(10^{-9})\omega^4 \end{vmatrix} = 0$$

Or

$$\omega^4 - 73.3(10^3)\omega^2 + 75.4(10^6) = 0$$

$$\omega_1 = 32.3 \text{ rad/s} \quad \text{or} \quad f_1 = 5.14 \text{ HZ}$$

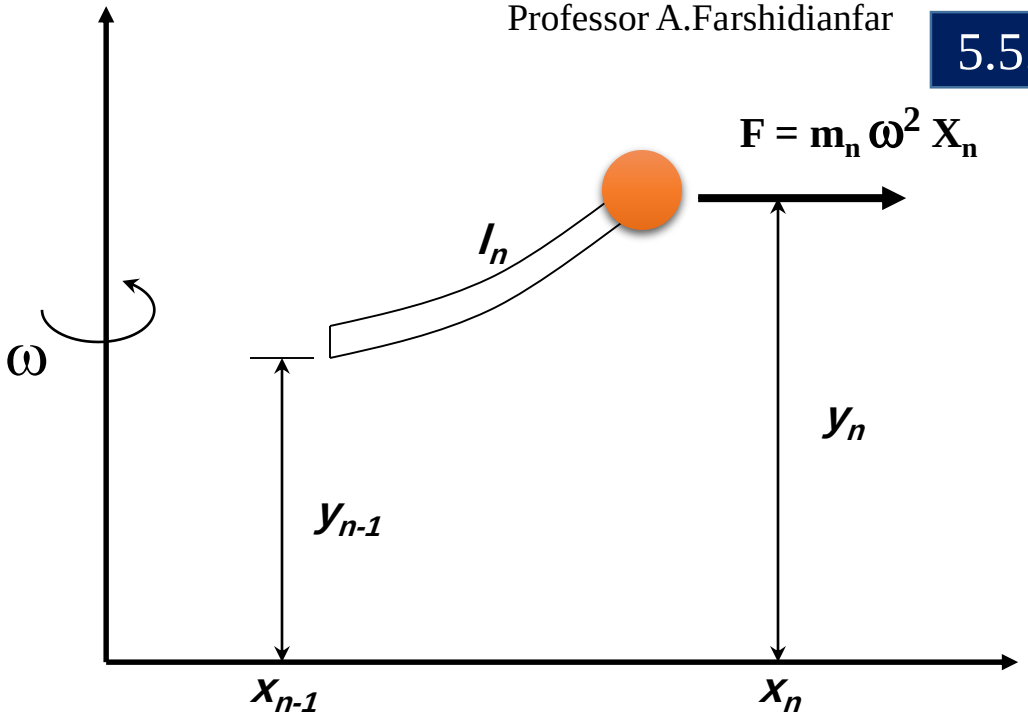
$$\omega_2 = 269 \text{ rad/s} \quad \text{or} \quad f_1 = 42.8 \text{ HZ}$$

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H.W.14:

A Turbine Blade :



Answer:

$$\begin{bmatrix} Y \\ \Phi \\ M \\ V \end{bmatrix}_{n-1}^{R} = \begin{bmatrix} 1 & -\left(L + FL^3/6EI\right) & L^2/2EI & L^3/6EI \\ 0 & \left(1 + FL^2/2EI\right) & -L/EI & -L^2/2EI \\ 0 & -\left(L + FL^3/6EI\right) & \left(1 + FL^2/2EI\right) & \left(L + FL^3/6EI\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n} \begin{bmatrix} Y \\ \Phi \\ M \\ V \end{bmatrix}_{n}^{L}$$

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$$\begin{bmatrix} Y \\ \Phi \\ M \\ N \end{bmatrix}_n^L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{bmatrix}_n \begin{bmatrix} Y \\ \Phi \\ M \\ N \end{bmatrix}_n^R$$

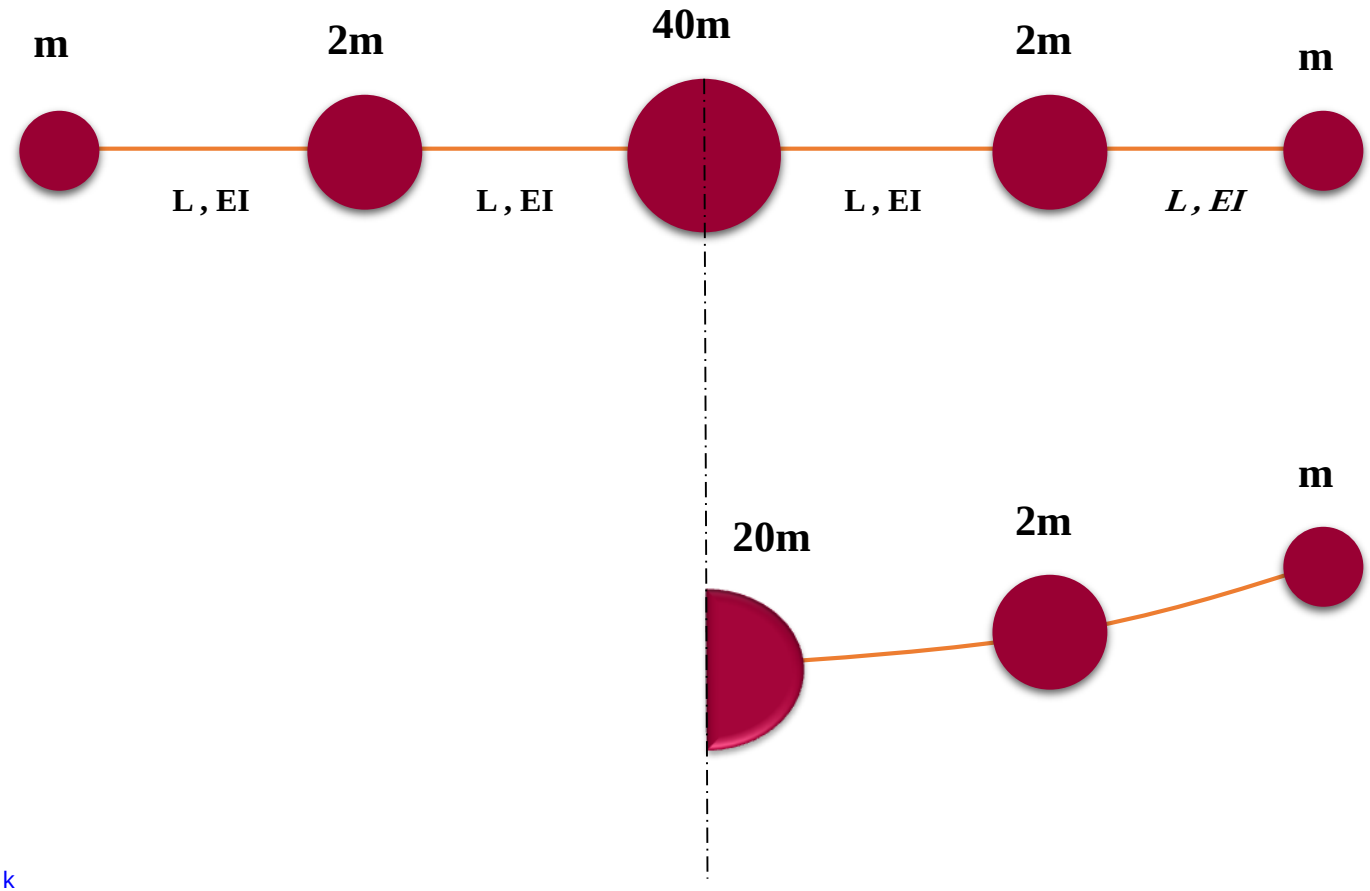
$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_{n-1}^R = \begin{bmatrix} (1+\frac{\omega^2 m l^3}{6EI}) & -(l+\frac{Fl^3}{6EI}) & \frac{l^2}{2EI} & \frac{l^3}{6EI} \\ (-\frac{\omega^2 m l^2}{2EI}) & (1+\frac{Fl^2}{2EI}) & -\frac{l}{EI} & -\frac{l^2}{2EI} \\ \omega^2 m(l+\frac{Fl^3}{6EI}) & -(l+\frac{Fl^3}{6EI}) & (1+\frac{Fl^2}{2EI}) & (l+\frac{Fl^3}{6EI}) \\ (\omega^2 m) & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_n^R$$

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H.W.15:

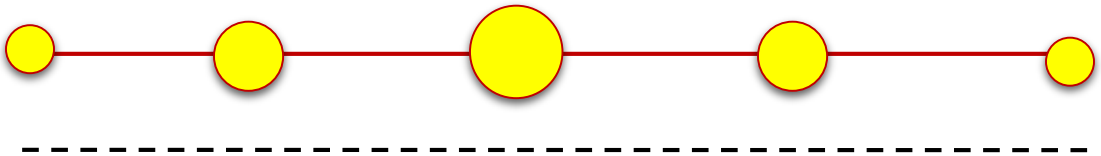
Compute the natural frequencies of the system:



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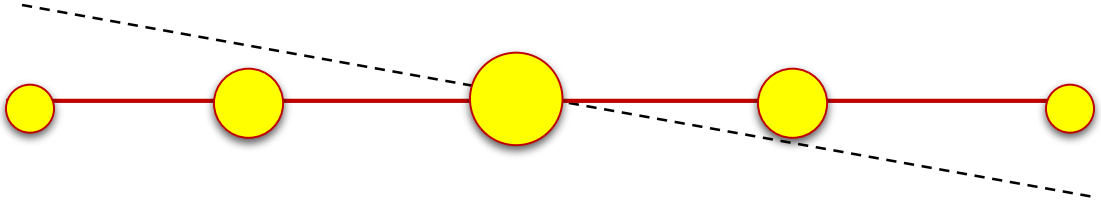
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$\omega_1^2 = 0$



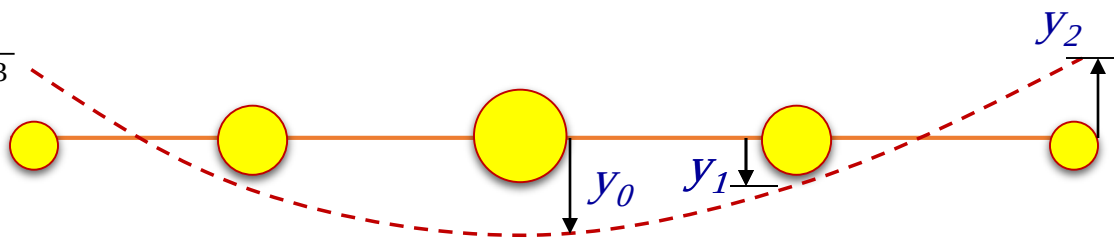
Rigid transformation

$\omega_2^2 = 0$



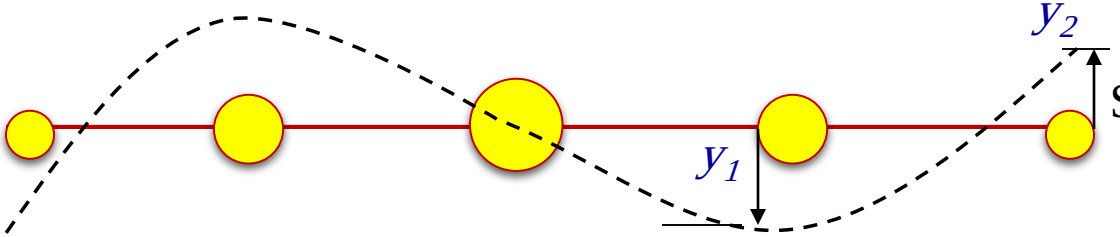
Rigid rotation

$$\begin{cases} \omega_1^2 = 0.632\dfrac{EI}{ml^3} \\ \dfrac{y_2}{y_0} = -17.549 \end{cases}$$



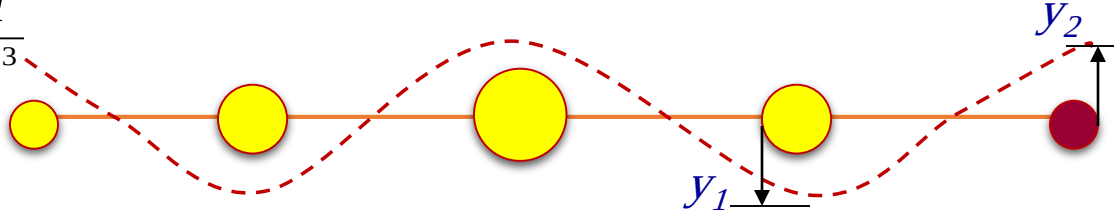
First flextural mode
(symmetrical)

$$\begin{cases} \omega_2^2 = 6\dfrac{EI}{ml^3} \\ \dfrac{y_2}{y_0} = 1.0 \end{cases}$$



Second flextural mode
(Anti - symmetrical)

$$\begin{cases} \omega_3^2 = 11.195\dfrac{EI}{ml^3} \\ \dfrac{y_2}{y_0} = 142.66 \end{cases}$$



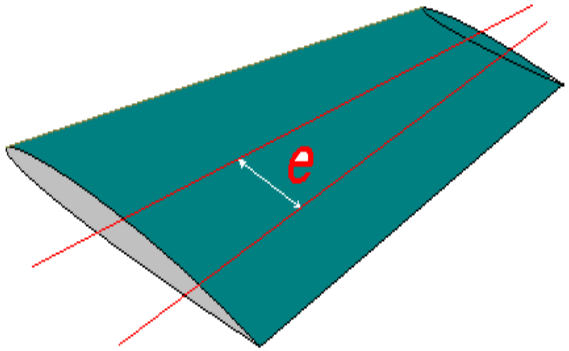
Third flextural mode
(symmetrical)

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Coupled flexure – Torsion Vibration

$$\begin{Bmatrix} y \\ \phi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_n^R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\omega^2 m & 0 & 0 & 1 & -\omega^2 me & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\omega^2 me & 0 & 0 & 0 & -\omega^2 m(R^2+e^2) & 1 \end{bmatrix}_n \times \begin{Bmatrix} y \\ \phi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_n^L$$



$\{z\}_n^R = [p]_n \{z\}_n^l$ The point matrix

$$\begin{Bmatrix} y \\ \phi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_n^L = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & -\frac{l^3}{6EI} & 0 & 0 \\ 0 & 1 & \frac{l}{EI} & -\frac{l^2}{2EI} & 0 & 0 \\ 0 & 0 & 1 & -l & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{GJ} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_n \times \begin{Bmatrix} y \\ \phi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_{n-1}^R$$

Airbus A350

$\{z\}_n^l = [F]_n \{z\}_{n-1}^R$ The field matrix

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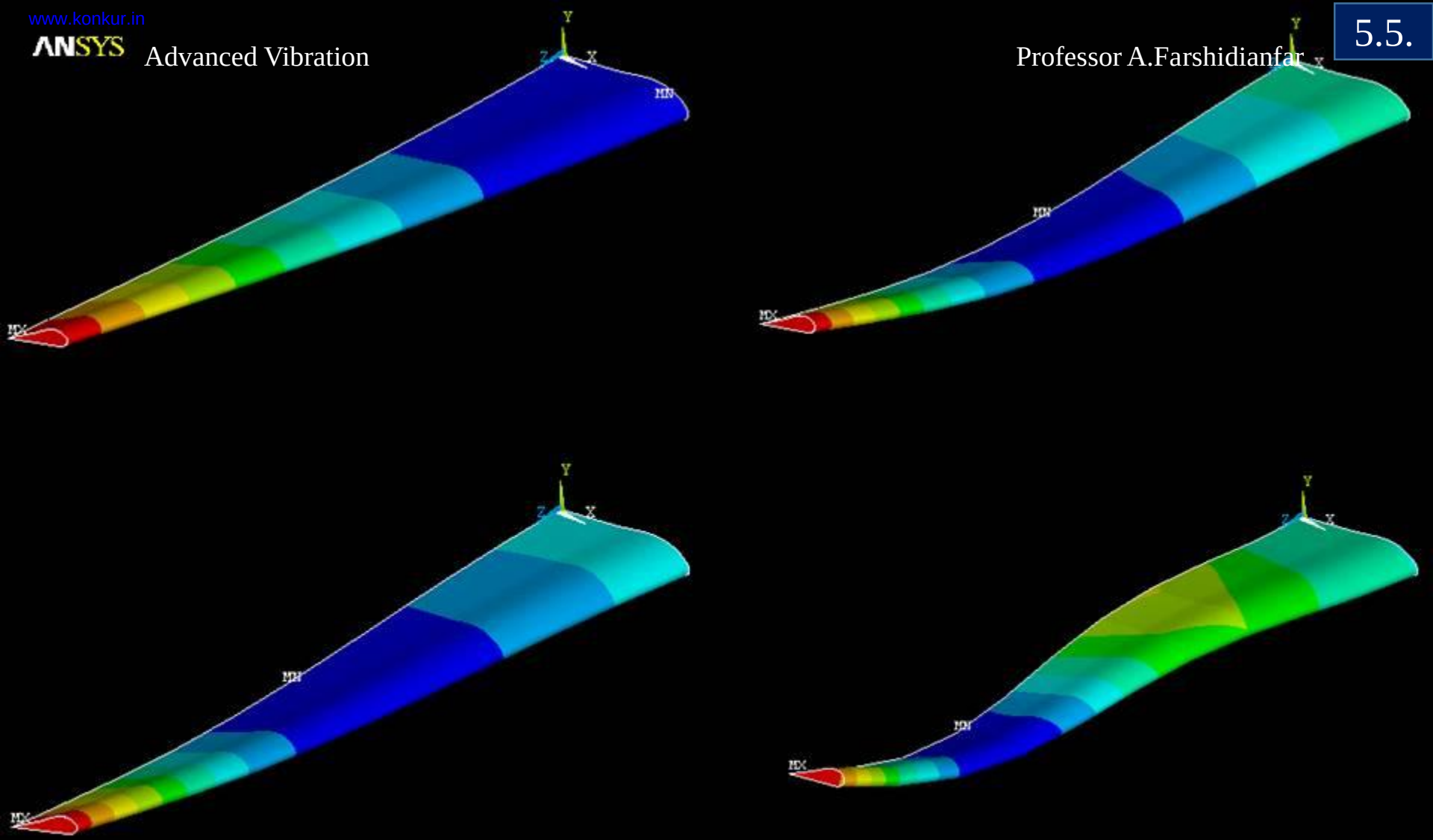
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$$\{z\}_n^R = [P]_n [F]_n \{z\}_{n-1}^R = [T]_n \{z\}_{n-1}^R$$

$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_n^R = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & -\frac{l^3}{6EI} & 0 & 0 \\ 0 & 1 & \frac{l}{EI} & -\frac{l^2}{2EI} & 0 & 0 \\ 0 & 0 & 1 & -l & 0 & 0 \\ -\omega^2 m & -\omega^2 ml & -\frac{\omega^2 ml^2}{2EI} & 1+\frac{\omega^2 ml^3}{6EI} & -\omega^2 me & -\omega^2 me\frac{l}{GJ} \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{GJ} \\ -\omega^2 me & -\omega^2 mel & -\omega^2 me\frac{l^2}{2EI} & \omega^2 me\frac{l^3}{6EI} & -\omega^2 m(e^2+R^2) & 1-\omega^2 m(e^2+R^2)\frac{l}{GJ} \end{bmatrix}_n \times \begin{Bmatrix} y \\ \varphi \\ M \\ V \\ \theta \\ T \end{Bmatrix}_{n-1}^R$$

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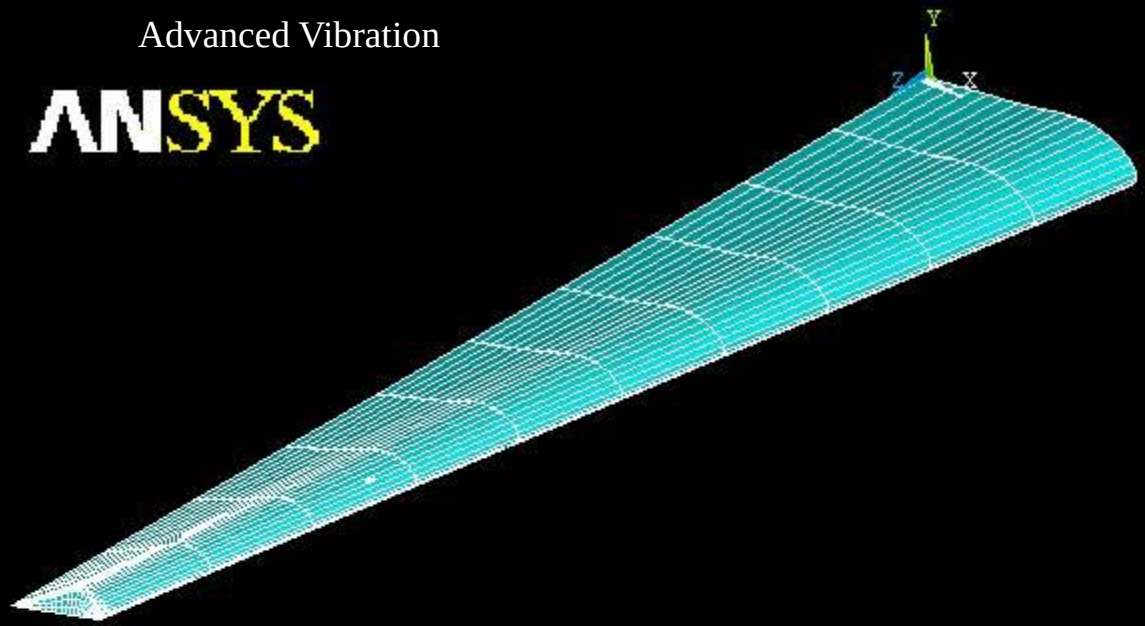
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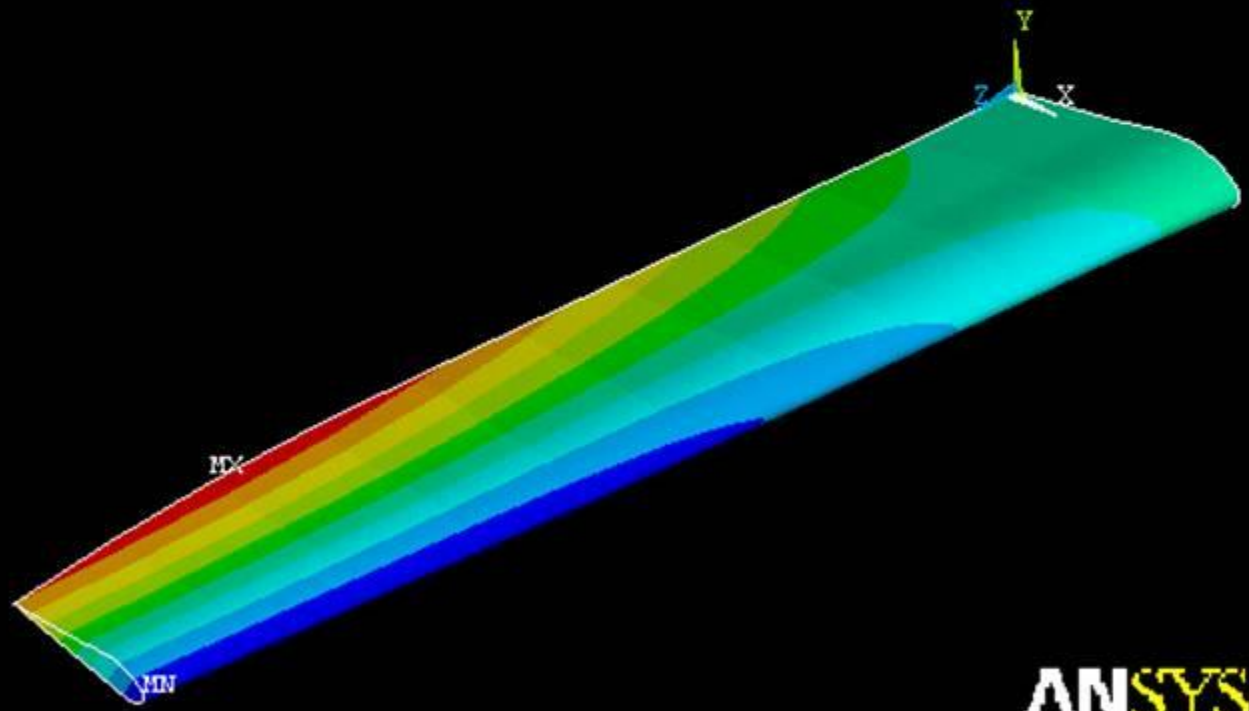


flextural mode

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Torsional mode

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5.5. Transfer matrix method (TMM) for continuous system

$$\frac{\partial^4 y}{\partial x^4}+\frac{1}{C^2}\frac{\partial^2 y}{\partial t^2}=0 \quad , \quad C^2=\frac{EI}{M} \quad , \quad y(x,t)=Y(x)q(t)$$

$$\frac{d^4Y(x)}{dx^4}-\beta^4 Y(x)=0 \quad , \quad \beta^4=\frac{m\omega^2}{EI}=\frac{\omega^2}{C^2}$$

$$\begin{cases} Y(x)=C_1\sinh\beta x+C_2\cosh\beta x+C_3\sin\beta x+C_4\cos\beta x \\ \varphi(x)=\dfrac{dY(x)}{dx}=\beta\left(C_1\cosh\beta x+C_2\sinh\beta x+C_3\cos\beta x-C_4\sin\beta x\right) \\ M(x)=EI\dfrac{dY^2(x)}{dx^2}=\beta^2 EI\left(C_1\sinh\beta x+C_2\cosh\beta x-C_3\sin\beta x-C_4\cos\beta x\right) \\ V(x)=-EI\dfrac{dY^3(x)}{dx^3}=-\beta^3 EI\left(C_1\cosh\beta x+C_2\sinh\beta x-C_3\cos\beta x+C_4\sin\beta x\right) \end{cases}$$

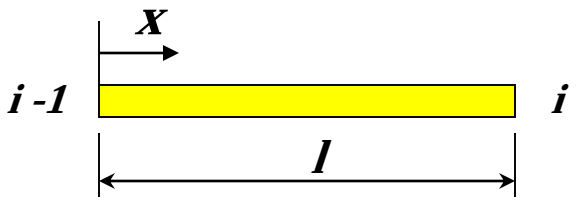
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$$\begin{Bmatrix} Y \\ \varphi \\ M \\ V \end{Bmatrix} = \begin{bmatrix} \sinh \beta x & \cosh \beta x & \sin \beta x & \cos \beta x \\ \beta \cosh \beta x & \beta \sinh \beta x & \beta \cos \beta x & -\beta \sin \beta x \\ \beta^2 EI \sinh \beta x & \beta^2 EI \cosh \beta x & -\beta^2 EI \sin \beta x & -\beta^2 EI \cos \beta x \\ -\beta^3 EI \cosh \beta x & -\beta^3 EI \sinh \beta x & \beta^3 EI \cos \beta x & -\beta^3 EI \sin \beta x \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

$$\{Z(x)\} = [T(x)]\{C\}$$

$$\begin{Bmatrix} y \\ \varphi \\ M \\ V \end{Bmatrix}_{i-1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ 0 & \beta^2 EI & 0 & -\beta^2 EI \\ \beta^3 EI & 0 & \beta^3 EI & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$



at $x=0$

$Z(0) = Z_{i-1}$ (1)

at $x=l$

$\{ Z(l) \} = \{ Z \}_i = [T(l)]\{C\}$

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$$\{Z\}_{i-1} = [T(0)]\{C\} \quad \rightarrow \quad \{C\} = [T(0)]^{-1}\{Z\}_{i-1}$$

2

$$[T(0)]^{-1} = \begin{bmatrix} 0 & \frac{1}{2\beta} & 0 & -\frac{1}{2EI\beta^3} \\ \frac{1}{2} & 0 & \frac{1}{2EI\beta^2} & 0 \\ 0 & \frac{1}{2\beta} & 0 & \frac{1}{2EI\beta^3} \\ \frac{1}{2} & 0 & -\frac{1}{2EI\beta^2} & 0 \end{bmatrix}$$

$$[T(L)] = \begin{bmatrix} \sinh\beta L & \cosh\beta L & \sin\beta L & \cos\beta L \\ \beta\cosh\beta L & \beta\sinh\beta L & \beta\cos\beta L & -\beta\sin\beta L \\ \beta^2 EI\sinh\beta L & \beta^2 EI\cosh\beta L & -\beta^2 EI\sin\beta L & -\beta^2 EI\cos\beta L \\ -\beta^3 EI\cosh\beta L & -\beta^3 EI\sinh\beta L & \beta^3 EI\cos\beta L & -\beta^3 EI\sin\beta L \end{bmatrix}$$

$$\xrightarrow{(1),(2)} \quad \{Z\}_i = [T(l)][T(0)]^{-1}\{Z\}_{i-1} = [H]_i\{Z\}_{i-1}$$

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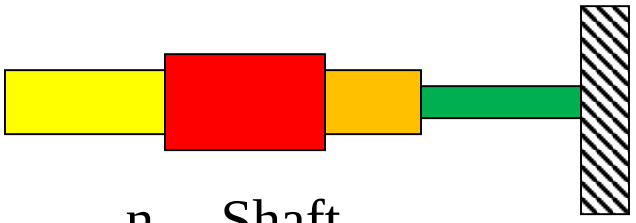
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$$[H]_i = \begin{bmatrix} A_0 & \frac{A_1}{\beta} & \frac{A_2}{EI\beta^2} & -\frac{A_3}{EI\beta^3} \\ \beta A_3 & A_0 & \frac{A_1}{EI\beta} & -\frac{A_2}{EI\beta^2} \\ \beta^2 EIA_2 & \beta EIA_3 & A_0 & -\frac{A_1}{\beta} \\ -\beta^3 EIA_1 & -\beta^2 EIA_2 & -\beta A_3 & A_0 \end{bmatrix}_i = [T(l)][T(0)]^{-1}$$

Transfer matrix

$$\begin{cases} A_0 = \frac{1}{2}(\cosh\beta l + \cos\beta l) \\ A_1 = \frac{1}{2}(\sinh\beta l + \sin\beta l) \\ A_2 = \frac{1}{2}(\cosh\beta l - \cos\beta l) \\ A_3 = \frac{1}{2}(\sinh\beta l - \sin\beta l) \end{cases}$$

$$\{Z\}_n = \underbrace{[H]_n\,[H]_{n-1}[H]_{n-2}\,\ldots.[H]_1}_{[H]}\{Z\}_1$$



n - Shaft Rotor

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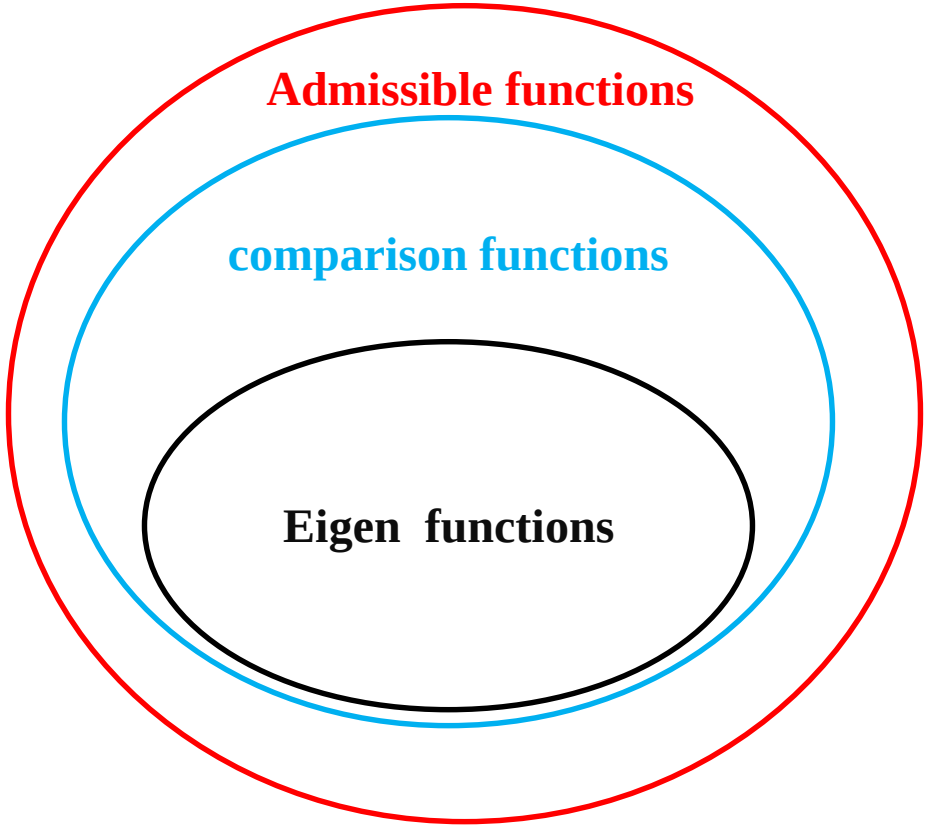
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5.6. The Rayleigh – Ritz Method

$$\omega^2 = R(z) = \frac{z^T k z}{z^T M z} \geq \omega_1^2$$

$\varphi_i =$ eigen vector

$z =$ arbitrary vector



B.C.

1– The geometric B.C. (Displacements & Slopes)

2 – The natural B.C. (Force & Moments)

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The procedure of the Rayleigh – Ritz Method

1 $\omega^2 = R(z) = \dfrac{U_{\max}}{T^*_{\max}}$, $T_{\max} = \dfrac{1}{2}\omega^2 \int_0^l m(x)\phi^2(x)dx = \omega^2 T^*_{\max}$

$T^*_{\max} =$ the reference kinetic energy

$y(x) = C_1\phi_1(x) + C_2\phi_2(x) + \ldots\ldots\ldots + C_n\phi_n(x) = \sum_{i=1}^{n} C_i\phi_i(x)$ deflection curve

$\phi_i(x) =$ admissible function

$$\begin{cases} U = \dfrac{1}{2}\sum_i\sum_j k_{ij}C_iC_j & \text{the potential energy} \\ T^* = \dfrac{1}{2}\sum_i\sum_j m_{ij}C_iC_j & \text{the kinetic energy} \end{cases}$$

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$$\text{for Beams}: \begin{cases} k_{ij} = \int_0^l EI\varphi_i''\varphi_j'' dx \\ m_{ij} = \int_0^l m\varphi_i\varphi_j dx \end{cases} \qquad \text{for Rods}: \begin{cases} k_{ij} = \int_0^l EA\varphi_i'\varphi_j' dx \\ m_{ij} = \int_0^l m\varphi_i\varphi_j dx \end{cases}$$

1

$$\frac{\partial \omega^2}{\partial C_i} = \frac{\partial}{\partial C_i}\left(\frac{U_{\max}}{T^*_{\max}}\right) = \frac{T^*_{\max}\dfrac{\partial U_{\max}}{\partial C_i} - U_{\max}\dfrac{\partial T^*_{\max}}{\partial C_i}}{T^{*\,2}_{\max}} = 0 \Rightarrow \frac{\partial U_{\max}}{\partial C_i} - \frac{U_{\max}}{T^*_{\max}}\frac{\partial T^*_{\max}}{\partial C_i} = 0$$

$$\therefore \quad \frac{\partial U_{\max}}{\partial C_i} - \omega^2 \frac{\partial T^*_{\max}}{\partial C_i} = 0 \quad , \quad \frac{\partial U_{\max}}{\partial C_i} = \sum_{j=1}^{n} k_{ij} C_i \quad , \quad \frac{\partial T^*_{\max}}{\partial C_i} = \sum_{j=1}^{n} m_{ij} C_i$$

$$C_1(k_{i1} - \omega^2 m_{i1}) + C_2(k_{i2} - \omega^2 m_{i2}) + \ldots\ldots\ldots + C_n(k_{in} - \omega^2 m_{in}) = 0$$

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$$\begin{bmatrix} (k_{11}-\omega^2 m_{11}) & (k_{12}-\omega^2 m_{12}) & \ldots\ldots\ldots\ldots & (k_{1n}-\omega^2 m_{1n}) \\ (k_{21}-\omega^2 m_{21}) & (k_{22}-\omega^2 m_{22}) & \ldots\ldots\ldots\ldots & (k_{2n}-\omega^2 m_{2n}) \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ (k_{n1}-\omega^2 m_{n1}) & (k_{n2}-\omega^2 m_{n2}) & \ldots\ldots\ldots\ldots & (k_{nn}-\omega^2 m_{nn}) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ \ldots \\ C_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{Bmatrix}$$

$$\left(\left[k\right]-\omega^2\left[m\right]\right)\{C\}=0 \quad \rightarrow \quad \det\left(\left[k\right]-\omega^2\left[m\right]\right)=0$$

$$\Rightarrow \boxed{\omega} \ \& \ \boxed{\text{modeshape}} \ \text{as a function}$$

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EXAMPLE



$y(x) = a_1 x^2 + a_2 x^3$ The admissible function

$\Phi_1(x) = x^2 \quad ,\Phi_2(x) = x^3 \quad ,\Phi_1''(x) = 2 \quad ,\Phi_2''(x) = 6x$

$$\begin{cases} k_{ij} = \int_0^l \mathrm{EI}\Phi_i''(x)\Phi_j''(x)dx \rightarrow k_{11} = 4EI\int_0^l \mathrm{dx} = 4EIL \\ k_{12} = EI\int_0^l \Phi_1''(x)\Phi_2''(x)dx = k_{21} = EI\int_0^l 2\times 6\mathrm{x}\ \mathrm{dx} = 12EI\int_0^l x dx = 12EI\ \frac{L^2}{2} = 6EIL^2 \\ k_{22} = EI\int_0^l \Phi_2''^2(x) = EI\times 36\int_0^l x^2 dx = 12EIL^3 \end{cases}$$

$$\begin{cases} \mathrm{m}_{ij} = \int_0^l m\Phi_i\Phi_j dx \rightarrow \mathrm{m}_{11} = \int_0^l \Phi_1^2 dx = m\int_0^l x^4 dx = \frac{mL^5}{5} \\ \mathrm{m}_{12} = \mathrm{m}_{21} = m\int_0^l \Phi_1\Phi_2 dx = m\int_0^l x^5 dx = \frac{mL^6}{6} \quad , \quad \mathrm{m}_{22} = m\int_0^l \Phi_2^2 dx = m\int_0^l x^6 dx = \frac{mL^7}{7} \end{cases}$$

$$\begin{bmatrix} 4EIL - \omega^2\frac{mL^5}{5} & 6EIL^2 - \omega^2\frac{mL^6}{6} \\ 6EIL^2 - \omega^2\frac{mL^6}{6} & 12EIL^2 - \omega^2\frac{mL^7}{7} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \det\left[\mathrm{k}\text{-}\omega^2 M\right] = 0 \Rightarrow \omega^4 - 1224(EI/mL^4)\omega^2 + 15120(EI/mL^4)^2 = 0$$

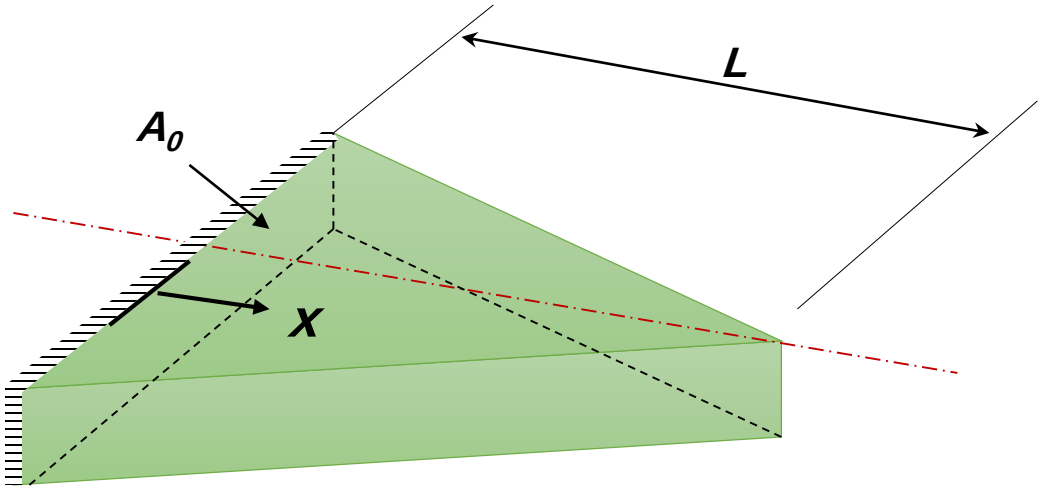
$$(\omega_1)_{\mathrm{app}} = \frac{3.533}{l^2}\sqrt{EI/m} \quad , \quad (\omega_1)_{\mathrm{exa}} = \frac{3.516}{l^2}\sqrt{EI/m}$$

$$(\omega_2)_{\mathrm{app}} = \frac{34.81}{l^2}\sqrt{EI/m} \quad , \quad (\omega_2)_{\mathrm{exa}} = \frac{22.03}{l^2}\sqrt{EI/m}$$

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H.W.16:



Answer :

$$\begin{cases} \omega_1 = 2.4062\sqrt{\dfrac{EA_0}{m_0 l^2}} \\ \omega_2 = 5.5297\sqrt{\dfrac{EA_0}{m_0 l^2}} \end{cases}$$

,

$$\begin{cases} u_1(x) = 1.0\sin\dfrac{\pi x}{2l} + 0.03689\sin\dfrac{3\pi x}{2l} \\ u_2(x) = -0.63819\sin\dfrac{\pi x}{2l} + 1.0\sin\dfrac{3\pi x}{2l} \end{cases}$$

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P.S. of H.W.16

1) $u_1(x)=C_1\underbrace{\sin(\pi x/2L)}_{\varphi_1(x)} \Rightarrow {}^{(1)}\omega_1 = 2.4146\sqrt{AE/mL^2}$

2) $u_2(x)=C_1\underbrace{\sin(\pi x/2L)}_{\varphi_1(x)}+C_2\underbrace{\sin(3\pi x/2L)}_{\varphi_2(x)}$

$$\Rightarrow \begin{cases} {}^{(2)}\omega_1 = 2.4062\sqrt{AE/mL^2} \\ {}^{(2)}\omega_2 = 5.5292\sqrt{AE/mL^2} \end{cases}, \quad \begin{cases} u_2^{(1)} = \sin(\pi x/2L) - 0.0369\sin(3\pi x/2L) \\ u_2^{(2)} = \sin(\pi x/2L) - 1.5651\sin(3\pi x/2L) \end{cases}$$

3) $u_3(x)=C_1\underbrace{\sin(\pi x/2L)}_{\varphi_1(x)}+C_2\underbrace{\sin(3\pi x/2L)}_{\varphi_2(x)}+C_3\underbrace{\sin(5\pi x/2L)}_{\varphi_3(x)}$

$$\Rightarrow \begin{cases} {}^{(3)}\omega_1 = 2.4049\sqrt{AE/mL^2} \\ {}^{(3)}\omega_2 = 5.5216\sqrt{AE/mL^2} \\ {}^{(3)}\omega_3 = 8.6646\sqrt{AE/mL^2} \end{cases}, \quad \begin{cases} u_3^{(1)} = \sin(\pi x/2L) - 0.0319\sin(3\pi x/2L) - 0.0072\sin(5\pi x/2L) \\ u_3^{(2)} = \sin(\pi x/2L) - 1.5440\sin(3\pi x/2L) + 0.0666\sin(5\pi x/2L) \\ u_3^{(3)} = \sin(\pi x/2L) - 1.2110\sin(3\pi x/2L) + 2.0270\sin(5\pi x/2L) \end{cases}$$

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P.S. of H.W.16

$$\left(\omega_1\right)_{\text{exact}} = 2.4048\sqrt{AE/mL^2} \le {}^{(3)}\omega_1 = 2.4049\sqrt{AE/mL^2} \le {}^{(2)}\omega_1 \le {}^{(1)}\omega_1$$

$$\left(\omega_2\right)_{\text{exact}} = 5.5201\sqrt{AE/mL^2} \le {}^{(3)}\omega_2 = 5.5216\sqrt{AE/mL^2} \le {}^{(2)}\omega_2$$

$$\left(\omega_3\right)_{\text{exact}} = 8.6537\sqrt{AE/mL^2} \le {}^{(3)}\omega_3 = 8.6646\sqrt{AE/mL^2}$$

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5.7. The Galerkin Method

$$L[W]=\lambda M[W] \qquad \text{The characteristic equation}$$

$$y=\sum_{j=1}^{n} C_i\varphi_i \qquad \varphi_i=\text{Comparison function}$$

$$L[W]-\lambda M[W]=0 \quad , \quad L[y]-\lambda M[y]=error=E$$

$$\int_D \varphi_i E dD=0 \qquad i=1,2,\ldots,n \qquad (1)$$

$$\int \varphi_i L[y]dD=\sum_{j=1}^{n} C_j \int \varphi_i L[\varphi_j]dD=\sum_{j=1}^{n} k_{ij}C_i$$

$$\triangleright\ k_{ij}=k_{ji}=\int_D \varphi_i L[\varphi_j]dD \qquad L=\text{self-adjoint oprrator} \qquad (2)$$

$$\int \varphi_i M[y]dD=\sum_{j=1}^{n} C_j \int \varphi_i M[\varphi_j]dD=\sum_{j=1}^{n} m_{ij}C_j$$

$$\triangleright\ m_{ij}=m_{ji}=\int_D \varphi_i M[\varphi_j]dD \qquad (\text{mass matrix}) \qquad (3)$$

$$\xrightarrow{(1),(2),(3)} \sum_{j=1}^{n}\left(k_{ij}-\lambda m_{ij}\right)C_j=0 \qquad (\text{Galerkin equation})$$

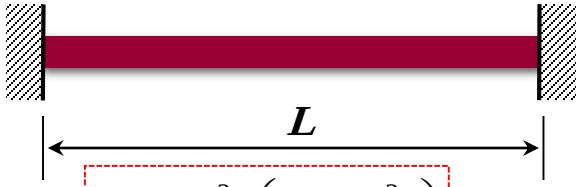
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EXAMPLE

Determine the natural frequencies of the beam with the Galerkin method.

$$y(x)=c_1\left(1-\cos(2\pi x/l)\right)+c_2\left(1-\cos(4\pi x/l)\right)=c_1\varphi_1(x)+c_2\varphi_2(x)$$



Solution:

$$L=\frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}\right)$$

$$k_{ij}=\int_0^l EI\varphi_i\frac{d^4\varphi_j}{dx^4}dx\ ,\ m_{ij}=\int_0^l m\varphi_i\varphi_j dx$$

$$k_{ij}=\int_D \phi_i L\left[\phi_j\right]dD=\int_0^l \phi_i\frac{d^2}{dx^2}\left(EI\frac{d^2\phi_j}{dx^2}\right)dx=$$

$$\phi_i\left[\frac{d}{dx}\left(EI\frac{d^2\phi_j}{dx^2}\right)\right]_0^l-\frac{d\phi_i}{dx}\left[\left(EI\frac{d^2\phi_j}{dx^2}\right)\right]_0^l+\int_0^l EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx$$

$$[K]=EI\begin{bmatrix}\frac{8\pi^4}{l^4} & 0\\ 0 & \frac{128\pi^4}{l^4}\end{bmatrix},\ [M]=m\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$\det\left([K]-\omega^2[M]\right)=0$$

$$\begin{cases}(\omega_1)_G=\dfrac{22.48}{l^2}\sqrt{\dfrac{EI}{m}} & ,c_1=\{23\quad 1\} \quad ,y_1(x)=23\left(1-\cos\dfrac{2\pi x}{l}\right)+1-\cos\dfrac{4\pi x}{l}\\ (\omega_2)_G=\dfrac{124.1}{l^2}\sqrt{\dfrac{EI}{m}} & ,c_2=\{-0.69\quad 1\} \quad ,y_2(x)=-0.69\left(1-\cos\dfrac{2\pi x}{l}\right)+1-\cos\dfrac{4\pi x}{l}\end{cases}$$

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The Reyleigh-Ritz method with same function :

$$\left(\frac{16\pi^4 EI}{l^4}\begin{bmatrix}1 & 0\\ 0 & 16\end{bmatrix}-\omega^2 m\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}\right)\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=0$$

$$\begin{cases}(\omega_1)_{\mathrm{R}}=\dfrac{22.35}{l^2}\sqrt{\dfrac{EI}{m}} & , \quad c_1=\{1.0 \quad 0.575\}\\ (\omega_2)_{\mathrm{R}}=\dfrac{124.0}{l^2}\sqrt{\dfrac{EI}{m}} & , \quad c_2=\{1.0 \quad -1.447\}\end{cases}$$

$$(\omega_1)_{\text{exact}}=\frac{22.4}{l^2}\sqrt{\frac{EI}{m}} \quad , \quad (\omega_2)_{\text{exact}}=\frac{61.6}{l^2}\sqrt{\frac{EI}{m}} \quad , \quad (\omega_3)_{\text{exact}}=\frac{120.9}{l^2}\sqrt{\frac{EI}{m}}$$

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H.W. 17:

Write transfer matrix for longitudinal vibration of rods with E, ρ, L and A parameters. Consider rod as a continuous system.

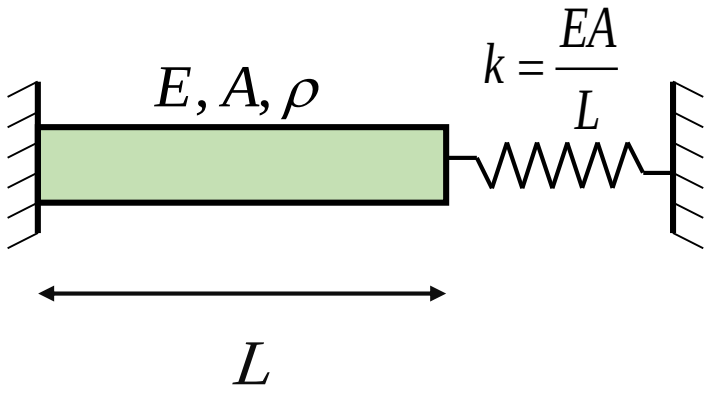
$$\begin{Bmatrix} P \\ U \end{Bmatrix}_1 = \begin{bmatrix} \cos\beta L & -EA\beta\sin\beta L \\ \dfrac{\sin\beta L}{EA\beta} & \cos\beta L \end{bmatrix} \begin{Bmatrix} P \\ U \end{Bmatrix}_2 \qquad \beta^2 = \frac{\rho}{E}\omega^2 = \frac{m}{EA}\omega^2$$

$$\frac{d^2U}{dx^2} + \beta^2 U(x) = 0, \qquad U(x) = C\sin(\beta x) + B\cos(\beta x)$$

H.W. 18:

Write transfer matrix for shown system.

$$\begin{Bmatrix} P \\ U \end{Bmatrix}_3 = \begin{bmatrix} 1 & 0 \\ 1/k & 1 \end{bmatrix} \begin{bmatrix} \cos\beta L & -EA\beta\sin\beta L \\ \dfrac{\sin\beta L}{EA\beta} & \cos\beta L \end{bmatrix} \begin{Bmatrix} P \\ U \end{Bmatrix}_1$$



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H.W. 19:

Solve previous H.W. with Rayleigh – Ritz method.

$$U(x)=C_1(x/L)+C_2(x/L)^2+C_3(x/L)^3+C_4(x/L)^4$$

Sol.:

$$\begin{cases}(\omega_1)_R=\dfrac{2.029}{L}\sqrt{E/\rho}, & U_1(x)=(x/L)+1.06346(x/L)^2-0.8993(x/L)^3+0.2805(x/L)^4\\ (\omega_2)_R=\dfrac{4.932}{L}\sqrt{E/\rho}, & U_2(x)=(x/L)-0.9260(x/L)^2-2.2310(x/L)^3+1.9780(x/L)^4\end{cases}$$

$$\begin{cases}(\omega_1)_{\text{exact}}=\dfrac{2.029}{L}\sqrt{E/\rho}, & U_1(x)=\sin(2.029x/L)\\ (\omega_2)_{\text{exact}}=\dfrac{4.913}{L}\sqrt{E/\rho}, & U_2(x)=\sin(4.913x/L)\end{cases}$$

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H.W. 20:

Regarding to given function, solve H.W. 18 with Rayleigh – Ritz method.

$$U(x)=C_1\sin(\pi x/2L)+C_2\sin(3\pi x/2L)$$

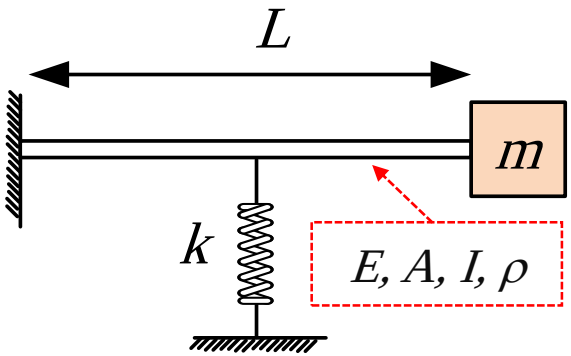
Sol.:

$$\begin{cases}(\omega_1)_R=\dfrac{2.0660}{L}\sqrt{E/\rho}, & U_1(x)=\sin(\pi x/2L)+0.1003\sin(3\pi x/2L)\\ (\omega_2)_R=\dfrac{4.9440}{L}\sqrt{E/\rho}, & U_2(x)=\sin(\pi x/2L)-9.9700\sin(3\pi x/2L)\end{cases}$$

H.W. 21:

Regarding to given function, find natural frequency of shown system with Rayleigh – Ritz method.

$$y(x)=C\left[3(x/L)^2-(x/L)^3\right]$$



Sol.:

$$\omega=\sqrt{\frac{12EI/L^3+0.3906k}{4m+0.9428\rho AL}}$$

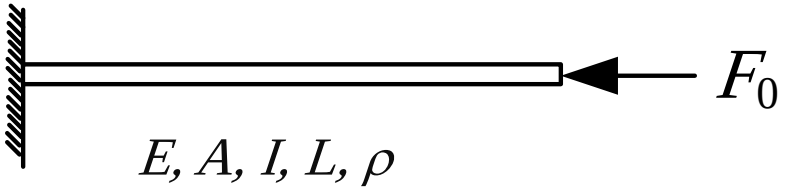
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H.W. 22:

The constant force (F_0) is applied to shown beam. Regarding to given function, find natural frequency of system with Rayleigh – Ritz method.

$$y(x)=C\left[3(x/L)^2-(x/L)^3\right]$$



Find approximation buckling force too.

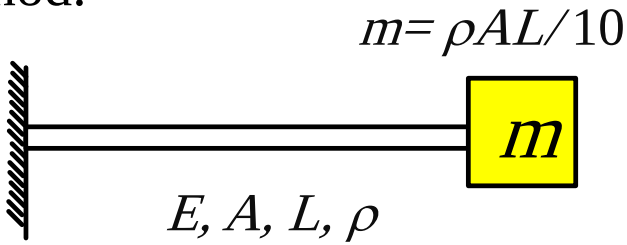
Sol.:

$$\omega=\sqrt{\frac{12EI/L^3+4.8F_0/L}{0.9428\rho AL}}$$

H.W. 23:

Regarding to given function, find two first natural frequencies and mode shapes of shown system with Rayleigh – Ritz method.

$$u(x)=C_1\sin(\pi x/2L)+C_2\sin(3\pi x/2L)$$



Sol.:

$$\begin{cases}\omega_1=\dfrac{1.431}{L}\sqrt{E/\rho}, & U_1(x)=\sin(\pi x/2L)-0.0208\sin(3\pi x/2L)\\ \omega_2=\dfrac{4.370}{L}\sqrt{E/\rho}, & U_2(x)=\sin(\pi x/2L)+5.354\sin(3\pi x/2L)\end{cases}$$

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5.8. Direct integration (Numerical Methods)

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}$$

I.C.

$$x(t=0)=x_0$$

$$\dot{x}(t=0)=\dot{x}_0$$

$$\ddot{x}(t=0)=\ddot{x}_0$$

t=0 , t=T , Δt=T/n

t=0,Δt,2Δt,. . . T

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1-Finite Difference Method

1.1-Forward Difference

1.2-Backward Difference

1.3-Central Difference(CDM)

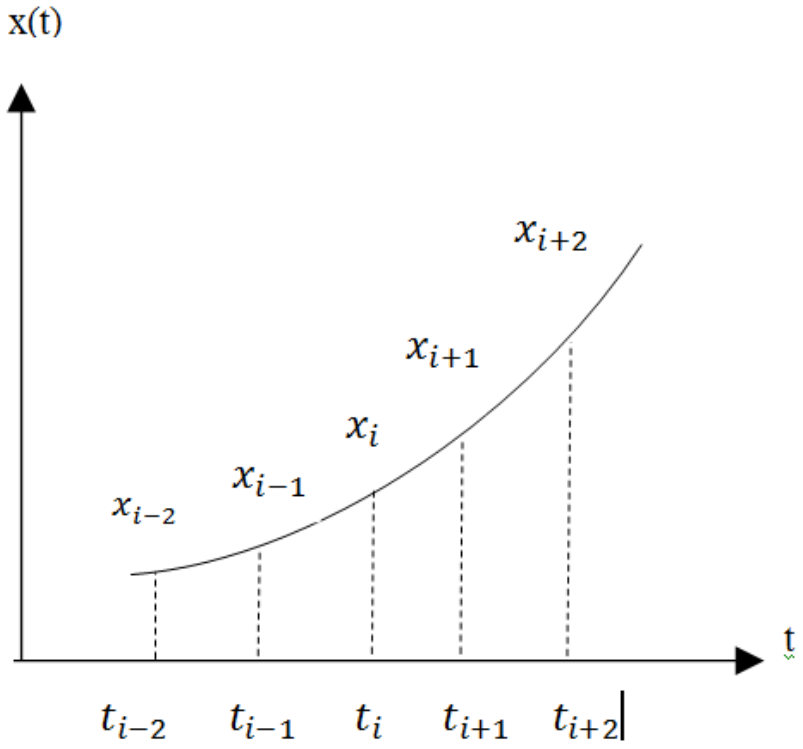
The Taylor expansion x_{i+1} , x_{i-1} about the pivotal point

$$x_{i+1} = x_i + h\dot{x}_i + \frac{h^2}{2!}\ddot{x}_i + \frac{h^3}{3!}\dddot{x}_i + \ldots$$

$$x_{i-1} = x_i - h\dot{x}_i + \frac{h^2}{2!}\ddot{x}_i - \frac{h^3}{3!}\dddot{x}_i + \ldots$$

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$$\dot{x}_i \cong \frac{1}{2h}(x_{i+1} - x_{i-1})$$

$$\ddot{x}_i \cong \frac{1}{h^2}(x_{i-1} - 2x_i + x_{i+1})$$

The ignored items are of order h^2

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The CDM for Lumped Systems:

$$[M]\{\ddot{x}\}_i + [C]\{\dot{x}\}_i + [K]\{x\}_i = \{F\}_i$$

$$[M]\left\{\frac{x_{i-1} - 2x_i + x_{i+1}}{(\Delta t)^2}\right\} + [C]\left\{\frac{x_{i+1} - x_{i-1}}{2\Delta t}\right\} + [K]\{x\}_i = \{F\}_i$$

$$\left(\frac{1}{(\Delta t)^2}[M] + \frac{1}{2\Delta t}[C]\right)\{x\}_{i+1} = \{F\}_i - \left([K] - \frac{2}{(\Delta t)^2}[M]\right)\{x\}_i - \left(\frac{1}{(\Delta t)^2}[M] - \frac{1}{2\Delta t}[C]\right)\{x\}_{i-1}$$

$$\Longrightarrow \quad \left[\hat{M}\right]\{x\}_{i+1} = \left\{\hat{R}\right\}_i \qquad \text{An explicit integration scheme}$$

Solution:

1:

$$\{\ddot{x}\}_0 = [M]^{-1}(\{F\}_0 - [C]\{\dot{x}\}_0 - [K]\{x\}_0)$$

$$\{x(t=0)\} = x_0 \;\&\; \{\dot{x}(t=0)\} = \dot{x}_0$$

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2: Choosing $\Delta t < \Delta t_{critical} = \Delta t_{cr}$, $\Delta t_{cr} = \frac{T_n}{\pi}$

(for stability)

$T_n =$ The smallest natural period in the system Hence the CDM is conditionally stable.

$h = \Delta t \le \frac{T_n}{10}$ A safe rule, $h = \Delta t = \frac{T_n}{20}$ Good

In practice, use for continuous element:

$\Delta t \le \frac{\Delta L}{C}$, $C = \sqrt{E/\rho}$ ΔL= The smallest distance between nodes

3: $\{x\}_{-1} = \{x\}_0 - \Delta t \{\dot{x}\}_0 + \frac{(\Delta t)^2}{2} \{\ddot{x}\}_0$

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4: Start i=0 $\{x\}_{i+1} = \{x(t = t_{i+1})\}$

$$\{F\}_i = \{F(t = t_i)\}$$

The recurrence formula

$$\{x\}_{i+1} = \left(\frac{1}{(\Delta t)^2}[M] + \frac{1}{2\Delta t}[C]\right)^{-1} \left(\{F\}_i - \left([K] - \frac{2}{(\Delta t)^2}[M]\right)\{x\}_i - \left(\frac{1}{(\Delta t)^2}[M] - \frac{1}{2\Delta t}[C]\right)\{x\}_{i-1}\right)$$

$$\{x\}_{i+1} = \left[\hat{M}\right]^{-1}\{\hat{R}\}_i$$

$$\{\dot{x}\}_i = \frac{1}{2\Delta t}(\{x\}_{i+1} - \{x\}_{i-1})$$

$$\{\ddot{x}\}_i = \frac{1}{(\Delta t)^2}(\{x\}_{i+1} - 2\{x\}_i + \{x\}_{i+1})$$

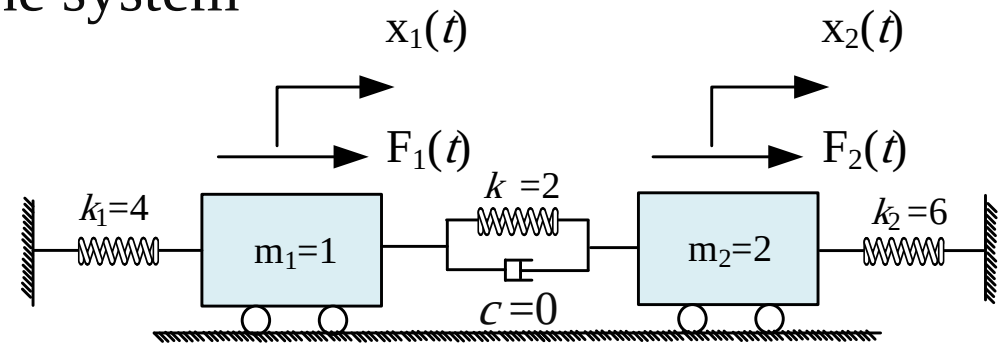
C.D.M $\Longrightarrow$ Conditionally stable error $\Longrightarrow$ order(h^2)

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EXAMPLE Find the response of the system

$$\begin{cases} F_1(t)=0 \\ F_2(t)=10 \end{cases}, \quad \begin{cases} \{x\}_0=\{x(t=0)\}=0 \\ \{\dot{x}\}_0=\{\dot{x}(t=0)\}=0 \end{cases}$$



Sol.:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$

$$[M]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad [k]=\begin{bmatrix} k_1+k & -k \\ -k & k_2+k \end{bmatrix}=\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$$

$$[C]=\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \{F(t)\}=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}=\begin{bmatrix} 0 \\ 10 \end{bmatrix}, \qquad \{X(t)\}=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The eigen value problem: $\left([k]-\omega^2[M]\right)\{X\}=0$

$$\left(\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}-\omega^2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=0$$

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EXAMPLE

$$\omega_1 = 1.807747 \text{ rad/sec} \quad , \quad \tau_1 = \frac{2\pi}{\omega_1} = 3.4757 \text{ sec} \quad , \quad \{x\}^1 = \begin{Bmatrix} 1.000 \\ 1.3661 \end{Bmatrix}$$

$$\omega_2 = 2.594620 \text{ rad/sec} \quad , \quad \tau_1 = \frac{2\pi}{\omega_2} = 2.4216 \text{ sec} \quad , \quad \{x\}^2 = \begin{Bmatrix} 1.000 \\ -0.3661 \end{Bmatrix}$$

$$h = \Delta t = \frac{T_n}{10} = \frac{\tau_2}{10} = 0.24216 \text{ sec}$$

$$\{\ddot{x}\}_0 = [M]^{-1}\left(\{F\}_0 - [C]\{\dot{x}\}_0 - [K]\{x\}_0\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

$$\{x\}_{-1} = \{x\}_0 - \Delta t\{\dot{x}\}_0 + \left(\Delta t^2/2\right)\{\ddot{x}\}_0 = \begin{Bmatrix} 0 \\ 0.1466 \end{Bmatrix}$$

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EXAMPLE

Using the recurrence formula

Time ($t_i=i\Delta t$)	Displacement	Time ($t_i=i\Delta t$)	Displacement
$t_1=\Delta t=0.24216$	$\begin{Bmatrix}0\\0.1466\end{Bmatrix}$	$t_6=6\Delta t$	$\begin{Bmatrix}0.9027\\2.5470\end{Bmatrix}$
$t_2=2\Delta t$	$\begin{Bmatrix}0.0172\\0.5520\end{Bmatrix}$	$t_7=7\Delta t$	$\begin{Bmatrix}1.2354\\2.6057\end{Bmatrix}$
$t_3=3\Delta t$	$\begin{Bmatrix}0.0931\\1.1222\end{Bmatrix}$	$t_8=8\Delta t$	$\begin{Bmatrix}1.4391\\2.4189\end{Bmatrix}$
$t_4=4\Delta t$	$\begin{Bmatrix}0.2678\\1.7278\end{Bmatrix}$	$t_9=9\Delta t$	$\begin{Bmatrix}1.4202\\2.0422\end{Bmatrix}$
$t_5=5\Delta t$	$\begin{Bmatrix}0.5510\\2.2370\end{Bmatrix}$	$t_{10}=10\Delta t$	$\begin{Bmatrix}1.1410\\1.5630\end{Bmatrix}$

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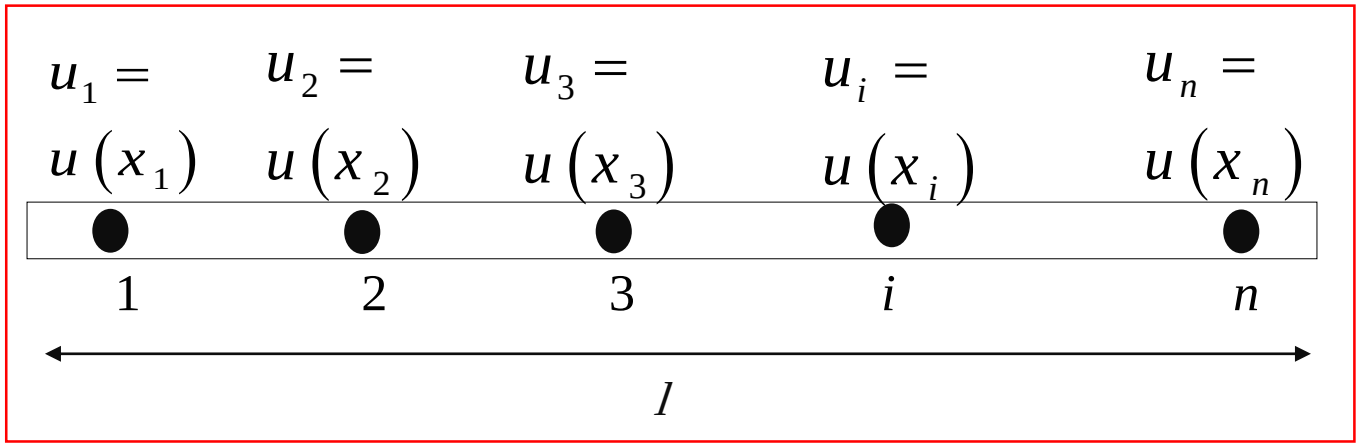
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EXAMPLE

The C.D.M for continuous systems:

1. Longitudinal vibration of bars

$$\frac{d^2u(x)}{dx^2}+\alpha^2u(x)=0 \qquad \alpha^2=\frac{\omega^2}{c^2}=\frac{\rho\omega^2}{E} \qquad h=\frac{l}{(n-1)}$$



$$\frac{1}{h^2}\left(u_{i+1}-2u_i+u_{i-1}\right)+\alpha^2u_i=0$$

$$u_{i+1}-\left(2-\lambda\right)u_i+u_{i-1}=0 \qquad \lambda=h^2\alpha^2$$

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$$\text{For } i=2 \quad \Longrightarrow \quad u_3-(2-\lambda)u_2+u_1=0$$

$$\vdots$$

$$\text{For } i=n\text{-}1 \quad \Longrightarrow \quad u_n-(2-\lambda)u_{n-1}+u_{n-2}=0$$

$$\begin{bmatrix} (2-\lambda) & -1 & 0 & \cdots & 0 \\ -1 & (2-\lambda) & -1 & 0 & \\ 0 & -1 & (2-\lambda) & & \vdots \\ & & & \ddots & -1 \\ 0 & & \cdots & -1 & (2-\lambda) \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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B.C : 1-Both end are fixed

$$u(x=0)=u(x=l)=0 \qquad or \qquad u_1=u_n=0$$

$$\begin{bmatrix} (2-\lambda) & -1 & 0 & \cdots & 0 \\ -1 & (2-\lambda) & -1 & 0 & \\ 0 & -1 & (2-\lambda) & & \vdots \\ & & & \ddots & -1 \\ 0 & & \cdots & -1 & (2-\lambda) \end{bmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\left([A]-\lambda[I]\right)\{u\}=0 \quad \Longrightarrow \quad \det\left([A]-\lambda[I]\right)=0$$

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$$[A]=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \\ 0 & -1 & 2 & -1 & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & -1 & 2 \end{bmatrix}_{(n-2)\times(n-2)}$$

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B.C : 2- One end is fixed and another one is free

At the free end $\rightarrow P = EA\left.\frac{dU}{dx}\right|_{x=0} \rightarrow \frac{dU_1}{dx} = \frac{U_2 - U_{-1}}{2h} = 0 \rightarrow U_{-1} = U_2$

$$U_2 - (2-\lambda)U_1 + U_{-1} = 0$$

$$U_{-1} = U_2 \rightarrow -(2-\lambda)U_1 + 2U_2 = 0$$

At the fixed end $\rightarrow U_n = 0$ $\qquad \left([A] - \lambda[I]\right)\{u\} = 0$

$$[A] = \begin{bmatrix} 2 & -2 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \\ 0 & -1 & 2 & -1 & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & -1 & 2 \end{bmatrix}, \quad \{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ . \\ . \\ U_{n-1} \end{Bmatrix}$$

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2. Torsional vibration of shafts

$$\frac{d^2\theta(x)}{dx^2}+\alpha^2\theta(x)=0, \qquad \alpha^2=\frac{\omega^2}{c^2}=\frac{\rho\omega^2}{G}=\frac{J\omega^2}{GI_pL}, \qquad J_{\text{shaft}}=\rho L I_p$$

$$\frac{1}{h^2}\left(\theta_{i+1}-2\theta_i+\theta_{i-1}\right)+\alpha^2\theta_i=0$$

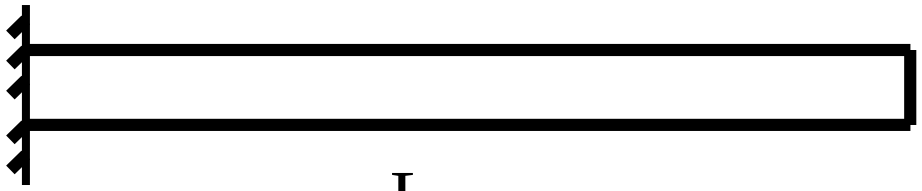
$$\theta_{i+1}-(2-\lambda)\theta_i+\theta_{i-1}=0 \qquad , \qquad \lambda=h^2\alpha^2$$

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EXAMPLE

3. Lateral vibration of beams



$$\frac{d^4 w(x)}{dx^4} - \beta^4 w(x) = 0 \qquad , \qquad \beta^4 = \frac{\rho A \omega^2}{EI} = \frac{m\omega^2}{EI}$$

$$\frac{d^4 w_i}{dx^4} = \frac{1}{h^4}\left(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right)$$

$$\Downarrow$$

$$w_{i+2} - 4w_{i+1} + (6 - \lambda) w_i - 4w_{i-1} + w_{i-2} = 0 \qquad , \qquad \lambda = h^4 \beta^4$$

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	i-3	i-2	i-1	I	i+1	i+2	i+3
$2hD =$			-1	0	1		
$h^2D^2 =$			1	-2	1		
$2h^3D^3 =$		-1	2	0	-2	1	
$h^4D^4 =$		1	-4	6	-4	1	

$[e=(h^2)]$

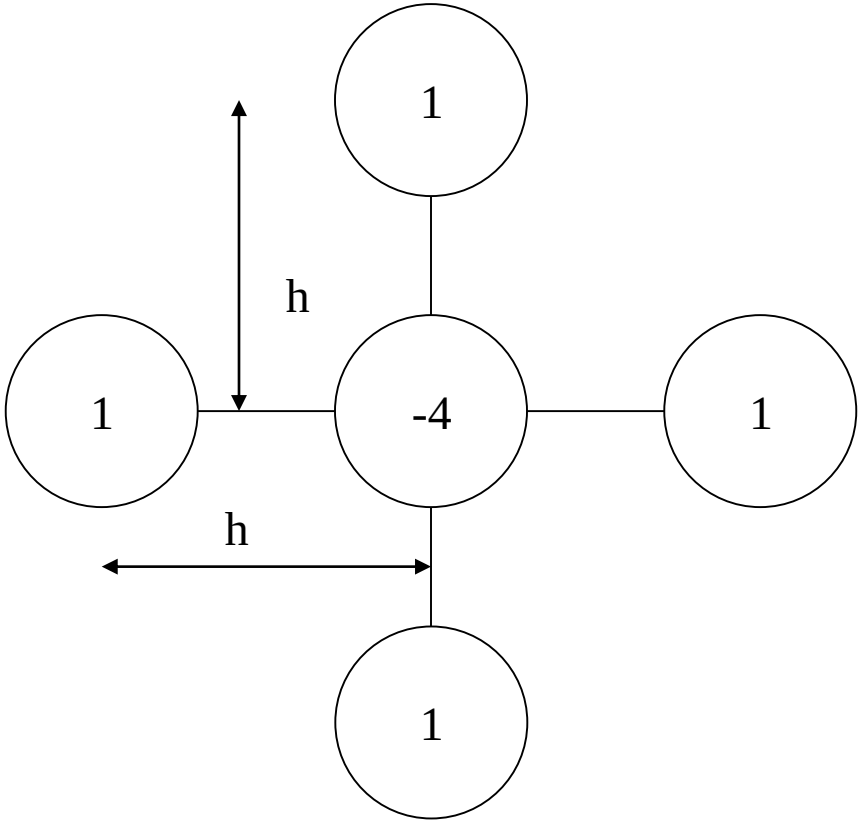
$$D = \frac{d}{dx}$$

Central difference operators

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$$h^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=h^2\,\nabla^2=$$

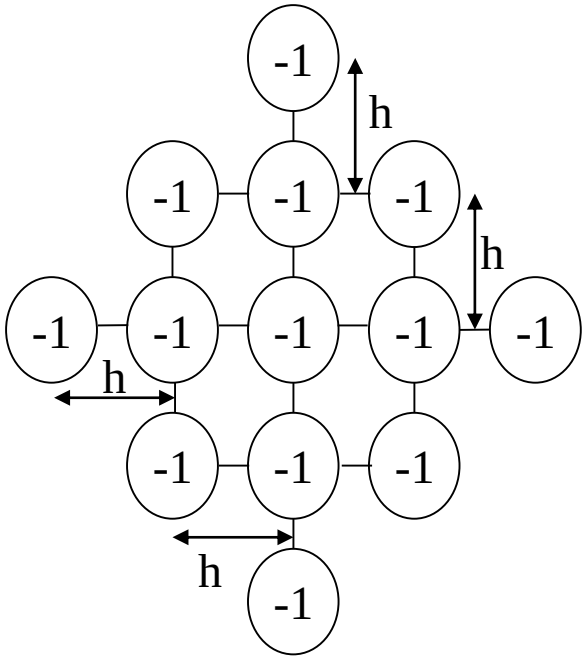


$$+h^2 o(h^2)$$

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$$h^4\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)=h^4\nabla^4=$$



$$+h^4o(h^2)$$

Central difference partial operators

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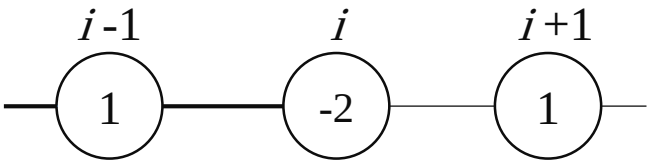
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Methods which increase precisions:

1. Decreasing h or increasing number of freedom
2. Using more accurate operators (some terms in Taylor's expansion is omitted)

$$h^2D^2x_i = x_{i-1} - 2x_i + x_{i+1}$$

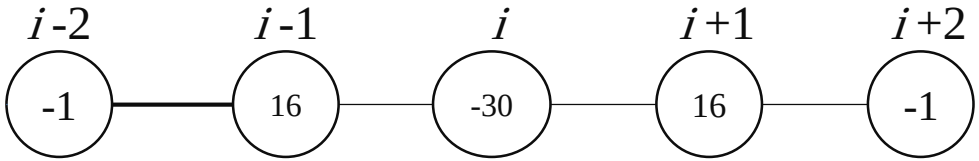
$h^2D^2 =$



$$\{e = o(h^2)\}$$

$$12h^2D^2x_i = -x_{i-2} + 16x_{i-1} - 30x_i + 16x_{i+1} - x_{i+2}$$

$12h^2D^2 =$



$$\{e = o(h^4)\}$$

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Methods which increase precisions:

3. Richardson Extrapolation:

$$\begin{cases} A_{n_1 n_2} = \alpha_1 A_{n_1} + \alpha_2 A_{n_2} \\ A_{n_1,n_2,n_3} = \beta_1 A_{n_1} + \beta_2 A_{n_2} + \beta_3 A_{n_3} \end{cases}$$

A_{n_1} = Answer for n_1

A_{n_2} = Answer for n_2

A_{n_3} = Answer for n_3

$$A_{n_1 n_2} = \alpha_1 A_{n_1} + \alpha_2 A_{n_2}$$

h^2-Extrapolation coefficient

n_2/n_1	α_1	α_2
2/1	-0.3333333333	1.3333333333
3/2	-0.8	1.8
4/3	-1.2857142867	2.2857142867
5/4	-1.7777777778	2.7777777778
6/5	-2.2727272727	3.2727272727
7/6	-2.7692307693	3.7692307693
8/7	-3.2666666667	4.2666666667
3/1	-0.125	1.125
5/3	-0.5625	1.5625
7/5	-1.0116666667	2.0116666667

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Richardson Extrapolation

$$A_{n_1,n_2,n_3} = \beta_1 A_{n_1} + \beta_2 A_{n_2} + \beta_3 A_{n_3}$$

A_{n_1} = Answer for n_1

A_{n_2} = Answer for n_2

A_{n_3} = Answer for n_3

(h^3, h^4)-Extrapolation coefficient

$n_3/n_2/n_1$	β_1	β_2	β_3
3/2/1	0.0416666667	-1.0666666667	2.025
4/2/1	0.0222222222	-0.4444444444	1.4222222222
4/3/2	0.2666666667	-2.3142857443	3.0176490176
5/4/2	0.6310206319	-2.3763703701	3.3069788069
5/4/3	0.7232142867	-4.0634920635	4.3102777773
6/5/4	1.4222222222	-6.3131313131	5.8909090909
7/6/5	2.3074242424	-9.0620870628	7.6956428515
8/7/6	3.5601399001	-12.3128206423	9.7523809524
8/3/1	0.005208888	-0.688826	1.6276611667
7/6/3	0.1265625	-1.6276011667	8.6040116667

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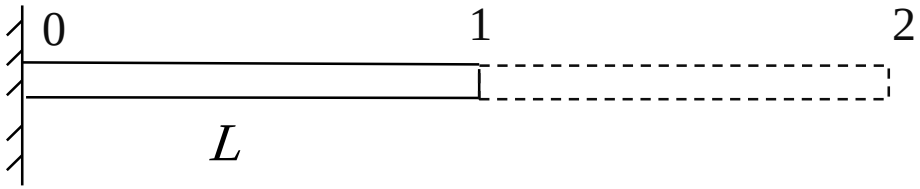
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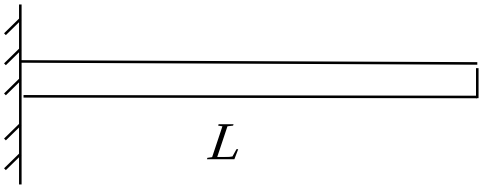
EXAMPLE

Richardson Extrapolation

Find natural torsional vibration of clamped-free shaft with length of L.

First: n=1





$$\text{For node 1} \Rightarrow \frac{d^2\theta}{dx^2} + L^2\beta^2\theta = 0 \Rightarrow \theta_0 - 2\theta_1 + \theta_2 = -{}_1\beta^2\theta_1 h^2$$

$$\text{B.C.} \Rightarrow \begin{cases} \theta_0 = 0 \quad (x = 0) \\ M_T = GJ \left.\dfrac{d\theta}{dx}\right|_{x=L} = 0 \Rightarrow \left(\dfrac{d\theta}{dx}\right)_1 = 0 \Rightarrow \theta_2 = \theta_0 \end{cases}$$

$$\left(2 - L_1^2\beta^2\right)\theta_1 = 0 \Rightarrow {}_1\beta^2 = 2/L^2 = 2/h^2 \Rightarrow {}_1\beta^2 L^2 = 2$$

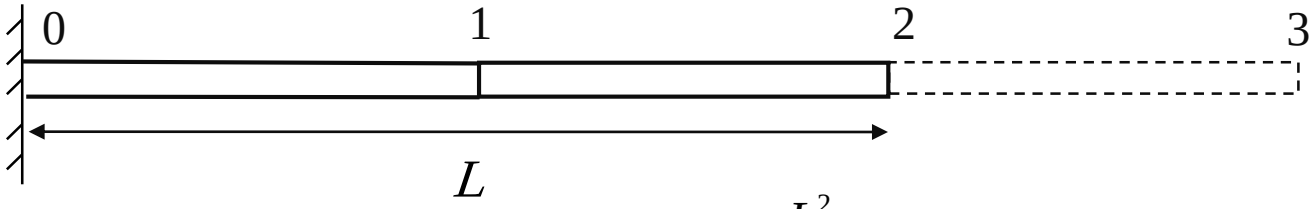
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EXAMPLE

Richardson Extrapolation

Second: n=2



For node1 $\Rightarrow \theta_0 - 2\theta_1 + \theta_2 = -{}_2\beta^2 \frac{L^2}{4}\theta_1$

For node 2 $\Rightarrow \theta_1 - 2\theta_2 + \theta_3 = -{}_2\beta^2 \frac{L^2}{4}\theta_2$

B.C. $\Rightarrow \begin{cases} \theta_0 = 0 \quad (x=0) \\ M_T = GJ \left.\frac{d\theta}{dx}\right|_{x=L} = 0 \Rightarrow \left(\frac{d\theta}{dx}\right)_2 = 0 \Rightarrow \theta_1 = \theta_3 \end{cases}$

$\Rightarrow {}_2\beta^2 L^2 = 8 - 4\sqrt{2} = 2.35$

Third: n=3 $\Rightarrow {}_3\beta^2 L^2 = 2.43$

$$\Rightarrow {}_{1,2,3}\beta^2 L^2 = \beta_1 \left({}_1\beta^2 L^2\right) + \beta_2 \left({}_2\beta^2 L^2\right) + \beta_3 \left({}_3\beta^2 L^2\right)$$

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EXAMPLE

Richardson Extrapolation

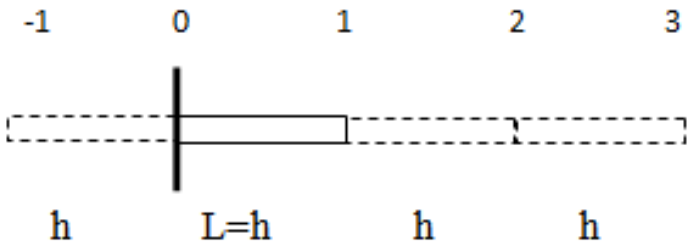
$$\Rightarrow {}_{1,2,3}\beta^2L^2 = \beta_1\left({}_1\beta^2L^2\right)+\beta_2\left({}_2\beta^2L^2\right)+\beta_3\left({}_3\beta^2L^2\right)$$

n	${}_n\beta^2L^2$	${}_{1,2}\beta^2L^2$	${}_{2,3}\beta^2L^2$	${}_{1,2,3}\beta^2L^2$	${}_{\text{Exact}}\beta^2L^2$
1	2				
		2.46			
2	2.35			2.48	2.48
			2.49		
3	2.43				

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1-The first approach:



i=1: $w_3 - 4w_2 + (6-\lambda)w_1 - 4w_0 + w_{-1} = 0 \quad (1)$

$$x = 0 \ , \ w_0 = 0 \quad (2) \ , \left(\frac{dw}{dx}\right)_0 = 0 \Longrightarrow \frac{w_1 - w_{-1}}{2h} = 0 \Longrightarrow w_1 = w_{-1} \quad (3)$$

$$x = L \ , \ M = 0 \ , \left(\frac{d^2 w}{dx^2}\right)_1 = 0 \Longrightarrow \frac{w_2 - 2w_1 + w_0}{h^2} = 0 \Longrightarrow w_2 = 2w_1 (4)$$

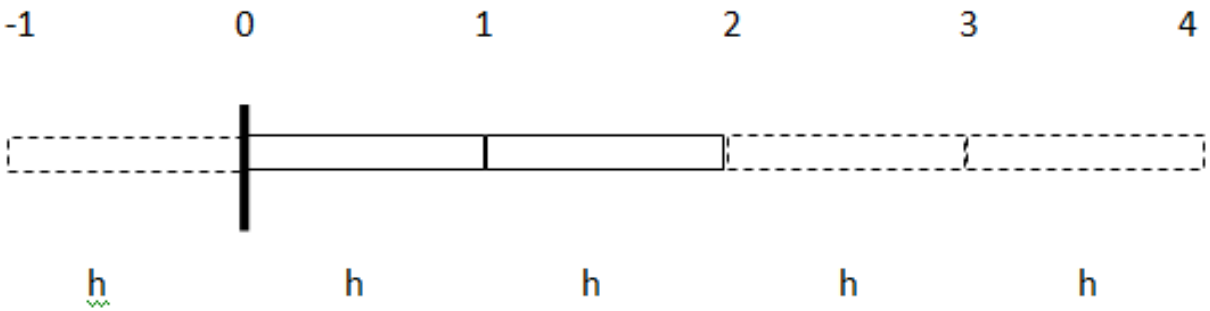
$$x = L \ , V = 0 \ , \left(\frac{d^3 w}{dx^3}\right)_1 = 0 \Rightarrow \frac{w_3 - 2w_2 + 2w_0 - w_{-1}}{2h^3} = 0 \Rightarrow w_3 = 5w_1 (5)$$

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$$(2),(3),(4),(5) \longrightarrow (1) \Longrightarrow \left(4-\lambda_1\right)w_1 = 0 \Longrightarrow {}^{(1)}\lambda_1 = 4$$

The second app.



$$1: \quad w_1 - 4w_0 + \left(6-\lambda\right)w_1 - 4w_2 + w_3 = 0$$

$$2: \quad w_0 - 4w_1 + \left(6-\lambda\right)w_2 - 4w_3 + w_4 = 0$$

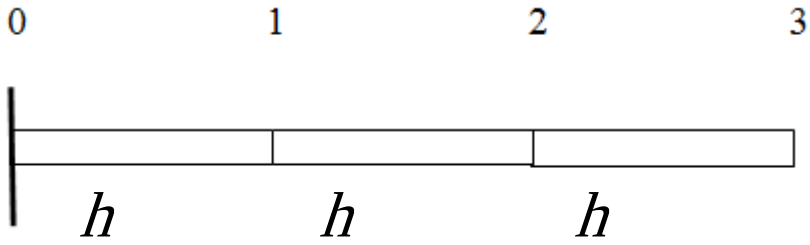
$$\text{B.C.} \quad w_0 = 0\ , w_1 = w_{-1}\ , w_3 = -w_1 + 2w_2\ ,\ w_4 = -4w_1 + 4w_2$$

$$\begin{cases} \left(6-\lambda\right)w_1 - 2w_2 = 0 \\ -4w_1 + \left(2-\lambda\right)w_2 = 0 \end{cases} \Longrightarrow \begin{cases} {}^{(2)}\lambda_1 = 0.536 \\ {}^{(2)}\lambda_2 = 7.464 \end{cases}$$

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The third app.



$$h = L/3$$

$$\begin{bmatrix} 7 & -4 & 1 \\ -4 & 5 & -2 \\ 2 & -4 & 2 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \lambda \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \Longrightarrow \quad {}^{(3)}\lambda_1 = 0.1278$$

1&2: $\lambda_1 = 1.333 \times 0.536 - 0.333 \times 4 = -0.6176$

2&3: $\lambda_1 = 1.8 \times 0.1278 - 0.8 \times 0.536 = 0.1988$

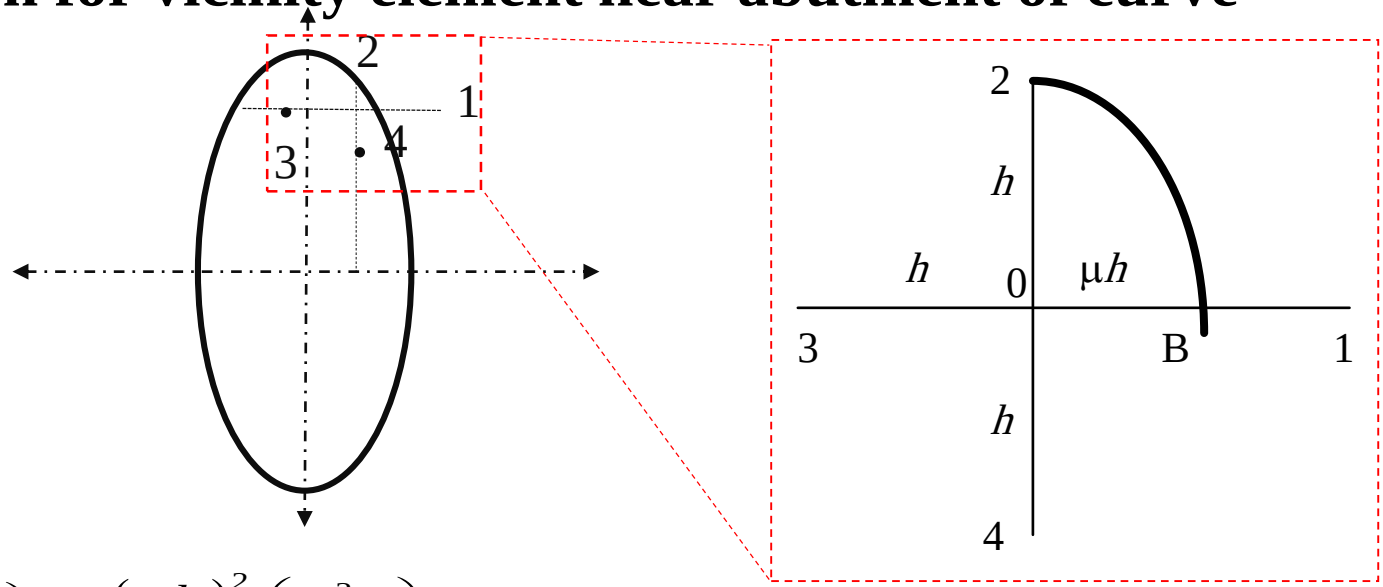
1&2&3: $\lambda_1 = 2.025 \times 0.1278 - 1.066 \times 0.536 - 0.042 \times 4 = -0.1446$

$$h = \frac{l}{3}\ ,\ \omega = \beta^2 \sqrt{\frac{EI}{m}} \Longrightarrow \omega_{\text{app}} = 3.42 \sqrt{\frac{EI}{ml^4}}\ ,\ \omega_{\text{exact}} = 3.516 \sqrt{\frac{EI}{ml^4}}$$

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Writing equation for vicinity element near abutment of curve



$$\phi_B = \phi_0 + (\mu h)\left(\frac{\partial \phi}{\partial x}\right)_0 + \frac{(\mu h)^2}{2!}\left(\frac{\partial^2 \phi}{\partial^2 x}\right)_0 + ... \quad (a)$$

$$\phi_3 = \phi_0 + (-h)\left(\frac{\partial \phi}{\partial x}\right)_0 + \frac{(-h)^2}{2!}\left(\frac{\partial^2 \phi}{\partial^2 x}\right)_0 + ... \quad (b)$$

$$(a)+\mu(b)$$

$0<\mu<1$

$$\phi_B + \mu\phi_3 = \phi_0(1+\mu) + \frac{\mu(1+\mu)h^2}{2!}\left(\frac{\partial^2 \phi}{\partial^2 x}\right)_0 + ...$$

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 \approx [\mu\phi_3 + \phi_B - \phi_0(1+\mu)]\frac{2!}{\mu(1+\mu)h^2}$$

$$\left(\frac{\partial^2 \phi}{\partial y^2}\right)_0 \approx \frac{\phi_2 - 2\phi_0 + \phi_4}{h^2}$$

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Writing equation for vicinity element near abutment of curve

In torsional vibration: $\nabla^2\phi + 2G\theta = 0$

$$\frac{2}{\mu(1+\mu)}\phi_B + \phi_2 + \frac{2}{1+\mu}\phi_3 + \phi_4 - 2\left(1+\frac{1}{\mu}\right)\phi_0 + 2G\theta h^2 = 0$$

$$(a)-\mu^2(b) \Rightarrow \left(\frac{\partial\phi}{\partial x}\right)_0 \approx \frac{\phi_B - \mu^2\phi_3 - (1-\mu^2)\phi_0}{\mu h(1+\mu)}$$

(a)-μ(b) can be used but it had less accuracy.

H.W. 24:

Find a relation for $\left(\frac{\partial^3\phi}{\partial x^3}\right)_0$.

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5.9. Implicit Integration Scheme

5.9.1.Houbolt Method (Backward difference)

$$\dot{x}_{i+1} = \frac{1}{6\Delta t}(11x_{i+1} - 18x_i + 9x_{i-1} - 2x_{i-2})$$

$$\text{Error} = O\left(h^3\right)$$

$$\ddot{x}_{i+1} = \frac{1}{(\Delta t)^2}(2x_{i+1} - 5x_i + 4x_{i-1} - x_{i-2})$$

Levy & Kroll , Unconditionally stable ,

For better accuracy: $\Delta t \le \frac{0.31}{\omega} = \frac{T_n}{20}$

ω is the biggest natural frequency

$$[M]\{\ddot{x}\}_{i+1} + [C]\{\dot{x}\}_{i+1} + [k]\{x\}_{i+1} = \{F\}_{i+1} = \{F(t = t_{i+1})\}$$

The recurrence formula:

$$\left(\frac{2}{(\Delta t)^2}[M] + \frac{11}{6\Delta t}[C] + [k]\right)\{x\}_{i+1} = \{F\}_{i+1} + \left(\frac{5}{(\Delta t)^2}[M] + \frac{3}{\Delta t}[C]\right)\{x\}_i$$

$$-\left(\frac{4}{(\Delta t)^2}[M] + \frac{3}{2\Delta t}[C]\right)\{x\}_{i-1} + \left(\frac{1}{(\Delta t)^2}[M] + \frac{1}{3\Delta t}[C]\right)\{x\}_{i-2}$$

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Steps:

1-I.Cs. $\Longrightarrow$ $\{x(t=0)\}=\{x\}_0$ & $\{\dot{x}(t=0)\}=\{\dot{x}\}_0$

$$\{\ddot{x}(t=0)\}=[M]^{-1}(\{F_0\}-[C]\{\dot{x}\}_0-[K]\{x\}_0)$$

2-Time step Δt is chosen based on accuracy consideration.

3- $$\{x\}_{-1}=\{x\}_0-\Delta t\{\dot{x}\}_0+\frac{(\Delta t)^2}{2}\{\ddot{x}\}_0$$

4- $\{x\}_1$ & $\{x\}_0 \rightarrow CDM$ $\Longrightarrow$

$$\{x\}_{i+1}=(\frac{1}{(\Delta t)^2}[M]+\frac{1}{2\Delta t}[C])^{-1}(\{F_i\}-([K]-\frac{2}{(\Delta t)^2}[M])\{x\}_i$$

$$-(\frac{1}{(\Delta t)^2}[M]-\frac{1}{2\Delta t}[C])\{x\}_{i-1})$$

Starting with $i=2$

5- $$\{x\}_{i+1}=(\frac{2}{(\Delta t)^2}[M]+\frac{11}{6\Delta t}[C]+[K])^{-1}(\{F\}_{i+1}+(\frac{5}{(\Delta t)^2}[M]+\frac{3}{\Delta t}[C])\{x\}_i$$

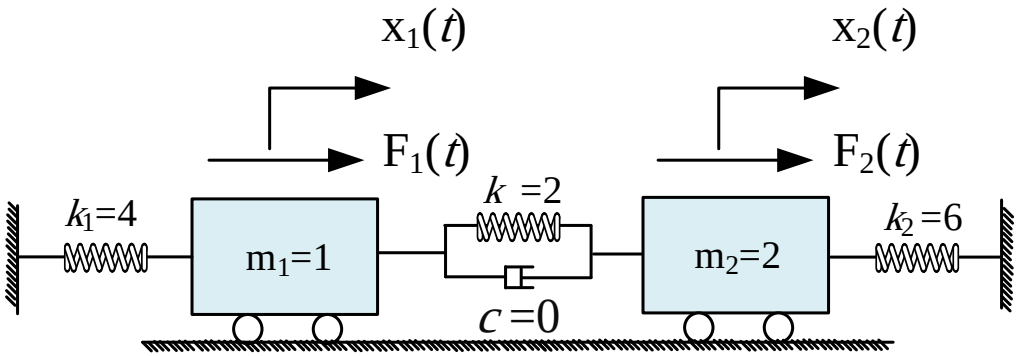
$$-(\frac{4}{(\Delta t)^2}[M]+\frac{3}{2\Delta t}[C])\{x\}_{i-1}+(\frac{1}{(\Delta t)^2}[M]+\frac{1}{3\Delta t}[C])\{x\}_{i-2})$$

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EXAMPLE

Solve with Houbolt method.



$$\begin{cases} F_1(t)=0 \\ F_2(t)=10 \end{cases}, \quad \begin{cases} \{x\}_0=\{x(t=0)\}=0 \\ \{\dot{x}\}_0=\{\dot{x}(t=0)\}=0 \end{cases}$$

$$\{\ddot{x}\}_0=\begin{Bmatrix}0\\5\end{Bmatrix}, \qquad \Delta t=0.24216 \text{ sec}$$

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EXAMPLE

Time ($t_i=i\Delta t$)	Displacement	Time ($t_i=i\Delta t$)	Displacement
$t_1=\Delta t=0.24216$	$\begin{Bmatrix}0.0000\\0.1466\end{Bmatrix}$	$t_6=6\Delta t$	$\begin{Bmatrix}0.7867\\2.5297\end{Bmatrix}$
$t_2=2\Delta t$	$\begin{Bmatrix}0.0172\\0.5520\end{Bmatrix}$	$t_7=7\Delta t$	$\begin{Bmatrix}1.0734\\2.6489\end{Bmatrix}$
$t_3=3\Delta t$	$\begin{Bmatrix}0.0917\\1.1064\end{Bmatrix}$	$t_8=8\Delta t$	$\begin{Bmatrix}1.2803\\2.5454\end{Bmatrix}$
$t_4=4\Delta t$	$\begin{Bmatrix}0.2501\\1.6906\end{Bmatrix}$	$t_9=9\Delta t$	$\begin{Bmatrix}1.3432\\2.2525\end{Bmatrix}$
$t_5=5\Delta t$	$\begin{Bmatrix}0.4924\\2.1941\end{Bmatrix}$	$t_{10}=10\Delta t$	$\begin{Bmatrix}1.2258\\1.8325\end{Bmatrix}$

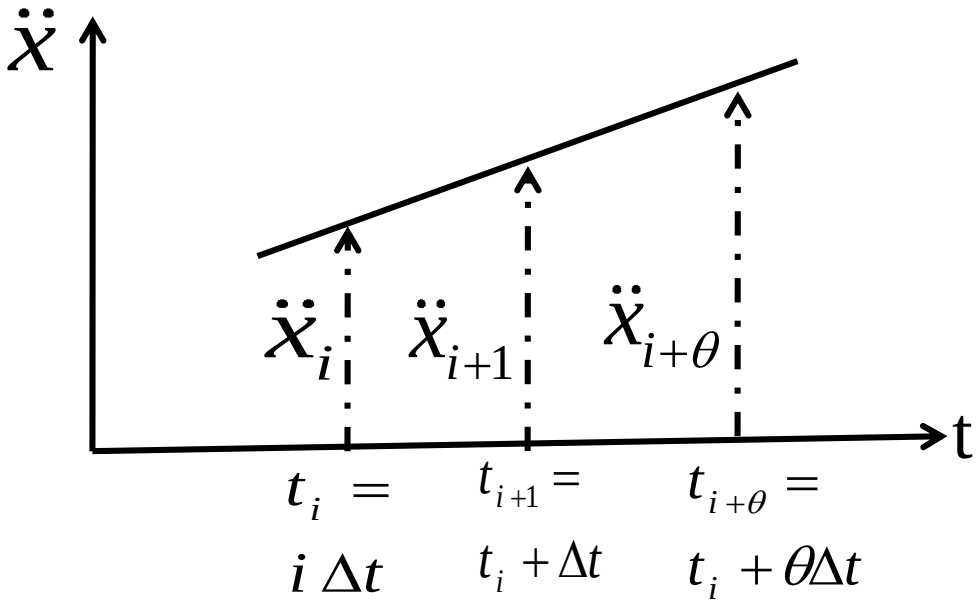
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5.9.2. Wilson Method

$$\ddot{x}_{i+\theta} = \left\{\ddot{x}\left(t_i + \tau\right)\right\} = \left\{\ddot{x}\left(t_i\right)\right\} + \frac{\tau}{\theta\Delta t}\left(\left\{\ddot{x}\left(t_{i+\theta}\right)\right\} - \left\{\ddot{x}\left(t_i\right)\right\}\right)$$

If $\theta \geq 1.37$ unconditionally Stable $0 \leq \tau \leq \theta\Delta t$



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$$\left\{\dot{x}\left(t_i+\tau\right)\right\}=\left\{\dot{x}\left(t_i\right)\right\}+\tau\left\{\ddot{x}\left(t_i\right)\right\}+\frac{\tau^2}{2\theta\Delta t}\left(\left\{\ddot{x}\left(t_{i+\theta}\right)\right\}-\left\{\ddot{x}\left(t_i\right)\right\}\right)$$

$$\left\{x\left(t_i+\tau\right)\right\}=\left\{x\left(t_i\right)\right\}+\tau\left\{\dot{x}\left(t_i\right)\right\}+\frac{1}{2}\tau^2\left\{\ddot{x}\left(t_i\right)\right\}+\frac{\tau^3}{6\theta\Delta t}\left(\left\{\ddot{x}\left(t_{i+\theta}\right)\right\}-\left\{\ddot{x}\left(t_i\right)\right\}\right)$$

$\left\{x\left(t_i\right)\right\}$ &$\left\{\dot{x}\left(t_i\right)\right\}$ = Integration constants , $\tau=\theta\Delta t$

$$\left\{\dot{x}\left(t_i+\theta\Delta t\right)\right\}=\left\{\dot{x}\left(t_i\right)\right\}+\theta\Delta t\left\{\ddot{x}\left(t_i\right)\right\}+\frac{\theta\Delta t}{2}\left(\left\{\ddot{x}\left(t_{i+\theta}\right)\right\}-\left\{\ddot{x}\left(t_i\right)\right\}\right) \quad (1)$$

$$\left\{x\left(t_i+\theta\Delta t\right)\right\}=\left\{x\left(t_i\right)\right\}+\theta\Delta t\left\{\dot{x}\left(t_i\right)\right\}+\frac{\theta^2\left(\Delta t\right)^2}{6}\left(\left\{\ddot{x}\left(t_{i+\theta}\right)\right\}+2\left\{\ddot{x}\left(t_i\right)\right\}\right)$$

$$\left\{x\left(t_i+\theta\Delta t\right)\right\}=\left\{x\left(t_{i+\theta}\right)\right\}=\left\{x\right\}_{i+\theta}$$

$$\left\{\ddot{x}\left(t_{i+\theta}\right)\right\}=\left\{\ddot{x}\right\}_{i+\theta}=\frac{6}{\theta^2\left(\Delta t\right)^2}\left(\left\{x\right\}_{i+\theta}-\left\{x\right\}_i\right)-\frac{6}{\theta\Delta t}\left\{\dot{x}\right\}_i-2\left\{\ddot{x}\right\}_i \quad (2)$$

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$$(1),(2)\quad \left\{\dot{x}\left(t_{i+\theta}\right)\right\}=\left\{\dot{x}\right\}_{i+\theta}=\frac{3}{\theta(\Delta t)}\left(\left\{x\right\}_{i+\theta}-\left\{x\right\}_{i}\right)-2\left\{\dot{x}\right\}_{i}-\frac{\theta\Delta t}{2}\left\{\ddot{x}\right\}_{i}$$

$$[M]\{\ddot{x}\}_{i+\theta}+[C]\{\dot{x}\}_{i+\theta}+[k]\{x\}_{i+\theta}=\{F\}_{i+\theta}=\{F\}_{i}+\theta\left(\{F\}_{i+1}-\{F\}_{i}\right)\quad (3)$$

$$\left(\frac{6}{\theta^{2}(\Delta t)^{2}}[M]+\frac{3}{\theta\Delta t}[C]+[k]\right)\{x\}_{i+\theta}=\{F\}_{i+\theta}+[M]\left(\frac{6}{\theta^{2}(\Delta t)^{2}}\{x\}_{i}+\frac{6}{\theta\Delta t}\{\dot{x}\}_{i}+2\{\ddot{x}\}_{i}\right)$$
$$+[C]\left(\frac{3}{\theta\Delta t}\{x\}_{i}+2\{\dot{x}\}_{i}+\frac{\theta\Delta t}{2}\{\ddot{x}\}_{i}\right)$$

The recurrence formula

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Procedure:

1-I.C. $\{x(t=0)\}=\{x\}_0$ & $\{\dot{x}(t=0)\}=\{\dot{x}\}_0$

$$\Longrightarrow \{\ddot{x}\}_0=[M]^{-1}\left(\{F\}_0-[C]\{\dot{x}\}_0-[K]\{x\}_0\right)$$

2-The time step(Δ t) is chosen on accuracy consideration only, normally $\theta=1.4$

3- $\{x\}_{i+\theta}=\{x(t+\theta\Delta t)\}$ ⟸ Using the recurrence formula

4- $$\{\ddot{x}\}_{i+1}=\{\ddot{x}\}_{t+\Delta t}=\frac{6}{\theta^3\Delta t^2}\left(\{x\}_{i+\theta}-\{x\}_i\right)-\frac{6}{\theta^2\Delta t}\{\dot{x}\}_i+\left(1-\frac{3}{\theta}\right)\{\ddot{x}\}_i$$

$$\{\dot{x}\}_{i+1}=\{\dot{x}\}_{t+\Delta t}=\{\dot{x}\}_i+\frac{\Delta t}{2}\left(\{\ddot{x}\}_{i+1}+\{\ddot{x}\}_i\right)$$

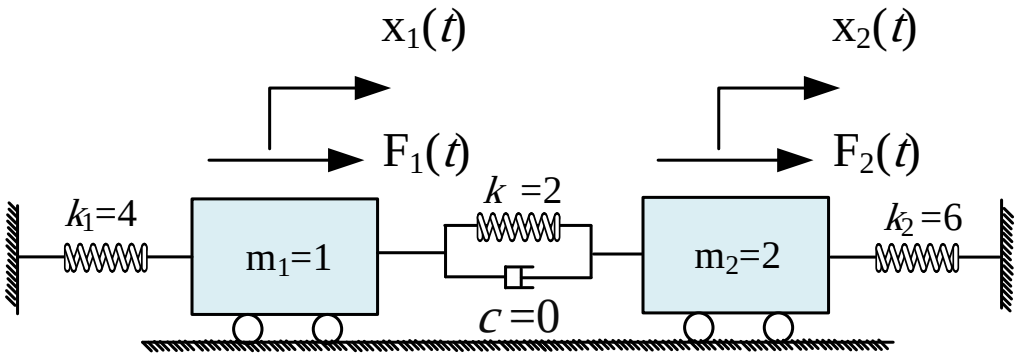
$$\{x\}_{i+1}=\{x\}_{t+\Delta t}=\{x\}_i+\Delta t\{\dot{x}\}_i+\frac{\Delta t^2}{6}\left(\{\ddot{x}\}_{i+1}+2\{\ddot{x}\}_i\right)$$

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EXAMPLE

Solve example with Wilson method.



$$\begin{cases} F_1(t)=0 \\ F_2(t)=10 \end{cases}, \quad \begin{cases} \{x\}_0=\{x(t=0)\}=0 \\ \{\dot{x}\}_0=\{\dot{x}(t=0)\}=0 \end{cases}$$

$$\{\ddot{x}\}_0=\begin{Bmatrix}0\\5\end{Bmatrix}, \qquad \Delta t=0.24216 \text{ sec}$$

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EXAMPLE

Time ($t_i=i\Delta t$)	Displacement	Time ($t_i=i\Delta t$)	Displacement
$t_1=\Delta t=0.24216$	$\begin{Bmatrix}0.0033\\0.1392\end{Bmatrix}$	$t_6=6\Delta t$	$\begin{Bmatrix}0.8074\\2.4981\end{Bmatrix}$
$t_2=2\Delta t$	$\begin{Bmatrix}0.0289\\0.5201\end{Bmatrix}$	$t_7=7\Delta t$	$\begin{Bmatrix}1.1035\\2.6191\end{Bmatrix}$
$t_3=3\Delta t$	$\begin{Bmatrix}0.1072\\1.0579\end{Bmatrix}$	$t_8=8\Delta t$	$\begin{Bmatrix}1.3158\\2.5056\end{Bmatrix}$
$t_4=4\Delta t$	$\begin{Bmatrix}0.2649\\1.6408\end{Bmatrix}$	$t_9=9\Delta t$	$\begin{Bmatrix}1.3688\\2.1929\end{Bmatrix}$
$t_5=5\Delta t$	$\begin{Bmatrix}0.5076\\2.1529\end{Bmatrix}$	$t_{10}=10\Delta t$	$\begin{Bmatrix}1.2183\\1.7503\end{Bmatrix}$

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5.9.3. Newmark Method

$$\{\dot{x}\}_{t+\Delta t} = \{\dot{x}\}_t + \left[(1-\beta)\{\ddot{x}\}_t + \beta\{\ddot{x}\}_{t+\Delta t}\right]\Delta t \qquad (1)$$

$$\{x\}_{t+\Delta t} = \{x\}_t + \{\dot{x}\}_t(\Delta t) + \left[\left(\frac{1}{2}-\alpha\right)\{\ddot{x}\}_t + \alpha\{\ddot{x}\}_{t+\Delta t}\right]\Delta t^2 \qquad (2)$$

If $\alpha = \frac{1}{2}$ & $\beta = \frac{1}{2}$ Wilson Method(Linear Acc. Method)

If $\alpha = \frac{1}{4}$ & $\beta = \frac{1}{2}$ (Averaging Acceleration Method)

$$\{\dot{x}\}_{t+\Delta t} = \{\dot{x}\}_{i+1} \& \quad \{x\}_{t+\Delta t} = \{x\}_{i+1}$$

$$\{\dot{x}\}_t = \{\dot{x}\}_i \quad \& \quad \{x\}_t = \{x\}_i$$

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(Equation of motion at t+Δt)

$$[M]\{\ddot{x}\}_{i+1}+[C]\{\dot{x}\}_{i+1}+[K]\{x\}_{i+1}=\{F\}_{i+1}$$

$$\{\ddot{x}\}_{i+1}=\frac{1}{\alpha(\Delta t)^2}\left(\{x\}_{i+1}-\{x\}_i-\{\dot{x}\}_i(\Delta t)-(\Delta t)^2\left(\frac{1}{2}-\alpha\right)\{\ddot{x}\}_i\right)$$

$$\{\dot{x}\}_{i+1}=\{\dot{x}\}_i+(1-\beta)\Delta t\{\ddot{x}\}_i$$

$$+\frac{\beta\Delta t}{\alpha(\Delta t)^2}\left(\{x\}_{i+1}-\{x\}_i-\{\dot{x}\}_i(\Delta t)-(\Delta t)^2\left(\frac{1}{2}-\alpha\right)\{\ddot{x}\}_i\right)$$

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The recurrence formula:

$$\left([K]+\frac{1}{\alpha(\Delta t)^2}[M]+\frac{\beta}{\alpha\Delta t}[C]\right)\{x\}_{i+1}=\{F\}_{i+1}+$$

$$[M]\left(\frac{1}{\alpha(\Delta t)^2}\{x\}_i+\frac{1}{\alpha\Delta t}\{\dot{x}\}_i+\left(\frac{1}{2\alpha}-1\right)\{\ddot{x}\}_i\right)+[C]\left(\frac{\beta}{\alpha\Delta t}\{x\}_i+\left(\frac{\beta}{\alpha}-1\right)\{\dot{x}\}_i+\frac{\Delta t}{2}\left(\frac{\beta}{\alpha}-2\right)\{\ddot{x}\}_i\right)$$

$$\{\ddot{x}\}_{i+1}=\frac{1}{\alpha(\Delta t)^2}(\{x\}_{i+1}-\{x\}_i)-\frac{1}{\alpha\Delta t}\{\dot{x}\}_i-(\frac{1}{2\alpha}-1)\{\ddot{x}\}_i$$

$$\{\dot{x}\}_{i+1}=\{\dot{x}\}_i+\Delta t(1-\beta)\{\ddot{x}\}_i+\beta\Delta t\{\ddot{x}\}_{i+1}$$

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The Newmark Procedure

1-Calculating [K] , [M] , [C]

2-Using Initial Conditions

$$\{x(t=0)\} = \{x\}_0 \quad \& \quad \{\dot{x}(t=0)\} = \{\dot{x}\}_0$$

$$\{\ddot{x}(t=0)\} = [M]^{-1}(\{F\}_0 - [C]\{\dot{x}\}_0 - [K]\{x\}_0)$$

3- The time step (Δt) is chosen based on accuracy consideration only (unconditionally stable)

Choosing :α , β

$$a_0 = \frac{1}{\alpha(\Delta t)^2} \qquad a_1 = \frac{\beta}{\alpha(\Delta t)} \qquad a_2 = \frac{1}{\alpha(\Delta t)} \qquad a_3 = \frac{1}{2\alpha} - 1$$

$$a_4 = \frac{\beta}{\alpha} - 1 \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\beta}{\alpha} - 2\right) \qquad a_6 = \Delta t(1-\beta) \qquad a_7 = \beta(\Delta t)$$

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4- $[\tilde{K}]=[K]+a_0[M]+a_1[C]$ The effective stiffness matrix

5-$$\{\tilde{F}\}_{i+1}=\{F\}_{i+1}+[M](a_0\{x\}_i+a_2\{\dot{x}\}_i+a_3\{\ddot{x}\}_i)$$
$$+[C](a_1\{x\}_i+a_4\{\dot{x}\}_i+a_5\{\ddot{x}\}_i)$$

The effective force vector

6- $[\tilde{K}]\{x\}_{i+1}=\{\tilde{F}\}_{i+1}$ Displacement

$\{\dot{x}\}_{i+1}=\{\dot{x}\}_i+a_6\{\ddot{x}\}_i+a_7\{\ddot{x}\}_{i+1}$ Velocity

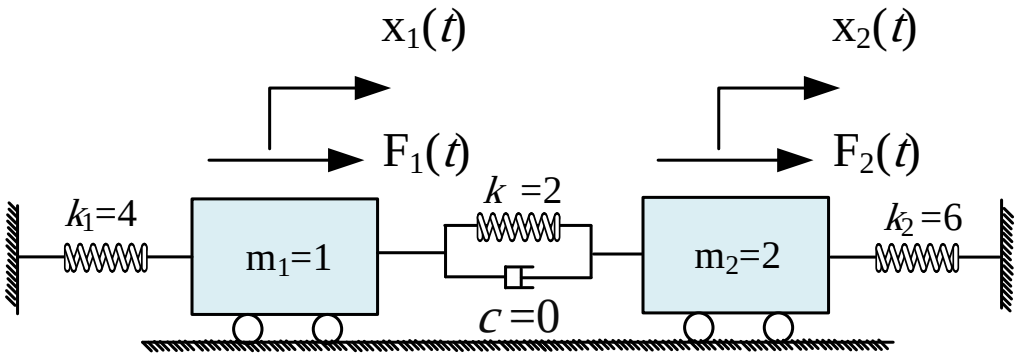
$\{\ddot{x}\}_{i+1}=a_0(\{x\}_{i+1}-\{x\}_i)-a_2\{\dot{x}\}_i+a_3\{\ddot{x}\}_i$ Acceleration

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EXAMPLE

Solve with Newmark method.



$$\begin{cases} F_1(t)=0 \\ F_2(t)=10 \end{cases}, \quad \begin{cases} \{x\}_0=\{x(t=0)\}=0 \\ \{\dot{x}\}_0=\{\dot{x}(t=0)\}=0 \end{cases}$$

$$\{\ddot{x}\}_0=\begin{Bmatrix}0\\5\end{Bmatrix}, \qquad \Delta t=0.24216 \text{ sec}$$

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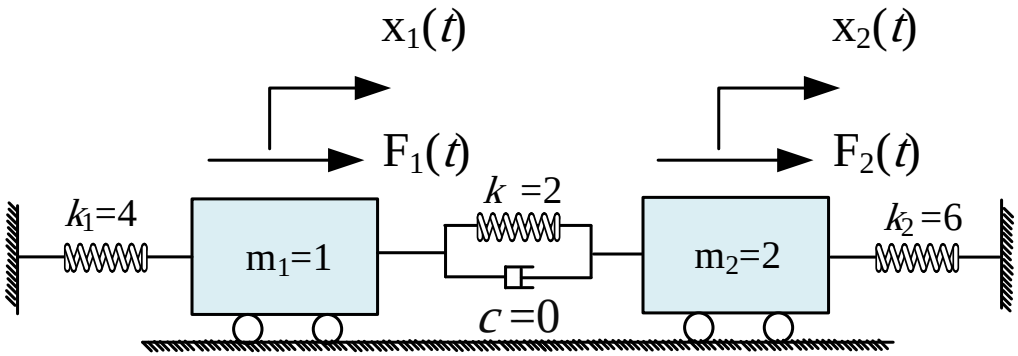
EXAMPLE

Time ($t_i=i\Delta t$)	Displacement	Time ($t_i=i\Delta t$)	Displacement
$t_1=\Delta t=0.24216$	$\begin{Bmatrix}0.0026\\0.1411\end{Bmatrix}$	$t_6=6\Delta t$	$\begin{Bmatrix}0.8530\\2.5336\end{Bmatrix}$
$t_2=2\Delta t$	$\begin{Bmatrix}0.0246\\0.5329\end{Bmatrix}$	$t_7=7\Delta t$	$\begin{Bmatrix}1.1730\\2.6229\end{Bmatrix}$
$t_3=3\Delta t$	$\begin{Bmatrix}0.1005\\1.0884\end{Bmatrix}$	$t_8=8\Delta t$	$\begin{Bmatrix}1.3892\\2.4674\end{Bmatrix}$
$t_4=4\Delta t$	$\begin{Bmatrix}0.2644\\1.6870\end{Bmatrix}$	$t_9=9\Delta t$	$\begin{Bmatrix}1.4134\\2.1137\end{Bmatrix}$
$t_5=5\Delta t$	$\begin{Bmatrix}0.5257\\2.2027\end{Bmatrix}$	$t_{10}=10\Delta t$	$\begin{Bmatrix}1.1998\\1.6426\end{Bmatrix}$

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Comparison of methods



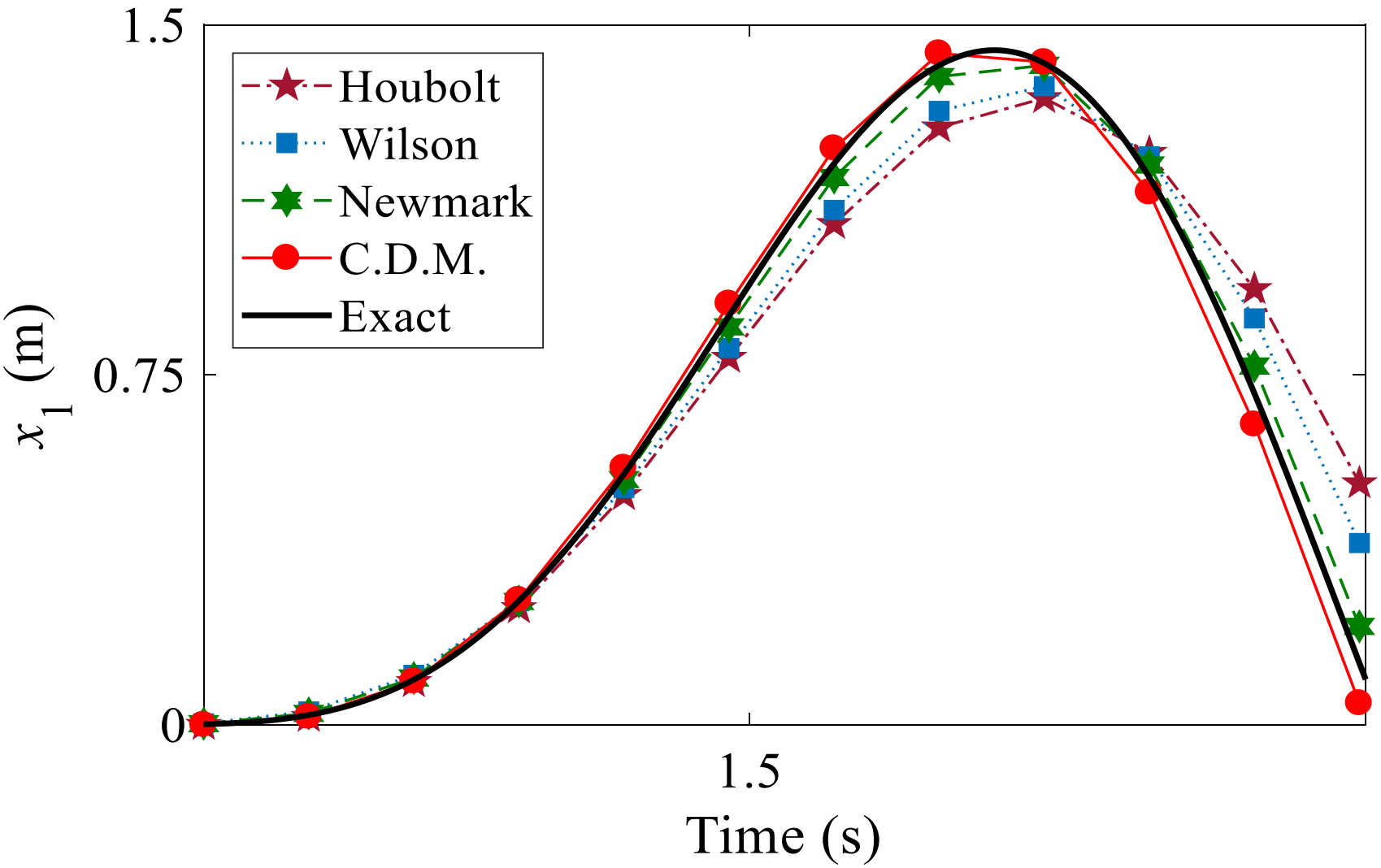
$$\begin{cases} F_1(t)=0 \\ F_2(t)=10 \end{cases}, \quad \begin{cases} \{x\}_0=\{x(t=0)\}=0 \\ \{\dot{x}\}_0=\{\dot{x}(t=0)\}=0 \end{cases}$$

$$\{\ddot{x}\}_0=\begin{Bmatrix}0\\5\end{Bmatrix}, \qquad \Delta t = 0.24216 \text{ sec}$$

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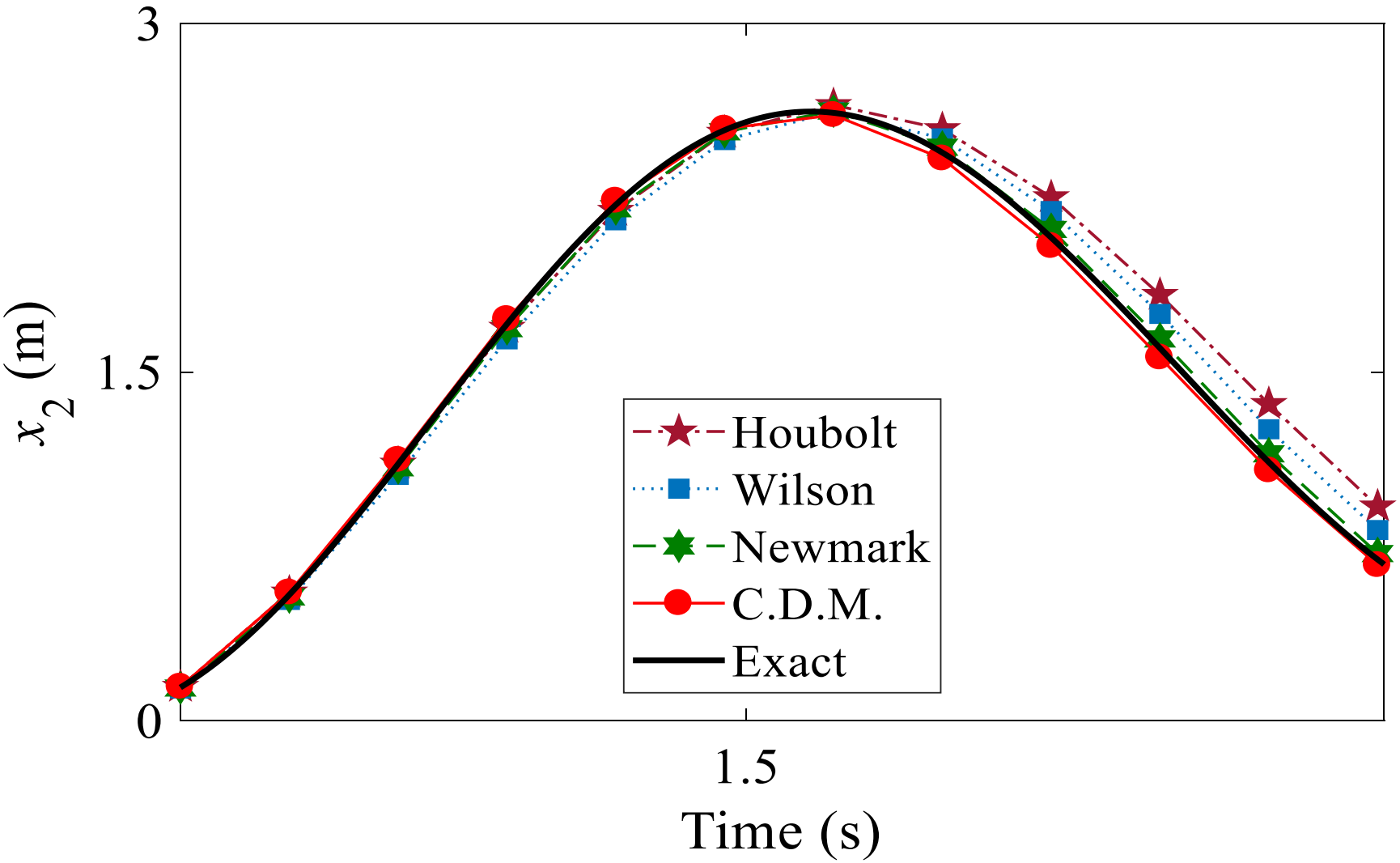
Comparison of methods



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Comparison of methods



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Advanced Vibration

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$$\varphi_1 = \alpha \quad , \quad \varphi_2 = -\alpha \quad , \quad \varphi_3 = +i\,\beta \quad , \quad \varphi_4 = -i\,\beta$$

$$Y = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 \cos \beta y + C_4 \sin \beta y$$

$$\omega_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho h}} \qquad m, n = 1, 2, \ldots.$$

$$w(x, y) = \sin \frac{m\pi}{a} x \ \sin \frac{n\pi}{b} y$$

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Beam

$$\omega^2 = R(y) = \frac{\int_0^l E\ I(x) \left[\frac{d^2 y(x)}{dx^2}\right]^2 dx}{\int_0^l \rho(x)\, y^2(x)\, dx}$$

Shaft

$$\omega^2 = R(\theta) = \frac{\int_0^l G\ J(x) \left[\frac{d\theta}{dx}\right]^2 dx}{\int_0^l I(x)\ \theta^2(x)\, dx}$$

Bar

$$\omega^2 = R(u) = \frac{\int_0^l E\ A(x) \left[\frac{du(x)}{dx}\right]^2 dx}{\int_0^l \rho(x) u^2(x)\, dx}$$

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shaft:

$$T_{n-1}^{R} = T_{n}^{L} \quad , \quad \theta_{n}^{L} - \theta_{n-1}^{R} = \frac{T_{n-1}^{R}}{K_{n}}$$

The field matrix

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n}^{L} = \begin{bmatrix} 1 & 1/K \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

$$\begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n}^{R} = \begin{bmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{bmatrix}_{n} \begin{bmatrix} 1 & 1/K \\ 0 & 1 \end{bmatrix}_{n} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

$$= \begin{bmatrix} 1 & 1/K \\ -J\omega^2 & 1-J\omega^2/K \end{bmatrix}_{n} \begin{Bmatrix} \theta \\ T \end{Bmatrix}_{n-1}^{R}$$

$$\{Z\}_{n}^{R} = [H]_{n} \{Z\}_{n-1}^{R}$$ The recurrence formula

$$\{Z\}_{n}^{R} = [H_n \; H_{n-1} \; H_{n-2} \; \dots \; H_2 \; H_1]_{n} \{Z\}_{0}^{R}$$

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